Odd-parity bipolar spherical harmonics

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Bipolar spherical harmonics (BiPoSHs) provide a general formalism for quantifying departures in the cosmic microwave background (CMB) from statistical isotropy and from Gaussianity. However, prior work has focused only on BiPoSHs with even parity. Here we show that there is another set of BiPoSHs with odd parity, and we explore their cosmological applications. We describe systematic artifacts in a CMB map that could be sought by measurement of these odd-parity BiPoSH modes. These BiPoSH modes may also be produced cosmologically through lensing by gravitational waves (GWs), among other sources. We derive expressions for the BiPoSH modes induced by the weak lensing of both scalar and tensor perturbations. We then investigate the possibility of detecting parity-breaking physics, such as chiral GWs, by cross-correlating opposite-parity BiPoSH modes with multipole moments of the CMB polarization. We find that the expected signal-to-noise of such a detection is modest.

I. INTRODUCTION

The detection of anisotropies in the cosmic microwave background (CMB) [1] has revolutionized the precision with which cosmological measurements can be made. Most of the information that has been obtained from the CMB so far has come from its power spectrum, the two-point correlation function, under the assumptions of isotropy and homogeneity. However, in recent years, attention has been paid to effects that go beyond the power spectrum, such as weak lensing [2], cosmic birefringence [3,4], and departures from statistical isotropy (SI) [5–7] and from Gaussianity [8–10].

Bipolar spherical harmonics (BiPoSHs) [11–13] provide an elegant and general formalism for quantifying a number of these physical effects. If the CMB map is Gaussian and statistically isotropic, then its statistics are specified entirely in terms of the power spectrum C_l, the expectation value of the squared magnitude of the spherical-harmonic coefficients a_{lm} for the map, and there are no correlations between different a_{lm}s. A wide variety of departures from SI and Gaussianity induce correlations between different a_{lm}s. The point of the BiPoSH formalism is to parametrize correlations between two different coefficients, a_{lm} and $a_{l'm'}$, that represent two different "angular-momentum" states, in terms of total angular momenta L and M. Bipolar spherical harmonics have been used to search for nonstandard cosmic topology [14], anisotropy in primordial power [7,15], and model-independent departures from SI [13,16–21]. They have also been used to test for asymmetric beams [22] and/or other systematic artifacts in WMAP [23]. BiPoSHs for polarization have been proposed to search for position-dependent rotation of the CMB polarization [24–26].

However, there is still more that can be done with bipolar spherical harmonics, and the purpose of this paper is to enumerate some of these opportunities. First and foremost, we point out here that almost all prior work on BiPoSHs has considered only BiPoSHs with even parity (Sec. II). There exists an entire other set of BiPoSHs that have the opposite parity, and these can provide probes of both cosmological effects and systematic artifacts that would remain elusive with the even-parity BiPoSHs that have been considered so far. We show, for example, that lensing by gravitational waves (GWs) can excite odd-parity BiPoSHs, and we describe a pointing error that could also excite these modes. In the process, we also show how gravitational lensing, by both density perturbations as well as GWs, can be described in terms of even- and odd-parity BiPoSHs (Sec. III). Finally, we discuss how odd-parity BiPoSHs could be used as probes of parity violation, and consider, in particular, the cross-correlation of opposite parity CMB lensing and polarization components (Sec. IV). We calculate the anticipated spectra and errors for such correlations, and determine that a large signal-to-noise is not expected for these cross-correlations, given the current upper bounds on a GW background.

II. REVIEW OF BIPOLAR SPHERICAL HARMONICS

A. Statistically isotropic and Gaussian maps

A CMB temperature map $T(\hat{n})$, as a function of position $\hat{n}$ on the sky, can be decomposed into spherical-harmonic coefficients

$$a_{lm} = \int d^2\hat{n}\, T(\hat{n}) Y_{lm}^*(\hat{n}).$$

If the map is statistically isotropic and Gaussian, then the statistics can be determined entirely in terms of the power spectrum C_l, defined by

$$\langle a_{lm}a^*_{l'm'}\rangle = C_l\delta_{ll'}\delta_{mm'}, \tag{1}$$

where the angle brackets denote an average over all realizations, and $\delta_{ll'}$ and $\delta_{mm'}$ are Kronecker deltas. Equation (1) states that all of the a_{lm} are uncorrelated, and Gaussianity further dictates that the probability distribution function for any a_{lm} to take on a particular value is a Gaussian distribution with variance C_l.

The spatial temperature autocorrelation function is defined to be $C(\hat{n}, \hat{n}') \equiv \langle T(\hat{n})T(\hat{n}')\rangle$. Most generally it is a function of the two directions $\hat{n}$ and $\hat{n}'$. However, if the map is statistically isotropic and Gaussian, then the spatial correlation function depends only on the angle θ, given by $\cos\theta = \hat{n} \cdot \hat{n}'$, between the two directions. In this case,

$$C(\hat{n}, \hat{n}') = \sum_l \frac{(2l+1)}{4\pi}C_lP_l(\hat{n} \cdot \hat{n}'),$$

where $P_l(x)$ are the Legendre polynomials.

B. Departures from Gaussianity/SI

Departures from Gaussianity and/or SI will induce correlations between different a_{lm}. The most general correlation between any two a_{lm} can be written,

$$\langle a_{lm}a^*_{l'm'}\rangle = C_l\delta_{ll'}\delta_{mm'}$$
$$+ \sum_{LM;L>0}(-1)^{m'}\langle lml', -m'|LM\rangle A^{LM}_{ll'}, \tag{2}$$

where C_l is the (isotropic) power spectrum, $\langle lml'm'|LM\rangle$ are Clebsch-Gordan coefficients, and the $A^{LM}_{ll'}$ are BiPoSH coefficients. The spatial two-point correlation function is then

$$C(\hat{n}, \hat{n}') = \sum_l \frac{(2l+1)}{4\pi}C_lP_l(\hat{n} \cdot \hat{n}')$$
$$+ \sum_{ll'LM}A^{LM}_{ll'}\{Y_l(\hat{n}) \otimes Y_{l'}(\hat{n}')\}_{LM}, \tag{3}$$

where

$$\{Y_l(\hat{n}) \otimes Y_{l'}(\hat{n}')\}_{LM} = \sum_{mm'}\langle lml'm'|LM\rangle Y_{lm}(\hat{n})Y_{l'm'}(\hat{n}') \tag{4}$$

are the bipolar spherical harmonics (BipoSHs). These BiPoSHs constitute a complete orthonormal basis for functions of $\hat{n}$ and $\hat{n}'$ in terms of total-angular-momentum states labeled by quantum numbers L and M composed of angular-momentum states with lm and $l'm'$; they are an alternative to the outer product of the $\{l, m\}$ and $\{l', m'\}$ bases.

C. Odd-parity bipolar spherical harmonics

It is instructive to decompose $A^{LM}_{ll'}$ into its odd and even parity parts,

$$A^{LM}_{ll'} = A^{\oplus LM}_{ll'}\frac{[1 + (-1)^{l+l'+L}]}{2} + A^{\ominus LM}_{ll'}\frac{[1 - (-1)^{l+l'+L}]}{2}, \tag{5}$$

where $A^{\oplus LM}_{ll'}$ ($A^{\ominus LM}_{ll'}$) are zero for the sum $l + l' + L$ being odd (even). It follows from the symmetry $C(\hat{n}, \hat{n}') = C(\hat{n}', \hat{n})$ that $A^{\oplus LM}_{ll'}$ ($A^{\ominus LM}_{ll'}$) are (anti) symmetric in l and l'. We also infer that $[A^{\oplus LM}_{ll'}]^* = (-1)^M A^{\oplus L-M}_{ll'}$ and $[A^{\ominus LM}_{ll'}]^* = (-1)^{M+1}A^{\ominus L-M}_{ll'}$. Thus, odd-parity BiPoSHs vanish for $l = l'$. Prior literature has considered physical effects (e.g., nontrivial topologies [27], SI violation [13,28]) that produce only $A^{\oplus LM}_{ll'}$, the even-parity BiPoSHs, and measurements have been carried out with WMAP data only for the $A^{\oplus LM}_{ll'}$ [16,29]. In this paper, we consider also the odd-parity BiPoSHs $A^{\ominus LM}_{ll'}$.

Estimators for the BiPoSH coefficients (both the $\oplus$ and $\ominus$ modes) can be constructed from a map of the CMB temperature field $T(\hat{n})$, as follows:

$$\widehat{A^{LM}_{ll'}} = \sum_{mm'}W_l^{-1}W_{l'}^{-1}a^{\text{map}}_{lm}a^{*\text{map}}_{l'm'}(-1)^{m'}\langle lml', -m'|LM\rangle, \tag{6}$$

and this estimator has a variance, under the null hypothesis (an SI Gaussian map),

$$\langle\widehat{A^{LM}_{ll'}}\widehat{A^{L'M'}_{\bar{l}\bar{l}'}}^*\rangle = \delta_{LL'}\delta_{MM'}[\delta_{l\bar{l}}\delta_{l'\bar{l}'} + (-1)^{l+l'+L}\delta_{l\bar{l}'}\delta_{\bar{l}l'}]$$
$$\times C_l^{\text{map}}C_{l'}^{\text{map}}W_l^{-2}W_{l'}^{-2}, \tag{7}$$

where $a^{\text{map}}_{lm} = W_l a_{lm} + a^{\text{noise}}_{lm}$ and $C_l^{\text{map}} = W_l^2 C_l + N_l$ are the temperature spherical-harmonic coefficients and power spectrum corrected for detector noise and finite resolution. The Gaussian detector window function, which encapsulates the effects of finite detector resolution, is given by $W_l \equiv \exp[-l^2\theta^2_{\text{FWHM}}/(16\ln2)]$, where θ_{FWHM} is the full width at half maximum of the detector. The instrumental noise contribution to the temperature power spectrum is given by

$$N_l = \frac{4\pi(\text{NET})^2}{t_{\text{obs}}\sqrt{f_{\text{sky}}}},$$

where f_{sky} is the fraction of the sky observed, NET is the noise equivalent temperature of the detector, and t_{obs} is the length of time over which the CMB was observed by a particular survey. We notice that the variance in Eq. (7) vanishes for odd parity and $l = l' = \bar{l} = \bar{l}'$, which is expected given that odd-parity BiPoSHs with $l = l'$ vanish.

The noise in any individual $A^{LM}_{ll'}$ is large, and so a search for a statistically significant departure from zero in one or a handful of $A^{LM}_{ll'}$ will probably not be too effective. It is better to consider specific models and/or parameterizations

for departures from SI/Gaussianity and then combine the $A_{ll'}^{LM}$ into a minimum-variance estimator for the SI/Gaussianity-violating parameters of those models. For example, [11,12] considered the bipolar power spectrum $\kappa_L \equiv \sum_{ll'M} |A_{ll'}^{LM}|^2$ as a parameterization for departures from SI. As another example, [15] combined $A_{ll'}^{LM}$ s with $L = 2$ and $l' = l$, $l \pm 2$ to derive minimum-variance estimators for the amplitude of an inflation-induced primordial-power quadrupole of the type considered in [30].

III. GRAVITATIONAL LENSING

A. Gradient and curl-type deflections

Consider a statistically isotropic and homogeneous Gaussian temperature map $T_g(\hat{n})$ on the sphere, where $\hat{n}$ is a position on the sky. Now suppose that each point on the sky $\hat{n}$ has been deflected from an original direction $\hat{n} + \vec{\Delta}(\hat{n})$ so that the observed temperature is $T(\hat{n}) = T_g(\hat{n} + \vec{\Delta}) \simeq T_g(\hat{n}) + \vec{\Delta} \cdot \vec{\nabla}_{\hat{\theta}} T_g(\hat{n})$. This deflection might come about cosmologically through weak gravitational lensing or may arise as an instrumental/measurement artifact (for example, if there are pointing errors).

The most general deflection field $\vec{\Delta}$ can be written in vector notation as

$$\vec{\Delta} = \vec{\nabla}_{\hat{\theta}} \phi(\hat{n}) + \vec{\nabla}_{\hat{\theta}} \times \Omega(\hat{n}), \qquad (8)$$

or in component notation, $\Delta_i = (\nabla_{\hat{\theta}})_i \phi(\hat{n}) + \epsilon_{ij}(\nabla_{\hat{\theta}})_j \Omega(\hat{n})$,[1] in terms of two scalar functions $\phi(\hat{n})$ and $\Omega(\hat{n})$ on the sphere, where $\vec{\nabla}_{\hat{\theta}}$ is the angular covariant derivative on the unit sphere. In other words, the most general vector field on a two-sphere can be written as the gradient of some scalar field $\phi(\hat{n})$ plus the curl of some other field $\Omega(\hat{n})$. Weak gravitational lensing by density perturbations gives rise, at linear order in the lensing potential, only to the gradient component. A curl component can arise cosmologically from second-order terms in the deflection field or from lensing by GWs. Systematic measurement effects may conceivably give rise to both types of deflections.

We now show that the $A_{ll'}^{\oplus LM}$ and $A_{ll'}^{\ominus LM}$ BiPoSH coefficients are induced, respectively, by the gradient and curl components of the deflection field. The change in the temperature moments induced by lensing is (at first order in ϕ and Ω),

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[1]Here, the Levi-Civita symbol on the unit sphere can be defined in terms of its three-dimensional equivalent as $\epsilon_{ij} = -\epsilon_{ijk} r_k$. The choice of sign here can be understood as the choice to have the spherical polar coordinates (θ, ϕ) form a right-handed coordinate system on the sky, since it will ensure that the basis vectors satisfy $\hat{e}_\theta \times \hat{e}_\phi = 1$.

$$\delta a_{lm} = \int d^2\hat{n} Y_{lm}^*(\hat{n})\{[\vec{\nabla}_{\hat{\theta}}\phi] \cdot [\vec{\nabla}_{\hat{\theta}}T(\hat{n})]$$
$$+ [\vec{\nabla}_{\hat{\theta}}\Omega(\hat{n})] \times [\vec{\nabla}_{\hat{\theta}}T(\hat{n})]\}$$
$$= \sum_{LM;L>0}\sum_{l'm'} a_{l'm'} \int d^2\hat{n} Y_{lm}^*(\hat{n})\{\phi_{LM}[\vec{\nabla}_{\hat{\theta}}Y_{LM}(\hat{n})]$$
$$\cdot [\vec{\nabla}_{\hat{\theta}}Y_{l'm'}(\hat{n})] + \Omega_{LM}[\vec{\nabla}_{\hat{\theta}}Y_{LM}(\hat{n})] \times [\vec{\nabla}_{\hat{\theta}}Y_{l'm'}(\hat{n})]\},$$

where in the second line we have decomposed

$$\phi(\hat{n}) = \sum_{L=1}^{\infty}\sum_{M=-L}^{L} Y_{LM}(\hat{n})\phi_{LM}, \qquad (9)$$

and similarly for $\Omega(\hat{n})$. We do not consider $L = 0$ modes of ϕ and Ω since they would not cause a deflection. In the notation of [31],

$$\vec{\nabla}_{\hat{\theta}}Y_{lm} = \sqrt{\frac{l(l+1)}{2}}[_1Y_{lm}\hat{m}_+ - _{-1}Y_{lm}\hat{m}_-],$$

where $_1Y_{lm}$ and $_{-1}Y_{lm}$ are spin-weighted spherical harmonics, the null coordinates $\hat{m}_\pm = (\hat{e}_\theta \mp i\hat{e}_\phi)/\sqrt{2}$, and the only nontrivial products of the null coordinates are $\hat{m}_+ \cdot \hat{m}_- = 1$, and $\hat{m}_+ \times \hat{m}_- = i$. Thus, it is obtained that

$$(\vec{\nabla}_{\hat{\theta}}Y_{LM}) \cdot (\vec{\nabla}_{\hat{\theta}}Y_{l'm'})$$
$$= -\frac{\sqrt{L(L+1)l'(l'+1)}}{2}[(_1Y_{LM})(_{-1}Y_{l'm'})$$
$$+ (_{-1}Y_{LM})(_1Y_{l'm'})], (\vec{\nabla}_{\hat{\theta}}Y_{LM}) \times (\vec{\nabla}_{\hat{\theta}}Y_{l'm'})$$
$$= -\frac{i\sqrt{L(L+1)l'(l'+1)}}{2}[(_1Y_{LM})(_{-1}Y_{l'm'}) - (_{-1}Y_{LM})(_1Y_{l'm'})].$$

Using the triple integral [31] of spin-weighted spherical harmonics, the δa_{lm} for the gradient and curl terms are obtained as

$$\delta a_{lm} = \sum_{LM;L>0}\sum_{l'm'}\frac{(-1)^{M+m}a_{l'm'}G_{ll'}^L}{\sqrt{(2L+1)l(l+1)}}$$
$$\times \left[\phi_{LM}\frac{[1 + (-1)^{l+l'+L}]}{2}\right.$$
$$\left. - i\Omega_{LM}\frac{[1 - (-1)^{l+l'+L}]}{2}\right]\langle lml', -m'|LM\rangle,$$

where

$$G_{ll'}^L \equiv \sqrt{\frac{L(L+1)l(l+1)l'(l'+1)(2l+1)(2l'+1)}{4\pi}}\langle l0l'1|L1\rangle.$$

Up to linear order in the deflection coefficients ϕ_{LM} and Ω_{LM}, the even- and odd-parity BiPoSH coefficients are then,

$$A_{ll'}^{\oplus LM} = \frac{\phi_{LM}}{\sqrt{2L+1}}\left[\frac{C_l G_{l'l}^L}{\sqrt{l'(l'+1)}} + \frac{C_{l'} G_{ll'}^L}{\sqrt{l(l+1)}}\right] = Q_{ll'}^{\oplus L}\phi_{LM},$$

(10)

$$A_{ll'}^{\ominus LM} = \frac{i\Omega_{LM}}{\sqrt{2L+1}}\left[\frac{C_l G_{l'l}^L}{\sqrt{l'(l'+1)}} - \frac{C_{l'} G_{ll'}^L}{\sqrt{l(l+1)}}\right] = Q_{ll'}^{\ominus L}\Omega_{LM},$$

(11)

where we have defined the quantities

$$Q_{ll'}^{\oplus L} = \frac{1}{\sqrt{2L+1}}\left[\frac{C_l G_{l'l}^L}{\sqrt{l'(l'+1)}} + \frac{C_{l'} G_{ll'}^L}{\sqrt{l(l+1)}}\right],$$

$$Q_{ll'}^{\ominus L} = \frac{i}{\sqrt{2L+1}}\left[\frac{C_l G_{l'l}^L}{\sqrt{l'(l'+1)}} - \frac{C_{l'} G_{ll'}^L}{\sqrt{l(l+1)}}\right].$$

Clearly, the gradient part contributes only to $A_{ll'}^{\oplus LM}$ and the curl part only to $A_{ll'}^{\ominus LM}$. Further, it is explicit that the gradient and curl parts of the deflection correspond, respectively, to the symmetric and antisymmetric (in $\{ll'\}$) parts of the total $A_{ll'}^{LM}$.

Suppose the $A_{ll'}^{LM}$ have been measured using the estimators in Eq. (6). If we then assume that lensing is the dominant source of BiPoSHs we can use Eqs. (7), (10), and (11) to construct maximum-likelihood estimators for the gradient and curl components of the deflection field,

$$\widehat{\phi_{LM}} = \frac{\sum\limits_{ll'} Q_{ll'}^{\oplus L*} \widehat{A_{ll'}^{\oplus LM}}/(W_l^{-2}W_{l'}^{-2}C_l^{\text{map}}C_{l'}^{\text{map}})}{\sum\limits_{ll'} |Q_{ll'}^{\oplus L}|^2/(W_l^{-2}W_{l'}^{-2}C_l^{\text{map}}C_{l'}^{\text{map}})},$$ (12)

$$\widehat{\Omega_{LM}} = \frac{\sum\limits_{ll'} Q_{ll'}^{\ominus L*} \widehat{A_{ll'}^{\ominus LM}}/(W_l^{-2}W_{l'}^{-2}C_l^{\text{map}}C_{l'}^{\text{map}})}{\sum\limits_{ll'} |Q_{ll'}^{\ominus L}|^2/(W_l^{-2}W_{l'}^{-2}C_l^{\text{map}}C_{l'}^{\text{map}})}.$$ (13)

The variance of these estimators, under the null hypothesis of no lensing, is given by

$$\langle\widehat{\phi_{LM}}\widehat{\phi_{L'M'}}^*\rangle \equiv \delta_{LL'}\delta_{MM'}(\sigma_L^\phi)^2$$

$$\equiv 2\delta_{LL'}\delta_{MM'}\left[\sum\limits_{ll'}|Q_{ll'}^{\oplus L}|^2/(W_l^{-2}W_{l'}^{-2}C_l^{\text{map}}C_{l'}^{\text{map}})\right]^{-1},$$ (14)

$$\langle\widehat{\Omega_{LM}}\widehat{\Omega_{L'M'}}^*\rangle \equiv \delta_{LL'}\delta_{MM'}(\sigma_L^\Omega)^2$$

$$\equiv 2\delta_{LL'}\delta_{MM'}\left[\sum\limits_{ll'}|Q_{ll'}^{\ominus L}|^2/(W_l^{-2}W_{l'}^{-2}C_l^{\text{map}}C_{l'}^{\text{map}})\right]^{-1},$$ (15)

where the sums in Eqs. (12) and (14) only include pairs of l, l' for which $l + l' + L$ is even, while those in Eqs. (13) and (15) only include pairs for which this quantity is odd.

B. Deflection field from metric perturbations

Cosmic shear, weak gravitational lensing due to density perturbations or GWs along the line of sight to the CMB,

will produce displacements like those in Eq. (8). Our goal here will be to calculate the displacement spherical-harmonic coefficients ϕ_{LM} and Ω_{LM} that arise from gravitational lensing due to density perturbations and GWs. There is a vast literature on lensing by density perturbations and also specifically on lensing of the CMB by density perturbations [2]. Our density-perturbation results follow most closely those of [31,32]. Lensing by GWs has been considered in [33]. We follow primarily the approach of [34,35], who calculated Ω_{LM} due to GWs, but extend their results to include ϕ_{LM} from GWs, reproducing the results of [36]. We make use in this Section of relevant work on lensing and/or differential analysis on the celestial sphere in [31,32,37,38].

We write the metric for the perturbed spacetime as

$$ds^2 = a^2(\eta)[-d\eta^2 + (\delta_{ij} + h_{ij})dx^i dx^j],$$

where h_{ij} is the metric perturbation in the synchronous gauge, and η is the conformal time. Now consider a photon that we observe to come from the direction $\hat{n}$ on the sky. In the absence of perturbations, this photon travels along a path $\vec{x}(\eta) = (\eta_0 - \eta)\hat{n}$ as a function of conformal time η, where η_0 is the conformal time today. Metric perturbations will induce perturbations in this trajectory, which we can calculate by integrating the geodesic equation back over the photon path to find the direction of propagation of the photon when it was emitted at a conformal time η. To first order in the metric perturbation h, we find the original direction of propagation of the photon on the sky to be $\hat{n} + \vec{\Delta}$, where [39]

$$\Delta^i(\hat{n}) = \frac{P_{im}}{\eta_0 - \eta}\int_{\eta_0}^{\eta} d\eta'\left[h_{mj}\hat{n}_j - \frac{1}{2}(\eta' - \eta)\hat{n}_k\hat{n}_l\partial_m h_{kl}\right]_{[\eta', (\eta_0-\eta')\hat{n}]}.$$ (16)

Here, we have ignored the observer terms $h_{ij}(\eta_0)$, and we have defined the projection tensor $P_{im} = \delta_{im} - n_i n_m$ onto the space perpendicular to the unit vector $\hat{n}$. The subscript indicates that the quantities in the integral are evaluated at time and space coordinates $(\eta, \vec{x}) = (\eta', (\eta_0 - \eta')\hat{n})$; i.e. they are evaluated along the unperturbed path of the photon. In our case, the source is the CMB, and $\eta = \eta_{\text{lss}}$ is the conformal time at the surface of last scatter. However, the calculation could also be applied to the lensing of galaxies in which case the relevant conformal time would be that corresponding to redshifts $z \sim 1$.

The functions $\phi(\hat{n})$ and $\Omega(\hat{n})$ in the decomposition in Eq. (8) can be obtained from

$$\nabla_{\hat{\theta}}^2\phi(\hat{n}) = \vec{\nabla}_{\hat{\theta}}\cdot\vec{\Delta}(\hat{n}), \qquad \nabla_{\hat{\theta}}^2\Omega(\hat{n}) = -\vec{\nabla}_{\hat{\theta}}\times\vec{\Delta}(\hat{n}),$$ (17)

whereas before $\vec{\nabla}_{\hat{\theta}}$ is the angular covariant derivative on the unit sphere. As [25] notes, the standard lensing

convergence is $\kappa = -(1/2)\nabla_{\tilde{\theta}}^2 \phi$ and the lensing rotation is $\omega = (1/2)\nabla_{\tilde{\theta}}^2 \Omega$.

The gradient component is obtained from

$$\nabla_{\tilde{\theta}}^2 \phi(\hat{n}) = \vec{\nabla}_{\tilde{\theta}} \cdot \vec{\Delta}$$

$$= -\frac{1}{\eta_0 - \eta} \left\{ \int_{\eta}^{\eta_0} d\eta' (\eta_0 - \eta')(\delta_{ik} - \hat{n}_i \hat{n}_k) \right.$$

$$\times \left[-\partial_k (h_{ij} n^j) + \frac{1}{2}(\eta' - \eta) \partial_i \partial_k (h_{lm} \hat{n}_l \hat{n}_m) \right]_{[\eta', (\eta_0 - \eta')\hat{n}]}$$

$$+ \int_{\eta}^{\eta_0} d\eta' [3\hat{n}_i \hat{n}_j h_{ij} - h_{ii} + (\eta' - \eta)$$

$$\left. \times (\hat{n}_j \partial_i h_{ij} - 2\hat{n}_i \hat{n}_j \hat{n}_k \partial_k h_{ij})]_{[\eta', (\eta_0 - \eta')\hat{n}]} \right\}, \quad (18)$$

where we have used the fact that $\vec{\nabla}_{\tilde{\theta}}$, which acts on the unit vector $\hat{n}$, behaves as $\nabla_{\tilde{\theta}}^i = (\eta_0 - \eta')(\delta_{ik} - \hat{n}_i \hat{n}_k)\partial_k$ inside the integral due to the dependence of $\vec{x}$ on $\hat{n}$ as defined in the integrand subscript.

Let us now consider the curl component. For this calculation we must use $\nabla_{\tilde{\theta}}^2 \Omega = -\vec{\nabla}_{\tilde{\theta}} \times \vec{\Delta}$ and then note that, as before, $\nabla_{\tilde{\theta}}^i = (\eta_0 - \eta')(\delta_{ik} - \hat{n}_i \hat{n}_k)\partial_k$ inside the integrand. Applying this to Eq. (16), we have [34]

$$\nabla_{\tilde{\theta}}^2 \Omega(\hat{n}) = -\int_{\eta}^{\eta_0} d\eta' (n_i n_l \epsilon_{ijk} \partial_j h_{kl})_{[\eta', \hat{n}(\eta_0 - \eta')]}. \quad (19)$$

C. Lensing by density (scalar metric) perturbations

Let us first consider scalar perturbations. In the conformal-Newtonian gauge in the absence of anisotropic stresses, the metric is given by

$$ds^2 = a^2(\eta)[-(1 - 2\Phi)d\eta^2 + (1 + 2\Phi)\delta_{ij} dx^i dx^j].$$

Noting that a conformal transformation preserves null geodesics, our calculations of the photon path will be unaffected if we work in a synchronous metric obtained from the conformal-Newtonian form through multiplication by $(1 + 2\Phi)$. Assuming that Φ is small and keeping terms only to linear order, we find the conformally-related metric,

$$ds^2 = a^2(\eta)[-d\eta^2 + (1 + 4\Phi)\delta_{ij} dx^i dx^j].$$

Using this metric perturbation $h_{ij} = 4\Phi \delta_{ij}$ in Eq. (18) above, we find that the first, third, and fourth terms vanish, giving for the gradient-type lensing caused by scalar perturbations,

$$\nabla_{\tilde{\theta}}^2 \phi^{\text{sca}}(\hat{n}) = -\frac{2}{\eta_0 - \eta} \int_{\eta}^{\eta_0} d\eta' (\eta' - \eta)$$

$$\times [(\delta_{ij} - \hat{n}_i \hat{n}_j)(\eta_0 - \eta') \partial_i \partial_j \Phi - 2\hat{n}_i \partial_i \Phi].$$

For small-scale fluctuations, the second term will be negligible compared with the first, so it can be dropped. We can rewrite the spatial derivatives in terms of $\vec{\nabla}_{\tilde{\theta}}$ to find

$$\nabla_{\tilde{\theta}}^2 \phi^{\text{sca}}(\hat{n}) = -\frac{2}{\eta_0 - \eta} \int_{\eta}^{\eta_0} d\eta' \frac{\eta' - \eta}{\eta_0 - \eta'} \nabla_{\tilde{\theta}}^2 \Phi(\eta', (\eta_0 - \eta')\hat{n}),$$

and we can remove the angular derivatives to obtain the usual expression for the projected potential

$$\phi^{\text{sca}}(\hat{n}) = -2 \int_{\eta}^{\eta_0} d\eta' \frac{\eta' - \eta}{(\eta_0 - \eta)(\eta_0 - \eta')} \Phi(\eta', (\eta_0 - \eta')\hat{n}).$$

We can once again decompose $\phi(\hat{n})$ in terms of its spherical-harmonic coefficients as in Eq. (9). We then find

$$\phi_{LM}^{\text{sca}} \equiv \int d^2\hat{n} Y_{LM}^*(\hat{n}) \phi^{\text{sca}}(\hat{n})$$

$$= -2 \int_{\eta}^{\eta_0} d\eta' \frac{\eta' - \eta}{(\eta_0 - \eta)(\eta_0 - \eta')}$$

$$\times \int d^2\hat{n} Y_{LM}^*(\hat{n}) \Phi(\eta', (\eta_0 - \eta')\hat{n}). \quad (20)$$

Thus, lensing by density perturbations with a given projected potential is characterized by nonzero even bipolar spherical harmonics $A_{ll'}^{\oplus LM}$ given by Eq. (10) with ϕ_{LM} given by ϕ_{LM}^{sca} above. Scalar perturbations cause no curl-type lensing, which we can see in several ways. For scalar perturbations, $h_{ij} \propto \Phi \delta_{ij}$, and so the left-hand side of Eq. (19) vanishes. Then, by taking a Laplacian of the mode expansion $\Omega_{LM} = \int d^2\hat{n}\Omega(\hat{n})Y_{LM}^*(\hat{n})$, and noting that the spherical harmonics are eigenfunctions of the Laplacian with eigenvalue $L(L + 1)$, we can write

$$\Omega_{LM} = \frac{1}{L(L + 1)} \int d^2\hat{n} Y_{LM}^*(\hat{n}) \nabla_{\tilde{\theta}}^2 \Omega(\hat{n}). \quad (21)$$

Thus, we find that all of the Ω_{LM}^{sca}, except possibly for the unphysical $L = 0$ mode, vanish. Equivalently, an argument can be made that scalar perturbations have no preferred direction, and so could not generate curl-modes, which do have a preferred direction. Thus, scalar modes produce no odd bipolar spherical harmonics $A_{ll'}^{\ominus LM}$.

We can go on to find the autocorrelation power spectrum of the ϕ_{LM}^{sca}. Starting from Eq. (20), we use the fact that the potential perturbations $\Phi(\eta, \vec{k})$ today are related to their primordial values $\Phi_P(\vec{k})$ by

$$\Phi(\eta, \vec{k}) = \frac{9}{10} \Phi_P(\vec{k}) T^{\text{sca}}(k) \frac{D_1(\eta)}{a(\eta)},$$

where $a(\eta)$ is the scale factor, $T^{\text{sca}}(k)$ is the scalar transfer function that describes the evolution of scalar modes through the epochs of horizon crossing and matter-radiation equality, and $D_1(\eta)$ is the growth function that captures the scale-independent evolution of scalar modes at later times [40]. The transfer function can be approximated using the fitting form of [41],

$$T^{\text{sca}}(x \equiv k/k_{\text{eq}})$$
$$= \frac{\ln(1 + 0.17x)}{0.171x}[1 + 0.284x + (1.18x)^2$$
$$+ (0.399x)^3 + (0.490x)^4]^{-0.25},$$

where k_{eq} is the wavenumber of the mode that crossed the horizon at matter-radiation equality, defined as $k_{\text{eq}} \equiv a_{\text{eq}}H(a_{\text{eq}}) = \sqrt{2}H_0 a_{\text{eq}}^{-1/2}$. We can write the growth function, under the assumption of cosmological-constant dark energy, as

$$D_1(\eta) = \frac{5\Omega_m}{2} \frac{H(\eta)}{H_0} \int_o^{a(\eta)} \frac{da'}{(a'H(a')/H_0)^3}.$$

We also write the autocorrelation of the primordial scalar fluctuations $\langle \Phi_P(\vec{k})\Phi_P^*(\vec{k'})\rangle = (2\pi)^3 \delta^3(\vec{k} - \vec{k'})P_\Phi(k)$, where the primordial power spectrum is given by

$$P_\Phi(k) = \frac{50\pi^2}{9k^3}\left(\frac{k}{H_0}\right)^{n_s-1}\Delta_R^2\left(\frac{\Omega_m}{D_1(a=1)}\right)^2.$$

With these ingredients, and after using the partial-wave decomposition,

$$e^{ik(\eta_0-\eta')\cos\theta} = \sum_{L=0}^{\infty} i^L(2L+1)j_L(k(\eta_0-\eta'))P_L(\cos\theta), \quad (22)$$

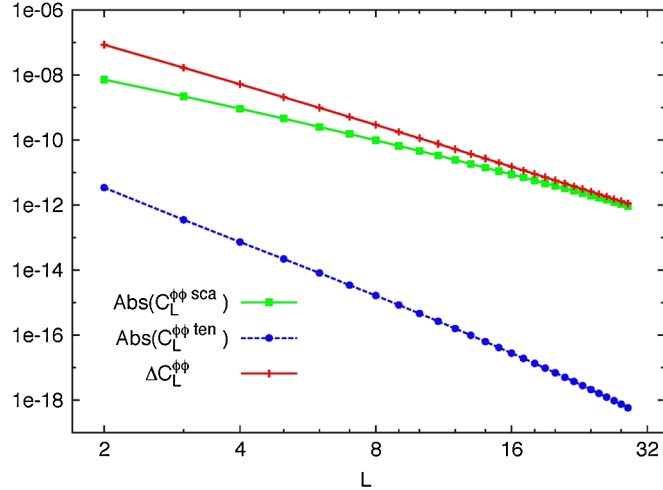we find the autocorrelation power spectrum to be

FIG. 1 (color online). Here we plot the autocorrelation power spectrum $C_L^{\phi\phi}$ of the gradient-type ϕ modes of cosmic shear. In green squares we show the autocorrelation of the ϕ modes from lensing by scalar perturbations, and in blue circles that of the ϕ modes induced by tensor perturbations. We use the WMAP-7 cosmological parameters, and assume the maximum allowable tensor-to-scalar ratio $r = 0.24$ from the WMAP-7 data combined with BAO and the H_0 measurement [42], to calculate the tensor contribution. The error with which these power spectra could be measured using the parameters of the Planck satellite is shown as red +s.

$$C_L^{\phi\phi\text{sca}} = \frac{2}{\pi}\left[\frac{9}{5(\eta - \eta_0)}\right]^2 \int dk k^2 P_\Phi(k)T^{\text{sca}}(k)^2$$
$$\times \left\{\int_\eta^{\eta_0} d\eta' \frac{(\eta' - \eta)}{(\eta_0 - \eta')} \frac{D_1(\eta')}{a(\eta')} j_L[(\eta_0 - \eta')k]\right\}^2. \quad (23)$$

To calculate the magnitude and shape of this autocorrelation function, we employ the WMAP 7-year cosmological parameters of [42]. We plot the result of our calculation in green squares in Fig. 1.

D. Lensing by GWs (tensor metric perturbations)

If the metric perturbation h_{ij} is caused by GWs, we can decompose it into plane waves,

$$h_{ij}(\vec{x}, \eta) = \int \frac{d^3k}{(2\pi)^3} e^{ik\cdot\vec{x}}T(k, \eta)\sum_{\alpha=+,\times} h^\alpha(\vec{k})\epsilon_{ij}^\alpha(\vec{k}), \quad (24)$$

where we sum over the two GW polarizations + and ×, the plane-wave amplitudes are $h^\alpha(\vec{k})$, and ϵ_{ij}^α are the polarization tensors, which are transverse, traceless matrices. Here, $T(k, \eta)$ is the GW transfer function, which gives the conformal-time evolution of the mode; [34] notes that it is well approximated by $T(k, \eta) = 3j_1(k\eta)/(k\eta)$.

Now consider a single GW mode propagating in the $\hat{z}$ direction with wavenumber k and + polarization. In this case the polarization tensor is

$$\epsilon_{ij}^+(k\hat{z}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The only nonzero metric-perturbation components are then $h_{xx} = -h_{yy} = h^+(\vec{k})e^{ikz}T(k, \eta)$. The unit vector $\hat{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$. The curl component of lensing of the CMB by tensor perturbations is then

$$\nabla_{\hat{\theta}}^2 \Omega^{\text{ten}}(\hat{n}) = ikh^+(\vec{k})\sin^2\theta \sin 2\varphi$$
$$\times \int_\eta^{\eta_0} d\eta' T(k, \eta')e^{ik(\eta_0-\eta')\cos\theta}. \quad (25)$$

A GW with the × polarization is the same as that with the + polarization, but rotated by 45° to the right. The $\Omega^{\text{ten}}(\hat{n})$ pattern is therefore the same, but with $\sin 2\varphi$ replaced by $-\cos 2\varphi$. We thus see that lensing by GWs will give rise to nonvanishing $A_{ll'}^{\Theta LM}$.

The gradient component of cosmic shear due to tensor perturbations is a bit more complicated; it is

$$\nabla^2_{\hat\theta}\phi^{\rm ten}(\hat n) = -\frac{h^+(\vec k)}{\eta_0 - \eta}\sin^2\theta\cos2\varphi\int_\eta^{\eta_0}d\eta' T(k,\eta')$$
$$\times\left\{3 - 2ik(\eta'-\eta)\cos\theta + (\eta_0-\eta')\right.$$
$$\left.\times\left[ik\cos\theta - \frac{k^2}{2}(\eta'-\eta)\sin^2\theta\right]\right\}e^{ik(\eta_0-\eta')\cos\theta}.$$

(26)

This can be further simplified by noting that

$$-ik\cos\theta e^{ik(\eta_0-\eta')\cos\theta} = \frac{\partial}{\partial\eta'}e^{ik(\eta_0-\eta')\cos\theta},$$

which then leads to

$$\nabla^2_{\hat\theta}\phi^{\rm ten}(\hat n) = -\frac{h^+(\vec k)}{\eta_0 - \eta}\sin^2\theta\cos2\varphi\int_\eta^{\eta_0}d\eta' T(k,\eta')$$
$$\times\left\{3 + 2(\eta'-\eta)\frac{\partial}{\partial\eta'} - (\eta_0-\eta')\left[\frac{\partial}{\partial\eta'}\right.\right.$$
$$\left.\left.+\frac{(\eta'-\eta)}{2}\left(k^2 + \frac{\partial^2}{\partial\eta'^2}\right)\right]\right\}e^{ik(\eta_0-\eta')\cos\theta}. \quad (27)$$

For the $\times$ polarization, we replace $\cos2\varphi$ by $\sin2\varphi$.

Note that the expressions for $\nabla^2_{\hat\theta}\phi^{\rm ten}$ and $\nabla^2_{\hat\theta}\Omega^{\rm ten}$ differ only in two ways: (1) The curl mode has a $\sin2\varphi$ dependence on the azimuthal angle φ, while the scalar mode has a $\cos2\varphi$ dependence (for the $+$ polarization). (2) The η' dependences of the two integrands differ.

We now find the spherical-harmonic coefficients $\phi^{\rm ten}_{LM} = \int d^2\hat n\,\phi^{\rm ten}(\hat n)Y^*_{LM}(\hat n)$ and $\Omega^{\rm ten}_{LM} = \int d^2\hat n\,\Omega^{\rm ten}(\hat n)Y^*_{LM}(\hat n)$. Taking the angular derivatives of this decomposition of the curl component, we find the result Eq. (21). We also expand these coefficients in terms of their polarization and $\vec k$ modes,

$$\Omega^{\rm ten}_{LM} = \int\frac{d^3k}{(2\pi)^3}\sum_{\alpha=+,\times}\Omega^{{\rm ten}\alpha}_{LM}(\vec k). \quad (28)$$

If we consider just one mode, with $\alpha = +$ and $\vec k = k\hat z$, and use Eq. (25), its amplitude simplifies into an angular and a conformal-time integral:

$$\Omega^{{\rm ten}+}_{LM}(k\hat z) = -\frac{ikh^+(\vec k)}{L(L+1)}\int_\eta^{\eta_0}d\eta' T(k,\eta')$$
$$\times\int d^2\hat n\,Y^*_{LM}(\hat n)\sin^2\theta\sin(2\phi)e^{ik(\eta_0-\eta')\cos\theta}.$$

The azimuthal integral is easily taken once the spherical harmonic is decomposed as

$$Y^*_{LM}(\hat n) = \sqrt{\frac{2L+1}{4\pi}\frac{(L-M)!}{(L+M)!}}e^{-iM\phi}P_{LM}(\cos\theta),$$

and yields the result that only $M = \pm2$ modes remain. The polar integral can then be taken by using the partial-wave

decomposition Eq. (22) and by converting associated Legendre polynomials into regular Legendre polynomials and using their orthogonality. The final result that we obtain for the spherical-harmonic coefficients of the curl mode is

$$\Omega^{{\rm ten}+}_{LM}(k\hat z) = i^L h^+(\vec k)(\delta_{M,2} - \delta_{M,-2})\sqrt{\frac{2L+1}{2}}F^\Omega_L(k), \quad (29)$$

where

$$F^\Omega_L(k) = \sqrt{\frac{2\pi(L+2)!}{(L-2)!}}\frac{1}{L(L+1)}\int_{k\eta}^{k\eta_0}dw\,T(w)\frac{j_L(k\eta_0-w)}{(k\eta_0-w)^2} \quad (30)$$

is a transfer function for Ω. Note that in writing Eq. (30) we have assumed that $T(k,\eta) = T(k\eta)$, and that for the $\times$ polarization the $\sin2\varphi$ dependence of $\Omega(\hat n)$ is replaced by $-\cos2\varphi$, so that the factor $(\delta_{M,2} - \delta_{M,-2})$ is replaced by $-i(\delta_{M,2} + \delta_{M,-2})$.

Likewise, noting the similarities between Eqs. (25) and (27), and decomposing $\phi^{\rm ten}_{LM}$ into modes as in Eq. (28)

$$\phi^{\rm ten}_{LM} = \int\frac{d^3k}{(2\pi)^3}\sum_{\alpha=+,\times}\phi^{{\rm ten}\alpha}_{LM}(\vec k), \quad (31)$$

the result for the amplitude of the gradient mode with $\alpha = +$ and $\vec k = k\hat z$ is

$$\phi^{{\rm ten}+}_{LM}(k\hat z) = i^L h^+(\vec k)(\delta_{M,2} + \delta_{M,-2})\sqrt{\frac{2L+1}{2}}F^\phi_L(k), \quad (32)$$

where

$$F^\phi_L(k) = -\sqrt{\frac{2\pi(L+2)!}{(L-2)!}}\frac{1}{L(L+1)}\int_{k\eta}^{k\eta_0}dw\,\frac{k\eta_0-w}{k(\eta_0-\eta)}T(w)$$
$$\times\left[\frac{\partial}{\partial w} + \frac{1}{2}(w-k\eta)\left(1 + \frac{\partial^2}{\partial w^2}\right)\right]\frac{j_L(k\eta_0-w)}{(k\eta_0-w)^2}. \quad (33)$$

Again, the factor $(\delta_{M,2} + \delta_{M,-2})$ is replaced by $-i(\delta_{M,2} - \delta_{M,-2})$ for the $\times$ polarization.

The contributions from this Fourier mode to the $\phi^{\rm ten}$ and $\Omega^{\rm ten}$ power spectra are $C^{\phi\phi{\rm ten}}_L(k\hat z)_+ = \sum_M\langle|\phi^{\rm ten}_{LM}|^2\rangle/(2L+1)$ and $C^{\Omega\Omega{\rm ten}}_L(k\hat z)_+ = \sum_M\langle|\Omega^{\rm ten}_{LM}|^2\rangle/(2L+1)$. Note that it is only the $M = \pm2$ modes that contribute. By rotational invariance, the contribution from the $\times$ polarization is the same, as is the contribution from any other mode with the same wavenumber k but pointing in a different direction. If the gravitational waves have power spectrum $P_T(k)$, defined by

$$\langle h^i_{\vec k}(h^j_{\vec k'})^*\rangle = (2\pi)^3\delta_D(\vec k - \vec k')\delta_{ij}P_T(k) \quad (34)$$

(with $\{i,j\} = \{\times, +\}$), then the ϕ and Ω power spectra are

$$C_L^{XX\text{ten}} = 2 \int \frac{d^3 k}{(2\pi)^3} P_T(k) [F_L^X(k)]^2 \qquad (35)$$

for $X = \{\phi, \Omega\}$. In this paper, we will assume a scale-invariant power spectrum

$$P_T(k) = \frac{\pi^2}{2k^3} \Delta_R^2 r, \qquad (36)$$

where we have neglected the spectral tilt and adopt the parameters of WMAP7 [42].

We calculate the variance in the measurement of these autocorrelation functions from an observed CMB temperature map, under the null hypothesis of no GWs, and obtain an expression in terms of the variance of the ϕ and Ω estimators, Eqs. (14) and (15)

$$\Delta C_L^{\phi\phi} = \sqrt{\frac{2}{2L+1}} (\sigma_L^{\phi 2} + C_L^{\phi\phi\text{sca}}), \qquad (37)$$

$$\Delta C_L^{\Omega\Omega} = \sqrt{\frac{2}{2L+1}} \sigma_L^{\Omega 2}. \qquad (38)$$

Here, $\sigma_L^{\phi 2}$ and $\sigma_L^{\Omega 2}$ are the variances of our estimators for ϕ_{LM} and Ω_{LM} as found in Eqs. (14) and (15). Note that under the null hypothesis of no GWs, there is no expected cosmological curl-type lensing signal, so this term is absent in Eq. (38).

To calculate these autocorrelation functions and their variances, we use the WMAP seven-year cosmological parameters of [42]. We plot the results of our calculations in Figs. 1 and 2, where lensing from scalar perturbations is plotted in green squares (absent for $C_L^{\Omega\Omega}$ since there are no
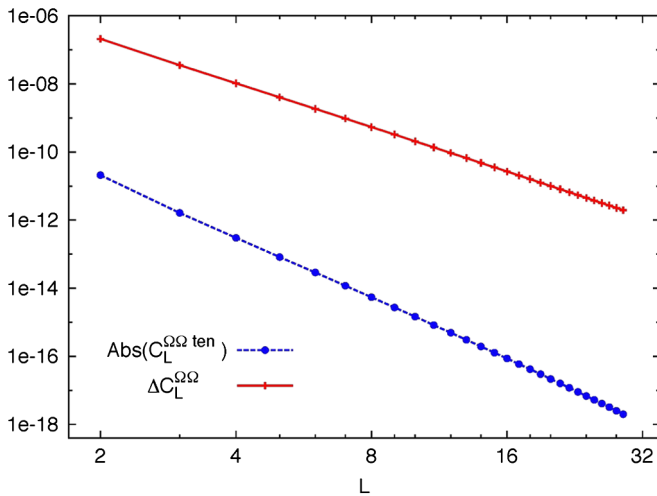

FIG. 2 (color online).   Here we plot the autocorrelation power spectrum $C_L^{\Omega\Omega}$ of the curl-type Ω modes of the weak lensing of the CMB temperature field. These modes can only be induced by tensor perturbations. We show the signal in blue circles and the error with which they could be measured using the parameters of the Planck satellite as red +s.

scalar contributions to the curl modes), lensing from tensor perturbations is plotted in blue circles, and the variance of these measurements is shown in red +s. We use the parameters of the Planck satellite, $NET = 62 \mu K s^{1/2}$, $t_{\text{obs}} = 2$ yr, $\theta_{\text{FWHM}} = 2.0635 \times 10^{-3}$ rad, and $f_{\text{sky}} \approx 1$. We can see that the scalar ϕ signal is several orders of magnitude greater than the tensor signal, and that the variance with which the ϕ-ϕ power spectrum could be measured with Planck is higher than the scalar signal for low multipoles. The corresponding variance with which the Ω-Ω power spectrum could be measured is also significantly larger than the signal. In both of these cases, therefore, the signal-to-noise of measuring the lensing from tensors using all multipoles with Planck is negligible, and remains negligible even in the case of the ideal CMB experiment with zero noise.

Thus, a stochastic background of GWs with power spectrum $P_T(k)$ predicts a spectrum of nonzero even and odd BiPoSHs given by Eqs. (10) and (11), with values of ϕ_{LM} and Ω_{LM} selected from Gaussian distributions with the variances $C_L^{\phi\phi\text{ten}}$ and $C_L^{\Omega\Omega\text{ten}}$ given by Eq. (35).

E. BiPoSHs from pointing errors

A telescope pointing error can be described as a process that causes the positions of points on the sky to be mislabeled. This then causes an effective deflection of the points on the sky $\hat{n}_{\text{obs}} = \hat{n} + \vec{\Delta}(\hat{n})$, where $\hat{n}_{\text{obs}}$ is the direction that the telescope believes it is pointed in and $\hat{n}$ is its actual pointing direction. As we saw in Sec. III A, we can decompose this deflection field $\vec{\Delta}(\hat{n})$ into gradient and curl components, which source even- and odd-parity BiPoSHs, respectively. Thus, from Eq. (17) we can see that any pointing error that has a nonzero curl component $\vec{\nabla}_{\hat{\theta}} \times \vec{\Delta}(\hat{n})$ will excite odd-parity BiPoSHs.

Imagine, for example, that a satellite such as Planck misestimates the rate with which it is precessing. Since it is this precession that builds up observations of subsequent rings of the sky, such a misestimation would cause a shearing of each ring relative to its neighbors. This type of a deflection has a nonzero curl component, and thus would excite odd-parity BiPoSHs. Measurement of these BiPoSHs, and, in particular, the odd-parity BiPoSHs, can therefore provide a useful check for such pointing errors.

IV. BIPOSHS AS PROBES OF PARITY VIOLATION

A. Correlation of opposite-parity lensing components

Since the $A_{ll'}^{\oplus LM}$ and $A_{ll'}^{\ominus LM}$ have opposite parity for the same L and M, a cross-correlation between the two can arise only if there is some parity-breaking in the physics responsible for producing the departures from SI/Gaussianity. Here we mention, by way of example, chiral GWs as a mechanism to produce such a parity-violating correlation [43–45].

The contribution to the cross-correlation power spectrum from a single Fourier mode in the $\hat{z}$ direction with $+$ polarization is $C_L^{\phi\Omega}(k\hat{z}) = \sum_m \langle \phi_{LM}\Omega_{LM}^* \rangle/(2L+1) = 0$; it vanishes as the contribution from $M = 2$ is canceled by that from $M = -2$. And if this is true, then by rotational invariance it is true for any other linearly-polarized GW. We thus conclude that a stochastic GW background predicts $C_L^{\phi\Omega} = 0$. In other words, there is no cross-correlation between ϕ and Ω, and thus no cross-correlation between the even and odd BiPoSHs, $A_{ll'}^{\oplus LM}$ and $A_{ll'}^{\ominus LM}$.

Following [44], however, consider a right-circularly-polarized GW: $h_R = h_+ + ih_\times$ (i.e., we sum a $+$ polarization wave with a $\times$ polarization wave out of phase by $90°$). The azimuthal-angle dependence for the wave is then $e^{2i\varphi}$, and Ω_{LM} and ϕ_{LM} have contributions only from $M = 2$. There is thus a nonzero cross-correlation between ϕ and Ω. Similarly for a left-circularly-polarized GW $h_L = h_+ - ih_\times$, the φ dependence is $e^{-2i\varphi}$, and only $M = -2$ modes are excited. There is again a cross-correlation between ϕ and Ω, but this time with the opposite sign.

In the standard inflationary scenario, there are equal numbers of right- and left-circularly-polarized GWs, and the cross-correlation between ϕ and Ω therefore vanishes. But if for some reason there is an asymmetry between the number of right- and left-circularly-polarized GWs [43–46], a manifestation of parity breaking, then there may be a parity-violating cross-correlation between ϕ and Ω, and thus between $A_{ll'}^{\oplus LM}$ and $A_{ll'}^{\ominus LM}$.

The chirality of the GW background can be parametrized by an amplitude A which can take values between -1 and 1, where $A = +1$ denotes that all of the GWs are right-circularly-polarized, and $A = -1$ denotes that they are all left-circularly-polarized. But we have seen that a right-handed GW contributes only to $M = 2$ modes, while a left-handed one contributes only to $M = -2$. We can denote this by weighting $M = 2$ components by $(A + 1)/2$ and $M = -2$ components by $(A - 1)/2$, so that our version of Eq. (29), for example, that is appropriate to the case of a chiral GW background will be

$$\Omega_{LM}^{\text{ten}+}(k\hat{z}) = i^L h^+(\vec{k})\left[\frac{(1+A)}{2}\delta_{M,2} - \frac{(1-A)}{2}\delta_{M,-2}\right]$$
$$\times \sqrt{\frac{2L+1}{2}}F_L^\Omega(k), \qquad (39)$$

and similarly for Eq. (32). In this way a fully right-circularly-polarized GW background will have only contributions from $M = 2$, a fully left-circularly-polarized background will have only contributions from $M = -2$, and if the amount of left and right-circularly-polarized waves is equal, that is if the GW background is nonchiral, the contributions from $M = 2$ and $M = -2$ cancel. The ϕ-Ω cross-correlation power spectrum is given by

$$C_L^{\phi\Omega} = A\int\frac{d^3k}{(2\pi)^3}P_T(k)F_L^\phi(k)F_L^\Omega(k). \qquad (40)$$

References [34,36,47] have shown that the amplitude of the stochastic gravitational-wave background is probably too small, even with the most optimistic assumptions, to produce a detectable gravitational-lensing signal in the CMB. The example of a chiral gravitational-wave background as a possible source of a detectable parity-breaking BiPoSH correlation is principally of academic interest. Still, [35] has recently argued that weak-lensing of the CMB by GWs may be detectable in its cross-correlation with the CMB-polarization pattern induced by these GWs [37,38,48–50]. We thus surmise that a chiral gravitational-wave background may still be able produce a detectable parity-breaking signal in BiPoSHs in cross-correlation with the CMB polarization, an idea we explore in the next section.

B. Large-angle CMB polarization spectra

We follow the work of [35], finding the multipole moments of the CMB E- and B-type polarization spectra for large angular scales by considering only those modes that are produced after reionization. The spherical-harmonic coefficients of B-type polarization modes can be decomposed as

$$B_{lm} = \int\frac{d^3k}{(2\pi)^3}\sum_{\alpha=+,\times}B_{lm}^\alpha(\vec{k}), \qquad (41)$$

where $B_{lm}^\alpha(\vec{k})$ is the amplitude of polarization B modes multipole moment lm in the direction $\vec{k}$. The general form of this amplitude is quite complicated, but we can simplify it if, as in Sec. III D, we consider only a single, $+$-polarized GW traveling in the $\hat{z}$ direction with wavenumber k. In this case, the B-mode amplitude can be written

$$B_{lm}^+(k\hat{z}) = i^l h^\alpha(\vec{k})(\delta_{m,2} - \delta_{m,-2})\sqrt{\frac{2l+1}{2}}F_l^B(k), \qquad (42)$$

$$F_l^B(k) = \frac{1}{2l+1}\sqrt{\frac{9\pi}{2}}\int_{\eta_{\text{re}}}^{\eta_0}d\eta\,\dot{\tau}(\eta)\{(l+2)j_{l-1}[k(\eta_0-\eta)]$$
$$-(l-1)j_{l+1}[k(\eta_0-\eta)]\}\int_{k\eta_{\text{lss}}}^{k\eta}dx\frac{-3j_2(x)}{x}\frac{j_2(k\eta-x)}{(k\eta-x)^2}, \qquad (43)$$

where the $h^\alpha(\vec{k})$ are the amplitudes of GW modes as defined in Eq. (24), $\dot{\tau}(\eta)$ is the scattering rate $\dot{\tau}(\eta) = n_e(\eta)\sigma_T a(\eta)$, with n_e the electron density, σ_T the Thompson scattering cross-section, and a the scale factor, and η_{re} and η_0 are the conformal times at reionization and today, respectively. Since we are only interested in small scales, we find the approximation $\eta_{\text{lss}} = 0$ is sufficient for our purposes, making the last integral significantly faster to

evaluate. The result above agrees with the results of [35], whose method we followed in its derivation, up to a factor of i.

We find that the corresponding E-type polarization multipoles from tensor perturbations take the same form as B_{lm} above, except for the opposite sign in front of $\delta_{m,-2}$ and a different factor in the curly brackets in Eq. (43). From [49] we find this alternative form to be $(2l + 1)/2\{-j_l(x) + j_l''(x) + 2j_l(x)/x^2 + 4j_l'(x)/x\}$, where here $x = [k(\eta_0 - \eta)]$, and derivatives are with respect to x. Employing spherical Bessel function identities, we can then write the E-type polarization multipoles as

$$E_{lm} = \int \frac{d^3k}{(2\pi)^3} \sum_{\alpha=+,\times} E_{lm}^\alpha(\vec{k}), \qquad (44)$$

$$E_{lm}^+(k\hat{z}) = i^l h^\alpha(\vec{k})(\delta_{m,2} + \delta_{m,-2})\sqrt{\frac{2l+1}{2}}F_l^E(k), \qquad (45)$$

$$\begin{aligned} F_l^E(k) = &\frac{1}{2l+1}\sqrt{\frac{9\pi}{2}}\int_{\eta_{\rm re}}^{\eta_0} d\eta\, \dot{\tau}(\eta)\Bigg\{\frac{(2l+1)}{[k(\eta_0-\eta)]^2}j_l[k(\eta_0-\eta)] \\ &- \frac{(2l+1)(3l^2+3l-4)}{(2l-1)(2l+3)}j_l[k(\eta_0-\eta)] \\ &+ \frac{l(l+3)}{2(2l-1)}j_{l-2}[k(\eta_0-\eta)] \\ &+ \frac{(l+1)(l-2)}{2(2l+3)}j_{l+1}[k(\eta_0-\eta)]\Bigg\} \\ &\times \int_{k\eta_{\rm lss}}^{k\eta} dx\frac{-3j_2(x)}{x}\frac{j_2(k\eta-x)}{(k\eta-x)^2}, \end{aligned} \qquad (46)$$

where the terms are defined as they were above for the B_{lm} amplitudes.

C. Parity-violating correlations from chiral GWs

We now want to calculate the expected cross-correlation between CMB-polarization multipole coefficients and weak-lensing-induced BiPoSHs of opposite parity. Note that these cross-correlations are directly related to the parity-odd three-point correlations discussed in [51]. As we mentioned above, if there is no parity-violating physics, then in the cross-correlation of a parity-even and a parity-odd observable, $M = 2$ terms and $M = -2$ terms will cancel each other, giving a net zero cross-correlation. However, if, for example, the GW background is chiral, then parity is broken and we can get a nonzero cross-correlation between opposite-parity observables. As we saw in Sec. IV A, a right-handed GW contributes only to $M = 2$ modes, while a left-handed one contributes only to $M = -2$. If we carry out a similar procedure for Eqs. (42) and (45) as we did in Eq. (39), weighting $M = 2$ components by $(A + 1)/2$ and $M = -2$ components by $(A - 1)/2$, we can calculate parity-violating correlations between polarization and lensing components while

accounting for the amplitude and handedness of a chiral GW background.

First considering the cross-correlation between B-modes of the CMB polarization and gradient-type modes of cosmic shear, we write

$$C_L^{\phi B} = \frac{1}{2L+1}\sum_M \langle \phi_{LM} B_{LM}^* \rangle.$$

As before, by rotational invariance we know that both $+$ and $\times$ polarizations will contribute equally to $C_L^{\phi B}$, as will modes with any wavenumber $\vec{k}$ whose magnitude k is the same. We can see that only $\phi_{LM}^{\rm ten}$ will contribute to this correlation, and not $\phi_{LM}^{\rm sca}$, as the scalar perturbation field is not correlated, on average, with the tensor perturbation field. Then using Eqs. (31), (32), (34), (41), and (42), we can write this cross-correlation as

$$C_L^{\phi B} = A \int \frac{d^3k}{(2\pi)^3} P_T(k) F_L^\phi(k) F_L^B(k). \qquad (47)$$

Similarly, we can write the cross-correlation between E-type polarization modes and curl-type modes of cosmic shear, using Eqs. (28), (29), (34), (44), and (45), as

$$C_L^{\Omega E} = A \int \frac{d^3k}{(2\pi)^3} P_T(k) F_L^\Omega(k) F_L^E(k), \qquad (48)$$

where the GW power spectrum is given by

$$P_T(k) = \frac{\pi^2 r \Delta_\mathcal{R}^2(k_0)}{2k^3}.$$

We want to calculate the magnitude and shape of such correlations, to determine whether such a signal is observable. We use the WMAP seven-year cosmological parameters and assume the maximum allowable level of GWs from early universe physics, with a tensor-to-scalar ratio $r = 0.24$, the limit from the WMAP-7 data combined with BAO and the H_0 measurement [42]. We also assume that the GW background is entirely right-circularly-polarized. As a first estimate, we calculate the level of such correlations while making several assumptions. We use the approximate form of the GW transfer function $T(k, \eta) \simeq 3j_1(k\eta)/(k\eta)$, assume that reionization happened instantaneously so that the electron density n_e is equal to a step function, and neglect contributions to the polarization modes that came from last scattering. The two last assumptions affect mostly the higher-L multipoles, which in this cross-correlation are suppressed since we see that $\phi_{LM}^{\rm ten}$ and $\Omega_{LM}^{\rm ten}$ fall off very fast with L.

With these assumptions, we have calculated the correlation functions $C_L^{\phi B}$ and $C_L^{\Omega E}$, and show them as the blue circles in Figs. 3 and 4. Note that the absolute value of the correlation functions are plotted, and that the cusps in the profiles result from sign changes. Note also that both correlation functions are linearly proportional to the chirality parameter A, so that they would flip in sign if the GW

background were left instead of right-circularly-polarized. We are only interested in low multipoles, since our assumptions break down for larger L, and such multipoles are strongly suppressed in correlation with the weak-lensing modes.

D. Variance of ϕ-B and Ω-E correlations

It is useful to know the variance with which we could measure such parity-violating cross-correlations. From [37] we see that the variance with which we could measure the cross-correlation C_L^{XY} of two distinct Gaussian random variables X and Y is given by

$$(\Delta C_L^{XY})^2 \equiv \langle (\widehat{C_L^{XY}} - C_L^{XY})^2 \rangle,$$

where $\widehat{C_L^{XY}} = 1/(2L + 1)\sum_M X_{LM}Y_{LM}^*$ is the estimator for the cross-correlation, and C_L^{XY} is its theoretical value under the null hypothesis. Reference [37] then evaluates this variance, assuming distinct X and Y, to be

$$(\Delta C_L^{XY})^2 = \frac{1}{2L + 1}[(C_L^{XY})^2 + C_L^{XX\mathrm{map}}C_L^{YY\mathrm{map}}], \quad (49)$$

where, as before, $C_L^{XX\mathrm{map}} = W_L^2 C_L + N_L^{XX}$, with W_L the window function defined in Sec. II C, and N_L^{XX} the noise in the measurement of C_L^{XX}.

In our case, the null hypothesis is that there is a GW background with the maximal tensor-to-scalar ratio, but it contains equal numbers of right- and left-circularly-polarized GWs, i.e., it is not chiral. In this case, the theoretical value of parity-violating cross-correlations is zero, so that the first term in Eq. (49) vanishes. Then, assuming that $\widehat{\phi_{LM}}$ and $\widehat{\Omega_{LM}}$ are Gaussian random variables, a reasonable assumption since many uncorrelated noise processes are likely to contribute to this measured value, we find for the variances,

$$(\Delta C_L^{\phi B})^2 = \frac{1}{2L + 1}C_L^{\phi\phi\mathrm{map}}C_L^{BB\mathrm{map}} \quad (50)$$

$$(\Delta C_L^{\Omega E})^2 = \frac{1}{2L + 1}C_L^{\Omega\Omega\mathrm{map}}C_L^{EE\mathrm{map}}. \quad (51)$$

To calculate these errors, we know that the instrumental errors on the polarization power spectra are given by

$$N_L^{EE} = N_L^{BB} = \frac{8\pi(\mathrm{NET})^2}{t_{\mathrm{obs}}\sqrt{f_{\mathrm{sky}}}}.$$

We use the Planck-satellite parameters, as in Sec. III D. We also use the CMB anisotropy calculator CAMB to calculate the temperature and polarization power spectra including effects at the surface of last scatter [52]. The resulting errors are shown as red +s in Figs. 3 and 4. This noise, which combines instrumental and cosmic-variance sources, is at least an order of magnitude above the corresponding maximum signal level at low multipoles, and drops less rapidly with l so that the low multipoles yield the highest signal-to-noise.
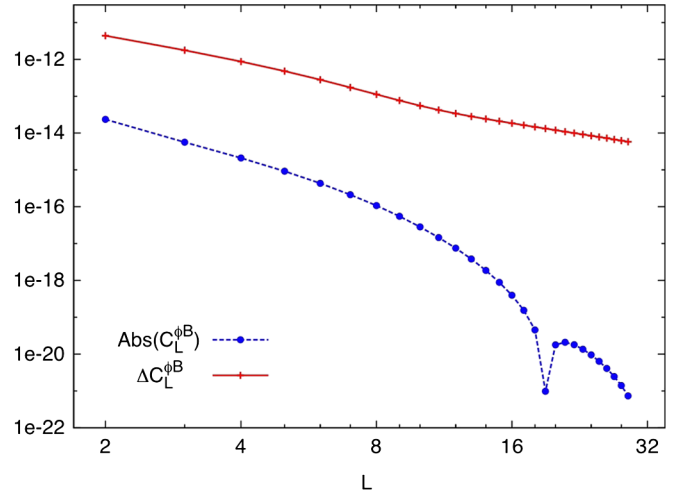

FIG. 3 (color online). Here we plot the cross-correlation $C_L^{\phi B}$ between the gradient ϕ modes of the weak lensing of cosmic shear with the curl-type B modes of the CMB-polarization in blue circles, and the noise on this measurement due to cosmic variance and Planck satellite instrumental noise in red +s. Since these quantities are of opposite parity, in the absence of parity-breaking physics we expect this cross-correlation to vanish. However, if we assume, for example, that the entire allowable GW background is right-circularly-polarized, such a cross-correlation could occur. The cross-correlation is linearly proportional to the chirality parameter A, defined such that $A = 1$ denotes a completely right-circularly-polarized GW background, $A = -1$ denotes completely left-circularly-polarized, and $A = 0$ denotes an unpolarized background. Here we assume the maximum allowable tensor-to-scalar ratio $r = 0.24$, the limit from WMAP-7 data combined with BAO and the H_0 measurement [42]. Cusps in the absolute value of the correlation function correspond to sign changes of the correlation function.

E. Signal-to-noise ratio of chiral GW background detection

We finally wish to calculate the achievable signal-to-noise of a measurement of the magnitude of such cross-correlations given our calculations of their shapes and variances. Such a measurement would tell us about the presence or absence of a chiral GW background, or of parity violation in the processes that caused departures from Gaussianity/SI in general. We can phrase the aim of this calculation as finding the error with which we could measure the chirality parameter A, which sets the amplitude of the cross-correlations relative to their maximum values in the case of a completely circularly-polarized GW background, as in Eqs. (47) and (48). Let us calculate this for the case of the ϕ-B cross-correlation; the Ω-E case will be similar.

We define a new quantity $C_{L\max}^{\phi B}$, defined such that

$$C_L^{\phi B} = AC_{L\max}^{\phi B}.$$

If we assume that the instrumental noise on $C_L^{\phi B}$ is Gaussian, so that $\widehat{C_L^{\phi B}} \equiv W_L^{-2}C_L^{\phi B}$ is a random variable drawn from a Gaussian probability distribution with
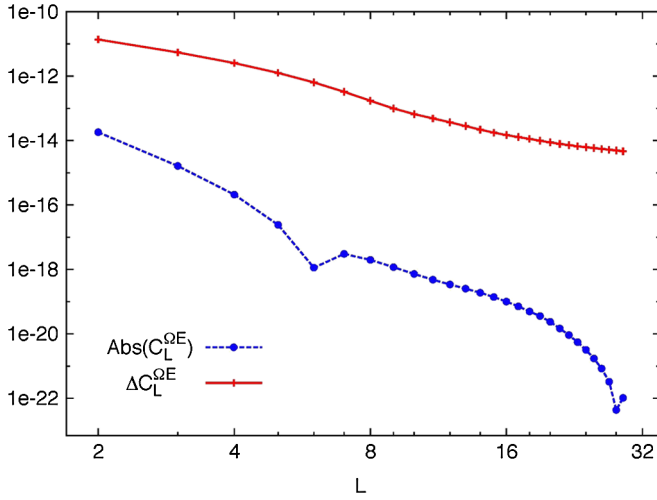
FIG. 4 (color online).   Here we plot the cross-correlation $C_L^{\Omega E}$ between the curl-type Ω modes of cosmic shear with the gradient-type E-modes of the CMB-polarization in blue circles, and the noise on this measurement due to cosmic variance and Planck-satellite instrumental noise in red +s. As with the $\phi - B$ correlation, we assume a completely right-circularly-polarized GW background, with the maximum currently permitted tensor-to-scalar ratio.

variance $(\Delta C_L^{\phi B})^2$ and mean $A C_{L\,\text{max}}^{\phi B}$, we can find the maximum-likelihood estimator for A to be

$$\hat{A} = \frac{\sum_L \widehat{C_L^{\phi B}} \, C_{L\,\text{max}}^{\phi B} (\Delta C_L^{\phi B})^{-2}}{\sum_L (C_{L\,\text{max}}^{\phi B})^2 (\Delta C_L^{\phi B})^{-2}}.$$

Then, assuming that the instrumental noise is uncorrelated between different multipoles, the variance of this estimator is given by

$$\langle \hat{A}^2 \rangle = \left[\sum_L (C_{L\,\text{max}}^{\phi B})^2 (\Delta C_L^{\phi B})^{-2} \right]^{-1}.$$

The maximum signal-to-noise with which we can measure this amplitude is given by

$$\left(\frac{S}{N} \right)_{\text{max}}^{\phi B} = \frac{\hat{A}_{\text{max}}}{\sqrt{\langle \hat{A}^2 \rangle}} = \left[\sum_L (C_{L\,\text{max}}^{\phi B})^2 (\Delta C_L^{\phi B})^{-2} \right]^{1/2}. \quad (52)$$

The same method can be used to calculate the obtainable signal-to-noise from the Ω-E cross-correlation, giving

$$\left(\frac{S}{N} \right)_{\text{max}}^{\Omega E} = \left[\sum_L (C_{L\,\text{max}}^{\Omega E})^2 (\Delta C_L^{\Omega E})^{-2} \right]^{1/2}. \quad (53)$$

Using the values of the cross-correlations and their errors calculated above, we find that the obtainable signal-to-noise from measurement of these cross-correlations is 0.002 for $C_L^{\Omega E}$ and 0.01 for $C_L^{\phi B}$. These numbers are too small for us to have any reasonable expectation of detection using the Planck satellite. Recalculating the above errors assuming

an ideal CMB experiment, with no instrumental noise and infinite resolution, the values of the signal-to-noise only change by a factor of two, indicating that this method is not likely to be a promising way to detect a chiral GW background.

V. CONCLUSIONS

BiPoSHs are a formalism to describe correlations between two different spherical-harmonic coefficients of the CMB temperature field, which can occur if the CMB temperature field is not exactly Gaussian or statistically isotropic. This paper introduces odd-parity BiPoSHs, a set of BiPoSHs that has not yet been studied, and details how they can be estimated from knowledge of the CMB temperature fluctuations.

We calculate the even- and odd-parity BiPoSHs that are sourced by gradient- and curl-type deflections of the CMB, respectively, and from this we obtain estimators for these deflections in terms of the BiPoSH coefficients. We show that lensing by scalar metric perturbations causes only gradient-type deflections, and thus only sources even-parity BiPoSHs. However, lensing by GWs produces both gradient- and curl-type deflections and thus sources both even- and odd-parity BiPoSHs. We calculate the expected power spectra of deflections due to scalar and tensor perturbations and their errors, and conclude that a reasonable signal-to-noise measurement of the amplitude of the GW background cannot be obtained from these autocorrelations even with the ideal CMB experiment, and thus from autocorrelations of the BiPoSH coefficients.

Although lensing by GWs produces both even- and odd-parity BiPoSHs, their opposite parity implies that they could not be correlated. However, in the presence of parity-violating physics, such as a chiral GW background, this parity argument breaks down and we might expect a correlation. We consider such a cross-correlation, and encourage its measurement even though the likelihood of observing a cosmological signal is low.

A GW background also produces signals in the E- and B-type CMB-polarization spectra, which are of even and odd parity, respectively. We consider the possibility that a chiral GW background would produce cross-correlations between opposite-parity components of lensing and polarization, and calculate the expected magnitude and errors of such cross-correlations. Although we find that the likelihood of observing a cosmological signal is low, we encourage the measurement of these cross-correlations since such a detection would provide evidence of important systematic errors or even new parity-breaking physics.

Here we have discussed BiPoSHs constructed from temperature multipole moments only, but the formalism can be generalized to include the polarization as well. It may also be that inclusion of the polarization improves the sensitivity to these parity-breaking, and other, signals. We plan to pursue this analysis in future work.

Finally, we note that weak-lensing distortions of distant galaxies can also be decomposed into curl and gradient components [32,53]. Similar tests for parity violation can thus also be carried out with weak lensing of galaxies.

ACKNOWLEDGMENTS

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