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Internal time, test clocks, and singularity resolution in dust-filled quantum cosmology

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The problem of time evolution in quantum cosmology is studied in the context of a dust-filled, spatially flat Friedmann-Robertson-Walker universe. In this model, two versions of the commonly-adopted notion of internal time can be implemented in the same quantization, and are found to yield contradictory views of the same quantum state: with one choice, the big-bang singularity appears to be resolved, but with another choice it does not. This and other considerations lead to the conclusion that the notion of internal time as it is usually implemented has no satisfactory physical interpretation. A recently proposed variant of the relational-time construction, using a test clock that is regarded as internal to a specific observer, appears to provide an improved account of time evolution relative to the proper time that elapses along the observer's worldline. This construction permits the derivation of consistent joint probability densities for observable quantities, which can be viewed either as evolving with proper time or as describing correlations in a timeless manner. It turns out that the observer whose sense of time originates in this test clock will find herself to be living in a bouncing universe.

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I. INTRODUCTION

It has long been appreciated that time evolution in generally-covariant theories such as general relativity is, on the face of it, a gauge transformation and, contrary to everyday experience, should therefore be unobservable. Reviews of this ''problem of time'' are given, for example, in [1–3], and textbook discussions may be found in [4,5]. In recent years, detailed studies of quantized models of cosmology (see, e.g. [6–10] for reviews) have demanded a practical solution to this problem, and a certain notion of ''internal time'' introduced by Rovelli [11–13] and further developed in, for example, [14–17] has been quite widely adopted. The ''evolving constant of the motion'' construction advocated in these papers is a particular implementation of the idea of relational time, according to which time evolution in covariant theories can be described only relative to the values assumed by some physical quantity that is chosen to serve as a clock. In simple cosmological models, the clock is typically a scalar field, say ϕ, and in the quantum theory one obtains a wave function $\psi(v;\varphi)$, where v is a variable describing the geometry of the model universe, as the solution of a Schrödinger-like equation in which the parameter φ plays the role of time. In this paper, we take v to be the volume of a spatially compact universe, in which case $|\psi(v;\varphi)|^2$ is interpreted as a probability density for the volume at the time when the scalar field ϕ takes the value φ.

Recently [18], we have argued that this interpretation is hard to sustain because, for the reasons summarized in Sec. IVA below, the values assigned to φ cannot be regarded as readings obtained by inspection of the ''clock'' ϕ. Indeed, according to the usual rules of quantum mechanics, the scalar field is not an observable at all, which is especially disconcerting if it constitutes the entire matter content of the universe. In view of this difficulty, we proposed a variant of the relational-time idea, which makes use of a small test clock, regarded as internal to some observer from whose point of view the universe is to be described. In this alternative construction, the test clock is unobservable, and we obtain a wave function[1] $\psi(v, \phi; \tau)$, evolving with a time parameter τ which is not a clock reading, but corresponds classically to the geometrical proper time that elapses along the observer's worldline. This new wave function yields a joint probability density, evolving in textbook fashion with proper time τ, for v and ϕ, which are now genuine observable quantities.

In this paper, we first wish to investigate whether, despite the above-mentioned difficulties, the internal-time wave function $\psi(v;\varphi)$ can be interpreted as expressing a correlation between two quantities (the volume and scalar field), both of which are observable in some suitably broadened sense. Specifically, we ask whether $|\psi(v;\varphi)|^2$ can be regarded as a *conditional* probability density for the volume, given that the scalar field has been determined to have the value φ. To that end, we study a simple model of an homogeneous universe filled with pressureless matter, described, following Brown and Kuchař [19], by a single scalar field. The classical version of this model is introduced in Sec. II, and we find that two complementary notions of internal time can be straightforwardly defined, using either the volume or the scalar field as a clock. These two internal times carry over to the quantized theory, as discussed in Sec. III, and we use them in Sec. IV to

[1]Later on, we will adjust the notation in which various wave functions are expressed, so as to maintain some important distinctions, which will be made precise in due course.

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compute two corresponding probability densities. *If* the conditional-probability interpretation is feasible, then these ought to represent, on the one hand the conditional-probability density for the volume given some value of the scalar field and, on the other hand, the conditional probability for the scalar field given some value of the volume. We find, however, that these probabilities are inconsistent. In fact, the two notions of internal time lead to two mutually contradictory views of time evolution: if the volume is used as internal time, the quantum theory appears to reproduce the classical big-bang singularity, whereas, if the scalar field is used as internal time, *exactly the same quantum state* appears to describe a universe in which the singularity is replaced by a bounce. This result serves to strengthen our conviction that the internal-time formalism, while resulting from perfectly sound mathematical manipulations, has no satisfactory physical interpretation.

In Sec. V, we augment the model by the addition of a test clock. As already indicated, the modified relational-time construction proposed in [18] leads to a joint probability density for the volume and scalar field, both of which are now *bona fide* observables, which evolves, according to a standard Schrödinger equation, with proper time τ. Like the usual time parameter in nonrelativistic physics, τ is not itself observable. In principle, no set of experimental results records the evolution of an observed quantity relative to the time parameter that appears in Newton's laws or in Schrödinger's equation. Rather, what is recorded is an observed correlation between the object of interest and the readings of some time-keeping device, and the rationale underlying the internal-time formalism is that taking this practical fact seriously is the key to solving the problem of time. Consequently, it is of some interest to recover a "timeless" interpretation, by supposing that the observer to whom our unobserved test clock is internal also possesses a time-keeping instrument that *is* observed. This issue is taken up in Sec. V C. For the model at hand, though not in general, it turns out that the role of a time-keeping instrument can conveniently be assigned to the scalar field (but not to the volume). When that is done, we obtain a genuine conditional-probability density for the volume, which coincides, to a good approximation, with the function $|\psi(v;\varphi)|^2$ found from the internal-time formalism with φ as the internal time. In that indirect and approximate sense, the Brown-Kuchař scalar field emerges as a preferred internal time, indicating a bouncing universe.

II. CLASSICAL DUST-FILLED COSMOLOGY

A. The model

As described in detail by Brown and Kuchař [19] (and in earlier work by Brown [20]), an effective hydrodynamic description of pressureless matter, or dust, is furnished by a collection of scalar fields $\{\phi, W_k, Z^k, M\}$ ($k = 1, 2, 3$) with a Lagrangian density of the form

$$\mathcal{L}_{\rm D} = -\tfrac{1}{2}\sqrt{-g}M(g^{\mu\nu}U_\mu U_\nu + 1), \tag{2.1}$$

where the 4-velocity field is $U_\mu = -\partial_\mu\phi + W_k\partial_\mu Z^k$. When the equations of motion are satisfied, the integral curves of U can be interpreted as the worldlines of dust particles, and the scalar field ϕ is linear in the geometrical proper time t that elapses along these worldlines. We consider the case of an homogeneous, spatially flat Friedmann-Robertson-Walker universe, with metric $g = \mathrm{diag}(-N^2, a^2, a^2, a^2)$. To be concrete, we take spatial sections of this universe to be compact, with coordinate volume $\int d^3x = 1$. Homogeneity implies that spatial derivatives of ϕ and Z^k vanish, and variation with respect to W_k leads also to $\partial_0 Z^k = 0$. In this special case, therefore, the dust is modeled by a single scalar field ϕ, along with the Lagrange multiplier M:

$$\mathcal{L}_{\rm D} = \tfrac{1}{2}Na^3M[N^{-2}(\partial_s\phi)^2 - 1], \tag{2.2}$$

where s is an arbitrary time coordinate. The momentum conjugate to ϕ is $p_\phi = N^{-1}a^3M\partial_s\phi$, and variation with respect to M yields the constraint $p_\phi^2 - (a^3M)^2 = 0$, which is second-class, because it fails to commute with the primary constraint $p_M = 0$. Up to a sign, this constraint is trivially solved for M, and we can construct the unconstrained dust Hamiltonian

$$H_{\rm D} = Np_\phi. \tag{2.3}$$

Evidently, p_ϕ is the total energy content of the dust, and we resolve the sign ambiguity by requiring this to be non-negative.

In the standard way, the Einstein-Hilbert action leads to a gravitational Hamiltonian, which we write in terms of the volume $v = a^3$ and its conjugate momentum $p_v = -(12\pi G)^{-1}N^{-1}\partial_s v/v$ as

$$H_{\rm grav} = -N(6\pi G)vp_v^2 =: NC_{\rm grav}, \tag{2.4}$$

G being the usual gravitational constant. The function $C_{\rm grav}$ defined by this equation is the gravitational contribution to the Hamiltonian constraint.

Taking the lapse function $N(s)$ to be a strictly positive, but otherwise arbitrary function, we introduce the invariant proper time

$$t(s) = \int_0^s N(s')ds' \tag{2.5}$$

and express the Hamilton equations of motion as

$$\dot\phi = N^{-1}\partial H_0/\partial p_\phi = 1 \tag{2.6}$$

$$\dot v = N^{-1}\partial H_0/\partial p_v = -(12\pi G)vp_v \tag{2.7}$$

$$\dot p_v = -N^{-1}\partial H_0/\partial v = (6\pi G)p_v^2, \tag{2.8}$$

where the overdot denotes differentiation with respect to t. The total Hamiltonian here is $H_0 = H_{\rm grav} + H_{\rm D}$

(distinguished by its subscript from the Hamiltonian of an extended model to be considered later) and the energy p_ϕ is a constant of the motion. Finally, the Hamiltonian constraint

$$C_0 := \partial H_0/\partial N = -(6\pi G) v p_v^2 + p_\phi = 0 \tag{2.9}$$

reproduces the Friedmann equation.

B. Dirac observables and internal time

The equations of motion (2.6), (2.7), and (2.8) are easily solved. Denoting by $\bar{\phi}(v, p_v, \phi, p_\phi; t)$, etc. the phase-space trajectory that passes through (v, p_v, ϕ, p_ϕ) at $t = 0$, we have

$$\bar{\phi}(v, p_v, \phi, p_\phi; t) = \phi + t \tag{2.10}$$

$$\bar{v}(v, p_v, \phi, p_\phi; t) = v(1 - 6\pi G p_v t)^2 \tag{2.11}$$

$$\bar{p}_v(v, p_v, \phi, p_\phi; t) = p_v(1 - 6\pi G p_v t)^{-1}. \tag{2.12}$$

Heuristically, given initial conditions that satisfy the constraint (2.9), these solutions provide a complete description of the evolution of this simple universe relative to the proper time t that elapses along the worldline of a comoving observer. Classically, this is possible because the equations of motion were derived *before* imposing the constraint. In a quantum-mechanical treatment, this time evolution cannot be reproduced, because the Hamiltonian $\hat{C}_0$ that is supposed to generate it vanishes when acting on a physically allowed state. The same difficulty arises classically in a more formal treatment of the constrained Hamiltonian dynamics. Here, one has to recognize that, while the proper time defined in (2.5) is invariant under reparametrizations of the coordinate s with $N'(s') = N(s)\mathrm{d}s/\mathrm{d}s'$ (the remnant, in this symmetry-reduced model, of the general coordinate invariance of general relativity), it is *not* invariant under an arbitrary change in the undetermined lapse function $N(s)$ unless this is accompanied by a compensating reparametrization. From this perspective, a change in $N(s)$, and hence evolution with respect to t, is a gauge transformation generated by the constraint function C_0. One has the option of fixing a gauge, by specifying once and for all a definite function $N(s)$. Classically, at least, the content of (2.10), (2.11), and (2.12) is independent of the actual function chosen, and can be regarded as physically meaningful. But in a manifestly gauge-independent approach, which we follow here, genuine physical information is carried only by gauge-invariant (Dirac) observables, which commute, in the sense of Poisson brackets, with C_0, and are therefore constants of the motion.

A construction due to Rovelli [11,13] allows us to obtain a 1-parameter family of gauge-invariant quantities—an "evolving constant of the motion"—as follows. Denote by t_φ the time at which $\bar{\phi}$ in (2.10) has the value φ. Then the quantity

$$V(\varphi) := \bar{v}(t_\varphi) = v[1 - 6\pi G p_v(\varphi - \phi)]^2, \tag{2.13}$$

in which we suppress the dependence of $\bar{v}$ on the phase-space coordinates (v, p_v, ϕ, p_ϕ), can be interpreted classically as "the volume at the time when the scalar field has the value φ." It is easy to check explicitly that $\{V(\varphi), C_0\} = 0$ for each value of the parameter φ.

A parameter such as φ is commonly referred to in the literature as a "relational," "emergent" or "internal" time, or simply as "time," the idea being that the scalar field serves as a physical clock, and the function $V(\varphi)$ describes the evolution of the volume with respect to the readings of this clock. In the present case, this may seem especially apt, in view of the linear dependence (2.10) of the scalar field on t. In [19], indeed, this idea is enshrined in the notation: these authors use T for the scalar field that we denote by ϕ. For reasons that will become clear later, however, we wish to maintain a clear distinction between the geometrical proper time defined by (2.5) and a scalar field ϕ whose equation of motion *happens to have the solution* (2.10).

We now wish to define a second family of Dirac observables, taking the volume, rather than the scalar field, as an internal time. This entails solving (2.11) for the time t_ν at which the volume has the value ν, and requires a little care with regard to signs. First, we take the volume always to be positive, in contrast to the loop-quantum-gravity-inspired treatment described in [21,22], where the physical volume is the absolute value of a more fundamental variable, whose sign reflects the orientation of a cotriad [23]. Next, it follows from the equation of motion (2.8) that the evolution preserves the sign of p_v, so the quantity σ that we provisionally identify as $\sigma = -\mathrm{sgn}(p_v)$, is a constant of the motion. Inspection of the solutions (2.11) and (2.12) then reveals that trajectories on which p_v is negative correspond to an expanding universe, starting from an initial singularity at $t = -(6\pi G|p_v|)^{-1}$, while those on which p_v is positive correspond to a contracting universe and terminate at a final singularity at $t = (6\pi G p_v)^{-1}$. Consequently, we can take the square root of (2.11), with $\bar{v} = \nu$, to obtain

$$\nu^{1/2} = v^{1/2} - 6\pi G v^{1/2} p_v t_\nu, \tag{2.14}$$

the sign of the square root being determined unambiguously by the requirement that ν is an increasing function of t_ν when p_v is negative. Thus, the Dirac observable that represents "the value of the scalar field when the volume is ν" is

$$\Phi(\nu) := \bar{\phi}(t_\nu)$$
$$= \phi + (6\pi G p_v)^{-1} - (6\pi G v^{1/2} p_v)^{-1} \nu^{1/2}. \tag{2.15}$$

Again, one may check that $\{\Phi(\nu), C_0\} = 0$ for every value of ν.

Finally, with a view to quantization, we define the Dirac observables

$$V := V(0) = v + 12\pi G v p_v \phi + (6\pi G)^2 v p_v^2 \phi^2 \quad (2.16)$$

$$Y := -(12\pi G)^{-1} V'(0) = v p_v + 6\pi G v p_v^2 \phi \quad (2.17)$$

$$\Phi := \Phi(0) = \phi + (6\pi G p_v)^{-1}, \quad (2.18)$$

and we note that C_{grav}, defined in (2.4), is a constant of the motion, and hence also a Dirac observable. We have the Poisson-bracket relations

$$\{V, Y\} = V, \qquad \{V, C_{\text{grav}}\} = -12\pi G Y,$$
$$\{Y, C_{\text{grav}}\} = C_{\text{grav}}, \qquad \{\Phi, C_{\text{grav}}\} = -1 \quad (2.19)$$

and the evolving observables can be expressed as

$$V(\varphi) = V - 12\pi G Y \varphi - 6\pi G C_{\text{grav}} \varphi^2 \quad (2.20)$$

$$\Phi(\nu) = \Phi + \sigma(-6\pi G C_{\text{grav}})^{-1/2} \nu^{1/2}, \quad (2.21)$$

where we now identify

$$\sigma := -\text{sgn}(v^{1/2} p_v). \quad (2.22)$$

The factor $v^{1/2}$ here, which follows from (2.15) appears inessential, but it will prove convenient to retain it.

We now have two notions of evolution with respect to internal time, and it will be useful to identify the generators of these evolutions. Indeed, we easily discover from (2.19), (2.20), and (2.21) that

$$\frac{dV(\varphi)}{d\varphi} = \{V(\varphi), H_\varphi\} \quad (2.23)$$

$$\frac{d\Phi(\nu)}{d\nu^{1/2}} = \{\Phi(\nu), H_\nu\} \quad (2.24)$$

with

$$H_\varphi := C_{\text{grav}} \quad (2.25)$$

$$H_\nu := \sigma(-C_{\text{grav}}/6\pi G)^{1/2}. \quad (2.26)$$

On any classical trajectory, $\Phi(\nu)$ is just the inverse of $V(\varphi)$, but we shall see that this inverse relationship is not preserved in the quantum theory.

III. QUANTUM DUST-FILLED COSMOLOGY

We consider a quantization scheme of the Wheeler-de Witt type in which, in the first instance, the canonical coordinates (v, p_v, ϕ, p_ϕ) are promoted to operators acting in an auxiliary (or kinematical) vector space. A convenient representation is that in which the operators $\hat{v}$ and $\hat{p}_\phi$ act by multiplication on wave functions $\Psi(v, \epsilon_D)$, while their conjugate variables act by differentiation:

$$\begin{aligned} \hat{v}\Psi(v, \epsilon_D) &= v\Psi(v, \epsilon_D), \\ \hat{p}_\phi \Psi(v, \epsilon_D) &= \epsilon_D \Psi(v, \epsilon_D) \\ \hat{p}_v \Psi(v, \epsilon_D) &= -i\hbar\partial_v \Psi(v, \epsilon_D) \\ \hat{\phi}\Psi(v, \epsilon_D) &= i\hbar\partial_{\epsilon_D} \Psi(v, \epsilon_D). \end{aligned} \quad (3.1)$$

The notation ϵ_D reflects the fact that $\hat{p}_\phi$ corresponds to the energy content of the dust. As in [18] we follow the authors of [21,22] in choosing for the gravitational constraint the operator ordering

$$\hat{C}_{\text{grav}} = -6\pi G \hat{p}_v \hat{v} \hat{p}_v. \quad (3.2)$$

Then the Hamiltonian constraint equation (2.9) reads

$$[(4/\lambda^2)\partial_v v \partial_v + \epsilon_D]\Psi(v, \epsilon_D) = 0, \quad (3.3)$$

where

$$\lambda := \left(\frac{2}{3\pi G\hbar^2}\right)^{1/2}. \quad (3.4)$$

With the definition

$$z := \lambda \epsilon_D^{1/2} v^{1/2}, \quad (3.5)$$

the general solution to this equation can be expressed as

$$\Psi(v, \epsilon_D) = \psi_+(\epsilon_D) H_+(z) + \psi_-(\epsilon_D) H_-(z), \quad (3.6)$$

where, in terms of the usual Hankel functions, we write $H_+(z) = -iH_0^{(2)}(z)$ and $H_-(z) = iH_0^{(1)}(z)$.

The classical phase space consists of two disjoint regions, distinguished by the discrete variable (2.22), containing expanding ($\sigma = 1$) and contracting ($\sigma = -1$) trajectories. [We ignore, for now, the hyperplane $p_v = 0$, on which the volume is constant, but see the comment following (4.14)]. The corresponding operator is

$$\hat{\sigma} = \text{sgn}\left(\frac{i\hbar\lambda\epsilon_D^{1/2}}{2} \frac{\partial}{\partial z}\right), \quad (3.7)$$

and we see from the integral representation

$$H_\pm(z) = \frac{2}{\pi} \int_0^\infty d\xi \exp(\mp iz \cosh\xi) \quad (3.8)$$

that $H_+(z)$ is a linear superposition of eigenfunctions of $i\partial_z$ with positive eigenvalues, while $H_-(z)$ is a superposition of eigenfunctions with negative eigenvalues. Consequently, the expression (3.7) defines an operator which acts on solutions to the constraint equation as

$$\hat{\sigma}\Psi(v, \epsilon_D) = \psi_+(\epsilon_D) H_+(z) - \psi_-(\epsilon_D) H_-(z), \quad (3.9)$$

though it does not necessarily have an unambiguous action on more general functions $f(v, \epsilon_D)$.

Clearly, a solution of the constraint equation is specified by a pair of functions $\psi_\pm(\epsilon_D)$, which we will now write as $\psi(\epsilon_D, \sigma)$, with $\sigma = \pm 1$. We construct a physical Hilbert space $\mathcal{H}_{\text{phys}}$ by equipping the set of such solutions with the inner product

$$(\psi_1, \psi_2)_{\rm phys} = \sum_{\sigma=\pm1} \int_0^\infty {\rm d}\epsilon_{\rm D} \bar{\psi}_1(\epsilon_{\rm D}, \sigma)\psi_2(\epsilon_{\rm D}, \sigma), \quad (3.10)$$

and restricting to functions that are normalizable under this inner product. Thus, $\mathcal{H}_{\rm phys}$ is the direct sum of two copies of $L^2(\mathbb{R}_+, {\rm d}\epsilon_{\rm D})$. Then the classical expression (2.22), which is a constant of the motion, is represented in $\mathcal{H}_{\rm phys}$, by the operator

$$\hat{\sigma}\psi(\epsilon_{\rm D}, \sigma) = \sigma\psi(\epsilon_{\rm D}, \sigma). \quad (3.11)$$

This operator is self-adjoint, since it acts by multiplication in the $(\epsilon_{\rm D}, \sigma)$ representation.

The operators $\hat{v}$, $\hat{p}_v$ and $\hat{\phi}$ defined in (3.1) do not have any well-defined action in $\mathcal{H}_{\rm phys}$; that is, they do not act on one solution of the constraint equation (3.3) to produce another solution. As one might expect, however, we can convert the classical Dirac observables V, Y and (with slightly more difficulty) Φ given in (2.16), (2.17), and (2.18) into operators that do act in $\mathcal{H}_{\rm phys}$. For $\hat{V}$ and $\hat{Y}$ we choose the operator orderings

$$\hat{V} = \hat{v} + 6\pi G(\hat{v}\hat{p}_v + \hat{p}_v\hat{v})\hat{\phi} - 6\pi G\hat{C}_{\rm grav}\hat{\phi}^2 \quad (3.12)$$

$$\hat{Y} := \frac{1}{2}(\hat{v}\hat{p}_v + \hat{p}_v\hat{v}) - \hat{C}_{\rm grav}\hat{\phi}, \quad (3.13)$$

with $\hat{C}_{\rm grav}$ given by (3.2). These three operators have the commutation relations $[\hat{A}, \hat{B}] = {\rm i}\hbar\widehat{\{A, B\}}$, with the Poisson brackets shown in (2.19), and it follows that the Heisenberg equation of motion (2.23) is promoted to

$${\rm i}\hbar\frac{{\rm d}\hat{V}(\varphi)}{{\rm d}\varphi} = [\hat{V}(\varphi), \hat{H}_\varphi] \quad (3.14)$$

with $\hat{H}_\varphi = \hat{C}_{\rm grav}$ and $\hat{V}(\varphi)$ the operator version of (2.20). With the use of Bessel's equation $(z\partial_z z\partial_z + z^2)H_\pm(z) = 0$, it is straightforward to find that each of these operators converts an expression of the form (3.6), namely, a linear combination of order-0 Hankel functions with z-independent coefficients, into another expression of the same form, the new coefficients being given in each case by

$$\hat{V}\psi(\epsilon_{\rm D}, \sigma) = -\frac{4}{\lambda^2}\frac{\partial}{\partial\epsilon_{\rm D}}\epsilon_{\rm D}\frac{\partial}{\partial\epsilon_{\rm D}}\psi(\epsilon_{\rm D}, \sigma) \quad (3.15)$$

$$\hat{Y}\psi(\epsilon_{\rm D}, \sigma) = {\rm i}\hbar\epsilon_{\rm D}^{1/2}\frac{\partial}{\partial\epsilon_{\rm D}}\epsilon_{\rm D}^{1/2}\psi(\epsilon_{\rm D}, \sigma) \quad (3.16)$$

$$\hat{H}_\varphi\psi(\epsilon_{\rm D}, \sigma) = \hat{C}_{\rm grav}\psi(\epsilon_{\rm D}, \sigma) = -\epsilon_{\rm D}\psi(\epsilon_{\rm D}, \sigma). \quad (3.17)$$

These equations specify the actions of the gauge-invariant operators in $\mathcal{H}_{\rm phys}$. With the inner product (3.10), $\hat{V}$ and $\hat{Y}$ are clearly symmetric, and $\hat{H}_\varphi$, which acts by multiplication, is self-adjoint.

Construction of an operator $\hat{\Phi}$ corresponding to (2.18) needs a little more thought, because $\hat{p}_v = -{\rm i}\hbar\partial_v$ does not have a unique inverse. That is, the antiderivative ${\rm i}\hbar^{-1}\int^v \Psi(v', \epsilon_{\rm D}){\rm d}v'$ does not yield a definite function until the lower limit of integration is specified.[2] Consider, however, a wave function $\Psi(v, \epsilon_{\rm D}) = \psi(\epsilon_{\rm D})\mathcal{C}_0(z)$, where $\mathcal{C}_0$ is any Bessel function of order 0. We define an operator $\widehat{p_v^{-1}}$ by writing

$$\hat{\Phi}\Psi(v, \epsilon_{\rm D}) = {\rm i}\hbar\frac{\partial\Psi(v, \epsilon_{\rm D})}{\partial\epsilon_{\rm D}} + \frac{{\rm i}\hbar\lambda^2}{4}\int_{v_0}^v \Psi(v', \epsilon_{\rm D}){\rm d}v'$$
$$= {\rm i}\hbar\frac{\partial\psi(\epsilon_{\rm D})}{\partial\epsilon_{\rm D}}\mathcal{C}_0(z) + \frac{{\rm i}\hbar}{2\epsilon_{\rm D}}\psi(\epsilon_{\rm D})\Delta(z),$$

where

$$\Delta(z) = z\partial_z\mathcal{C}_0(z) + \int_{z_0}^z \mathcal{C}_0(z')z'{\rm d}z',$$

z_0 is a constant, and $v_0 = z_0^2/\lambda^2\epsilon_{\rm D}$. By virtue of Bessel's equation, we have

$$\Delta(z) = z\partial_z\mathcal{C}_0(z) - \int_{z_0}^z \partial_{z'}[z'\partial_{z'}\mathcal{C}_0(z')]{\rm d}z' = z_0\mathcal{C}_0'(z_0).$$

This function is actually independent of z, and cannot be expressed as a linear combination of $H_\pm(z)$. It can, however, be made to vanish if we choose z_0 to be a zero of $z\mathcal{C}_0'(z)$, which is at a complex infinity in the case of the Hankel functions. In this way, we obtain

$$\hat{\Phi}\psi(\epsilon_{\rm D}, \sigma) = {\rm i}\hbar\partial_{\epsilon_{\rm D}}\psi(\epsilon_{\rm D}, \sigma). \quad (3.18)$$

It is perhaps worth noting that although this operator appears superficially to have the same action as the kinematical operator $\hat{\phi}$, it is a different operator. Thus, the derivative of (3.6) with respect to $\epsilon_{\rm D}$ contains, in addition to the derivatives of $\psi_\pm(\epsilon_{\rm D})$, terms proportional to $z\partial_z H_\pm(z)$, which are not linear combinations of $H_\pm(z)$. With the above construction, these unwanted terms are precisely cancelled by the action of $\widehat{p_v^{-1}}$, arising from the last term in (2.18). Correspondingly, while the kinematical operator $\hat{\phi}$ commutes with $\hat{v}$ and $\hat{p}_v$, the operator $\hat{\Phi}$ does not commute with $\hat{V}$ or $\hat{Y}$. In fact, it can be expressed in terms of the gravitational variables as

$$\hat{\Phi} = \hat{C}_{\rm grav}^{-1}(\tfrac{1}{2}{\rm i}\hbar - \hat{Y}), \quad (3.19)$$

and does not represent an independent physical degree of freedom. Alternatively, of course, the gravitational

[2] If, as is often done, we were to define a kinematical *Hilbert* space $\mathcal{H}_{\rm kin}$, say with the inner product $(\Psi_1, \Psi_2)_{\rm kin} = \int{\rm d}v{\rm d}\epsilon_{\rm D}\bar{\Psi}_1\Psi_2$, we would find that the restriction of $\hat{p}_v$ to $\mathcal{H}_{\rm kin}$ does have a unique inverse. However, as often happens, solutions of the constraint equation are not normalizable under this inner product, and do not belong to $\mathcal{H}_{\rm kin}$, so the Hilbert-space inverse of $\hat{p}_v$ has no well-defined action on these solutions.

variables can be expressed in terms of $\hat{\Phi}$ and its conjugate momentum, which is multiplication by $\epsilon_{\rm D}$.

The operator $\hat{\Phi}$ is symmetric under the inner product (3.10), and has the commutator $[\hat{\Phi}, \hat{C}_{\rm grav}] = -i\hbar$, in agreement with the last Poisson-bracket relation of (2.19). Consequently, the equation of motion (2.24) becomes the operator equation

$$i\hbar \frac{d\hat{\Phi}(\nu)}{d\nu^{1/2}} = [\hat{\Phi}(\nu), \hat{H}_\nu], \tag{3.20}$$

where $\hat{H}_\nu$ is the operator in $\mathcal{H}_{\rm phys}$ whose action is specified by

$$\hat{H}_\nu \psi(\epsilon_{\rm D}, \sigma) = \left(\frac{2\epsilon_{\rm D}}{3\pi G}\right)^{1/2} \sigma \psi(\epsilon_{\rm D}, \sigma). \tag{3.21}$$

This operator is self-adjoint, since it acts by multiplication in the $(\epsilon_{\rm D}, \sigma)$ representation, and the evolution it generates is therefore unitary.

IV. INTERPRETATION OF INTERNAL TIME

A. Conditional and joint probabilities

We argued in [18] that, while an operator such as $\hat{V}(\varphi)$ is a perfectly good Dirac observable, the internal-time parameter φ cannot bear the interpretation that one would like to place on it (and which indeed is placed on it, in a significant part of the literature).

At the level of the classical equations of motion (2.6), (2.7), and (2.8) and their solutions (2.10), (2.11), and (2.12), it seems quite feasible to say that $V(\varphi)$ is the volume "when" the scalar field has the value φ, provided that one does not enquire too closely about the instant of time "when" this pair of values is realized. But even classically, if one follows the systematic procedure of constructing a reduced, physical phase space, whose points are gauge orbits in the constraint manifold, one finds that this physical phase space is 2-dimensional. The corresponding configuration space is 1-dimensional: there do not exist two independent physical quantities which might simultaneously be determined to have the values φ and $V(\varphi)$.

Quantum mechanically, the configuration space on which states in $\mathcal{H}_{\rm phys}$ are defined is again 1-dimensional. The parameter φ, which labels the family of Dirac observables $\hat{V}(\varphi)$, *cannot* be construed as a value obtained by observation of a physical clock (the scalar field),[3] because there is no operator acting in $\mathcal{H}_{\rm phys}$, independent of $\hat{V}$, to represent any such observable clock. In particular, the operator $\hat{\Phi}$ cannot serve this purpose, for two related reasons. First, $\hat{\Phi}$ does not commute with $\hat{V}(\varphi)$ for any value of φ, so the rules of quantum mechanics do not allow simultaneous measurements of the quantities represented by these two operators. Second, the reason these operators do not commute is that they were constructed through the "evolving constant of the motion" algorithm, and are thus quite different from the kinematical operators $\hat{v}$ and $\hat{\phi}$. Classically, to interpret φ as a value of Φ is to interpret $V(\varphi)$ as "the volume at the time when the value φ is assumed by 'the scalar field at the time when the volume is zero.'" This is, of course, incoherent, and is made no less so by the transition to quantum mechanics. Similar remarks apply, of course to the parameter ν that labels the family of observables $\hat{\Phi}(\nu)$: it *cannot* be construed as the result of a measurement of the volume. Given that these parameters cannot be construed as values obtained from measurements, it is hard to see that they have any physical meaning at all.

We now wish to strengthen this conclusion by considering the possibility that, notwithstanding the arguments just given, the internal-time formalism might be construed as yielding a joint probability distribution that represents a correlation between observable quantities, in this case the volume and the scalar field. That is to say, we will try to extend the notion of "observables" by dropping the requirement that they be represented by mutually commuting operators in $\mathcal{H}_{\rm phys}$. This is, in particular, a timeless interpretation, in which one might decide to ignore conundrums concerning the times at which specific values of the observables are realized. We will show, though, that in general no such probability distribution exists.

The idea is this. If $\hat{V}(\varphi)$ is regarded as the Heisenberg-picture operator associated with evolution in an internal time φ, generated by the Hamiltonian $\hat{H}_\varphi$, then we can construct the corresponding Schrödinger-picture wave function

$$\tilde{\psi}(\tilde{\nu}, \sigma; \varphi) = \exp(-i\hat{H}_\varphi \varphi/\hbar)\tilde{\psi}(\tilde{\nu}, \sigma; 0), \tag{4.1}$$

where $\tilde{\psi}(\tilde{\nu}, \sigma; 0)$ is a suitable transform of $\psi(\epsilon_{\rm D}, \sigma)$ on which $\hat{V}$ (that is, $\hat{V}(0)$) acts by multiplication: $\hat{V}\,\tilde{\psi}(\tilde{\nu}, \sigma; 0) = \tilde{\nu}\,\tilde{\psi}(\tilde{\nu}, \sigma; 0)$. According to the usual interpretation, the object

$$\tilde{\mathcal{P}}(\tilde{\nu}; \varphi) := \sum_\sigma |\tilde{\psi}(\tilde{\nu}, \sigma; \varphi)|^2 \tag{4.2}$$

is the time-dependent probability density for obtaining the value $\tilde{\nu}$ from a measurement of the volume performed at "time" φ. In particular, this probability has the time-independent normalization $\int_0^\infty \tilde{\mathcal{P}}(\tilde{\nu}; \varphi)d\tilde{\nu} = 1$. According to the foregoing discussion, the problem with this is that we have no idea what is meant by "performing the measurement at time φ."

The problem might be alleviated if we could reinterpret (4.2) as a *conditional* probability density $\tilde{\mathcal{P}}(\tilde{\nu}|\varphi)$ for the

[3] That is, the usual rules of quantum mechanics do not allow it to be construed in this way. One might perhaps adopt the view that the usual rules of quantum mechanics should be adjusted in some way to accommodate the notion of internal time. One possibility for such an adjustment is considered in this section, and we discuss this issue more generally in Sec. VI.

volume, given that some other quantity (which we hope to identify with the scalar field) has been determined to have the value φ, even though that other quantity does not appear explicitly in our formalism. An interpretation of that kind requires the existence of a *joint* probability density $\mathcal{P}(\tilde{\nu}, \varphi)$, with the normalization

$$\int_0^\infty \mathrm{d}\tilde{\nu} \int_{-\infty}^\infty \mathrm{d}\varphi \mathcal{P}(\tilde{\nu}, \varphi) = 1, \tag{4.3}$$

such that

$$\tilde{\mathcal{P}}(\tilde{\nu}|\varphi) = \frac{\mathcal{P}(\tilde{\nu}, \varphi)}{\int_0^\infty \mathcal{P}(\tilde{\nu}', \varphi)\mathrm{d}\tilde{\nu}'}. \tag{4.4}$$

If this joint probability density does exist, we can use it to find the conditional-probability density

$$\underset{\sim}{\mathcal{P}}(\varphi|\tilde{\nu}) = \frac{\mathcal{P}(\tilde{\nu}, \varphi)}{\int_{-\infty}^\infty \mathcal{P}(\tilde{\nu}, \varphi')\mathrm{d}\varphi'} \tag{4.5}$$

for the scalar field, given that the volume has been determined to have the value $\tilde{\nu}$.

For the model under consideration, it happens that we already have such a probability density to hand, namely

$$\underset{\sim}{\mathcal{P}}(\underset{\sim}{\varphi}|\nu) := \sum_\sigma |\underset{\sim}{\psi}(\underset{\sim}{\varphi}, \sigma; \nu)|^2, \tag{4.6}$$

where $\underset{\sim}{\psi}(\underset{\sim}{\varphi}, \sigma; \nu)$ is the wave function evolved in the internal-time parameter ν, and expressed in the representation where $\hat{\Phi}$ acts by multiplication: $\hat{\Phi}\, \underset{\sim}{\psi}(\underset{\sim}{\varphi}, \sigma; \nu) = \underset{\sim}{\varphi}\, \underset{\sim}{\psi}(\underset{\sim}{\varphi}, \sigma; \nu)$. More precisely, *if* the expression on the right of (4.2) can be interpreted as a conditional probability, then it must be possible to interpret (4.6) in the same way. Up to this point, we have maintained a notational distinction between quantities that *prima facie* have quite different meanings, namely $\{\nu, \varphi\}$, which are internal-time parameters, and $\{\tilde{\nu}, \underset{\sim}{\varphi}\}$, which are configuration-space coordinates. However, if the internal-time formalism is to be interpreted as expressing an observable correlation between, in this case, the volume and the scalar field, then it must be possible to use these variables interchangeably, and we will now do so, until further notice. In particular, we would like to identify the probability density (4.6) with the one displayed in (4.5).

In the light of the arguments summarized at the beginning of this section, one might expect the conditional-probability interpretation to present some difficulty. Let us, indeed, attempt to construct the required joint probability $\mathcal{P}(\nu, \varphi)$ from the conditional probabilities $\tilde{\mathcal{P}}(\nu|\varphi)$ and $\underset{\sim}{\mathcal{P}}(\varphi|\nu)$, which can be calculated once a wave function is specified. Denote by $f(\varphi)$ the denominator in (4.4) and by $g(\nu)$ the denominator in (4.5). Then we have

$$f(\varphi)\tilde{\mathcal{P}}(\nu|\varphi) = \mathcal{P}(\nu, \varphi) = g(\nu)\underset{\sim}{\mathcal{P}}(\varphi|\nu). \tag{4.7}$$

Up to constants $f_0 := f(0)$ and $g_0 := g(0)$, which are fixed by normalization, the unknown functions are determined by

$$f(\varphi) = \frac{g_0 \underset{\sim}{\mathcal{P}}(\varphi|0)}{\tilde{\mathcal{P}}(0|\varphi)} \tag{4.8}$$

$$g(\nu) = \frac{f_0 \tilde{\mathcal{P}}(\nu|0)}{\underset{\sim}{\mathcal{P}}(0|\nu)}. \tag{4.9}$$

We see that the joint probability density is well defined (the first and last expressions in (4.7) are consistent) only if

$$R := \frac{f_0 \underset{\sim}{\mathcal{P}}(\varphi|\nu)\tilde{\mathcal{P}}(\nu|0)\tilde{\mathcal{P}}(0|\varphi)}{g_0 \tilde{\mathcal{P}}(\nu|\varphi)\underset{\sim}{\mathcal{P}}(0|\nu)\underset{\sim}{\mathcal{P}}(\varphi|0)} = 1. \tag{4.10}$$

In the following, we will calculate the ratio R for a specific state, and find that it is not equal to 1. This counterexample is sufficient to demonstrate that R is not equal to 1 in general, but we will also argue that our example is not especially atypical.

B. Inconsistent probabilities

We begin by constructing the wave function $\tilde{\psi}(\tilde{\nu}, \sigma; \varphi)$ that appears in (4.1) and (4.2). To simplify matters, we consider a state for which $\psi(\epsilon_\mathrm{D}, -1) = 0$, a condition that is preserved by evolution with respect to both $\hat{H}_\varphi$ and $\hat{H}_\nu$. That is, we focus on the sector that corresponds classically to an expanding universe, and we will drop the label $\sigma = 1$ that indicates this explicitly. The representation in which the volume operator $\hat{V}$ acts by multiplication is obtained by the Hankel transform

$$\tilde{\psi}(\nu; \varphi) = \frac{\lambda}{2} \int_0^\infty \mathrm{d}\epsilon_\mathrm{D} J_0(\lambda \epsilon_\mathrm{D}^{1/2} \nu^{1/2}) \mathrm{e}^{\mathrm{i}\epsilon_\mathrm{D}\varphi/\hbar} \psi(\epsilon_\mathrm{D}), \tag{4.11}$$

where J_0 is the Bessel function of the first kind. Bessel's equation $(z\partial_z z\partial_z + z^2)J_0(z) = 0$ implies that the operator $\hat{V}$ defined in (3.15) does indeed act on this function by multiplication, if $\psi(\epsilon_\mathrm{D}; \varphi) = \mathrm{e}^{\mathrm{i}\epsilon_\mathrm{D}\varphi/\hbar}\psi(\epsilon_\mathrm{D})$ belongs to the domain on which $\hat{V}$ is symmetric. Provided that $\psi(\epsilon_\mathrm{D}, \varphi)$ possesses an invertible Hankel transform, it is straightforward to verify that this transform preserves the normalization, $\int_0^\infty |\tilde{\psi}(\nu; \varphi)|^2 \mathrm{d}\nu = \int_0^\infty |\psi(\epsilon_\mathrm{D})|^2 \mathrm{d}\epsilon_\mathrm{D}$.

We consider normalized states of the form

$$\psi(\epsilon_\mathrm{D}) = [(2\beta)^{2n+1}/(2n)!]^{1/2} \epsilon_\mathrm{D}^n \mathrm{e}^{-\beta\epsilon_\mathrm{D}}, \tag{4.12}$$

where n is a positive integer and β a real, positive constant, which satisfy both of these requirements. Our main motivation for this choice is that the Hankel transform can be calculated analytically. In these states, the mean energy of the dust is $\bar{\epsilon}_\mathrm{D} = (n + \frac{1}{2})\beta^{-1}$ and its dispersion is $\Delta\epsilon_\mathrm{D} = (2n + 1)^{-1/2}\bar{\epsilon}_\mathrm{D}$, so for large values of n the energy distribution becomes quite sharply peaked. However, the analytic expressions for $\tilde{\psi}(\nu; \varphi)$ become very cumbersome for large n, so we focus on the case $n = 1$. In that case, we find

$$\tilde{\mathcal{P}}(\nu|\varphi) = (\lambda^2/\beta\gamma^3)(\gamma - \bar{\nu} + \tfrac{1}{4}\bar{\nu}^2)e^{-\bar{\nu}/\gamma}, \qquad (4.13)$$

with

$$\bar{\nu} := \lambda^2\nu/2\beta, \qquad \gamma := 1 + (\varphi/\beta\hbar)^2.$$

It is easy to check that $\int_0^\infty \tilde{\mathcal{P}}(\nu|\varphi)d\nu = 1$.

Evolution in the internal time $\nu^{1/2}$ is generated by $\hat{H}_\nu$, whose action on the $\sigma = +1$ subspace, and in the ϵ_D representation is multiplication by $(2\epsilon_D/3\pi G)^{1/2} = \lambda\hbar\epsilon_D^{1/2}$. The representation in which $\hat{\Phi}$ acts by multiplication is obtained by Fourier transformation:

$$\underset{\sim}{\psi}(\varphi, \nu) = (2\pi\hbar)^{-1/2}\int_0^\infty d\epsilon_D e^{i\epsilon_D\varphi/\hbar}e^{-i\lambda\epsilon_D^{1/2}\nu^{1/2}}\psi(\epsilon_D). \qquad (4.14)$$

Again, one may easily check that $\hat{\Phi}\,\underset{\sim}{\psi}(\varphi, \nu) = \varphi\underset{\sim}{\psi}(\varphi, \nu)$, provided that $\psi(\epsilon_D) = 0$ when ϵ_D is 0 or ∞, which is the condition for the operator (3.18) to be symmetric. We note in passing that classically, when the constraint is satisfied, $\epsilon_D = 0$ implies $p_v = 0$. On this hypersurface, the volume is constant and the Dirac observable (2.15) is not well defined. The need to restrict attention here to wave functions that vanish at $\epsilon_D = 0$ is therefore not surprising.

Using the wave function (4.12) with $n = 1$, we obtain

$$\underset{\sim}{\mathcal{P}}(\varphi|\nu) = \frac{2\beta^3}{\pi\hbar}\frac{|\chi(\rho)|^2}{(\beta^2 + \varphi^2/\hbar^2)^2}, \qquad (4.15)$$

where $\rho := \lambda^2\nu/(\beta - i\varphi/\hbar)$ and

$$\chi(\rho) := \frac{1}{8}\Big\{8 - 2\rho + i\sqrt{\pi\rho}(\rho - 6)e^{-\rho/4} \times\Big[1 - \mathrm{erf}\Big(\frac{i\sqrt{\rho}}{2}\Big)\Big]\Big\}. \qquad (4.16)$$

We do not know (and neither does any computer-algebra package available to us) how to compute the integral $\int_{-\infty}^\infty \underset{\sim}{\mathcal{P}}(\varphi|\nu)d\varphi$ analytically, but numerical evaluation yields the value 1 for randomly selected values of ν.

With these probabilities in hand, we find the ratio (4.10) to be

$$R = \frac{\gamma e^{-\bar{\nu}}|\chi(\rho)|^2(1 - \bar{\nu} + \frac{1}{4}\bar{\nu}^2)}{e^{-\bar{\nu}/\gamma}|\chi(2\bar{\nu})|^2(\gamma - \bar{\nu} + \frac{1}{4}\bar{\nu}^2)}. \qquad (4.17)$$

While R is equal to 1 by construction when $\varphi = 0$ or $\nu = 0$, it is not equal to 1 elsewhere. Consequently, the functions $\tilde{\mathcal{P}}(\nu|\varphi)$ and $\underset{\sim}{\mathcal{P}}(\varphi|\nu)$ *cannot* consistently be interpreted as conditional-probability densities arising from some underlying joint probability distribution. Certainly, this conclusion is based on a single counterexample, but it seems that our sample wave function has no special pathological feature, and the inconsistency is very likely to be generic. This will become a little clearer on examination of the kinds of evolution that are associated with the two internal times φ and ν.

C. Singularity resolution

It is of considerable interest to see exactly what is implied by the two probability distributions. To this end, we evaluate them using a wave function of the form (4.12) with $n = 4$. The resulting analytic expressions are lengthy and unilluminating, but the somewhat more sharply peaked energy distribution leads to probability densities whose nature is more readily apparent to the eye.

Figure 1 shows the probability density $\tilde{\mathcal{P}}(\tilde{\nu};\varphi)$ for the volume, evolved with the internal-time parameter φ. (We revert to the notation of (4.2), since the conditional-probability notation has proved to be inappropriate). Clearly, the classical singularity at $v = 0$ is resolved, in the sense that it has been replaced by a bounce in the quantum theory. (Since the generator $\hat{H}_\varphi$ of evolution in φ is independent of $\hat{\sigma}$, the probability density for the corresponding state in the classically contracting sector $\sigma = -1$ is exactly the same). This agrees qualitatively with the results of Amemiya and Koike [24], who studied a similar model, adopting the Brown-Kuchař scalar field as an internal time, though the quantization schemes they considered differ in detail from ours. In loop-quantum-gravity-inspired treatments, such as those described in [21,22,25], the singularity is also seen to be resolved, but the mechanism appears to be different. In particular, it is found in [21] (where the matter content is a conventional massless scalar field, which is also used as internal time) that the minimum volume at the bounce corresponds to a density $\rho_{\text{crit}} = 3/(16\pi^2\zeta^3G^2\hbar)$, ζ being the Barbero-Immirzi parameter, independent of the details of the quantum state. In the present treatment, by contrast, the minimum value of $\langle\hat{V}(\varphi)\rangle$ is just $\langle\hat{V}\rangle$, which is given by

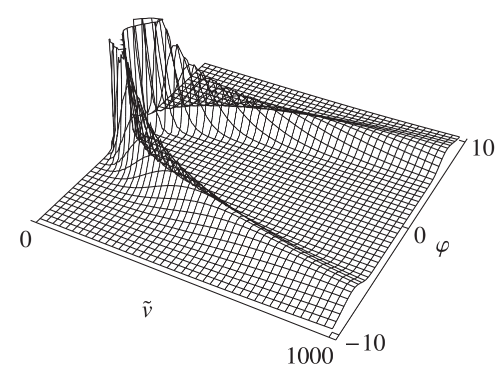

FIG. 1. The probability density for the volume $\tilde{\nu}$ evolved in the internal-time parameter φ. The wave function is of the form (4.12) with $n = 4$, and inessential constants have the values $\hbar = \lambda = \beta = 1$.

$2\beta/\lambda^2$ in any state of the form (4.12). (That the bounce occurs at $\varphi = 0$ is a consequence of choosing β to be real in (4.12); it apparent from (4.11) that an imaginary part of β simply shifts φ by a constant). This gives a maximum density $\rho_{\rm bounce} = \lambda^2 \bar{\epsilon}_{\rm D}^2/(2n+1)$. Physically, a universal critical density of the order of the Planck density seems more reasonable, so it is worth emphasizing that our purpose here is to explore the nature of time evolution, not to construct an optimal model of cosmology, or an optimal quantization scheme.

Figure 2 shows the probability density $\underset{\sim}{\mathcal{P}}(\underset{\sim}{\varphi}; \nu)$ for the scalar field, evolved with the internal-time parameter ν. It clearly indicates a universe expanding from the initial singularity: the scalar field, which classically increases linearly with proper time, here increases with the volume. In the same sense, a state in the $\sigma = -1$ sector would be seen to contract towards a final singularity. [Specifically, replacing $\epsilon_{\rm D}^{1/2}$ with $-\epsilon_{\rm D}^{1/2}$ in (4.14) leads to the mirror image of Fig. 2] From this perspective, the two sectors of the quantum theory reproduce the expanding and contracting regions of the classical phase space. A similar result is found in [21] for a Wheeler-de Witt quantization of the model whose matter content is a massless scalar field, and more recently in [26], where the same model is treated in a consistent-histories approach.

It is obvious from these figures that the two probability densities they depict cannot arise as conditional probabilities from an underlying joint probability density. We emphasize that they are computed from *exactly the same quantum state*; the difference arises solely from the two different notions of internal time used to construct the families of Dirac observables $\hat{V}(\varphi)$ and $\hat{\Phi}(\nu)$ and their associated Schrödinger-picture wave functions. It seems clear that these two notions of internal time ''see'' the expanding and contracting sectors of the theory very differently, and that the difference is unlikely to be attributable to specific features of our chosen states.[4]

D. Difficulties with internal time

The probability densities shown in Figs. 1 and 2 underline in a rather striking way (which we did not anticipate at the outset of this investigation) the difficulties of interpretation of the internal-time formalism discussed in Sec. IVA. It seems to us that the two notions of internal time must stand or fall together, since they are merely two different implementations of the same algorithm. That is to say, since the two families of Dirac observables are constructed in the same way, one cannot consistently maintain that $\hat{V}(\varphi)$ represents the volume when the scalar field has the value φ without accepting that, by the same token, $\hat{\Phi}(\nu)$ must represent the scalar field when the volume is ν. Since the two associated probability distributions, which arise in *exactly the same quantum state* are seen to be in gross conflict, we conclude that both viewpoints cannot simultaneously be correct, and therefore that neither of them is correct. As argued above, the underlying reason for this is that neither φ nor ν can be interpreted as a value assumed by any physically observable quantity.

Some might wish to claim that the conflict should be resolved in favor of φ as a preferred internal time, because the model can be ''deparametrized'' in this variable: that is, the constraint (2.9) can be solved to obtain $p_\phi = -\Theta(v, p_v)$, where the function Θ is independent of ϕ, and serves as the generator of evolution in the internal time φ. We do not think that this is a good argument in itself, but defer discussion of this point to Sec. VI.

It is perhaps also worth pointing out that the difficulties discussed here are quite distinct from a well-known technical problem that arises, for example, in the double-oscillator model studied by Rovelli [11,27], where no single internal-time variable can uniquely parametrize the whole of a classical trajectory. In the quantum theory, one finds that the evolution in any one internal time is at best approximately unitary in some restricted range. Here, by contrast, every classical trajectory is completely parametrized either by the volume or by the scalar field, and in the quantum theory both evolutions are exactly unitary over the whole parameter range $-\infty < \varphi < \infty$ or $0 \le \nu < \infty$ corresponding to the configuration space on which the theory is defined.

A pressing question raised by the above results is, of course, does the quantum theory resolve the classical singularity or not? More generally, since in this example

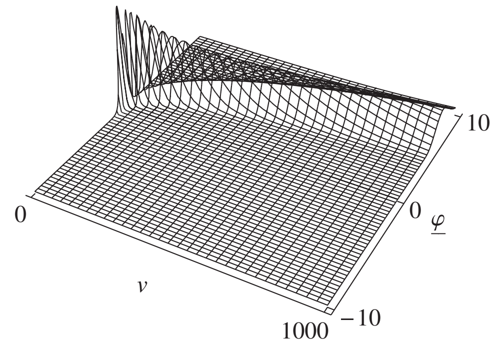

FIG. 2. The probability density for the scalar field $\underset{\sim}{\varphi}$ evolved in the internal-time parameter ν. The quantum state is exactly the same as in FIG. 1.

[4]Both classically and quantum mechanically, various functions of the internal-time parameter ν, including the classical Dirac observable (2.15) and the wave function (4.14), can be smoothly continued to negative values of $\sqrt{\nu}$. Figure 2 is obtained by using only positive values of $\sqrt{\nu}$, as required by (2.14), which account for all possible values of the volume, regarded as a physical clock. One might perhaps wonder whether continuation of this probability density to negative values of $\sqrt{\nu}$ would show the classical singularity to be resolved, as in Fig. 1. A detailed discussion of this question is given in the appendix, where we find that the quantum-mechanical situation exactly parallels the classical one, and the singularity is not resolved.

the answer seems to depend on an arbitrary choice of the variable that is to serve as internal time, do we have a reliable way of deciding whether or not the singularity is resolved in the context of *any* cosmological model and quantization scheme? A possible answer is that quantum mechanics is actually ambivalent on this question. It could be, for example, that given some definite quantum state, different classes of observer will unavoidably disagree on whether that state involves a singularity or not. We proposed in [18], following earlier work in [28], a variant of the idea of relational time that seems to avoid the difficulties of interpretation we have stressed up to now, by introducing a "test clock," which provides a preferred notion of time evolution *from the point of view of a specific observer*. As described in the following section, this preferred description, from the point of view of a comoving observer internal to the model universe, effectively coincides with that furnished by φ as an internal time.

V. DUST-FILLED COSMOLOGY WITH A TEST CLOCK

A. Quantization of an extended model

As explained at greater length in [18], we supplement the model studied in previous sections with a rough-and-ready description of a small clock, which we consider to be *internal* to a specific observer, and thus localized on that observer's worldline. In a complete description, the coordinate functions, say $x^\mu(s)$, that locate this worldline should appear as extra phase-space coordinates (as should a complete set of metric and matter fields, $g_{\mu\nu}(x)$ and $\phi_a(x)$), but in the spirit of simplified cosmological models of the sort considered here, we assume that these degrees of freedom can be neglected. If the observer is comoving, then the proper time along the worldline is the same t as appears in (2.5). From the set of variables q_i that describe the internal workings of the clock, we suppose that a function $r(\{q\})$ can be constructed, which constitutes the reading of the clock. Crucially, however, we take the view that this reading, being internal to the observer in question, is in principle inaccessible to that observer, and for that reason need not feature in the physical phase space that describes the universe from that observer's point of view. Rather, it provides the context for observations of other physical quantities to be made. Solution of the equations of motion for the variables $\bar{q}_i(\{q\}, t)$ yields the clock reading $r(\{q\}, t) := r(\{\bar{q}(\{q\}, t)\})$ at the proper time t. This reading need not be linear in t, but, if the clock is fit for our purpose, there must be, classically, a unique function $t_0(\{q\})$, such that $r(\{q\}, t_0) = 0$. That is, $t_0(\{q\})$ is the proper time at which the clock reads 0. It is not hard to show that $\{t_0, h\} = -1$, where Nh is the clock's Hamiltonian. Because the system described by h is (ideally) localized on a single worldline, h and t_0 are independent of the cosmological variables (v, p_v, ϕ, p_ϕ). The total Hamiltonian constraint is

$$C := C_0 + h = C_{\rm grav} + p_\phi + h \tag{5.1}$$

and, as shown in [18], the quantity

$$V(\tau) := \bar{v}(v, p_v, \phi, p_\phi; t_0 + \tau) \tag{5.2}$$

is, for each τ, a Dirac observable, $\{V(\tau), C\} = 0$. So, of course, are $\Phi(\tau)$, etc., defined in the same way. Thus, we obtain a new set of evolving constants of the motion, which can be interpreted classically as the volume, etc., when a proper time τ has elapsed along the observer's worldline since the clock read 0. The second crucial feature of this construction is that τ is *not* to be interpreted as a reading obtained from observation of a physical clock. From the point of view of the dynamical system governed by the constraint (5.1), it is an external, "Heraclitian" time, in the sense of Unruh and Wald [29].

We quantize this enlarged model using essentially the same scheme as in Sec. III, adopting a kinematical representation in which $\hat{v}$, $\hat{p}_\phi$ and $\hat{h}$ act by multiplication on wave functions $\Psi(v, \epsilon_{\rm D}, \epsilon_{\rm c})$, where $\epsilon_{\rm c}$ is the energy of the clock. The kinematical operator representing the fiducial proper time t_0 is then $\hat{t}_0 = -i\hbar\partial/\partial\epsilon_{\rm c}$. Since we are no longer using the notion of evolution in the internal-time parameter ν, no operator of interest distinguishes the two sectors $\sigma = \pm 1$, and we will focus on a single sector, writing a solution to the constraint equation as

$$\Psi(v, \epsilon_{\rm D}, \epsilon_{\rm c}) = \psi(\epsilon, \epsilon_{\rm D})\mathcal{C}_0(\lambda\epsilon^{1/2}v^{1/2}), \tag{5.3}$$

where $\epsilon := \epsilon_{\rm D} + \epsilon_{\rm c}$ is the total energy, and $\mathcal{C}_0$ is any Bessel function of order 0. The physical Hilbert space is now $\mathcal{H}_{\rm phys} = L^2(\mathbb{R}_+^2, {\rm d}\epsilon_{\rm D}{\rm d}\epsilon_{\rm c})$, with the inner product

$$\begin{aligned}(\psi_1, \psi_2)_{\rm phys} &= \int_0^\infty {\rm d}\epsilon_{\rm D}\int_0^\infty {\rm d}\epsilon_{\rm c}\bar{\psi}_1(\epsilon_{\rm D} + \epsilon_{\rm c}, \epsilon_{\rm D}) \\ &\quad\times \psi_2(\epsilon_{\rm D} + \epsilon_{\rm c}, \epsilon_{\rm D}) \\ &= \int_0^\infty {\rm d}\epsilon \int_0^\epsilon {\rm d}\epsilon_{\rm D}\bar{\psi}_1(\epsilon, \epsilon_{\rm D})\psi_2(\epsilon, \epsilon_{\rm D}). \end{aligned}\tag{5.4}$$

The intention is that $\epsilon_{\rm c}$ should be small compared with $\epsilon_{\rm D}$ in the same sense that the biological system responsible for an astronomer's circadian rhythm, say, is small compared with the energy content of the visible universe, and we implement this by restricting attention to *states* such that $|\psi|^2$ is very small unless $\epsilon_{\rm c} \ll \epsilon_{\rm D}$. The resulting limitations on the resolution with which τ-dependent observables might be determined are examined in some detail in [18] for a universe whose matter content is a massless scalar field, and we will not repeat the analysis for the present model.

Acting in $\mathcal{H}_{\rm phys}$, we find families of Dirac observables

$$\hat{V}(\tau) = \hat{V} - 8(\hbar\lambda)^{-2}\hat{Y}\tau - 4(\hbar\lambda)^{-2}\hat{C}_{\rm grav}\tau^2 \tag{5.5}$$

$$\hat{\Phi}(\tau) = \hat{\Phi} + \tau, \tag{5.6}$$

where, acting on $\psi(\epsilon, \epsilon_{\rm D})$,

$$\hat{V} = -(4/\lambda^2)\partial_\epsilon \epsilon \partial_\epsilon \tag{5.7}$$

$$\hat{Y} = i\hbar \epsilon^{1/2}\partial_\epsilon \epsilon^{1/2} \tag{5.8}$$

$$\hat{C}_{\text{grav}} = -\epsilon \tag{5.9}$$

$$\hat{\Phi} = i\hbar \partial_{\epsilon_D}. \tag{5.10}$$

Note, in particular, that in (5.10), the derivative is with respect to ϵ_D keeping the total energy ϵ fixed. Treating ϵ_D and ϵ_c as independent variables, we have $\hat{\Phi} = \hat{\phi} + \hat{t}_0 = i\hbar(\partial_{\epsilon_D} - \partial_{\epsilon_c})$. Consequently, the observables $\hat{V}(\tau)$ and $\hat{\Phi}(\tau)$ now commute for every τ and, according to the usual rules of quantum mechanics, can simultaneously be assigned measured values $\underline{\nu}$ and $\underline{\varphi}$. A domain on which all of these operators are symmetric is provided by the boundary conditions

$$\psi(\epsilon, \epsilon) = \psi(\epsilon, 0) = \partial_\epsilon \psi(\epsilon, \epsilon_D)|_{\epsilon=\epsilon_D} = 0. \tag{5.11}$$

Of course, $\epsilon = \epsilon_D$ means that the clock energy ϵ_c vanishes, and we always take $\psi(\epsilon, \epsilon_D)$ to vanish fast enough at large arguments to ensure square integrability.

B. Joint probability density

Evolution of the Heisenberg-picture operators (5.5) and (5.6) is generated by the self-adjoint Hamiltonian $\hat{H}_\tau = \hat{C}_{\text{grav}} + \hat{p}_\phi = \epsilon_D - \epsilon$, and states satisfying the boundary conditions (5.11) can be expressed in a representation in which both $\hat{V}$ and $\hat{\Phi}$ act by multiplication, through combined Fourier and Hankel transformation. Thus, we can now define a *bona fide* joint probability density, which evolves in the usual way with proper time τ, namely

$$\mathcal{P}(\underline{\nu}, \underline{\varphi}; \tau) = |\psi(\underline{\nu}, \underline{\varphi}; \tau)|^2, \tag{5.12}$$

where

$$\psi(\underline{\nu}, \underline{\varphi}; \tau) = \frac{\lambda}{2\sqrt{2\pi\hbar}} \int_0^\infty d\epsilon \int_0^\epsilon d\epsilon_D e^{i\epsilon_D \underline{\varphi}/\hbar} e^{i(\epsilon-\epsilon_D)\tau/\hbar} \times J_0(\lambda \epsilon^{1/2}\underline{\nu}^{1/2})\psi(\epsilon, \epsilon_D). \tag{5.13}$$

In particular, this probability density has the τ-independent normalization $\int_0^\infty d\underline{\nu} \int_{-\infty}^\infty d\underline{\varphi}\, \mathcal{P}(\underline{\nu}, \underline{\varphi}; \tau) = 1$.

To make contact with the probabilities discussed in Sec. IV, consider a state that factorizes as

$$\psi(\epsilon, \epsilon_D) = \psi(\epsilon)\psi_c(\epsilon_c), \tag{5.14}$$

where ψ_c is a wave function for the clock and, as above, $\epsilon_c = \epsilon - \epsilon_D$ is the clock's energy. (Recall, though, that the clock is *not* represented by any operator independent of $\hat{V}$ and $\hat{\Phi}$ acting in $\mathcal{H}_{\text{phys}}$, and τ is *not* a value obtained from observation of the clock). The wave function (5.13) becomes

$$\psi(\underline{\nu}, \underline{\varphi}; \tau) = \frac{\lambda}{2} \int_0^\infty d\epsilon e^{i\epsilon \underline{\varphi}/\hbar} J_0(\lambda \epsilon^{1/2}\underline{\nu}^{1/2})\psi(\epsilon) \times \frac{1}{\sqrt{2\pi\hbar}} \int_0^\epsilon d\epsilon_c e^{i\epsilon_c(\tau-\underline{\varphi})/\hbar} \psi_c(\epsilon_c). \tag{5.15}$$

The idea of a test clock described previously implies that $\psi(\epsilon)$ should be peaked around a value $\bar{\epsilon}$ of the order of the energy content of the visible universe, while $\psi_c(\epsilon_c)$ should be peaked around a value $\bar{\epsilon}_c$ somewhat smaller than the mass of an astronomer. Under these circumstances, it is an excellent approximation to extend the upper limit of the ϵ_c integral to infinity, in which case the wave function (5.15) and the probability density (5.12) also factorize. In fact, we have

$$\mathcal{P}(\underline{\nu}, \underline{\varphi}; \tau) \simeq \tilde{\mathcal{P}}(\underline{\nu}|\underline{\varphi})\mathcal{P}(\underline{\varphi}; \tau), \tag{5.16}$$

where $\tilde{\mathcal{P}}(\underline{\nu}|\underline{\varphi})$ coincides with the probability density defined in (4.2), except that we do not now need to distinguish the sectors $\sigma = \pm 1$. Here, however, the arguments $\underline{\nu}$ and $\underline{\varphi}$ genuinely stand for values obtained from observation of the quantities represented by the commuting operators $\hat{V}$ and $\hat{\Phi}$, and $\tilde{\mathcal{P}}(\underline{\nu}|\underline{\varphi})$ is a genuine conditional probability. The function

$$\mathcal{P}(\underline{\varphi}; \tau) := \frac{1}{2\pi\hbar} \left| \int_0^\infty d\epsilon_c e^{i\epsilon_c(\tau-\underline{\varphi})/\hbar} \psi_c(\epsilon_c) \right|^2 \tag{5.17}$$

is just the probability density for obtaining the value $\underline{\varphi}$ from a measurement of the scalar field at proper time τ. Given a suitable choice of $\psi_c(\epsilon_c)$, it will be sharply peaked on a trajectory of the form $\underline{\varphi} = \Phi_0 + \tau$, consistent with the solution (5.6) of the Heisenberg equation of motion.

C. Timeless interpretation

We have emphasized that the parameter τ, which labels the families of Dirac observables $\hat{V}(\tau)$ and $\hat{\Phi}(\tau)$ is an unobservable, external time parameter. Classically, it coincides with the arc length of an observer's worldline, on which the fiducial event, that the unobserved clock reads 0, serves to define an origin. Naturally, the time-dependent observations recorded by an astronomer who wishes to test a theoretical expression such as (5.12) will not refer either to τ or to the internal, unobserved clock that we have pictured, for the sake of argument, as a biological clock. Instead, they will refer to the observed readings of some time-keeping device, which we will call (taking a cautious view of the generosity of the relevant funding agency) a "wristwatch."

From an operational point of view (more or less in the sense of Bridgman [30]) substantive physics is contained only in correlations between measured values of observable quantities. The idea that a solution to the problem of time in constrained systems is to be sought in such correlations is explicitly developed in, for example, [31–33], and is implicit in much of the recent literature. We have argued that the required correlations are *not* directly provided by a wave function such as (4.1), but in the model considered here, it is straightforward to see that this wave function does indirectly lead to an estimate of the desired correlation, to an approximation that might be extremely good. In principle, we would like to study correlations between observed values of the volume, the scalar field and the astronomer's wristwatch. Since the wristwatch is another small clock, which classically follows, for practical purposes, the same worldline as the unobserved test clock, we could incorporate it into our model by adding its energy $h_{\rm w}$ to the total constraint:

$$C_{\rm total} := C_{\rm grav} + p_\phi + h_{\rm w} + h. \tag{5.18}$$

This presents no technical difficulty, but since, for this particular model, p_ϕ and $h_{\rm w}$ appear in the same way in $C_{\rm total}$, it is clear that we can usefully economize on clocks by deleting $h_{\rm w}$ and treating the scalar field as the astronomer's timekeeping device. (Here, we profit from the enormous simplification that results from the assumption of homogeneity. In a more general setting, we envisage that Dirac observables depending on τ can be constructed only from fields in the observer's immediate locality. Technical difficulties aside, this is no significant limitation, since the local fields include, for example, the cosmic microwave background radiation photons entering the astronomer's telescope.)

Suppose, then, that simultaneous measurements of volume and scalar field are performed at a sequence of proper times τ_i, distributed according to some density function $\eta(\tau)$, with $\int_{-\infty}^{\infty}\eta(\tau)d\tau = 1$. [$\eta(\tau)$ may have support only in a subinterval of $(-\infty, \infty)$ if, for example, the observer's worldline terminates at a singularity.] The function

$$\mathcal{P}(\underline{\nu}, \underline{\varphi}) := \int_{-\infty}^{\infty} \mathcal{P}(\underline{\nu}, \underline{\varphi}; \tau)\eta(\tau)d\tau \tag{5.19}$$

is a correctly normalized joint probability density, which furnishes a timeless description of the correlation between these measured quantities.[5] In the approximation that the joint probability density factorizes as in (5.16), we find

$$\mathcal{P}(\underline{\nu}, \underline{\varphi}) \simeq \tilde{\mathcal{P}}(\underline{\nu}|\underline{\varphi})\mathcal{P}(\underline{\varphi}), \tag{5.20}$$

where $\mathcal{P}(\underline{\varphi}) = \int_{-\infty}^{\infty}\mathcal{P}(\underline{\varphi};\tau)\eta(\tau)d\tau$ is the probability for finding the measurement of the scalar field on a randomly selected occasion to have yielded the value φ. For the state studied in Sec. IV C, $\tilde{\mathcal{P}}(\underline{\nu}|\underline{\varphi})$ is precisely the function depicted in Fig. 1, but the introduction of an unobserved test clock now allows us to interpret this function as a *bona fide* conditional probability. In that indirect and approximate sense, we identify φ as a preferred internal time, and conclude that the singularity *is* resolved in this quantum theory—or at least that the observer whose unobserved clock is represented by h will discover herself to be living in a bouncing universe.

In part, this conclusion is specific to our somewhat artificial model, in which the matter content of the universe is provided by the Brown-Kuchař scalar field. More generally, consider a Hamiltonian constraint of the form

$$C := C_{\rm grav+matt} + p_{\rm w} + h = 0. \tag{5.21}$$

In this expression, $C_{\rm grav+matt}$ is the contribution of metric and matter fields, h is the Hamiltonian of an unobserved test clock, and $p_{\rm w}$ is the momentum conjugate to the pointer reading $r_{\rm w}$ of a wristwatch following essentially the same worldline as the test clock. By taking the Hamiltonian of the wristwatch to be just $p_{\rm w}$, we model a time-keeping device that is manufactured so as to supply a linear measure of the proper time along its worldline. Following the same steps as before, we expect to obtain a physical Hilbert space of functions $\psi(\boldsymbol{\gamma}, \boldsymbol{\epsilon}, \boldsymbol{\epsilon}_{\rm w})$, where $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}_{\rm w} + \boldsymbol{\epsilon}_{\rm c}$ is the total energy of the wristwatch and the unobserved clock, while $\boldsymbol{\gamma}$ collectively denotes the remaining metric and matter variables. Operators on $\mathcal{H}_{\rm phys}$ representing Dirac observables include

$$\hat{C}_{\rm grav+matt} = -\boldsymbol{\epsilon} \tag{5.22}$$

$$\hat{r}_{\rm w} = i\hbar\frac{\partial}{\partial\boldsymbol{\epsilon}_{\rm w}}, \tag{5.23}$$

as in (5.9) and (5.10). Taking a wave function that factorizes as $\psi(\boldsymbol{\gamma}, \boldsymbol{\epsilon}, \boldsymbol{\epsilon}_{\rm w}) = \psi(\boldsymbol{\gamma}, \boldsymbol{\epsilon})\psi_{\rm c}(\boldsymbol{\epsilon}_{\rm c})$, we will obtain a Schrödinger-picture wave function analogous to (5.15), of the form

$$\psi(\underline{\boldsymbol{\gamma}}, \underline{r}_{\rm w}; \tau) = \int_0^\infty d\boldsymbol{\epsilon} e^{i\boldsymbol{\epsilon}\underline{r}_{\rm w}/\hbar}\psi(\underline{\boldsymbol{\gamma}}, \boldsymbol{\epsilon}) \times \int_0^{\boldsymbol{\epsilon}} d\boldsymbol{\epsilon}_{\rm c} e^{i\boldsymbol{\epsilon}_{\rm c}(\tau - \underline{r}_{\rm w})/\hbar}\psi_{\rm c}(\boldsymbol{\epsilon}_{\rm c}). \tag{5.24}$$

From this, we obtain a joint probability $\mathcal{P}(\underline{\boldsymbol{\gamma}}, \underline{r}_{\rm w}; \tau) = |\psi(\underline{\boldsymbol{\gamma}}, \underline{r}_{\rm w}; \tau)|^2$ and, if we wish, the timeless version $\mathcal{P}(\underline{\boldsymbol{\gamma}}, \underline{r}_{\rm w}) = \int\mathcal{P}(\underline{\boldsymbol{\gamma}}, \underline{r}_{\rm w}; \tau)\eta(\tau)d\tau$, which describe correlations between observed values of the cosmological quantities $\boldsymbol{\gamma}$ and the wristwatch pointer $r_{\rm w}$. However, because the energy $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}_{\rm w} + \boldsymbol{\epsilon}_{\rm c}$ is now the total energy of two small

[5] This argument takes no account of any "collapse of the wave function" occasioned by performance of the measurements. Consequently, in the spirit of the Copenhagen interpretation, the sequence of measurements should strictly be understood as being performed on an ensemble of identically prepared universes. We find this deeply unsatisfactory, but the problem concerns the interpretation of quantum mechanics, especially as applied to the universe as a whole, about which we have nothing useful to say, rather than the interpretation of time, about which we believe we do have something useful to say. We adopt a Copenhagen-like point of view, not because we find it convincing, but because we find rival interpretations to be more cumbersome without being any more convincing. While recognizing that many physicists would disagree with this, we do not think that the present paper would be enhanced by a digression on this issue.

clocks, the wide separation of energy scales which led to the factorization in (5.16) is no longer present, and the cosmological field φ is replaced by the wristwatch pointer $r_{\rm w}$. Generically, therefore, we do *not* automatically recover a conditional-probability interpretation of any internal time chosen from among the cosmological variables $\boldsymbol{\gamma}$.

VI. DISCUSSION

We argued in [18] that the notion of internal time, especially as commonly implemented in models of quantum cosmology, is unsatisfactory, for two reasons. First, it offers no account of the passage of time as it is ordinarily conceived. Ordinarily, there seems to be a clear sense in which a well-constructed clock reads "10s" seven seconds after it read "3s," and this does not appear merely to result from a conspiracy amongst the manufacturers of timepieces. That is, a time-keeping device can be said to work accurately (or not), because there is a time to be kept, not merely because its readings tend to agree (or not) with those of other devices of the same sort. In general relativity, this notion of time is provided, for some specific observer, by the proper time that elapses along that observer's worldline, but it is not recovered in a treatment that describes evolution by correlating, for example, the volume of a spatial region with the value of a scalar field. Second, as summarized in Sec. IVA, the meaning of a parameter that serves as an "internal time" is unclear; in particular, it cannot be construed, within the usual rules of quantum mechanics, as a value obtained from the observation of any physical quantity that is to be regarded as a clock, because no such quantity is represented by any operator acting in the physical Hilbert space $\mathcal{H}_{\rm phys}$.

Various points of view might, perhaps, be adopted with regard to the meaning of internal-time parameters, and to the adjustments to the rules of quantum mechanics that one might be willing to contemplate in order to accommodate them. Let us consider some possibilities, starting from the need to find a meaning for the wave function $\tilde{\psi}(\tilde{\nu}, \sigma; \varphi)$ constructed in (4.1) and for the parameter φ that appears in it. This wave function is a solution of the Schrödinger-like equation

$$i\hbar\partial_\varphi \tilde{\psi}(\tilde{\nu}, \sigma; \varphi) = \hat{H}_\varphi \tilde{\psi}(\tilde{\nu}, \sigma; \varphi), \tag{6.1}$$

and it is expressed in the representation in which the Dirac observable $\hat{V}$ associated with the volume acts by multiplication, $\hat{V}\,\tilde{\psi}(\tilde{\nu}, \sigma; \varphi) = \tilde{\nu}\,\tilde{\psi}(\tilde{\nu}, \sigma; \varphi)$. It is worth pointing out that an equation of the same form arises directly from the constraint equation (3.3) if this is expressed in a representation in which the kinematical operator $\hat{\phi}$ acts by multiplication on kinematical wave functions $\Psi(v, \phi)$. Some authors might choose to work directly with this constraint equation, constructing the physical Hilbert space as a subspace of functions $\Psi(v, \phi)$ which solve the constraint, and are normalizable in some ϕ-independent inner product. For the model at hand, the resulting quantum theory would be broadly similar to the one obtained in this paper, though it might differ in detail, depending on the various choices to be made. For example, our inner product (3.10) respects the positivity of the dust energy, and to reproduce this feature, the Fourier transforms with respect to ϕ of wave functions $\Psi(v, \phi)$ included in $\mathcal{H}_{\rm phys}$ should be restricted to those having support only on the positive real axis. In either theory, the physical wave function $\tilde{\psi}(\tilde{\nu}, \sigma; \varphi)$ or $\Psi(v, \phi)$ is defined on a 1-dimensional configuration space, with coordinate $\tilde{\nu}$ or v, while the scalar-field variable φ or ϕ appears in the final physical theory merely as a parameter, and it is the status of this parameter that needs to be elucidated. The following possibilities would seem to present themselves.

Viewpoint 1: internal time is not observable Suppose that the internal-time parameter φ (or ϕ) is interpreted *not* as the value of some observed quantity, but rather as a "Heraclitian" variable [29] which, like the Newtonian time of nonrelativistic quantum mechanics, specifies the circumstances under which observations are made. From this point of view, there is no formal difficulty in applying the usual interpretation of the wave function. One might phrase this interpretation by saying that $\sum_\sigma |\tilde{\psi}(\tilde{\nu}, \sigma; \varphi)|^2$ is the probability density for obtaining the result $\tilde{\nu}$ from a measurement of the volume at internal-time φ. What, though, are the circumstances specified by saying that the measurement was made "at internal-time φ"? Since φ is a possible value of a scalar field, this can only mean that the measurement coincided with the scalar field actually having the value φ (or ϕ). According to the point of view we are considering, this means that the scalar field is unobservable *in principle* (not merely owing to the lack of a suitable observing instrument) and also that it is not subject to quantum indeterminacy, since we take it to assume the definite value φ (or ϕ) without being observed. The existence of a field with these unusual properties is not ruled out by observation. Its theoretical status is, however, very awkward. In the collection of kinematical operators (3.1) and in the constraint equation (3.3), expressed in the representation-independent form $(\hat{C}_{\rm grav} + \hat{C}_{\rm matter})\Psi = 0$ with $\hat{C}_{\rm matter} = \hat{p}_\phi$, the scalar field appears on exactly the same footing as the volume. In a more general model, any field that is treated as giving rise to an internal-time parameter, whether it be a matter field or a metric field, appears on exactly the same footing as any other quantum field. Its special "Heraclitian" status is conferred purely by the decision to construct a particular family of gauge-invariant quantities, such as $\hat{V}(\varphi)$, or to assign a special role to ϕ in writing a solution to the constraint equation. This status might instead be conferred on some other variable, as we have done for the volume in the present example. It seems unreasonable to us that this arbitrary choice should suffice to render unobservable some field whose energy nevertheless contributes to the constraint (or

Friedmann) equation, so we conclude that viewpoint 1 is hard to sustain consistently. The alternative is

Viewpoint 2: internal time is observable We now suppose that φ *is* a value obtained from observation of a physical scalar field. What can now be made of the statement that $\sum_\sigma|\tilde{\psi}(\tilde{\nu},\sigma;\varphi)|^2$ is the probability density for obtaining the result $\tilde{\nu}$ from a measurement of the volume at internal-time φ? Once again, it is necessary to ascertain what is meant by "at internal-time φ". Since $\tilde{\nu}$ and φ are both values obtained from observation, this phrase must mean that the two observations coincide, in some suitable sense. (In ordinary language, one would like to say that this coincidence consists in the volume and the field being observed at the same time, but if the notion of time is provided by φ itself, such an explanation is of doubtful value.) The scalar field either is or is not indeterminate. Suppose that it is *not* subject to quantum-mechanical indeterminacy. We might then envisage the history of the universe as a sequence of states, labeled by determinate values of the scalar field, and say that "at internal-time φ" means "in the state labeled by the value φ", regardless of whether that value is actually determined by observation or not. Here, we encounter the same difficulty as with viewpoint 1, namely, that the scalar field is deprived of its quantum indeterminacy only by an arbitrary choice in the way the mathematics is presented, and it still appears unreasonable to us that this should be so. Suppose, then that both $\tilde{\nu}$ and φ are indeterminate. The only method known to us of quantifying the results of joint observations of a pair of indeterminate variables is a joint probability density, and the only way we know of restricting probabilistic statements to a specified value of one of these variables is the associated conditional probability. If we wish to retain some aspect of the probability interpretation of a wave function, then, it seems that the probabilities must be given in these terms. We considered in Sec. IV the possibility of modifying the usual probability interpretation in order to do that. The object $\tilde{\mathcal{P}}(\tilde{\nu};\varphi)=\sum_\sigma|\tilde{\psi}(\tilde{\nu},\sigma;\varphi)|^2$ treats $\tilde{\nu}$ and φ asymmetrically: because of its φ-independent normalization $\int\tilde{\mathcal{P}}(\tilde{\nu};\varphi)\mathrm{d}\tilde{\nu}=1$, it can serve as a probability density for $\tilde{\nu}$ at a fixed value of φ, but not the other way round. It is therefore a candidate (and, in fact, the only available candidate) for a conditional-probability density. Assuming that the possible results of joint measurements of two indeterminate variables are governed by some joint probability density, there is a well-defined way of obtaining conditional probabilities for either of them, and if the two variables appear in the quantum theory on the same footing, then the same quantum-mechanical procedure for obtaining these conditional probabilities must work for both of them. In this paper, we have studied an model for which two candidates for the conditional-probability densities can be found explicitly by the same procedure, and we have shown that they are *not* consistent with any joint probability density. This counterexample is enough to show that the joint probability interpretation is not valid in general. We conclude that viewpoint 2 is also hard to sustain consistently.

We think that the possibilities considered above cover all the readily apparent strategies by which one might attempt to supply a clear meaning for the notion of internal time and, for the reasons given, we do not think that any of them is successful. Moreover, we believe it would be hard to find one that does not fall foul of one or more of the difficulties we have identified. One would have, for example, to find the physical principle that explains why some particular quantum field takes on determinate values, while other fields that appear in one's theory on the same footing retain their quantum indeterminacy; or to explain why the statistics of several indeterminate variables need not involve the existence of a joint probability density; or, perhaps, to explain why internal time is neither observable nor unobservable, so that neither of the above viewpoints applies.

As described in Sec. V, a variant of the relational-time construction proposed in [18,28] is capable of circumventing these difficulties. We augmented the model by including an idealized description of a small test clock, which we imagine to be internal to a specific observer, in this case a comoving observer. The time parameter τ that labels families of Dirac observables is, classically, at least, the proper time that elapses along the observer's worldline. Its value is not to be regarded as obtained from observation of any physical clock and, from the point of view of the observer in question, plays the same role as the *external* time in textbook Newtonian or quantum mechanics. The test clock itself we take to be unobservable in principle by the observer in question, by virtue of being internal to that observer. Using this construction, we could obtain *bona fide* joint probabilities, which describe genuine correlations between cosmological observables and whatever time-keeping device the observer might use in the course of recording observations.

In the case that the observer uses the cosmological scalar field as a (large) clock, the resulting conditional-probability density for the volume essentially coincides with the evolving probability density obtained using the scalar field as an internal time. This is a special feature of the Brown-Kuchař field, whose contribution to the Hamiltonian constraint (2.9) is just its canonical momentum. For the same reason, this model is deparametrizable in the variable ϕ: as indicated above, by substituting $p_\phi\to-\mathrm{i}\hbar\partial_\phi$, one converts the constraint equation into a Schrödinger-like equation that governs evolution in the internal time ϕ. We do not think, however, that deparametrizability is in itself sufficient to identify a preferred internal time, though it does, of course, make the implementation straightforward. In general, a model is deparametrizable in a variable ϕ if the constraint $C(\phi,p_\phi,\boldsymbol{\omega})=0$ can be solved to obtain $p_\phi=-\Theta(\boldsymbol{\omega})$, where Θ is independent of ϕ, and $\boldsymbol{\omega}$ denotes the remaining canonical variables. Again, one can substitute $p_\phi \to -i\hbar\partial_\phi$ to obtain a Schrödinger-like equation, but, again, the parameter φ cannot be interpreted as the observed value of some physical quantity, and it is hard to see what other meaning it might have. Moreover, this strategy suffers from what Kuchař [2] calls the ''Hilbert space problem''. That is, the Schrödinger-like evolution can be implemented only if the inner product on $\mathcal{H}_{\text{phys}}$ is chosen so as to make $\hat{\Theta}$ self-adjoint. The same inner product will not necessarily confer self-adjointness on the generators of evolution with respect to other candidates for an internal time, and it seems somewhat unreasonable that the inner product, and hence the quantum theory as a whole, should depend on this arbitrary choice of an internal time. In the present example, ϕ has a preferred status not because of deparametrizability as such, but rather because its contribution to the constraint is linear in p_ϕ with a constant coefficient.

We believe that the idea of a test clock, internal to some specific observer and localized on that observer's worldline, as implemented here provides an improved notion of time evolution, but it would be at best premature to suggest that it yields a definitive solution to the problem of time in general. Among the limitations of the proposal as we have so far described it are the following. (1) While the construction appears to be successful in spatially homogeneous models with a single constraint, it does not follow that a similar construction will work in more general spacetimes. (2) In particular, we have bypassed any explicit description of the observer's worldline by considering only a comoving observer in a Friedmann-Robertson-Walker universe. In more general situations, it would be essential to retain as further quantum degrees of freedom the coordinates $\hat{x}^\mu(\tau)$ that specify the worldline, and it remains to be seen whether this more general construction can be implemented successfully. A scheme of this kind is likely to yield time-dependent Dirac observables constructed only from fields in the vicinity of the observer's worldline. This, however, is in principle sufficient to account for the time-dependent observations made by an (ideally very long-lived) astronomer, since these local fields would include all the photons entering the astronomer's telescope. (3) The time parameter τ survives the passage to quantum mechanics as a c-number parameter, but it is not obvious that this parameter can be unambiguously described in the quantum theory as the arc length of a worldline. (4) In the context of simple cosmological models, at least, it seems to be an inevitable consequence of the constraint that some object to which one is inclined to attribute a real physical existence turns out to be unobservable. We think it is plausible that a clock which is internal to some observer should turn out to be unobservable in a description of the universe ''from that observer's point of view''. However, this plausible form of words is not directly mandated by the formalism, and some other way of understanding the unobservability of whatever quantity is eliminated by solution of the constraint may turn out to be better founded. We plan to address these issues in future work.

APPENDIX: EXTENSION OF EVOLUTION TO NEGATIVE VALUES OF $\sqrt{\nu}$

The unitary evolution of, say, the wave function (4.14) in the parameter $\sqrt{\nu}$ is smooth at $\sqrt{\nu} = 0$, and can be continued to negative values of this parameter. Here, we examine what, if anything, this continuation might mean and, in particular, whether it leads to a resolution of the classical singularity, contrary to what is suggested by Fig. 2.

Observe first that this continuation can equally well be carried out at the classical level. In terms of the proper time coordinate t, the volume $\bar{v}(t)$ given in (2.11) vanishes at $t_s = (6\pi G p_v)^{-1}$, but both $\bar{v}(t)$ and $\bar{\phi}(t)$, given in (2.10) are smooth functions over the whole range of proper time values $-\infty < t < \infty$. Eliminating t from these two smooth functions, we obtain the Dirac observable $\Phi(\nu)$ given in (2.15), which is a smooth function of the parameter $\sqrt{\nu}$, for $-\infty < \sqrt{\nu} < \infty$. The ''big-bang'' singularity is not apparent in any of these smooth functions, but it *is* apparent in the canonical momentum $\bar{p}_v(t)$ given in (2.12), which has an infinite discontinuity at $t = t_s$. A short calculation shows that the Dirac observable $P_v(\nu) := \bar{p}_v(t_\nu)$ (where, as in Sec. II B, t_ν is the time at which $\bar{v}(t_\nu) = \nu$) can be expressed as

$$P_v(\nu) = \frac{1}{6\pi G[\Phi(0) - \Phi(\nu)]}, \tag{A1}$$

and of course it exhibits an infinite discontinuity at $\sqrt{\nu} = 0$.

To be clear about the meaning of this discontinuity, consider the projection of the phase-space trajectories onto the (v, p_v) plane, depicted schematically in Fig. 3. As we have remarked previously, these trajectories fill out two disjoint regions of the phase plane, which we have

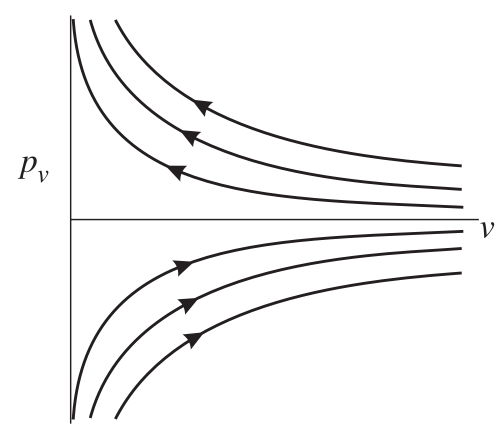

FIG. 3. Schematic depiction of the classical phase-space trajectories projected onto the (v, p_v) plane. Arrows indicate the direction of increasing t.

labeled by $\sigma = -\text{sgn}(v^{1/2}p_v)$. Each expanding trajectory in the lower quadrant of the phase plane, $\sigma = 1$, is completely parametrized by the range of values $t_s < t < \infty$. It has an initial singularity, in the precise sense that it reaches the end of the available phase space at a finite parameter value $t = t_s$. Similarly, each contracting trajectory in the upper quadrant of the phase plane, $\sigma = -1$ is completely parametrized by $-\infty < t < t_s$ and has a final singularity at $t = t_s$. We emphasize that the expanding trajectories are distinct from the contracting ones: no trajectory passes from one of the regions $\sigma = \pm 1$ to the other. Thus, although the two functions $\bar{v}(t)$ corresponding to an expanding trajectory and its mirror-image contracting trajectory can be combined to form a single smooth function,[6] they nevertheless still describe two distinct trajectories. In terms of the Dirac observables, the infinite discontinuity (and the associated sign change) in $P_v(\nu)$, regarded as a function of $\sqrt{\nu}$, indicates that positive and negative values of $\sqrt{\nu}$ correspond to distinct trajectories, notwithstanding the fact that $\Phi(\nu)$ is smooth at $\sqrt{\nu} = 0$. In fact, we see that allowing for both signs of $\sqrt{\nu}$ is completely redundant; the classical dynamics is fully described by restricting $\sqrt{\nu}$ to positive values, and using the discrete parameter σ to distinguish expanding and contracting trajectories. Indeed, the continuation to negative values of $\sqrt{\nu}$ is potentially misleading: it suggests that the system can pass from a contracting state to an expanding state by dynamical evolution, which is not true.

The preceding discussion involves an assumption that deserves to be made explicit, namely, that the points of the half-plane depicted in Fig. 3 are in one-to-one correspondence with distinct physical states of the system in question. In particular, (i) the geometrical state of the universe is completely specified by the values of v and p_v, so that the positive and negative square roots of v specify exactly the same state, and (ii) the limits $p_v \to +\infty$ (the Hubble parameter $\dot{a}/a$ approaches $-\infty$) and $p_v \to -\infty$ (the Hubble parameter approaches $+\infty$) are physically *different* limits. To see the importance of this assumption, consider the change of variable $v = u^2$, and suppose that positive and negative values of $u = \sqrt{v}$ specify physically *distinct* states. The momentum conjugate to u is $p_u = 2v^{1/2}p_v$, and the solutions (2.11) and (2.12) to the Hamilton equations become

$$\bar{u}(t) = u - 3\pi G p_u t \tag{A2}$$

and $\bar{p}_u(t) = p_u =$ constant. For this system, the phase space is the whole (u, p_u) plane. It is a different system from the one described by (v, p_v) because, although the dynamical equations of the two systems are locally related by a canonical transformation, there is no one-to-one correspondence between their distinct physical states. For the (u, p_u) system, the distinct trajectories are simply the lines $p_u =$ constant, each smoothly parametrized by t in the whole range $-\infty < t < \infty$. The "big bang" at $v = 0$ corresponds to the line $u = 0$, but this is now in the interior of the phase space, and there is no singularity there. As with any dynamical system, one is free to contemplate quantities such as $p_v := p_u/2u$, which diverge at $u = 0$, but that is a different matter. Each trajectory of the (u, p_u) system is divided, at $\bar{u} = 0$, into two segments; on one segment, the volume $\bar{u}(t)^2$ is an increasing function of t, while on the other it is a decreasing function. Thus, although there is no unique correspondence between states of the (v, p_v) and (u, p_u) systems in general, one can map a mirror-image pair of (v, p_v) trajectories into a single (u, p_u) trajectory. The image of this map *excludes* the point $u = 0$, unless one adds to the (v, p_v) phase space the "points at infinity" $(v, p_v) = (0, \pm\infty)$, and stipulates that both of these points map to the single point $(0, p_u)$ for the appropriate value of p_u.

It seems fairly clear that classical Friedman-Robertson-Walker cosmology corresponds to the (v, p_v) system rather than to the (u, p_u) system. The basic metric variable is the square of the scale factor, $a^2 = v^{2/3}$, and real, non-negative values of a^2 correspond straightforwardly to real, non-negative values of v. Were we to write instead $a^2 = u^{1/3}$, then allowing u to assume negative values entails allowing a^2 to assume complex values—a possibility not conventionally embraced by cosmologists. We might protect the reality of the scale factor by writing instead $a^2 = |u|^{1/3}$. In that case, negative values of u specify exactly the same geometry as positive values, and should be excluded, to avoid overcounting of states. With that restriction, expanding and contracting states once more belong to *distinct* classical trajectories, which terminate at finite parameter values at initial or final singularities: this *restriction* of the (u, p_u) system is equivalent to the (v, p_v) system. This, of course, accords with the conventional view that the "big bang" of classical FRW cosmology is an initial singularity, rather than the midpoint of a bouncing history, at which the Hubble parameter happens to change sign via an infinite discontinuity. According to this conventional view (which we adopt throughout), avoidance of the singularity must involve a nonzero minimum volume.

In this paper, we describe a quantization of the (v, p_v) system. One might perhaps attempt to quantize the (u, p_u) system, and to interpret it as a quantum cosmology in which complex values of the scale factor are allowed, on the grounds that a quantum system can exist in states that are classically inaccessible. These are not necessarily good grounds, because quantization typically preserves the configuration space (in this case, $\sqrt{v} > 0$) of the original

[6]Specifically, for $p_v > 0$, say, the function $\bar{v}(t) = v(1 - 6\pi G p_v t)^2$ describes, for $t < (6\pi G p_v)^{-1}$, the volume on the contracting trajectory that passes through (v, p_v) at $t = 0$. Writing $t = 2(6\pi G p_v)^{-1} + t'$, we find that for $t > (6\pi G p_v)^{-1}$, the same function $\bar{v}(t) = v(1 + 6\pi G p_v t')^2$ describes the volume on the expanding trajectory that passes through $(v, -p_v)$ at $t' = 0$.

classical system. (For example, the wave function of a quantum mechanical particle in a box may have support at points inside the box which a classical particle cannot reach, but it is not defined outside the box). Because we deal specifically with the (v, p_v) system, it is meaningful to ask whether the classical singularity is resolved in the quantum theory. This question would not arise for the (u, p_u) system, because it has no classical singularity.

The issue this appendix is intended to address can now be stated clearly. In nonrelativistic quantum mechanics, evolution with respect to an *external* time parameter t, implemented by a unitary operator $\hat{U}(t) = \exp(-i\hat{H}t/\hbar)$, can be extended to the whole range of parameter values $-\infty < t < \infty$. An impediment to this evolution typically indicates a failure of unitarity. That is, if we find a state Ψ such that $\hat{U}(t)\Psi$ is not well defined for all t, then $\hat{U}$ is not a genuine unitary operator. In constructing the wave function (4.14) and exhibiting the corresponding probability density in Fig. 2, we have terminated the evolution at $\sqrt{\nu} = 0$. Two questions might be raised: (i) does this termination indicate some failure of unitarity and, consequently, a flaw in our quantization scheme? (ii) by terminating the evolution, do we miss some important feature of the quantum theory, such as the resolution of the singularity that seems to be apparent from Fig. 1, where the internal-time parameter is φ?

The answer to both of these questions is no. The evolution in $\sqrt{\nu}$ is perfectly unitary, because the Hamiltonian $\hat{H}_\nu$ is self-adjoint, and can perfectly well be continued to negative values of $\sqrt{\nu}$. We have actually terminated it at $\sqrt{\nu} = 0$, because ν is not an external time, but an *internal* time parameter. That is, it is supposed to correspond in some way to a value of the volume v, and all physically distinct, positive values of v are accounted for by the corresponding *positive* values of $\sqrt{\nu}$.

As we have discussed at length above, negative values of $\sqrt{\nu}$ are redundant, and potentially misleading, in the classical theory. To ascertain whether the same is true in the quantum theory, we first replot, in Fig. 4, the probability density of Fig. 2 as a function of $\sqrt{\nu}$, allowing for negative values. It is peaked on a function $\underset{\sim}{\varphi}(\sqrt{\nu})$ corresponding to the family of Dirac observables (2.15), with a negative value of p_v (because we chose a wave function $\psi(\epsilon_D, \sigma)$ that vanishes for $\sigma = -1$). This function corresponds to a contracting universe for $\sqrt{\nu} < 0$, and to an expanding universe for $\sqrt{\nu} > 0$ but, as discussed above, the fact that this smooth function can be constructed does not imply that the singularity is resolved. In particular, $\underset{\sim}{\varphi}(\sqrt{\nu})$ is just a straight line, passing smoothly through $(\sqrt{\nu}, \underset{\sim}{\varphi}) = (0, 0)$, so there is no minimum volume. Moreover, having in hand the quantum Dirac observable $\hat{\Phi}$, given in (3.18), we can construct the operator $\hat{P}_v(\nu)$ from the classical expression (A1), and it clearly diverges at $\sqrt{\nu} = 0$. Thus, the view of this quantum-mechanical state furnished by ν as an internal-time parameter exactly parallels the classical situation. The $\sqrt{\nu} < 0$ region of Fig. 4 simply reproduces the probability density that would be obtained by choosing an initial wave function in the $\sigma = -1$ sector instead of the $\sigma = +1$ sector. Classically, the fact that $\Phi(\nu)$ can be smoothly continued to artificial negative values of $\sqrt{\nu}$ does not imply that the system can pass dynamically between the $\sigma = \pm 1$ sectors. The fact that a wave function peaked at this same function can be constructed gives no reason to suppose that the quantum-mechanical situation is any different.

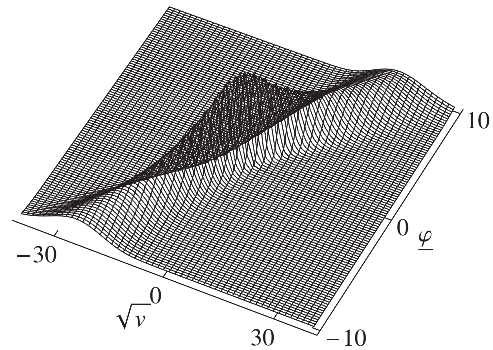

FIG. 4. The probability density for the scalar field $\underset{\sim}{\varphi}$ evolved in the internal-time parameter $\sqrt{\nu}$.

How does this affect the comparison of Fig. 2 with Fig. 1? Recall that, while Fig. 2 depicts the evolution with the internal-time parameter ν of the probability density for values $\underset{\sim}{\varphi}$ of the scalar field, Fig. 1 shows the evolution with internal-time parameter φ of the probability density for values $\tilde{\nu}$ of the volume. To make a meaningful comparison with Fig. 4, we convert the latter to a probability density for values $\sqrt{\tilde{\nu}}$ of the square root of the volume. According to the standard formula, the required expression is

$$\tilde{\mathcal{P}}(\tilde{\nu}; \varphi) \left| \frac{d\tilde{\nu}}{d\sqrt{\tilde{\nu}}} \right| = 2\tilde{\mathcal{P}}((\sqrt{\tilde{\nu}})^2; \varphi)|\sqrt{\tilde{\nu}}|. \qquad \text{(A3)}$$

This probability density is clearly an even function of $\sqrt{\tilde{\nu}}$, and is shown in Fig. 5. We observe again a clear qualitative difference between Figs. 4 and 5. Thus, the expedient of continuing to negative values of $\sqrt{\nu}$ does not remove the contradiction between the views provided of the same quantum state by the two choices of internal time.

For completeness, we mention that there is a strategy (though one that we believe to be unsound) which serves to lessen the discrepancy apparent to the eye between the pairs of probability densities exhibited in Figs. 1, 2, 4, and 5. By contemplating both positive and negative values of $\sqrt{\nu}$, we implicitly adopt the view that these values correspond to physically distinct states of the internal clock that supplies this notion of internal time. As we have argued, this view is misguided in the context of the (v, p_v) system we are dealing with. To adopt this view

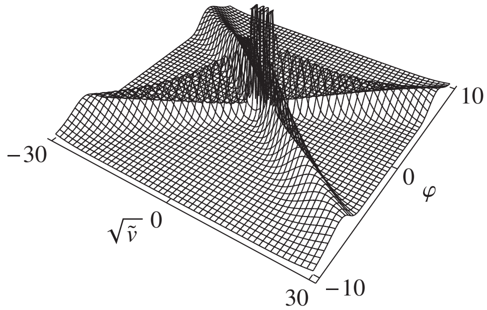

FIG. 5. The probability density for the square root of the volume, $\sqrt{\tilde{\nu}}$ evolved in the internal-time parameter φ.

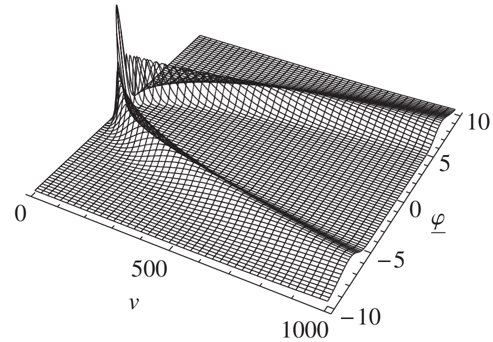

FIG. 6. The probability density for the scalar field $\underset{\sim}{\varphi}$ evolved in the internal-time parameter ν. In contrast to FIG. 2, we suppose that different signs of $\sqrt{\nu}$ correspond to distinct states of the internal clock.

consistently, one should instead quantize the (u, p_u) system which, we have argued, does not have a satisfactory cosmological interpretation. The relationship between the two resulting quantum systems is ambiguous, not only because of the operator-ordering choices that must be made in quantizing each system separately, but also because the equation $p_u = v^{1/2} p_v$ that relates them classically has itself an ordering ambiguity if we try to regard it as a quantum-mechanical statement. Let us, however, set these important difficulties aside, and suppose that, although on the occasion of any particular measurement our clock reading actually supplies a value of $\sqrt{\nu}$, which may have either sign, we record only the value $\nu = (\sqrt{\nu})^2$. Assuming that positive and negative clock readings are equally likely, we obtain a probability density for $\underset{\sim}{\varphi}$ at the internal time ν, namely

$$\underset{\sim}{\mathcal{P}}(\underset{\sim}{\varphi}; \nu) = \frac{1}{2}\left[\underset{\sim}{\mathcal{P}}(\underset{\sim}{\varphi}; \sqrt{\nu}) + \underset{\sim}{\mathcal{P}}(\underset{\sim}{\varphi}; -\sqrt{\nu})\right], \tag{A4}$$

where, on the right-hand side, $\underset{\sim}{\mathcal{P}}(\underset{\sim}{\varphi}; \sqrt{\nu})$ is the function depicted in Fig. 4.

This new probability density is shown in Fig. 6, which appears similar, though not identical, to Fig. 1. In particular, because Fig. 6 merely presents Fig. 4 in a different form, it depicts a probability density peaked on a classical trajectory that passes *through* the point $(\nu, \underset{\sim}{\varphi}) = (0, 0)$, indicating that the singularity is not resolved. Exact agreement is, of course, not to be expected, because Fig. 1 is the probability for the volume, evolved with internal time φ, while Figs. 2 and 6 are probability densities for the scalar field, evolved with internal time ν. To make a quantitative comparison, we can ask whether the new probability density (A4) satisfies the consistency condition (4.10) for these two probability densities to arise from one underlying joint probability density. The answer is that it does not. Thus, even if one is willing to accept the construction of (A4) as plausible, the two views of the quantum evolution furnished by the two choices of internal time remain inconsistent.

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