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Critical phenomena and thermodynamic geometry of Reissner-Nordström-anti-de Sitter black holes

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The phase transition of Reissner-Nordström black holes in $(n + 1)$-dimensional anti–de Sitter spacetime is studied in detail using the thermodynamic analogy between a RN-AdS black hole and a van der Waals liquid-gas system. We first investigate critical phenomena of the RN-AdS black hole. The critical exponents of relevant thermodynamical quantities are evaluated. We find identical exponents for a RN-AdS black hole and a van der Waals liquid-gas system. This suggests a possible universality in the phase transitions of these systems. We finally study the thermodynamic behavior using the equilibrium thermodynamic state space geometry and find that the scalar curvature diverges exactly at the van der Waals–like critical point where the heat capacity at constant charge of the black hole diverges.

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I. INTRODUCTION

The black hole is one of the most interesting objects in physics. The study of black hole thermodynamics [1,2] is therefore quite important. Black holes are indeed thermodynamical objects with a physical temperature and an entropy. It has been known over the past few decades that the thermodynamics of black holes provides an important tool for understanding several issues involving quantum theories of gravity. These have been intensely discussed in the recent past. However, there is no microscopic or statistical description behind their thermodynamical behavior, although thermodynamic studies of black holes do indicate extremely rich phase structures and critical phenomena in these systems. We can consider black holes as states in a thermodynamical ensemble and we can study phase transition in black holes. A well-known example comes from Hawking and Page [3]. Motivated by these ideas, much work has been done on the phase structure of black holes, and quite rich phase structure and critical phenomena have been found [4,5].

Recently, the study of phase transitions of black holes in asymptotically anti–de Sitter (AdS) spacetime [6–13] has focused much interest since these transitions have been related to holographic superconductivity [14–16] in the context of the AdS/CFT correspondence (see relevant reviews in [17]). In this paper, we first review a thermodynamic analogy between an $(n + 1)$-dimensional Reissner-Nördstrom (RN)-AdS black hole and a van der Waals liquid-gas system first discovered in [7,8]. From this analogy we calculate the critical exponents of relevant thermodynamical quantities and discuss the scaling symmetry of the free energy. Our main result is that we find all critical exponents of the $(n + 1)$-dimensional RN-AdS black hole are the same as van der Waals liquid-gas system and are independent of the spacetime dimension. This means there maybe exist some possible universality in the phase transitions of these systems. Based on our results, we find that the critical exponents of the four-dimensional RN-AdS black hole in [8] have some errors. In [8], the author has not used the standard method [18] to calculate the value of β and this may be the reason why the author got the wrong value. Furthermore we study the phase transition using a geometrical perspective of equilibrium thermodynamics. This approach has been developed over the last few decades [12,13,19–21]. We find the scalar curvature diverges precisely at the van der Waals–like critical point where the heat capacity at constant charge of the black hole diverges.

It is well-known that the Hawking-Page phase transition is a very important phenomenon, which originally described the transition between a stable phase and an unstable phase of an asymptotic AdS black hole [22]. Based on the principle of AdS/CFT correspondence, such a phase transition has been used to describe the confinement/deconfinement phase transition of QCD. Another important black hole phase transition is the Davies phase transition [23], which describes the transition between stable and unstable phases of a charged black hole. In our case, the above two kinds of phase transitions exist when the temperature is above the critical temperature T_c. As the black hole approaches the critical point, these two kinds of phase structure combine to form a van der Waals–like critical point.

This paper is organized as follows. In Sec. II we briefly discuss the thermodynamics of the $(n + 1)$-dimensional RN-AdS black hole, mainly using it to establish our notations and obtain formulas of thermodynamic functions for later use. In Sec. III, we study the critical behavior of the

$(n+1)$-dimensional RN-AdS black hole at the van der Waals–like critical point. All critical exponents have been obtained. Further, in Sec. IV, we study the state space scalar curvature of the $(n+1)$-dimensional RN-AdS black hole in detail. Finally, Sec. V contains a discussion of our results.

II. PHASE STRUCTURE OF A $(n+1)$-DIMENSIONAL RN-ADS BLACK HOLE

The theme of the present section is to give an overview of the singular behavior of the heat capacity at constant charge of a $(n+1)$-dimensional RN-AdS black hole which forms the background of this work. For more details of the spherical case, see [7]. Recently, motivated by the study of holographic superconductivity [14,15], plane symmetric and hyperbola symmetric cases in special dimensions have also been discussed [24].

Now we consider general $(n+1)$-dimensional RN-AdS black holes ($n \geq 3$). The form of the spacetime metric is

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{n-1}^2, \tag{1}$$

where

$$f(r) = k - \frac{8\tilde{\Gamma}M}{r^{n-2}} + \frac{Q^2}{r^{2n-4}} + \frac{r^2}{l^2},$$

we have defined $\tilde{\Gamma} = \Gamma(\frac{n}{2})/(n-1)\pi^{(n/2)-1}$ for convenience, and $\Lambda = -\frac{n(n-1)}{2l^2}$ is the cosmological constant. (Throughout we shall adopt Planck units in which $G = \hbar = c = k_B = 1$, where all symbols have their usual meanings. Here Q is the charge parameter which is equal to the electric charge only in -the four-dimensional case; in the general case there is a dimension-dependent factor between them.) Here $k = 1, 0, -1$ corresponds to the sphere, plane, and hyperbola symmetric cases, respectively, and $d\Omega_{n-1}$ is the metric of the associated $(n-1)$-dimensional base manifold.

The mass of the black hole is given by

$$M = \frac{1}{8\tilde{\Gamma}}\left(kr_+^{n-2} + \frac{Q^2}{r_+^{n-2}} + \frac{r_+^n}{l^2}\right), \tag{2}$$

where r_+ is the value of r at the horizon.

Using the Bekenstein-Hawking formula, we have

$$S = \frac{A_{n-1}}{4} = \frac{\pi}{2(n-1)\tilde{\Gamma}} r_+^{n-1}. \tag{3}$$

It is now possible to determine the other thermodynamic entities using the basic thermodynamic relations

$$\delta M = T\delta S + \Phi \delta Q. \tag{4}$$

These are defined as

$$T = \left(\frac{\partial M}{\partial S}\right)_Q = \frac{1}{4\pi} \frac{-\frac{2\Lambda}{n-1} r_+^{2n-2} + (n-2)kr_+^{2n-4} - (n-2)Q^2}{r_+^{2n-3}}, \tag{5}$$

$$\Phi = \left(\frac{\partial M}{\partial Q}\right)_S = \frac{1}{4\tilde{\Gamma}} \frac{Q}{r_+^{n-2}}, \tag{6}$$

$$C_Q = T\left(\frac{\partial S}{\partial T}\right)_Q = \frac{2\pi^2}{\tilde{\Gamma}} \frac{r_+^{3n-4}T}{-\frac{2\Lambda}{n-1} r_+^{2n-2} - (n-2)kr_+^{2n-4} + (n-2)(2n-3)Q^2}, \tag{7}$$

where Φ is the potential difference between the horizon and infinity, T is the Hawking temperature, S is the entropy, and C_Q is the heat capacity at constant charge of the black hole.

For a nonextreme black hole, it can be seen from (7) that C_Q is always positive and regular for the $k = 0, -1$ cases, which tells us that there is no phase transition happening. However, for the spherically symmetric case ($k = 1$), C_Q will become singular for a certain set of black hole parameters (M, Q) at which

$$-\frac{2\Lambda}{n-1} r_+^{2n-2} - (n-2)r_+^{2n-4} + (n-2)(2n-3)Q^2 = 0. \tag{8}$$

Considering Eq. (8), we then find that the critical points are given in terms of the radius of the event horizon as r_1, $r_2 (r_1 < r_2)$ when $Q^2 < (-\frac{(n-2)^2}{2\Lambda})^{n-2} \frac{1}{(n-1)(2n-3)} =: Q_c^2$. For the special value $Q^2 = Q_c^2$, the two horizons degenerate, so we denote $r_c := r_1 = r_2 = (n-2)/\sqrt{-2\Lambda}$.

For fixed Q so that $Q^2 < Q_c^2$, $C_Q < 0$ when $r_1 < r_+ < r_2$ and $C_Q > 0$ when $r_+ < r_1$ and $r_+ > r_2$, so across the

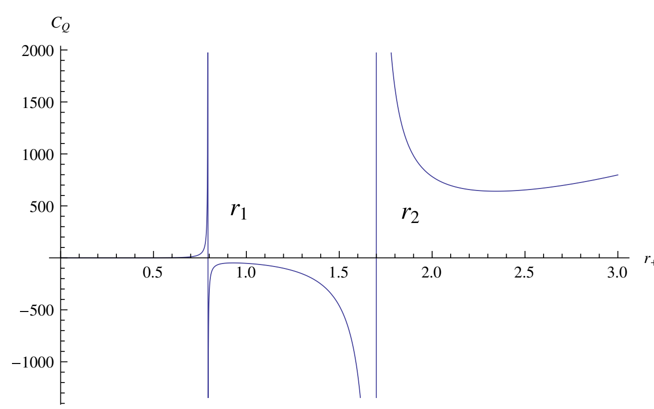

FIG. 1 (color online). Heat capacity at constant charge with r_+ for $Q < Q_c$.

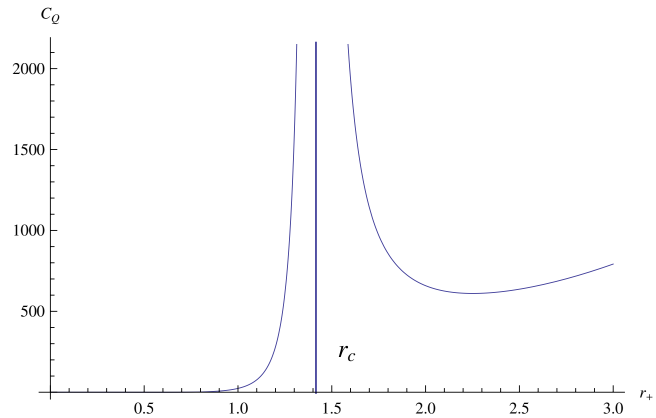

FIG. 2 (color online). Heat capacity at constant charge with r_+ for $Q = Q_c$.

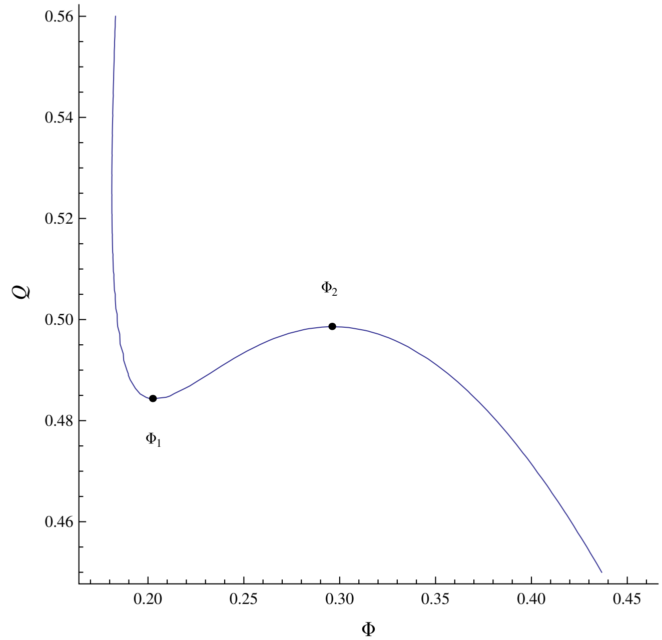

FIG. 3 (color online). The isotherm of a RN-AdS black hole along which $T > T_c$. The local maxima and minima located, respectively, at Φ_1 and Φ_2 are critical points of C_Q. For $\Phi \in (\Phi_1, \Phi_2)$, the black hole is unstable with $(\frac{\partial Q}{\partial \Phi})_T > 0$.

critical points at r_1 and r_2, there is a change of thermodynamic stability of a black hole (see Fig. 1).

When the limit Q^2 approaches the critical value Q_c^2, both r_1 and r_2 degenerate into r_c. In this case, C_Q remains positive and the unstable phase of a black hole disappears (see Fig. 2). When Q^2 is greater than Q_c^2, the heat capacity C_Q of the RN-AdS black hole is always regular.

III. CRITICAL BEHAVIOR AT THE VAN DER WAALS–LIKE CRITICAL POINT

As described in Sec. II, when the charge of a RN-AdS black hole reaches the critical value Q_c, the critical points at r_1 and r_2 degenerate into a single critical point located at r_c. The thermally unstable phase of a RN-AdS black hole disappears (see Fig. 2). The theme of this section is to study the critical thermodynamic behavior of a RN-AdS black hole near r_c. To this end, we shall first review a thermodynamic analogy between a RN-AdS black hole and a van der Waals liquid-gas system. The analogy, though incomplete, will still serve as a very useful guide in the study of the critical behavior of a RN-AdS black hole in the vicinity of r_c.

A. Thermodynamic analogy with a van der Waals liquid-gas system

Given the potential at the event horizon $\Phi = \frac{1}{4\tilde{\Gamma}} \frac{Q}{r_+^{n-2}}$, the equation of state (5) can be rewritten as

$$T = \frac{1}{4\pi} \frac{(n-2)(4\tilde{\Gamma}\Phi)^{2/(n-2)} - (n-2)(4\tilde{\Gamma}\Phi)^{((2n-2)/(n-2))} - \frac{2\Lambda}{n-1} Q^{2/(n-2)}}{(4\tilde{\Gamma}Q\Phi)^{1/(n-2)}}. \tag{9}$$

In terms of the thermodynamical variables (Q, Φ), we have

$$C_Q = \frac{\pi}{2\tilde{\Gamma}} \frac{(n-2)Q^{((n-1)/(n-2))}(4\tilde{\Gamma}\Phi)^{2/(n-2)} - (n-2)Q^{((n-1)/(n-2))}(4\tilde{\Gamma}\Phi)^{((2n-2)/(n-2))} - \frac{2\Lambda}{n-1} Q^{((n+1)/(n-2))}}{(n-2)(2n-3)(4\tilde{\Gamma}\Phi)^{((3n-3)/(n-2))} - (n-2)(4\tilde{\Gamma}\Phi)^{((n+1)/(n-2))} - \frac{2\Lambda}{n-1} Q^{2/(n-2)}(4\tilde{\Gamma}\Phi)^{((n-1)/(n-2))}} \tag{10}$$

and

$$\left(\frac{\partial Q}{\partial \Phi}\right)_T = \frac{Q}{\Phi} \frac{(n-2)(2n-3)(4\tilde{\Gamma}\Phi)^{((2n-2)/(n-2))} - (n-2)(4\tilde{\Gamma}\Phi)^{2/(n-2)} - \frac{2\Lambda}{n-1} Q^{2/(n-2)}}{(n-2)(4\tilde{\Gamma}\Phi)^{((2n-2)/(n-2))} - (n-2)(4\tilde{\Gamma}\Phi)^{2/(n-2)} - \frac{2\Lambda}{n-1} Q^{2/(n-2)}}. \tag{11}$$

It may be inferred from (11) that, like a subcritical isotherm of a van der Waals liquid-gas system in the (P, V) phase plane, an isotherm of a RN-AdS black hole with $T > T_c$ also has a local maxima and minima located, respectively, at Φ_1 and Φ_2. Along the segment of the isotherm between Φ_1 and Φ_2, a RN-AdS black hole is in a thermally unstable phase with $(\frac{\partial Q}{\partial \Phi})_T > 0$ (see Fig. 3).

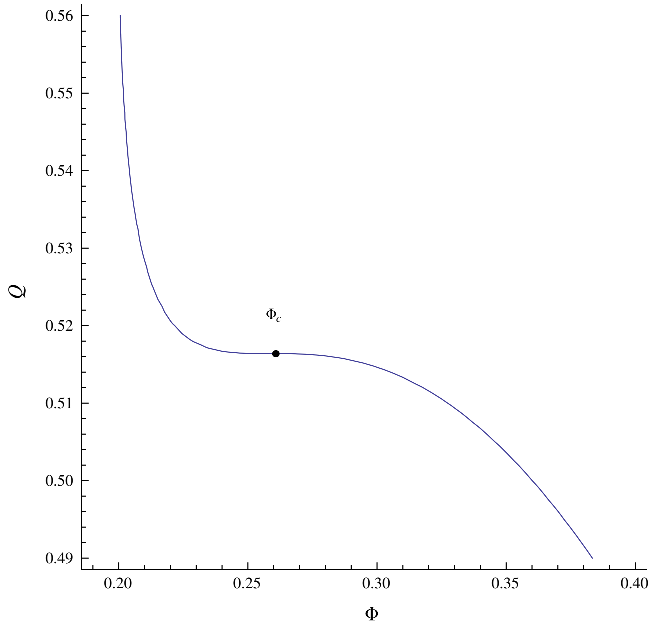

FIG. 4 (color online). The critical isotherm along which $T = T_c$. The point of inflection located at Φ_c is a critical point of C_Q, $C_Q > 0$ along the critical isotherm.

In the limit when T_c is reached, the shape of the isotherm undergoes noticeable change (see Fig. 4) and the critical points located at Φ_1 and Φ_2 on a subcritical isotherm coalesce into a single critical point located at $\Phi_c := \frac{1}{4\tilde{\Gamma}} \frac{Q_c}{r_c^{n-2}} = \frac{1}{4\tilde{\Gamma}}[1/\sqrt{(n-1)(2n-3)}]$ at the critical isotherm. The critical point at Φ_c coincides with that located at r_c on the critical isocharge curve with $Q = Q_c$.

Like the case of the van der Waals liquid-gas system, the critical point at the critical isotherm (along which $T = T_c$) of a RN-AdS black hole is also a point of inflection of the critical isotherm and may be characterized by

$$\left(\frac{\partial Q}{\partial \Phi}\right)\bigg|_c = 0, \qquad \left(\frac{\partial^2 Q}{\partial \Phi^2}\right)\bigg|_c = 0,$$

where the subscript c denotes the corresponding quantity evaluated at the critical point at r_c from now on. In view of the above similarities, if we formally identify the variables (Q, Φ) of a RN-AdS black hole with (P, V) of a van der Waals liquid-gas system, then we see that, at least at a qualitative level, the phase structure of a RN-AdS black hole does bear certain remarkable resemblances to that of a van der Waals liquid-gas system.

B. The introduction of an order parameter

In analogy to a van der Waals liquid-gas system, an order parameter in the RN-AdS context which measures the phase change across the critical point at r_c may also be defined in terms of the Maxwell equal-area law. To do so, in the (Q, Φ) phase plane, fix a subcritical isotherm and draw a horizontal line which intersects the subcritical isotherm at points a, d, b (see Fig. 5) such that the area bounded by the horizontal line segment ad and the isotherm is equal to that bounded by the line segment db and the isotherm.

As in the case of a van der Waals liquid-gas system, define

$$\eta = \Phi_b - \Phi_a \tag{12}$$

as the order parameter to describe the phase change of a RN-AdS black hole near r_c.

C. Critical exponents

Near the critical point at the critical isotherm, the critical behavior of a van der Waals liquid-gas system may be described in terms of

$$\begin{aligned}
&(1)\; P - P_c \sim (V - V_c)^{\delta},\\
&(2)\; \frac{V_g - V_l}{V_c} \sim (-\epsilon)^{\beta},\\
&(3)\; C_P \sim (-\epsilon)^{-\alpha'} \qquad (T < T_c)\\
&\qquad \sim \epsilon^{-\alpha} \qquad (T > T_c),\\
&(4)\; \kappa_T \sim (-\epsilon)^{-\gamma'} \qquad (T < T_c)\\
&\qquad \sim \epsilon^{-\gamma} \qquad (T > T_c).
\end{aligned}$$

With the formal correspondence $(Q, \Phi) \leftrightarrow (P, V)$ as described in the preceding subsection, analogous quantities may also be defined for a RN-AdS black hole. The concrete values of the corresponding critical exponents in the case of a RN-AdS black hole can also be worked out as follows.

1. The degree of the critical isotherm δ

Using the equation of state (9), we have

$$Q^{1/(n-2)} = -\frac{n-1}{4\Lambda}\left(4\pi(4\tilde{\Gamma}\Phi)^{1/(n-2)}T - \sqrt{16\pi^2(4\tilde{\Gamma}\Phi)^{2/(n-2)}T^2 - \frac{8\Lambda(n-2)}{n-1}((4\tilde{\Gamma}\Phi)^{((2n-2)/(n-2))} - (4\tilde{\Gamma}\Phi)^{2/(n-2)})}\right). \tag{13}$$

In order to examine the neighborhood of the critical point, we introduce expansion parameter $\epsilon = (T/T_c) - 1$ and $\omega = (\Phi/\Phi_c) - 1$. In the neighborhood of the critical point, (13) can be written

$$Q = a_{00} + a_{10}\epsilon + a_{01}\omega + a_{11}\epsilon\omega + a_{20}\epsilon^2 + a_{02}\omega^2 + a_{21}\epsilon^2\omega + a_{12}\epsilon\omega^2 + a_{30}\epsilon^3 + a_{03}\omega^3 + \cdots, \tag{14}$$

where $a_{\mu\nu}$ is the coefficient of $\epsilon^{\mu}\omega^{\nu}$ in the expansion.

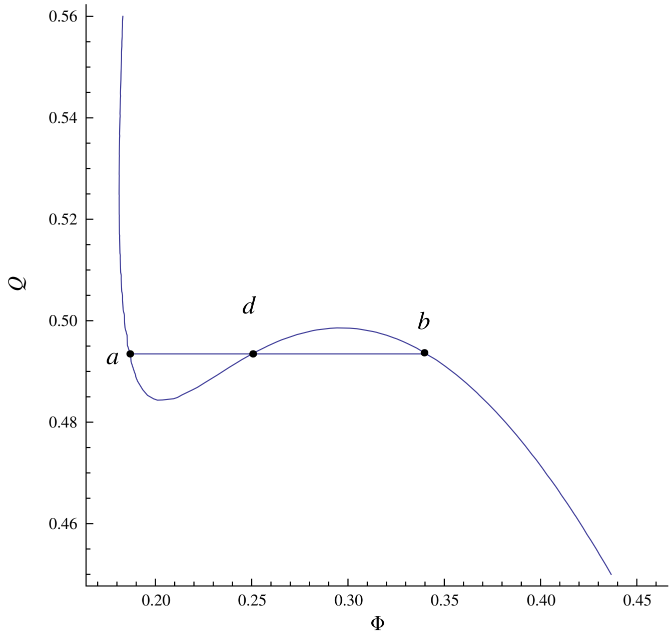

FIG. 5 (color online). A horizontal line is drawn which connects points a and b of the subcritical isotherm. The area bounded by the line segment ad and the isotherm is equal to that bounded by the line segment db and the isotherm.

Let $\epsilon = 0$ in (14). This gives

$$Q = a_{00} + a_{01}\omega + a_{02}\omega^2 + a_{03}\omega^3 + \cdots, \tag{15}$$

where

$$a_{00} = Q_c, \qquad a_{01} = a_{02} = 0, \qquad a_{03} \neq 0.$$

This means

$$\delta = 3.$$

2. The degree of the coexistence curve β

In the neighborhood of the critical point, we have (14). The values of ω on either side of the coexistence curve can be found from the conditions that, along the isotherm,

$$\int_{\Phi_a}^{\Phi_b} \Phi dQ = 0 \tag{16}$$

and

$$Q(\Phi_a) = Q(\Phi_b). \tag{17}$$

Let $\Phi_a = \Phi_c(1 - \omega_a)$ and $\Phi_b = \Phi_c(1 + \omega_b)$. Substituting (14) into (16) and (17), we have

$$a_{11}\epsilon(\omega_b + \omega_a) + a_{21}\epsilon^2(\omega_b + \omega_a) + \tfrac{1}{2}(a_{11} + 2a_{12})\epsilon(\omega_b^2 - \omega_a^2) + a_{03}(\omega_b^3 + \omega_a^3) = 0 \tag{18}$$

and

$$a_{11}\epsilon(\omega_b + \omega_a) + a_{21}\epsilon^2(\omega_b + \omega_a) + a_{12}\epsilon(\omega_b^2 - \omega_a^2) + a_{03}(\omega_b^3 + \omega_a^3) = 0. \tag{19}$$

In order for (18) and (19) to be consistent, we must have $\omega_a = \omega_b$. If we plug this into (18) or (19), we get $\omega_a = \omega_b = \omega$. This gives

$$\omega^2 = -\frac{1}{a_{03}}(a_{11}\epsilon + a_{21}\epsilon^2).$$

Thus,

$$\omega_b \approx \omega_a = \sqrt{-\frac{a_{11}}{a_{03}}\epsilon} = (n-2)\sqrt{6\epsilon}.$$

This means

$$\beta = \tfrac{1}{2}.$$

3. The heat capacity exponent α

From (3), (5), and (6), we have

$$C_\Phi = \frac{\pi}{2\tilde{\Gamma}} \frac{(n-2)Q^{((n-1)/(n-2))}(4\tilde{\Gamma}\Phi)^{2/(n-2)} - (n-2)Q^{((n-1)/(n-2))}(4\tilde{\Gamma}\Phi)^{((2n-2)/(n-2))} - \frac{2\Lambda}{n-1}Q^{((n+1)/(n-2))}}{(n-2)(4\tilde{\Gamma}\Phi)^{((3n-3)/(n-2))} - (n-2)(4\tilde{\Gamma}\Phi)^{((n+1)/(n-2))} - \frac{2\Lambda}{n-1}Q^{2/(n-2)}(4\tilde{\Gamma}\Phi)^{((n-1)/(n-2))}}. \tag{20}$$

From (20), we see that C_Φ display no singular behavior at the critical point. Therefore

$$\alpha = \alpha' = 0.$$

4. The isothermal compressibility exponent γ

Let us compute $(\partial Q/\partial\omega)_\epsilon$. We obtain

$$\left(\frac{\partial Q}{\partial \omega}\right)_\epsilon = a_{11}\epsilon + a_{21}\epsilon^2 + 2a_{12}\epsilon\omega + 3a_{03}\omega^2.$$

For $T < T_c$ one approaches the critical point along the critical isochore. Then setting $\omega = 0$, we obtain

$$\left(\frac{\partial Q}{\partial \omega}\right)_\epsilon = a_{11}\epsilon = 2^{2-(n/2)}(n-2)^{n-2}\sqrt{(n-1)(2n-3)}(-\Lambda)^{1-(n/2)}\epsilon$$

for $\omega = 0$. Therefore

$$\gamma' = 1.$$

For $T > T_c$ one approaches the critical point along the coexistence curve. Then setting $\omega = \sqrt{-(a_{11}/a_{03})\epsilon}$, we obtain

$$\left(\frac{\partial Q}{\partial \omega}\right)_\epsilon = -2a_{11}\epsilon = -2^{3-(n/2)}(n-2)^{n-2}\sqrt{(n-1)(2n-3)}(-\Lambda)^{1-(n/2)}\epsilon$$

for $\omega = \sqrt{-(a_{11}/a_{03})\epsilon}$. Therefore

$$\gamma = 1.$$

D. Scaling symmetry for the Gibbs free energy near criticality

In the case of a van der Waals liquid-gas system, scaling symmetry exists for the singular part of the Gibbs free energy near the critical point located at the critical isotherm and the critical exponents may all be expressed in terms of the two independent homogeneity degrees of the Gibbs energy [25]. In this subsection, we shall show that similar scaling symmetry also exists for a RN-AdS black hole. The behavior of Gibbs free energy near the critical point is similar to the van der Waals system, from which scaling laws for the critical exponents can be derived. Scaling symmetry in the black hole critical phenomena was first discussed in [26] in the context of of Kerr-Newman black holes.

Sufficiently close to r_c, the Gibbs free energy for a RN-AdS black hole may be written as $G = G_r + G_s$. Here G_r is the regular part of the Gibbs free energy whose second order partial derivatives are well behaved at the critical point at r_c, and G_s is the part of the Gibbs free energy responsible for the singular thermodynamic behavior of a RN-AdS black hole near r_c. G_s can be further worked out to be

$$G_s = a\epsilon^2 + b\omega^{4/3} \tag{21}$$

for some constant a, b dependent on Λ. From (21), we find

$$G(\lambda^p\epsilon, \lambda^q\omega) = \lambda G(\epsilon, \omega) \tag{22}$$

with $p = \frac{1}{2}$, $q = \frac{3}{4}$, and λ a real constant. As in the case of a van der Waals liquid-gas system, the critical exponents derived in the previous section can be expressed in terms of p, q as

$$\alpha = 2 - \frac{1}{p}, \qquad \beta = \frac{1-q}{p}, \qquad \gamma = \frac{2q-1}{p}, \qquad \delta = \frac{q}{1-q}. \tag{23}$$

From (23), it may also be seen that the critical exponents in the critical regime of r_c are not independent. They are related by following Eqs. [18,25]:

$$\alpha + 2\beta + \gamma = 2, \qquad \alpha + \beta(\delta - 1) = 2, \qquad \gamma(\delta - 1) = (2-\alpha)(\delta - 1), \qquad \gamma = \beta(\delta - 1). \tag{24}$$

Apart from obtaining the algebraic relations among the critical exponents, (23) or (24) also enable us to give a consistency check of the validity of the critical exponents obtained in Sec. III C.

IV. STATE SPACE SCALAR CURVATURE FOR RN-ADS BLACK HOLE

The thermal geometry method is a very important method to study phase structures of a thermal system. This method was first studied by Weinhold [19] and has been developed over the last few decades [12,13,20,21]. This method can be seen as an independent check of the phase structure which we have obtained in the previous section. The thermodynamic geometry of $(3 + 1)$-dimensional RN-AdS black holes and their thermodynamic instability were discussed in [5,27]. In this section, we will study the critical phenomena of $(n + 1)$-dimensional RN-AdS black holes using thermodynamic geometry. The Hessian of the entropy function (or other thermodynamical potentials) can be thought of as a metric tensor on the state space [12,13,19–21]. In the context of the thermodynamical fluctuation theory, Ruppeiner has argued that the Riemannian geometry of this metric gives insight into the underlying statistical mechanical system. In this picture the occurrence of a van der Waals critical point is connected with the divergence of the state space scalar curvature. The metric as defined by Ruppeiner [20] is given by

$$g_{ij} = -\frac{\partial^2 S(x)}{\partial x^i \partial x^j}, \tag{25}$$

where the coordinates x^i are chosen to be the extensive variables of the system. In fact, it is convenient to use the Weinhold metric which is defined in the following way [19],

$$g^W_{ij} = -\frac{\partial^2 U(x)}{\partial x^i \partial x^j}, \tag{26}$$

where we use U to denote the internal energy. It is well-known that the line elements in Ruppeiner geometry and the Weinhold geometry are conformally related by [28,29]

$$ds^2_R = \frac{1}{T} ds^2_W, \tag{27}$$

where T is the temperature of the RN-AdS black hole. In this picture we will consider $U = M - Q\Phi$, $x^1 = S$, and $x^2 = \Phi$.

From (2), (3), and (5)–(7), we have

$$M = \frac{(n-1)S^{((n-2)/(n-1))}}{4\pi}\left(\frac{\pi}{2(n-1)\tilde{\Gamma}}\right)^{1/(n-1)}\left(1 + 16\tilde{\Gamma}^2\Phi^2 - \frac{2\Lambda}{n(n-1)}\left(\frac{2(n-1)\tilde{\Gamma}S}{\pi}\right)^{2/(n-1)}\right), \tag{28}$$

$$T = \frac{1}{4\pi}\frac{-16(n-2)\tilde{\Gamma}^2(\frac{\pi}{2(n-1)\tilde{\Gamma}})^{2/(n-1)}\Phi^2 + (n-2)(\frac{\pi}{2(n-1)\tilde{\Gamma}})^{2/(n-1)} - \frac{2\Lambda}{n-1}S^{2/(n-1)}}{(\frac{\pi S}{2(n-1)\tilde{\Gamma}})^{1/(n-1)}}, \tag{29}$$

$$C_Q = (n-1)S\frac{-16(n-2)\tilde{\Gamma}^2(\frac{\pi}{2(n-1)\tilde{\Gamma}})^{2/(n-1)}\Phi^2 + (n-2)(\frac{\pi}{2(n-1)\tilde{\Gamma}})^{2/(n-1)} - \frac{2\Lambda}{n-1}S^{2/(n-1)}}{16(n-2)(2n-3)\tilde{\Gamma}^2(\frac{\pi}{2(n-1)\tilde{\Gamma}})^{2/(n-1)}\Phi^2 - (n-2)(\frac{\pi}{2(n-1)\tilde{\Gamma}})^{2/(n-1)} - \frac{2\Lambda}{n-1}S^{2/(n-1)}}. \tag{30}$$

Now using (28) and (29) we can easily calculate the Ruppeiner metric

$$g_{SS} = -\frac{1}{(n-1)S}\frac{-16(n-2)\tilde{\Gamma}^2(\frac{\pi}{2(n-1)\tilde{\Gamma}})^{2/(n-1)}\Phi^2 + (n-2)(\frac{\pi}{2(n-1)\tilde{\Gamma}})^{2/(n-1)} + \frac{2\Lambda}{n-1}S^{2/(n-1)}}{-16(n-2)\tilde{\Gamma}^2(\frac{\pi}{2(n-1)\tilde{\Gamma}})^{2/(n-1)}\Phi^2 + (n-2)(\frac{\pi}{2(n-1)\tilde{\Gamma}})^{2/(n-1)} - \frac{2\Lambda}{n-1}S^{2/(n-1)}},$$

$$g_{S\Phi} = \frac{-32(n-1)\tilde{\Gamma}^2(\frac{\pi}{2(n-1)\tilde{\Gamma}})^{2/(n-1)}S}{-16(n-2)\tilde{\Gamma}^2(\frac{\pi}{2(n-1)\tilde{\Gamma}})^{2/(n-1)}\Phi^2 + (n-2)(\frac{\pi}{2(n-1)\tilde{\Gamma}})^{2/(n-1)} - \frac{2\Lambda}{n-1}S^{2/(n-1)}} = g_{\Phi S},$$

$$g_{\Phi\Phi} = \frac{-32(n-2)\tilde{\Gamma}^2(\frac{\pi}{2(n-1)\tilde{\Gamma}})^{2/(n-1)}\Phi}{-16(n-2)\tilde{\Gamma}^2(\frac{\pi}{2(n-1)\tilde{\Gamma}})^{2/(n-1)}\Phi^2 + (n-2)(\frac{\pi}{2(n-1)\tilde{\Gamma}})^{2/(n-1)} - \frac{2\Lambda}{n-1}S^{2/(n-1)}}.$$

Observe that all the metric components have an identical denominator which appears in the expression of the temperature.

Our concern is the scalar curvature of the Ruppeiner metric, which is

$$R = \frac{C(S,\Phi)}{A(S,\Phi)B^2(S,\Phi)},$$

where

$$A(S,\Phi) = -16(n-2)\tilde{\Gamma}^2\left(\frac{\pi}{2(n-1)\tilde{\Gamma}}\right)^{2/(n-1)}\Phi^2 + (n-2)\left(\frac{\pi}{2(n-1)\tilde{\Gamma}}\right)^{2/(n-1)} - \frac{2\Lambda}{n-1}S^{2/(n-1)}, \tag{31}$$

$$B(S,\Phi) = 16(n-2)(2n-3)\tilde{\Gamma}^2\left(\frac{\pi}{2(n-1)\tilde{\Gamma}}\right)^{2/(n-1)}\Phi^2 - (n-2)\left(\frac{\pi}{2(n-1)\tilde{\Gamma}}\right)^{2/(n-1)} - \frac{2\Lambda}{n-1}S^{2/(n-1)}, \tag{32}$$

and $C(S,\Phi)$ is a regular function whose explicit form is irrelevant to the singular behavior of R. The function $A(S,\Phi)$ is always positive due to the nonextremal condition, as can be seen from (29). The function $B(S,\Phi)$ is identical with the denominator of the heat capacity at constant charge (30). Hence the scalar curvature will diverge exactly at those points at which the heat capacity diverges. It is easy to see that there are two singular points when the temperature is above T_c and these two points coincide as $T = T_c$, so the thermal geometry method gives the same result which we have found in Sec. III.

V. DISCUSSIONS

In the present work, we have obtained different thermodynamic entities like temperature, potential, and heat capacity at constant charge for a $(n+1)$-dimensional RN-AdS black hole from the first law of black hole thermodynamics. The heat capacity shows a divergence at the van der Waals–like critical point. Moreover, we have investigated the critical behavior of the $(n+1)$-dimensional RN-AdS black hole at the van der Waals–like critical point. One of the striking characteristics of the phase transition is the fact that many measures of a system's behavior near a critical point are independent of the details of the interactions between the particles making up the system. The universal features are not only independent of the numerical details of the interparticle interactions, but are also independent of the most fundamental aspects of the structure of the system. The critical exponents of a $(n+1)$-dimensional RN-AdS black hole and a van der Waals liquid-gas system are exactly the same. This result is quite interesting because of the differences in the physical property of the two systems. As we have said in the Introduction, such a van der Waals–like critical point can be seen as forming by the combination of a Hawking-Page critical point and a Davies critical point; both of them are very important and have been studied quite well, so considering the relation between these phase structures must be very helpful to exploring the microcosmic mechanics of the black hole phase transition.

We have also studied the phase transition using the geometrical perspective of equilibrium thermodynamics. We find that both the scalar curvature and the heat capacity at constant charge have a common denominator and hence diverge at identical points. This shows that a divergence in the scalar curvature corresponds to a divergence in the heat capacity at constant charge, thereby suggesting the occurrence of a phase transition. Moreover, there is another factor in the denominator of the scalar curvature which is the same expression arising in the expression for the temperature (29). So we cannot put it equal to zero due to the nonextremal condition. Therefore we can easily get information about the occurrence of a phase transition from the scalar curvature.

We have found that the van der Waals–like critical behavior does not present in the planar or hyperbolic RN-AdS black holes. But we know that there is another type of phase transition associated with a scalar hair in the planar case [30]. Based on the AdS/CFT correspondence, this type of phase transition describes the superconductivity phase transition in the dual boundary system (see [17] and references therein), which has been proved a powerful tool to study superconductivity phenomena. In our case, the boundary system dual to the spherical RN-AdS black hole should also have critical behavior dual to the van der Waals–like critical behavior in the bulk. So it is quite natural to ask whether this dual boundary theory also describes some important, realistic phenomena of phase transition in physics.

We have also observed that RN-AdS black holes do not posses the van der Waals–like phase transition, while the Hawking-Page phase transition can occur in this kind of background [31]. So we see that the asymptotic AdS background is crucial for the van der Waals–like phase transition. It is interesting to investigate the underlying mechanism of this phenomenon.

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