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Consistent 3D quantum gravity on lens spaces

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We study nonperturbative quantization of the first order formulation of three-dimensional gravity with positive cosmological constant (de Sitter space being the prototype vacuum solution, whose Euclideanization gives the three sphere) on the background topology of lens space, which is a three sphere modulo a discrete group. Instead of the strategy followed by a recent work [A. Castro, N. Lashkari, and A. Maloney, Phys. Rev. D **83**, 124027 (2011); A. Castro, N. Lashkari, and A. Maloney Phys. Rev. D **84**, 089904(E) (2011)], which compares results in the second and first order formulations of gravity, we concentrate solely on the latter . We note, as a striking feature, that the quantization which relies heavily on the axiomatic of topological quantum field theory, can only be consistently carried by augmenting the conventional theory by an additional topological term coupled through a dimensionless parameter. More importantly the introduction of this additional parameter renders the theory finite.

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I. INTRODUCTION

Most of the nontrivial results in three-dimensional (3D) gravity, including the famous Banados-Teitelboim-Zanelli BTZ black hole solution, are known for the negative cosmological constant sector. Also, there is a definite trace of AdS/CFT correspondence when the space-time is asymptotically anti-de Sitter (AdS). On the other hand, the study of 3D gravity with a positive cosmological constant has generated considerable interest only recently [1]. This involves evaluation of the 1-loop partition function in the metric formulation in order to find the de Sitter (dS) vacuum, namely, the Hartle Hawking state. They showed for the first time the equivalence of the Chern-Simons (CS) framework of gravity with Einstein theory up to the 1-loop level in the quantum regime. In addition, topologically massive gravity (TMG), which unlike pure gravity consists of propagating modes, has been thoroughly studied in [2]. The main question these studies aim to address is how one can make sense of 3D de Sitter quantum gravity through the vacuum state. Surprisingly, the pure topological gravity theory fails to give any satisfactory answer to it in the sense that the partition function (both in 1-loop and nonperturbative computations) tend to diverge unregularizably when one considers the sum over the infinitely large class of lens spaces (a typical solution of 3D de Sitter gravity, corresponding to saddle points in the path integral); whereas the answer for TMG containing local degrees of freedom is in the affirmative. The latter is tame under the sum over topologies.

The pure gravity and TMG calculations have been considered in the Euclidean signature with the motivation that Euclideanized de Sitter gravity is "thermal." This has been made precise in terms of the Euclidean de Sitter geometry in [1]. Moreover, in the Einstein-Hilbert theory path the integral is sensible in the Euclidean picture. On the other hand, if one prefers to study the theory in the first order formulation, in the Chern-Simons framework, Euclideanization is not an obvious idea that one would come across. This is because CS theory is manifestly topological and does not rely on a background metric as long as perturbative analysis is not the primary goal. But once one tries to make contact with the metric formulation through $\langle e_\mu, e_\nu\rangle = g_{\mu\nu}$, Euclideanization can be viewed from the choice in the internal metric on the frame bundle (of vielbeins), and hence the structure group. This change reflects upon the choice of gauge group of the CS theory. The gauge group changes from noncompact $SO(3,1)$ to compact $SU(2)\times SU(2)$, thus making the problem tractable from the gauge theory perspective. The action then becomes the difference of two $SU(2)$ CS theories.

This is the motivation for us to look at the Euclideanized version. In this case, we do not need a Wick rotation in space-time, and our partition functions keeps the formal expression

$$Z=\int DA\exp\biggl(i\frac{k}{4\pi}\int \mathrm{tr}\biggl(A\wedge dA+\frac{2}{3}A^3\biggr)\biggr),$$

where "tr" stands for the metric over $\mathfrak{su}(2)$. We can see that this form of the path integral will help us in the end so that the trouble of working with imaginary coupling of CS will not also get in our way.[1] Since we would be confined in the first order regime, our concern about the background appears only through its possible topologies. The choice of

[1]In another important work Witten [3] recently pointed out how quantization of the CS theory with complex coupling can be carried out by suitably deforming the functional integral contour. However, for this case one still has to study the possibility of associating a finite dimensional Hilbert space of CS theory on a compact Riemann surface, which we need for quantization here.

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topology is however motivated strongly from the fact that Euclideanized de Sitter space can be identified with S^3, through its metric and topology. We choose it to be of the form S^3/Γ, or lens spaces to be precise. Γ is a suitable discrete group with known action on S^3 as in [1]. Of course feasible solutions are always locally dS.

Now at this point it may seem that we are free to choose any of the standard quantization techniques for this theory. This may involve directly evaluating the partition function or taking recourse to geometric quantization [4]. As is well known, the former is well suited for perturbative calculations, and computation of the determinant of the elliptic operator that arises is well understood in terms of the analytic torsion even for noncompact gauge groups [5]. But once we are interested in nonperturbative results, we must investigate the gauge moduli space of solutions, upon which a suitable canonical quantization may be carried out. However, on the given topology of lens space the solution space modulo the gauge transformations give only a collection of finite points, which certainly is not a symplectic manifold. We therefore use the standard surgery and gluing prescription for the construction of the space and using axioms of TQFT find the partition function as [6][2]

$$Z = \langle \psi | U | \psi \rangle. \tag{1}$$

Here $|\psi\rangle \in \mathcal{H}_{T^2}$ is a state of quantized CS theory on the boundary of a solid torus, gluing two of which we construct a lens space. U is an element of the T^2 mapping class group, specifying which gives us a class of lens spaces. This is where the "conventional wisdom" of viewing first order gravity as the difference between two $SU(2)$ CS theories fails. This failure becomes manifest when one looks at the CS levels $\pm\frac{l}{8G}$ (l being the inverse of the root of the cosmological constant and G the Newton's constant).

But we see that in the famous work of Witten [7], a plausible approach of viewing first order gravity theory as a difference between two CS with *unequal levels* were presented. *Ab initio* this action does not have a metric interpretation. Nevertheless, it gives same equations of motion as that of ordinary CS gravity, which are equivalent to Einstein's equation for the class of invertible vierbeins. The crux is that as one solves the torsionless condition (half of the equations of motion) and substitutes in the action, it becomes metric TMG and gains local excitations. This is very much unlike the case of CS gravity with equal and opposite couplings. However, within the arena of first order gravity alone one could still get back metric interpretation through the construction of a dual conformal field theory (CFT), especially in the negative cosmological constant sector. The first step towards this exciting result was taken in [8] in the metric framework. In a more recent work (although for a negative cosmological constant) [9] this approach has been proved to work well in terms of holomorphically factorizable dual CFTs for CS gravity with unequal couplings. Chiral and antichiral central charges are presented there in terms of the CS couplings (also see [10]) and quantum Banados-Teitelboim-Zanelli black holes are studied. The resulting CFT has been shown to be rich in content in reference to the monster group.

The same theory of gravity as two $SU(2)$ CS with unequal couplings has been studied in [11] where geometrical observables like area and length are quantized for their spectra. This illuminates that one can study quantum theories involving a metric even without starting with a theory of metric variable. The new parameter enters the spectra in a way that makes the spectra physically meaningful.

The problem with equal and opposite couplings is that the CS part corresponding to the negative level is ill-defined and cannot be quantized on T^2 [12]. We need to extend the theory in the way described in [7,9,11,12] so that the couplings of the CS theories can be tuned to be positive. This is a necessary condition since dim ($\mathcal{H}_{T^2}$) equals the product of shifted CS couplings. When both the couplings are positive integers, we get a situation that we regard as *consistent quantization*. At the same time it is worth mentioning that such an extension does not alter the equations of motion. Hence, the gravitational interpretation of the theory remains intact.

Furthermore, due to this extension (through introduction of a new dimensionless parameter) we get a finite answer for the partition function, as opposed to [1]. We exhibit the finiteness explicitly at a certain limit of this new parameter. This is certainly an improvement towards finding an answer about how meaningful 3D de Sitter quantum gravity is.

II. THE EXTENDED THEORY

The functional of the $SU(2)$ vielbein and connection, the conventional Euclidean theory describing first order 3D gravity with positive cosmological constant $\Lambda = \frac{1}{l^2}$ is (in the units where $16\pi G = 1 = c$)

$$S[e,\omega] = 2\int\Big(e^I \wedge (2d\omega_I + \epsilon_{IJK}\omega^J\omega^K) + \frac{1}{3l^2}\epsilon_{IJK} e^I \wedge e^J \wedge e^K\Big). \tag{2}$$

In terms of $SU(2)$ CS connections $A^{(\pm)} = \omega \pm e/l$ this action reads

$$S = l(I[A^{(+)}] - I[A^{(-)}])$$

with

$$I[A] = \int\Big(A^I \wedge dA_I + \frac{1}{3}\epsilon_{IJK}A^I \wedge A^J \wedge A^K\Big)$$

as the integral of the CS form. The variational problem is perfectly well defined on the topology of the lens space S^3/Γ. Equations of motion are the well-known ones:

[2] Choosing framing of surgery suitably.

$$\text{flat CS connections} \quad 2dA_I^{(\pm)} + \epsilon_{IJK}A^{(\pm)J} \wedge A^{(\pm)K} = 0$$

or in terms of variables pertaining to gravity:

$$\text{torsionless condition} \quad de^I + \epsilon^{IJK}e_J \wedge \omega_K = 0 \quad \text{and} \tag{3a}$$

$$\text{curvature equation} \quad 2d\omega^I + \epsilon^{IJK}\omega_J \wedge \omega_K = -\frac{1}{l^2}\epsilon^{IJK}e_J \wedge e_K. \tag{3b}$$

Now the observation that the action

$$\begin{aligned}\tilde{S}[e,\omega] &= 2l\int\Big(\omega^I \wedge d\omega_I + \frac{1}{l^2}e^I \wedge de_I \\ &\quad + \frac{1}{3}\epsilon_{IJK}\omega^I \wedge \omega^J \wedge \omega^K + \frac{1}{l^2}\epsilon_{IJK}\omega^I \wedge e^J \wedge e^K\Big) \\ &= l(I[A^{(+)}] + I[A^{(-)}])\end{aligned} \tag{4}$$

on a closed manifold also gives the same equations of motion (3) motivates one to linearly combine (4) to (2). In terms of the CS variables, one therefore constructs the action with the introduction of a new parameter γ:

$$\tilde{I}[A^{(+)}, A^{(-)}] = S + \frac{1}{\gamma}\tilde{S} = \frac{k_{(+)}}{2\pi}I[A^{(+)}] + \frac{k_{(-)}}{2\pi}I[A^{(-)}], \tag{5}$$

where $k_{(\pm)} = \frac{l(1/\gamma \pm 1)}{8G}$. Here we have restored G so that Einstein's equation is satisfied.

It now calls for a short discussion for interpreting (5). Most interestingly it gives the same equations of motion (3) [for manifolds without boundary], independent of the couplings $k_{(\pm)}$. This feature was first noted in the celebrated paper in [7]. When (3a) is solved for ω and substituted in (3b), one exactly gets the Einstein equation of the metric theory for the invertible class of vierbeins from (3). A more detailed description about this theory and its relationship with Einstein-Hilbert theory and TMG is available in [12,13].

Although the solution space for this extended theory remains same, we find that the phase space structures are different. The presymplectic structure of the theory given in terms of two arbitrary vector fields tangential to the space of solutions is

$$\Omega(\delta_1,\delta_2) = \frac{k_{(+)}}{\pi}\int_\Sigma \delta_1 A^{(+)} \wedge \delta_2 A^{(+)} + \frac{k_{(-)}}{\pi}\int_\Sigma \delta_1 A^{(-)} \wedge \delta_2 A^{(-)}. \tag{6}$$

Σ is a suitable Cauchy foliation of the base manifold. It is clear that the situation $k_{(-)} \to 0$ as $\gamma \to 1$, is comparable to the "chiral point" of the theory in the AdS case, which has a well understood dual CFT. At this point the presymplectic structure automatically becomes degenerate in the $\delta A^{(-)}$ directions (leaving apart its original gauge degeneracy). This degeneracy is evident if one considers the equal Euclidean time Poisson brackets:

$$\begin{aligned}\{\omega_i^I(x,\tau), e_j^J(y,\tau)\} &= 4\pi G\frac{\gamma^2}{\gamma^2-1}\varepsilon_{ij}\delta^{IJ}\delta^2(x,y) \\ \{\omega_i^I(x,\tau), \omega_j^J(y,\tau)\} &= -4\pi G\frac{\gamma/l}{\gamma^2-1}\varepsilon_{ij}\delta^{IJ}\delta^2(x,y) \\ \{e_i^I(x,\tau), e_j^J(y,\tau)\} &= -4\pi G\frac{\gamma l}{\gamma^2-1}\varepsilon_{ij}\delta^{IJ}\delta^2(x,y);\end{aligned} \tag{7}$$

δ^{IJ} is the $\mathfrak{su}(2)$ metric.

III. PROBLEMS WITH CANONICAL QUANTIZATION ON LENS SPACE

Since we are interested in the nonperturbative evaluation of the partition function, the information about the lens space that suffices is its algebraic topology. This is given by[3] $L(p,q) = S^3/\mathbb{Z}_p$. The physical phase space of this theory containing only flat connections, is given by (hom: $\pi_1(L(p,q)) \to SU(2))/\sim$, (moduli space of flat $SU(2)$ connections modulo gauge transformations) where $\sim$ denotes gauge equivalence classes. For the lens space $L(p,q)$, the fundamental group is isomorphic to $\mathbb{Z}_p$, which is freely generated by a single generator, say α; i.e. the group consists of the elements $\{\alpha^n | n = 0,\ldots,p-1\}$. The homomorphisms to $SU(2)$, which we denote by h must satisfy $h[\alpha^p] = (h[\alpha])^p = 1$. In the defining representation (using the freedom of group conjugation) of $SU(2)$, this gives

$$h[\alpha] = e^{2\pi i\sigma_3/p}.$$

Hence, the moduli space consists of only p distinct points and therefore can in no way be a symplectic manifold. In physical terms, these points represent holonomies of the p disjoint noncontractible loops around the p marked points on $L(p,q)$.

In this connection we wish to emphasize that the configuration corresponding to $n = 0$ above, is unique to first order gravity only. It represents the holonomy of the connection $A^{(\pm)} = 0$ or its gauge equivalent class. This means that we are taking the $e = 0 = \omega$ solution in our phase space. These configurations do not give rise to any physically meaningful metric, as elucidated in [9]. But while

[3]Role of $q(\text{mod}\,p)$ coprime to p comes through the action $\mathbb{Z}_p$: $S^3 \to S^3$. This is most easily viewed by considering S^3 as a unit sphere in $\mathbb{C}^2$ and specifying the $\mathbb{Z}_p$ action as $(z_1, z_2) \mapsto (e^{2\pi i/p}z_1, e^{2\pi iq/p}z_2)$.

doing nonperturbative quantization of the first order theory we must include them in the phase space.

IV. APPROPRIATE QUANTIZATION

A. $\mathcal{H}_{T^2}$

Since we have seen that direct attempts to quantize the theory on $L(p, q)$ fails, we should resort to indirect means as exemplified in (1). In this respect, we construct $L(p, q)$ by gluing two solid tori through their boundaries using an element of the mapping class group

$$U = \begin{pmatrix} q & b \\ p & d \end{pmatrix} \in SL(2, \mathbb{Z}). \tag{8}$$

The quantization strategy [6] as outlined in the Introduction requires associating two quantum Hilbert spaces of the CS theory with the boundary of the solid tori. We therefore have to find $\mathcal{H}_{T^2}$. Although this can be found in various places, for example, in [4,12,14,15], for completeness we would like to give a simple and short description of it.

Since we are quantizing CS theory on T^2 (the third dimension may be taken as $\mathbb{R}$, the whole 3 manifold being viewed as a trivial line bundle over T^2), we have as the starting point, the moduli space: (hom: $\pi_1(T^2) \rightarrow SU(2))/\sim$.

Now $\pi_1(T^2) = \mathbb{Z} \oplus \mathbb{Z}$ and is a freely generated Abelian group with two generators α, β having the relation $\alpha\beta\alpha^{-1}\beta^{-1} = 1$. Taking advantage of the group conjugacy as before, we take the two-dimensional (2D) representation of the homomorphism maps as

$$h[\alpha] = e^{i\sigma_3\theta} \quad h[\beta] = e^{i\sigma_3\phi}\theta, \quad \phi \in [-\pi, \pi]. \tag{9}$$

This endows the 2D moduli space $\mathcal{M}$ with the topology of T^2 (parameterized by θ, ϕ). Note that this simple construction of $\mathcal{M}$ is motivated from the rigorous point of viewing it as $\mathcal{M} = T \times T/W$, where T is the torus of maximal dimension (for $SU(2)$ which is 1 and $T = S^1$) and W is the Weyl group with Ad action on the group. Our strategy will be to first quantize $T \times T$ and then take Weyl invariant "parallel" sections of the line bundle on it.

The "pushed down" symplectic structure on $\mathcal{M}$ is

$$\omega = \frac{k}{2\pi} d\theta \wedge d\phi.$$

An appeal to Weil's integrality criterion

$$\int_{\mathcal{M}} \frac{\omega}{2\pi} \in \mathbb{Z} \tag{10}$$

now assures that k must be an integer. At the stage of prequantization a prequantum line bundle is chosen over $\mathcal{M}$ and before choosing the polarization for this line bundle we pick a complex structure τ for $\mathcal{M}$ (induced by that on the surface of the solid torus). This gives us the holomorphic coordinate: $z = \frac{1}{\pi}(\theta + \tau\phi)$ on $\mathcal{M}$. We re-express

$$\omega = \frac{ik\pi}{4\tau_2} dz \wedge d\bar{z}.$$

We thus work with a Kähler structure on $\mathcal{M}$ and a line bundle on it with a connection whose curvature is $-i\omega$. The rest of the prequantization technique can be analogously constructed as given in [12]. This equips us with prequantized Hamiltonian functions $\hat{\theta}' = -\frac{2i}{k+2}\tau\partial_z + \pi z$ and $\hat{\phi}' = \frac{2i}{k+2}\partial_z$. It is important to note the shift of k by the dual Coxeter number of $SU(2)$ to $k + 2$ which originates from the nontrivial Polyakov-Wiegman factor [16] for non-Abelian compact gauge groups. In a more rigorous fashion its appearance is explained due to nonanomalous connection construction on the Hilbert bundle in [4], which guarantees finally the quantum Hilbert space to be independent of the complex structure initially chosen for quantization.

We finally impose the quantization conditions on the polarized wave functions $\psi(z)$[4]:

$$e^{i(k+2)m\hat{\theta}'} e^{-i(k+2)n\hat{\phi}'} \psi(z) = \psi(z).$$

This is solved by level $r = k + 2$ theta functions:

$$\vartheta_{j,r}(z, \tau) = \sum_{n \in Z} \exp\left[2\pi i r\tau\left(n + \frac{j}{2r}\right)^2 + 2\pi i r z\left(n + \frac{j}{2r}\right)\right]$$

with $j = -r + 1, \ldots, r$ (since $\vartheta_{j+2r,r}(z, \tau) = \vartheta_{j,r}(z, \tau)$). We will now construct the Weyl invariant subspace of this $2r$ dimensional vector space. Weyl invariance on $\mathcal{M}$ means identification of z with $-z$.[5] Observing that $\vartheta_{j,r}(-z, \tau) = \vartheta_{-j,r}(z, \tau)$, we project to the Weyl-odd subspace consisting of the $r - 1 = k + 1$ vectors:

$$\vartheta^-_{j,r}(z, \tau) = \vartheta_{j,r}(z, \tau) - \vartheta_{-j,r}(z, \tau) j = 1, \ldots r - 1.$$

As per [4], one should now consider a "quantum bundle" over the space of complex structures τ with fibers as the Hilbert space we have just found. The physical states should be parallel sections of this new bundle with respect to a projectively flat connection of the "quantum bundle." Those vectors turn out to be

$$\psi_{j,k}(z, \tau) = \frac{\vartheta^-_{j+1,r}(z, \tau)}{\vartheta^-_{1,2}(z, \tau)} j = 0, \ldots k. \tag{11}$$

By taking the ratio of two Weyl-odd functions we thus found the Weyl invariant vector space as desired. This space is orthonormal and serves as the required Hilbert space.

[4]The apparent operator ordering ambiguity is unphysical, costing only up to a phase in the wave function

[5]This is so because the traces of the holonomies (9) are gauge invariant rather than $h[\alpha]$, $h[\beta]$ themselves and the traces do not distinguish between (θ, ϕ) and $(-\theta, -\phi)$. This is another statement of Weyl invariance.

B. Gluing and $L(p, q)$

We know that the mapping class group $SL(2, \mathbb{Z})$ or rather $SL(2, \mathbb{Z})/\mathbb{Z}_2$ of T^2 is "generated" by two modular transformation elements T, S. Any general element U of $SL(2, \mathbb{Z})$ can be expressed as

$$U = S\prod_{s=1}^{t-1}(T^{m_s}S).$$

In its 2D representation U produces $L(p, q)$ by gluing two solid tori for [17]

$$U = \begin{pmatrix} q & b \\ p & d \end{pmatrix}.$$

The above representation of U in terms of T, S implies the following identity [6]:

$$p/q = -m_{t-1} + \frac{1}{m_{t-2} - \frac{1}{\cdots - \frac{1}{m_1}}}. \qquad (12)$$

The Chern-Simons-Witten invariant or the partition function is given by [18]

$$Z(r)_{L(p,q)} = \langle \psi_{0,k}|U|\psi_{0,k}\rangle$$

and it is independent of the parameters b, d [6]. From the knowledge of the action of S and T on theta functions we can evaluate these matrix elements. In the canonical 2-framing this was evaluated to be

$$Z(r)_{L(p,q)} = -\frac{i}{\sqrt{2rp}}\exp(6\pi i s(q, p)/r)\sum_{\pm}\sum_{n=1}^{p} \times \exp\left(\frac{2\pi i q r n^2}{p} + \frac{2\pi i n(q \pm 1)}{p} \pm \frac{\pi i}{rp}\right), \qquad (13)$$

where

$$s(q, p) = \sum_{l=1}^{p-1}\frac{l}{p}\left(\frac{lq}{p} - \left[\frac{lq}{p}\right] - \frac{1}{2}\right).$$

is the Dedekind sum defined in terms of the floor function [ ].

C. Sum over topologies and finiteness of the partition function

We note from the construction of $\mathcal{H}_{T^2}$ (11) that the dimension of the Hilbert space is $r_{(\pm)} - 1$ corresponding, respectively, to the "+" type and "−" type CS sectors. This is meaningful only when $r_{(\pm)} - 1 \in \mathbb{N}$ (excluding zero). These conditions come out to be stringent and restrict the parameters of the theory. Since $r_{(\pm)} - 2 = k_{(\pm)} = \frac{l(1/\gamma \pm 1)}{8G}$, we have (when $\hbar$ and c are restored suitably)[6] the following restrictions:

$$a := \frac{l}{8l_p} = s/2s \in \mathbb{N} \quad \text{and} \quad \gamma = \frac{a}{(a-1)+t}t \in \mathbb{N}. \qquad (14)$$

[6] l_p is the 3D Planck length $l_p = G\hbar/c^3$

These restrictions are the prototypes of any topological field theory [14]. One may be tempted to compare these with those appearing in [9] for $k_{(\pm)}$, where the unequal CS parameters are prescribed with discrete values in context of gravity. The apparent difference is due the choice of a different background topology used in [9].

These nontrivial restrictions which validate the quantization (through positivity of the dimension of the Hilbert space) does not allow $\gamma \to \infty$ which was again the starting point of the ordinary theory (2). It is also interesting to see that the set of allowed value of γ also includes 1, the "chiral" point for $t = 1$. This motivates us strongly to study the corresponding chiral limit of the underlying dual CFT, if any.

Leaving those issues for later discussion we now return to our original problem and express the gravity partition function (henceforth by gravity partition function we mean the partition function for the first order gravity) as the product of the partition functions of "+" type and the "−" type theories (5):

$$Z^{\mathrm{Grav}}_{L(p,q)} = Z(r_{(+)})_{L(p,q)}Z(r_{(+)})_{L(p,q)}. \qquad (15)$$

Full gravity partition function would on the other hand be stated after summing over all topologies, i.e.

$$Z^{\mathrm{tot}} = \sum_{p=1}^{\infty}\sum_{\substack{q(\mathrm{mod}\,p)\\(q,p)=1}} Z^{\mathrm{Grav}}_{L(p,q)}.$$

This final sum is where one encounters the divergence as explained in [1] through sums of kind $\sum_{\substack{q(\mathrm{mod}\,p)\\(q,p)=1}} 1 = \phi(p)$, the Euler totient function. For the purpose of comparison with [1] and to study the convergence property of our partition function we choose a particular classical saddle for which the sum over n in (13) is replaced by a particular value of $n = \frac{q\pm1}{2}$, respectively, for the "+" and the "−" type theory instead of taking the corresponding sum in (15). For clarity, further simplification is made through assuming a to take only integral values and $a/\gamma \in 2\mathbb{N}$. However, these simplifications do not alter the final convergence properties of the sum. Using (13) in a more illuminating form[7] we have explicitly:

$$Z^{\mathrm{tot}} = -\frac{1}{2\sqrt{r_{(+)}r_{(-)}}}\sum_{p=1}^{\infty}\frac{1}{p}\sum_{\substack{q(\mathrm{mod}\,p)\\(q,p)=1}}\exp(6\pi i s(q, p)/R_+) \times \exp\left(\frac{\pi i}{p}(2a + (q + q^*)(a/\gamma + 2))\right) \times [e^{(\pi i/pR_+)+(2\pi i/p)(q+1)} + e^{-(\pi i/pR_+)+(2\pi i/p)(q-1)} - e^{(\pi i/pR_-)+(4\pi i/p)} - e^{(-\pi i/pR_-)}], \qquad (16)$$

where

[7] Let A be the set of all such integers $q(\mathrm{mod}\,p)$ with $(q, p) = 1$. It is easy to see that $\{q^*(\mathrm{mod}\,p)|qq^* = 1(\mathrm{mod}\,p)\} = A$. This property has been used.

$$\frac{1}{R_\pm} = \frac{1}{r_{(+)}} \pm \frac{1}{r_{(-)}}.$$

It is now easy to see that all the terms in the q summand are q dependent and the divergence producing totient function does not occur. However since no closed form of the q sum is available, for the purpose of explicit checking we go to the limit where $\gamma > 0$ is small ($\ll 1$). Since the coupling constants become effectively large in this limit the partition function contains the expressions up to one loop. From (14) one observes that this limit is consistent with our quantization program by fixing a and pushing the integer t large. In this limit $\frac{1}{R_+} \sim \frac{2\gamma}{a}$ and $\frac{1}{R_-} \sim \frac{2\gamma^2}{a}$ are both small. Out of the γ terms appearing as polynomials in the exponentials of (16) i.e., $\frac{1}{\gamma}$, 1, γ, γ^2 we keep $\frac{1}{\gamma}$, 1 and neglect the last two. This implies

$$Z^{\rm tot} = -\frac{\gamma}{a}\left(1 - \frac{2\gamma}{a}\right)\sum_{p=1}^{\infty}\frac{1}{p}\,{\rm e}^{2\pi i a/p}\cos(2\pi/p) \times\left[S\left(\frac{a}{2\gamma}+2, \frac{a}{2\gamma}+1; p\right) - {\rm e}^{2\pi i/p}S\left(\frac{a}{2\gamma}+1, \frac{a}{2\gamma}+1; p\right)\right] \tag{17}$$

$$S(\alpha, \beta; p) = \sum_{\substack{q({\rm mod}\,p)\\(q,p)=1}} \exp(2\pi i(\alpha q + \beta q^*)/p).$$

Expanding the exponential and the cosine functions in the inverse power of p, we obtain an infinite series of Kloosterman zeta functions defined by

$$L(m, n; s) = \sum_{p=1}^{\infty} p^{-2s} S(m, n; p).$$

The Kloosterman zeta function is again analytic in the region $\Re s > 1/2$.

Now, as we are in the *small* γ regime, the summand in (17) can well be approximated as

$$\sum_{p=1}^{\infty}\frac{1}{p}\,{\rm e}^{2\pi i a/p}\cos(2\pi/p)\left(1-{\rm e}^{2\pi i/p}\right)S\left(\frac{a}{2\gamma}, \frac{a}{2\gamma}; p\right) = \sum_{m,n,r=0}^{\infty}\frac{(2\pi i)^{r+n+2m+1}}{r+1}\frac{a^n}{(2m)!n!r!}\sum_{p=1}^{\infty}p^{-(r+n+2m+2)} \times S\left(\frac{a}{2\gamma}, \frac{a}{2\gamma}; p\right) = \sum_{m,n,r=0}^{\infty}\frac{(2\pi i)^{r+n+2m+1}}{r+1}\frac{a^n}{(2m)!n!r!} \times L\left(\frac{a}{2\gamma}, \frac{a}{2\gamma}; \frac{r+n+2m}{2}+1\right). \tag{18}$$

The good news is that we get a series of $L(\frac{a}{2\gamma}, \frac{a}{2\gamma}; s)$ with $s \geq 1$. Hence, the partition function is free from divergences. Had we set $a/\gamma + 2 = 0$, the second Kloosterman sum would have reduced to the totient function. That is a potential source of singularity, which is obvious since its zeta function is expressed in terms of the Riemann zeta function and $\zeta(1)$ is singular. We again see that the finiteness of the parameter γ saves us from having a meaningless quantization.

Here we wish to point out that we are evaluating the partition function in the case of small γ. This again corresponds to large CS couplings $k_{((\pm))}$. However, quantum CS theory dictates that large coupling means quantum correction first [6]. In that sense (17) or (18) corresponds to the one loop result.

V. THE METRIC COUNTERPART AND THE TMG STORY

The key relation connecting the first order formalism and metric regime is $\langle e_\mu, e_\nu\rangle = g_{\mu\nu}$. It should be supplemented with the torsionless condition ensuring the geometry to be Riemannian. If one starts with the action (5), one gets this condition (3b) as an equation of motion. Solving this equation makes (4) the well-known gravitational Chern-Simons and (2) the Einstein-Hilbert action provided we use only the invertible subset of vierbeins from (3b). The action (5) becomes TMG, with γ playing the role of topological mass. It is not surprising that dynamics of TMG and that of (5) are quite different; including equations of motion and canonical structures. The most important feature perhaps is that TMG has a local degree of freedom that is absent in the theory described by (5) and one should not expect similarity in their quantum theories. However, TMG being the closest kin to our theory in metric version, for completeness we present a comparative study with quantum TMG focussing its convergence properties as worked out in detail in [2].

To be more precise, we first focus on what is meant by quantum dS TMG. This issue, as we have already mentioned, has been exhaustively studied in [2]. The one loop partition function is shown there to converge. If we denote the contributions coming from Einstein-Hilbert theory by E and the contributions from the massive modes by MG (massive graviton), their result shows that

$$\sum_{p=1}^{\infty}\sum_{\substack{q({\rm mod}\,p)\\(q,p)=1}} Z_E^{(0)}Z_{\rm MG}^{(0)}Z_E^{(1)} \sim \sum_{r=0}^{\infty}\frac{(2\pi a)^r}{r!}L\left(\frac{a}{2\gamma}, \frac{a}{2\gamma}; \frac{r}{2}+\frac{1}{2}\right) + \text{trivially analytic terms.} \tag{19}$$

One can now compare this with (18). The interesting fact is that here the term corresponding to $r = 0$ in the sum of the right-hand side is the source of divergence since it corresponds to Kloosterman zeta function with $s = 1/2$. But it is also showed in [2] that when one includes $Z_{\rm MG}^{(1)}$ as the product and then performs the sum over p, the divergence is eaten up. This means that up to one loop calculation they have

$$Z = \sum_{p=1}^{\infty} \sum_{\substack{q(\mathrm{mod}\,p)\\(q,p)=1}} Z_E^{(0)} Z_{\mathrm{MG}}^{(0)} Z_E^{(1)} Z_{\mathrm{MG}}^{(1)}.$$

The expression of $Z_{\mathrm{MG}}^{(1)}$ as given in [2] is far too complicated for the above expression to be analytically simplified and compared with (18). But the mechanism through which the divergence in the above expression is controlled by $Z_{\mathrm{MG}}^{(1)}$ is very similar to the way in which we showed (18) to be finite. In essence both our topological theory of gravity and TMG (dynamical) have finite and *similarly* convergent partition functions. Since these theories are classically different, this fact seems to be quite surprising. That TMG is derived as a metric version of our theory may however qualitatively explain this similarity in partition functions up to one loop. We conclude that although the finiteness of TMG could be ascribed to its propagating graviton modes, our theory (5), being devoid of massive gravitons still yields a reasonably similar convergent partition function.

VI. CONCLUSION

Our analysis can be summarized as follows:

(1) Construction of the associated Hilbert spaces on the torus surfaces is correct only for finite γ. These constructions spell out the set of allowed values of γ and this does not include $\gamma \to \infty$.

(2) That finite values of γ can make the partition function divergence free is shown explicitly for $\gamma \ll 1$. This is most important from the point of view of the quantization of lens space gravity.

The fact that pure Einstein gravity has a divergent partition function even at one loop and TMG is finite may seem to be a lucrative point of discussion in the context of the work we present here. One can pass over to TMG (essentially dynamical) from action (5), which is topological, by imposing the torsionless condition. Hence, they share the same parameter content. In the AdS sector, however, this similarity is more pronounced as they have same dual CFTs. Whereas in the present case, such an analogy is premature, since dual CFT in 3D de Sitter gravity is yet to be understood. Any progress on this front would surely shed light on the proposed dS/CFT [19] correspondence (which works in four dimensions) in three dimensions and on its gravitational interpretation.

On the other hand, the finiteness brought in by the gravitational Chern-Simons term of TMG also may be interpreted in light of (5). This being parity odd, there are phases in the partition function. Control of the divergence can be ascribed to this fact. This explanation works in the perturbative regime for TMG at least, as shown in [2]. Our result being finite is in conformity with TMG.

Another point of interest, which we leave for future study, is the interpretation of the theory when $\gamma \to 1$. In the AdS paradigm an analogous point in parameter space has been shown to have critical CFT dual [20,21]. In light of the proposed dS/CFT [19] framework, this may serve as exciting evidence for dual critical CFT.

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