Measuring γ with $B \to K\pi\pi$ and $B \to KK\bar{K}$ decays

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We present a method for cleanly extracting the CP phase γ from the Dalitz plots of $B^+ \to K^+\pi^+\pi^-$, $B_d^0 \to K^+\pi^0\pi^-$, $B_d^0 \to K^0\pi^+\pi^-$, $B_d^0 \to K^+K^0K^-$, and $B_d^0 \to K^0K^0\bar{K}^0$. The $B \to K\pi\pi$ and $B \to KK\bar{K}$ decays are related by flavor SU(3) symmetry, but SU(3) breaking is taken into account. Most of the experimental measurements have already been made—what remains is a Dalitz-plot analysis of $B_d^0 \to K^0K^0\bar{K}^0$ (or $B_d^0 \to K_SK_SK_S$). We (very) roughly estimate the error on γ to be $\sim$25%. This is somewhat larger than the error in two-body decays, but it would be the first clean measurement of γ in three-body decays. Furthermore, at the super-B factory, it is possible that γ could be measured more precisely in three-body decays than in two-body decays.

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In the past, most of the theoretical work looking at clean methods for extracting weak-phase information in the B system focused on two-body decays. This is essentially because (i) final states such as ψK_S, $\pi^+\pi^-$, etc. are CP eigenstates, and (ii) if there is a second decay amplitude, with a different weak phase, it has been possible to find methods to remove this "pollution," and cleanly get at the weak phases. On the other hand, in three-body B decays, final states such as $K_S\pi^+\pi^-$ are not CP eigenstates—the value of its CP depends on whether the relative $\pi^+\pi^-$ angular momentum is even ($CP+$) or odd ($CP-$). Furthermore, even if the CP of the final state were determined in some way, one still has the problem of removing the pollution due to additional decay amplitudes. For these reasons, it has generally been thought that it is not possible to obtain clean weak-phase information from three-body decays [1].

Recently, it was shown that this is not true. By doing a diagrammatic analysis of the three-body amplitudes, one can resolve these two problems [2]. First, a Dalitz-plot analysis can be used to experimentally separate the $CP+$ and $-$ components of the three-particle final state. Second, one can often remove the pollution of additional diagrams and cleanly measure the CP phases. In fact, in Ref. [3], it was shown how to extract the weak phase γ from $B \to K\pi\pi$ decays. We briefly describe this method below.

In $B \to K\pi\pi$ decays, the isospin state of the $\pi\pi$ pair must be symmetric (antisymmetric) if the relative angular momentum is even (odd). As we will see below, it is the symmetric case which is most interesting. Here there are six possible decays: $B^+ \to K^+\pi^+\pi^-$, $B^+ \to K^+\pi^0\pi^0$, $B^+ \to K^0\pi^+\pi^0$, $B_d^0 \to K^+\pi^-\pi^0$, $B_d^0 \to K^0\pi^+\pi^-$, and $B_d^0 \to K^0\pi^0\pi^0$. The first step is to express the amplitudes for these processes in terms of diagrams. The diagrams are as in two-body B decays [4]: the color-favored and color-suppressed tree amplitudes T and C, the gluonic-penguin amplitudes P_{tc} and P_{uc}, and the color-favored and color-suppressed electroweak-penguin (EWP) amplitudes P_{EW} and P_{EW}^C. (We neglect annihilation- and exchange-type diagrams.) Furthermore, for three-body decays, it is necessary to "pop" a quark pair from the vacuum. The diagrams are written with subscripts, indicating that the popped quark pair is between two (nonspectator) final-state quarks (subscript "1"), or between two final-state quarks including the spectator (subscript "2"). (For $B \to K\pi\pi$ decays, the popped quark pair is $u\bar{u}$ or $d\bar{d}$. Under isospin, these amplitudes are equal.)

In addition, some time ago it was shown that, under flavor SU(3) symmetry, there are relations between the EWP and tree diagrams in $B \to K\pi$ decays [5,6]. In Ref. [3], it was shown that similar EWP-tree relations hold for $B \to K\pi\pi$ decays. Taking $c_1/c_2 = c_9/c_{10}$ for the Wilson coefficients (which holds to about 5%), these take the simple form

$$P'_{EW1} = \kappa T'_1, \qquad P'_{EW2} = \kappa T'_2,$$
$$P'^C_{EW1} = \kappa C'_1, \qquad P'^C_{EW2} = \kappa C'_2, \tag{1}$$

where

$$\kappa \equiv -\frac{3}{2}\frac{|\lambda_t^{(s)}|}{|\lambda_u^{(s)}|}\frac{c_9 + c_{10}}{c_1 + c_2}, \tag{2}$$

with $\lambda_p^{(s)} = V_{pb}^* V_{ps}$.

Now, the EWP-tree relations assume SU(3) symmetry (and the approximate ratio of Wilson coefficients). The expected error due to SU(3)-breaking effects is $O(30\%)$. However, the dominant diagram in $\bar{b} \to \bar{s}$ decays is P'_{tc}, so that EWPs and trees are subleading effects. Thus SU(3) breaking is subdominant—the net theoretical error due to the use of the EWP-tree relations is only $O(5\%)$. This is

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consistent with the error estimates given in Ref. [5] (for EWP-tree relations in $B \to K\pi$).

In addition, there is an important caveat. Under SU(3), the final state in $B \to K\pi\pi$ involves three identical particles, so that the six permutations of these particles (the group S_3) must be taken into account. That is, the three particles are in a totally symmetric state, a totally antisymmetric state, or one of four mixed states. However, the EWP-tree relations hold only for the totally symmetric state. Thus, the analysis must be carried out for this state. Now, the expressions for the $B \to K(\pi\pi)_{\rm sym}$ amplitudes in terms of diagrams hold even under full SU(3) symmetry [3]. It is therefore only necessary to produce observables for the totally symmetric states. This is doable, and below we present the details of how this is carried out.

With the above EWP-tree relations, the six $B \to K(\pi\pi)_{\rm sym}$ amplitudes can be written in terms of 5 effective diagrams (i.e. linear combinations of the diagrams) [3]. There are therefore 10 theoretical parameters in the amplitudes[1]: 5 magnitudes of effective diagrams, 4 relative (strong) phases, and γ. On the other hand, there are 11 experimental observables. Given that $B^+ \to K^0\pi^+\pi^0$ is not independent (its amplitude is proportional to that of $B_d^0 \to K^+\pi^0\pi^-$), these are the branching ratios and direct CP asymmetries of $B^+ \to K^+\pi^+\pi^-$, $B^+ \to K^+\pi^0\pi^0$, $B_d^0 \to K^+\pi^0\pi^-$, $B_d^0 \to K^0\pi^+\pi^-$, and $B_d^0 \to K^0\pi^0\pi^0$, and the indirect CP asymmetry of $B_d^0 \to K^0\pi^+\pi^-$ (the indirect CP asymmetry of $B_d^0 \to K^0\pi^0\pi^0$ will essentially be impossible to measure). Since there are more observables than theoretical parameters, γ can be extracted by doing a fit.[2]

The disadvantage of this method is that it involves the decays $B^+ \to K^+\pi^0\pi^0$ and $B_d^0 \to K^0\pi^0\pi^0$. With two π^0 mesons in the final state, both of these decays will be extremely difficult to measure. We are therefore motivated to see if the $B \to K\pi\pi$ method can be modified, avoiding these two decays. As we show below, this can indeed be done—things can be considerably improved by using $B \to KK\bar{K}$ decays. The use of these decays is quite natural since they, like $B \to K\pi\pi$, are also $\bar{b} \to \bar{s}$ transitions.

First, consider $B \to K\pi\pi$ decays with the $\pi\pi$ pair in a symmetric isospin state. We leave aside $B^+ \to K^+\pi^0\pi^0$, $B_d^0 \to K^0\pi^0\pi^0$ and $B^+ \to K^0\pi^+\pi^0$ (since, as mentioned above, its amplitude is not independent). The amplitudes of the remaining three processes are

$$\begin{aligned}
2A(B_d^0 \to K^+\pi^0\pi^-)_{\rm sym} &= T_1' e^{i\gamma} + C_2' e^{i\gamma} - P_{EW2}' - P_{EW1}'^C,\\
\sqrt{2}A(B_d^0 \to K^0\pi^+\pi^-)_{\rm sym} &= -T_1' e^{i\gamma} - C_1' e^{i\gamma} - \tilde{P}_{uc}' e^{i\gamma} + \tilde{P}_{tc}'\\
&\quad + \frac{1}{3}P_{EW1}' + \frac{2}{3}P_{EW1}'^C - \frac{1}{3}P_{EW2}'^C,\\
\sqrt{2}A(B^+ \to K^+\pi^+\pi^-)_{\rm sym} &= -T_2' e^{i\gamma} - C_1' e^{i\gamma} - \tilde{P}_{uc}' e^{i\gamma} + \tilde{P}_{tc}'\\
&\quad + \frac{1}{3}P_{EW1}' - \frac{1}{3}P_{EW1}'^C + \frac{2}{3}P_{EW2}'^C.
\end{aligned} \tag{3}$$

In the above, $\tilde{P}' \equiv P_1' + P_2'$. (As $B \to K\pi\pi$ is a $\bar{b} \to \bar{s}$ transition, the diagrams are written with primes.) Here we have explicitly written the weak-phase dependence (this includes γ and the minus sign from $V_{tb}^* V_{ts}$ [$\tilde{P}_{tc}'$ and EWPs]), while the diagrams contain strong phases.

Second, consider $B \to KK\bar{K}$ decays. For the case in which the final KK pair is in a symmetric isospin state, there are four such processes: $B^+ \to K^+K^+K^-$, $B^+ \to K^+K^0\bar{K}^0$, $B_d^0 \to K^+K^0K^-$, and $B_d^0 \to K^0K^0\bar{K}^0$. Here, $B^+ \to K^+K^+K^-$ and $B^+ \to K^+K^0\bar{K}^0$ are not independent—their amplitudes are proportional to those of $B_d^0 \to K^+K^0K^-$ and $B_d^0 \to K^0K^0\bar{K}^0$, respectively. These are

$$\begin{aligned}
&\sqrt{2}A(B_d^0 \to K^+K^0K^-)_{\rm sym}\\
&\quad = -T_{2,s}' e^{i\gamma} - C_{1,s}' e^{i\gamma} - \hat{P}_{uc}' e^{i\gamma} + \hat{P}_{tc}' + \frac{2}{3}P_{EW1,s}' - \frac{1}{3}P_{EW1}'\\
&\qquad + \frac{2}{3}P_{EW2,s}'^C - \frac{1}{3}P_{EW1}'^C,\\
&A(B_d^0 \to K^0K^0\bar{K}^0)_{\rm sym}\\
&\quad = \hat{P}_{uc}' e^{i\gamma} - \hat{P}_{tc}' + \frac{1}{3}P_{EW1,s}' + \frac{1}{3}P_{EW1}' + \frac{1}{3}P_{EW2,s}'^C + \frac{1}{3}P_{EW1}'^C,
\end{aligned} \tag{4}$$

where $\hat{P}' \equiv P_{2,s}' + P_1'$. In the above, certain diagrams are written with the subscript "s." This indicates that the popped quark pair is $s\bar{s}$. When the diagram has no subscript s (the penguin or EWP diagrams), this means that the popped quark pair is $u\bar{u}$ or $d\bar{d}$, but the virtual particle decays to $s\bar{s}$.

We now assume flavor SU(3) symmetry. This has two consequences. First, the amplitude with a popped $s\bar{s}$ quark pair is equal to that with a popped $u\bar{u}$ or $d\bar{d}$. That is, we no longer need the subscript s on diagrams. This means that the diagrams in $B \to KK\bar{K}$ decays are the same as those in $B \to K\pi\pi$ decays. Second, the EWP-tree relations of Eq. (1) hold.

Thus, under SU(3) the amplitudes of Eqs. (3) and (4) take the form

[1]In fact, the expression for any indirect CP asymmetry contains another theoretical parameter—the phase of B_d^0-$\bar{B}_d^0$ mixing, β. However, its value can be taken from the indirect CP asymmetry in $B_d^0 \to J/\psi K_S$ [7].

[2]There is a complication in that the diagrams are momentum dependent, as are the observables. In obtaining the best-fit "values" of the diagrams, one will determine the momentum dependence of their magnitudes and relative strong phases. On the other hand, γ is independent of the particles' momenta. Later in the paper, we detail how such a fit is done.

$$\begin{aligned}
2A(B_d^0 \to K^+\pi^0\pi^-)_{\rm sym} &= T_1' e^{i\gamma} + C_2' e^{i\gamma} - \kappa(T_2' + C_1'),\\
\sqrt{2}A(B_d^0 \to K^0\pi^+\pi^-)_{\rm sym} &= -T_1' e^{i\gamma} - C_1' e^{i\gamma} - \tilde{P}'_{uc} e^{i\gamma} + \tilde{P}'_{tc} + \kappa\left(\frac{1}{3}T_1' + \frac{2}{3}C_1' - \frac{1}{3}C_2'\right),\\
\sqrt{2}A(B^+ \to K^+\pi^+\pi^-)_{\rm sym} &= -T_2' e^{i\gamma} - C_1' e^{i\gamma} - \tilde{P}'_{uc} e^{i\gamma} + \tilde{P}'_{tc} + \kappa\left(\frac{1}{3}T_1' - \frac{1}{3}C_1' + \frac{2}{3}C_2'\right),\\
\sqrt{2}A(B_d^0 \to K^+K^0K^-)_{\rm sym} &= -T_2' e^{i\gamma} - C_1' e^{i\gamma} - \tilde{P}'_{uc} e^{i\gamma} + \tilde{P}'_{tc} + \kappa\left(\frac{1}{3}T_1' - \frac{1}{3}C_1' + \frac{2}{3}C_2'\right),\\
A(B_d^0 \to K^0K^0\bar{K}^0)_{\rm sym} &= \tilde{P}'_{uc} e^{i\gamma} - \tilde{P}'_{tc} + \kappa\left(\frac{2}{3}T_1' + \frac{1}{3}C_1' + \frac{1}{3}C_2'\right).
\end{aligned} \tag{5}$$

Note that this implies that $A(B^+ \to K^+\pi^+\pi^-)_{\rm sym} = A(B_d^0 \to K^+K^0K^-)_{\rm sym}$. Further, we reiterate that the above expressions for the amplitudes hold also for the totally symmetric final state, to which the EWP-tree relations apply.

We now define the following five effective diagrams:

$$\begin{aligned}
T_a' &\equiv T_1' - T_2',\\
T_b' &\equiv C_2' + T_2',\\
P_a' &\equiv \tilde{P}'_{uc} + T_2' + C_1',\\
P_b' &\equiv \tilde{P}'_{tc} + \kappa\left(\frac{1}{3}T_1' + \frac{2}{3}C_1' - \frac{1}{3}C_2'\right),\\
C_a' &\equiv \kappa(C_1' - C_2').
\end{aligned} \tag{6}$$

The amplitudes can be written in terms of these five diagrams:

$$\begin{aligned}
2A(B_d^0 \to K^+\pi^0\pi^-)_{\rm sym} &= T_a' e^{i\gamma} + T_b' e^{i\gamma} - C_a' - \kappa T_b',\\
\sqrt{2}A(B_d^0 \to K^0\pi^+\pi^-)_{\rm sym} &= -T_a' e^{i\gamma} - P_a' e^{i\gamma} + P_b',\\
\sqrt{2}A(B^+ \to K^+\pi^+\pi^-)_{\rm sym} &= -P_a' e^{i\gamma} + P_b' - C_a',\\
\sqrt{2}A(B_d^0 \to K^+K^0K^-)_{\rm sym} &= -P_a' e^{i\gamma} + P_b' - C_a',\\
A(B_d^0 \to K^0K^0\bar{K}^0)_{\rm sym} &= P_a' e^{i\gamma} - T_b' e^{i\gamma} - \frac{1}{\kappa}C_a' e^{i\gamma} - P_b'\\
&\quad + \kappa T_a' + \kappa T_b' + C_a'.
\end{aligned} \tag{7}$$

As with the $B \to K\pi\pi$ method, five effective diagrams corresponds to 10 theoretical parameters: 5 magnitudes of diagrams, 4 relative phases, and γ. But there are 11 (momentum-dependent) experimental observables: the decay rates and direct asymmetries for the four decays $B_d^0 \to K^+\pi^0\pi^-$, $B_d^0 \to K^0\pi^+\pi^-$, $B_d^0 \to K^+K^0K^-$, and $B_d^0 \to K^0K^0\bar{K}^0$ (we ignore $B^+ \to K^+\pi^+\pi^-$ since its amplitude is not independent), and the indirect asymmetries of $B_d^0 \to K^0\pi^+\pi^-$, $B_d^0 \to K^+K^0K^-$, and $B_d^0 \to K^0K^0\bar{K}^0$. With more observables than theoretical parameters, γ can be extracted from a fit.

We now present the details of how the fit is carried out. Consider the decay $B \to P_1P_2P_3$, in which the three pseudoscalar mesons P_i $(i = 1 - 3)$ have momenta p_i. From these, we can construct the three Mandelstam variables:

$$s_{12} \equiv (p_1 + p_2)^2, \quad s_{13} \equiv (p_1 + p_3)^2, \quad s_{23} \equiv (p_2 + p_3)^2. \tag{8}$$

These are not independent, but obey

$$s_{12} + s_{13} + s_{23} = m_B^2 + m_1^2 + m_2^2 + m_3^2. \tag{9}$$

Experimentally, the Dalitz plot of this decay is measured. Its events are given in terms of two Mandelstam variables, say s_{12} and s_{13}. Now, the great advantage of a Dalitz-plot analysis is that it allows one to extract the full amplitude of the decay. We write

$$\mathcal{M}(B \to P_1P_2P_3) = \sum_j c_j e^{i\theta_j} F_j(s_{12}, s_{13}), \tag{10}$$

where the sum is over all decay modes (resonant and nonresonant). c_j and θ_j are the magnitude and phase of the j contribution, respectively, measured relative to one of the contributing channels. The distributions F_j, which depend on s_{12} and s_{13}, describe the dynamics of the individual decay amplitudes, and take different (known) forms for the various contributions. The key point is that a maximum likelihood fit over the entire Dalitz plot gives the best values of the c_j and θ_j. Thus, the decay amplitude can be obtained, up to an overall normalization. This normalization is fixed by the constraint of the measured partial rate [7]:

$$\Gamma = \frac{1}{(2\pi)^3}\frac{1}{32m_B^3}\int |\mathcal{M}|^2 ds_{12}ds_{13}. \tag{11}$$

With this, the decay amplitude $\mathcal{M}(s_{12}, s_{13})$ is known.

As will be seen below, we rely heavily on $\mathcal{M}(s_{12}, s_{13})$. In particular, we use it to obtain the observables for the $B \to P_1P_2P_3$ decay. As such, the errors on these observables come entirely from the uncertainty in $\mathcal{M}(s_{12}, s_{13})$. While, as noted above, it is possible to obtain the best-fit values of the Dalitz-plot variables c_j and θ_j, there are errors associated with these values. This is due to two sources. First, one has the statistical error in the experimental Dalitz plot. Second, there is a systematic uncertainty related to the choice of the F_j in Eq. (10). In addition, there is a statistical error in the overall normalization [coming from Eq. (11)]. All of these must be carefully taken into account in order to obtain conservative errors on the Dalitz-plot variables.

As noted earlier, the EWP-tree relations hold only for the totally symmetric SU(3) decay amplitude. But this can be found from the above:

$$\mathcal{M}_{\text{fully sym}} = \frac{1}{\sqrt{6}}[\mathcal{M}(s_{12}, s_{13}) + \mathcal{M}(s_{13}, s_{12}) + \mathcal{M}(s_{12}, s_{23}) + \mathcal{M}(s_{23}, s_{12}) + \mathcal{M}(s_{23}, s_{13}) + \mathcal{M}(s_{13}, s_{23})]. \tag{12}$$

Using this, it is possible to compute the $B \to P_1P_2P_3$ observables. However, recall that the method involves a fit using the observables from several different decays ($B_d^0 \to K^+\pi^0\pi^-$, $B_d^0 \to K^0\pi^+\pi^-$, $B_d^0 \to K^+K^0K^-$, and $B_d^0 \to K^0K^0\bar{K}^0$). All observables must involve the same Mandelstam variables. On the other hand, the numbering of final-state particles is arbitrary, so that s_{12} for one decay might equal s_{13} for a different decay. All of this makes it somewhat confusing to ensure that observables in different decays have the same Mandelstam variables. For this reason, it is useful at this stage to change notation (but the physics is unchanged). In any decay there are three Mandelstam variables. We define s_{++}, s_+ and s_- to be the largest, second-largest, and smallest of these, respectively. The identities of the particles which are associated with s_{++}, s_+, and s_- are irrelevant (e.g. s_{++} can correspond to s_{12}, s_{13}, or s_{23}). This is consistent with the assumption of SU(3) and the fully symmetric decay amplitude. With these Mandelstam variables, we have

$$\mathcal{M}_{\text{fully sym}} = \frac{1}{\sqrt{6}}[\mathcal{M}(s_{++}, s_+) + \mathcal{M}(s_+, s_{++}) + \mathcal{M}(s_{++}, s_-) + \mathcal{M}(s_-, s_{++}) + \mathcal{M}(s_-, s_+) + \mathcal{M}(s_+, s_-)]. \tag{13}$$

Since s_{++}, s_+, and s_- are not independent, this gives the fully symmetric amplitude as a function of two Mandelstam variables, say s_{++} and s_+.

The observables are obtained as follows. First, one forms the totally symmetric SU(3) decay amplitudes as in Eq. (13) for each $B \to P_1P_2P_3$ decay ($\mathcal{M}_{\text{fully sym}}$) and its CP conjugate ($\bar{\mathcal{M}}_{\text{fully sym}}$). Second, using these, for specific values of s_{++} and s_+, one computes the partial rates:

$$\Gamma_{s_{++},s_+} = \frac{1}{(2\pi)^3}\frac{1}{32m_B^3}|\mathcal{M}_{\text{fully sym}}(s_{++}, s_+)|^2,$$
$$\bar{\Gamma}_{s_{++},s_+} = \frac{1}{(2\pi)^3}\frac{1}{32m_B^3}|\bar{\mathcal{M}}_{\text{fully sym}}(s_{++}, s_+)|^2. \tag{14}$$

These allow the computation of the CP-averaged branching ratio and direct CP asymmetry:

$$BR_{s_{++},s_+} = \frac{1}{\Gamma_B}(\Gamma_{s_{++},s_+} + \bar{\Gamma}_{s_{++},s_+}),$$
$$A_{s_{++},s_+} = \frac{\Gamma_{s_{++},s_+} - \bar{\Gamma}_{s_{++},s_+}}{\Gamma_{s_{++},s_+} + \bar{\Gamma}_{s_{++},s_+}}. \tag{15}$$

Third, for those decays in which the final state is accessible to both B_d^0 and $\bar{B}_d^0$ mesons, one has an indirect (mixing-induced) CP asymmetry. It is given by

$$S_{s_{++},s_+} = \text{Im}\left[e^{-2i\beta}\frac{\bar{\mathcal{M}}_{\text{fully sym}}(s_{++}, s_+)}{\mathcal{M}_{\text{fully sym}}(s_{++}, s_+)}\right]. \tag{16}$$

As discussed earlier, in all cases, the error on the observables is found by propagating the errors on the Dalitz-plot variables. These include both statistical and systematic effects.

Now, given that the method assumes flavor SU(3) symmetry, one would like to know how SU(3) breaking affects the analysis, and what is its size. Leaving aside the EWP-tree relations, in which SU(3)-breaking effects are subdominant, there are two areas where the breaking may be significant. First, under SU(3), the diagrams in $B \to KK\bar{K}$ and $B \to K\pi\pi$ are the same. Since both decays are $\bar{b} \to \bar{s}$ transitions, the difference between them is that $B \to KK\bar{K}$ decays have an $s\bar{s}$ quark pair in the final state, hadronizing to $K\bar{K}$, while $B \to K\pi\pi$ decays have $u\bar{u}$ or $d\bar{d}$, hadronizing to $\pi\pi$. This is essentially the same for each diagram. (The SU(3)-breaking effect associated with an $s\bar{s}$ pair being popped from the vacuum may not be exactly equal to that when $s\bar{s}$ is produced in the decay of a virtual particle, but the difference is small.) Thus, including SU(3) breaking, the amplitudes of Eq. (7) can be written

$$\begin{aligned}
2A(B_d^0 \to K^+\pi^0\pi^-)_{\text{sym}} &= T'_a e^{i\gamma} + T'_b e^{i\gamma} - C'_a - \kappa T'_b,\\
\sqrt{2}A(B_d^0 \to K^0\pi^+\pi^-)_{\text{sym}} &= -T'_a e^{i\gamma} - P'_a e^{i\gamma} + P'_b,\\
\sqrt{2}A(B^+ \to K^+\pi^+\pi^-)_{\text{sym}} &= -P'_a e^{i\gamma} + P'_b - C'_a,\\
\sqrt{2}A(B_d^0 \to K^+K^0K^-)_{\text{sym}} &= (1 + f_{SU(3)})[-P'_a e^{i\gamma} + P'_b - C'_a],\\
A(B_d^0 \to K^0K^0\bar{K}^0)_{\text{sym}} &= (1 + f_{SU(3)})\Big[P'_a e^{i\gamma} - T'_b e^{i\gamma} - \frac{1}{\kappa}C'_a e^{i\gamma} - P'_b + \kappa T'_a + \kappa T'_b + C'_a\Big],
\end{aligned} \tag{17}$$

where $f_{SU(3)}$ is the SU(3)-breaking factor. Second, under SU(3), π's and K's are identical particles, so that there is no difference between the Mandelstam variables for the processes $B \to KK\bar{K}$ and $B \to K\pi\pi$. There is therefore an SU(3)-breaking effect between the fully symmetric decay amplitudes for the two types of decay. However, it can be included in $f_{SU(3)}$.

The addition of $f_{SU(3)}$ brings the number of unknown theoretical parameters to 11. In principle, these can all be determined from a fit to the 11 experimental observables, albeit with discrete ambiguities. However, we can do better. Above it was noted that, in the limit of perfect SU(3), $A(B^+ \to K^+\pi^+\pi^-)_{\text{sym}} = A(B_d^0 \to K^+K^0K^-)_{\text{sym}}$. This means that $f_{SU(3)}$ can be determined by a comparison of these two decays. In particular,

$$\frac{\tau_+}{\tau_0}\frac{\mathcal{B}(B_d^0 \to K^+K^0K^-)_{\text{sym}}}{\mathcal{B}(B^+ \to K^+\pi^+\pi^-)_{\text{sym}}} = (1 + f_{SU(3)})^2. \quad (18)$$

In fact, this comparison can be performed now since the decays have been measured: $B^+ \to K^+\pi^+\pi^-$ in Ref. [8], $B_d^0 \to K^+K^0K^-$ in Ref. [9]. Now, since the EWP-tree relations Eq. (1) have been used to derive the expressions for the amplitudes, Eq. (18) holds only for totally symmetric states. Using the technique described above, one can obtain $A(B^+ \to K^+\pi^+\pi^-)_{\text{fully sym}}$ and $A(B_d^0 \to K^+K^0K^-)_{\text{fully sym}}$. In order to get the branching ratios, we compute the integral of the square of the fully symmetric amplitudes over the Dalitz plot (taking care to avoid sextuple counting). Doing this gives

$$\begin{aligned}\mathcal{B}(B^+ \to K^+\pi^+\pi^-)_{\text{fully sym}} &= 0.19\mathcal{B}(B^+ \to K^+\pi^+\pi^-),\\ \mathcal{B}(B_d^0 \to K^+K^0K^-)_{\text{fully sym}} &= 0.50\mathcal{B}(B_d^0 \to K^+K^0K^-).\end{aligned} \quad (19)$$

From Ref. [10], we have τ_0/τ_+=0.93, $\mathcal{B}(B^+ \to K^+\pi^+\pi^-)$= $(51.0\pm2.9)\times10^{-6}$ and $\mathcal{B}(B_d^0 \to K^+K^0K^-)=(24.7\pm2.3)\times 10^{-6}$. Equation (18) then gives

$$f_{SU(3)} = 0.17 \pm 0.06. \quad (20)$$

(This error does not include the errors in the parameters obtained from the Dalitz-plot analyses of the two decays.)

We can now put all the pieces together to describe how the fit is to be performed. The fully symmetric amplitudes for the decays $B_d^0 \to K^+\pi^0\pi^-$, $B_d^0 \to K^0\pi^+\pi^-$, $B_d^0 \to K^+K^0K^-$, and $B_d^0 \to K^0K^0\bar{K}^0$ are given in Eq. (17). They are a function of 10 unknown parameters, including γ. The value of $f_{SU(3)}$ is taken from Eq. (20). The 11 observables and their errors are computed as described above—the (fully symmetric) branching ratios and direct CP asymmetries are given in Eq. (15), and the indirect CP asymmetries in Eq. (16). Note that these are for specific values of s_{++} and s_+. One has a different set of observables for each (s_{++}, s_+) pair. With 10 unknowns and 11 constraints, one can now perform the fit. This will determine the magnitudes and relative strong phases of the five effective diagrams, as well as γ, all for the chosen values of (s_{++}, s_+). This is to be repeated for each independent (s_{++}, s_+) pair.[3] This has two effects. First, one will be able to fix the momentum dependence of the diagrams. Second, and more importantly, since γ is momentum independent, one can average over all the (s_{++}, s_+) fits. This will reduce its error, perhaps considerably.

Now, we already have experimental information about most of the required $B \to K\pi\pi$ and $B \to KK\bar{K}$ decays. In particular, the measurements of the Dalitz plots of $B_d^0 \to K^+\pi^0\pi^-$, $B_d^0 \to K^0\pi^+\pi^-$ and $B_d^0 \to K^+K^0K^-$ are described in Refs. [9,11,12], respectively. On the other hand, we do not yet have the Dalitz plot of $B_d^0 \to K^0K^0\bar{K}^0$. The branching ratio and CP asymmetries of $B_d^0 \to K_SK_SK_S$ are given in Ref. [13]. While the use of the final state $K_SK_SK_S$ is excellent—it is proportional to the fully symmetric state of $K^0K^0\bar{K}^0$—the observables are momentum independent. That is, an integration over the Dalitz plot has been performed. However, the method described in this paper requires the momentum-dependent observables. Once the Dalitz plot for $B_d^0 \to K_SK_SK_S$ is known, this method for extracting γ can be carried out.

Even though all the experimental data is not yet available, we can still attempt to estimate the precision with which γ can be obtained. Consider first $B_d^0 \to K^+K^0K^-$. According to the BaBar measurement in Ref. [9], the largest contributions to this decay come from the ϕK^0 and f_0K^0 resonances, and the $(K^+K^-)_{\text{NR}}K^0$, $(K^+K^0)_{\text{NR}}K^-$ and $(K^-K^0)_{\text{NR}}K^+$ nonresonant pieces. They find

$$\begin{aligned}\phi K^0\colon c_j &= 0.0085 \pm 0.0010,\\ f_0K^0\colon c_j &= 0.622 \pm 0.046,\\ (K^+K^-)_{\text{NR}}K^0\colon c_j &= 1\ (\text{fixed}),\\ (K^+K^0)_{\text{NR}}K^-\colon c_j &= 0.33 \pm 0.07,\\ (K^-K^0)_{\text{NR}}K^+\colon c_j &= 0.31 \pm 0.08,\end{aligned} \quad (21)$$

where c_j is defined in Eq. (10). The errors, which are statistical only, range from 7% to 25%. The above method describes how to obtain $\mathcal{M}_{\text{fully sym}}(B_d^0 \to K^+K^0K^-)$ from the amplitude given in Ref. [9], and from this the $B_d^0 \to K^+K^0K^-$ observables. A full numerical analysis is needed to do this, properly taking into account the errors on the c_j above, as well as the errors on the θ_j and F_j of Eq. (10), and the other resonances. However, a rough guess is that the errors on the observables will be about 20%. Similarly, we (guess)timate that the errors on the observables of the other decays, including those of $B_d^0 \to K^0K^0\bar{K}^0$, will be $\sim$20%. In order to obtain γ, a fit to the observables must be performed, taking into account the SU(3)-breaking factor of Eq. (20) (the error on $f_{SU(3)}$ will increase once the errors in the Dalitz-plot parameters are included), and one must average over the independent (s_{++}, s_+) pairs. It is impossible to predict with any accuracy what the error on γ will be, but an error of $O(25\%)$ does not seem unreasonable.

How does this compare with the precision on γ measured in two-body decays? The answer is: not that badly. The standard way of directly measuring γ uses $B \to D^0/\bar{D}^0K$ decays within the GLW [14] or ADS [15] methods. The latest measurement yields $\gamma = (68^{+10}_{-11})^\circ$ [16], i.e. the error is $\sim$15%. To be sure, our estimated error of $O(25\%)$ on the value of γ as extracted from three-body decays is worse than 15%. However, it is still roughly the

[3]The two pairs $(s_{++}, s_+)_1$ and $(s_{++}, s_+)_2$ are considered as independent if $|\mathcal{M}_{\text{fully sym}}((s_{++}, s_+)_1)|$ and $|\mathcal{M}_{\text{fully sym}}((s_{++}, s_+)_2)|$ do not overlap when one takes into account the errors on the Dalitz-plot parameters of Eq. (10).

same size, and if a full analysis were done, the real error might turn out to be smaller than our estimate. More to the point, when the Dalitz-plot measurements are done at the super-B factory, the Dalitz-plot parameters will be obtained with a smaller statistical error. This will have two effects. First, the error on γ will be reduced for each (s_{++}, s_{+}) pair. Second, one will have more independent (s_{++}, s_{+}) pairs, so the error will be further reduced when one averages over all the (s_{++}, s_{+}) fits. [See the discussion following Eq. (20).] Thus, the extraction of γ from three-body B decays may turn out to be more precise than that from two-body decays.

Compared to its original version, this paper has been considerably modified with the addition of the detailed discussion of how the fit is done, and the guesstimate of the error on the extracted value of γ. We are grateful to Jim Smith for asking the key question which led to this revision. This work was financially supported by NSERC of Canada.

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