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Heavy-meson masses in the ϵ regime of heavy-meson chiral perturbation theory

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The pseudoscalar and vector heavy-meson masses are calculated in the ϵ-regime of Heavy Meson Chiral Perturbation Theory to order ϵ^4. The results presented will allow the determination of low-energy coefficients (LECs) directly from Lattice QCD calculations of the heavy-mesons masses for lattices that satisfy the ϵ-regime criteria. In particular, the LECs that parametrize the next-to-leading order volume dependence of the heavy-meson masses are necessary for evaluating the light-pseudoscalar meson (π, K, η) and heavy meson ($\{D^0, D^+, D_s^+\}$, $\{B^-, \bar{B}^0, \bar{B}_s^0\}$) scattering phase shifts.

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I. INTRODUCTION

Understanding the properties of systems composed of heavy mesons, containing a single heavy quark, and the pseudo-Goldstone bosons (pGB) of Quantum Chromodynamics (QCD) is currently a topic of high interest. This interest has been partly triggered by the renaissance of charmonium and open-charm studies. A resonance that has initiated much discussion is the narrow $D_{s0}^*(2317)$, first observed by the *BABAR* collaboration [1]. This resonance couples to the S-wave DK continuum scattering state [2,3]. At low energies the strength of the DK interaction is predominantly parametrized by the scattering length. This has resulted in several theoretical studies that have attempted to determine the S-wave scattering lengths in the pGB-heavy meson scattering channels [4–8]. The determination of these scattering lengths would not only help discern the heavy-meson spectrum, but is needed in order to evaluate transport coefficients of systems containing heavy-light mesonic species, e.g. the hadronic phase of heavy ion collisions.

Currently, a combination of effective field theories (EFTs) and Lattice Quantum Chromodynamics (LQCD) provides the best option for performing reliable calculations of low-energy QCD observables (reviews on these topics include [9–15]). Heavy Meson Chiral Perturbation Theory (HMχPT) [16–18] is the low-energy EFT for studying strong-interaction quantities of mesons containing a single heavy quark and a single light antiquark. The nonperturbative QCD contributions to HMχPT are parametrized by low-energy coefficients (LECs). The predictive power of HMχPT is currently limited by the poor determination of these LECs, e.g. currently next-to-leading order (NLO) LECs are determined within a factor of three of precision, resulting in scattering lengths that are known within a factor of three [6,7]. The results outlined in the work will help reduce the uncertainties of LECs needed in the evaluation of pGB-heavy meson scattering.

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Historically, LQCD calculations have used moderate volumes and unphysically large pion masses. With advances in computing technology, performing LQCD calculation at the physical point ($m_\pi \approx 140$ MeV) of QCD is now a reality. However, limited computer resources require state of the art calculation with physical pion masses to be performed with small physical volumes. This leads to sizable volume effects contributing to the quantities of interest, and while it is natural to want to remove them, these effects can hold physically important information. More specifically, volume effects are parametrized by the LECs of the EFT, therefore by evaluating physical observables in small volumes one can determine the LECs.

In an infinite volume, the expansion parameters of HMχPT are p/Λ_χ, m_l/Λ_χ, and $\Lambda_{\mathrm{QCD}}/m_{\mathrm{Q}}$, where p is the characteristic momentum of the interaction, m_l is the mass of the light Goldstone bosons, m_{Q} is the heavy-quark mass, Λ_χ is the chiral symmetry breaking scale, and Λ_{QCD} is the characteristic scale of QCD. In a finite volume this expansion scheme is consistent in the p-regime [19,20].

However, for volumes smaller than the Compton wavelength of the Goldstone bosons, the zero momentum mode is enhanced with respect to the non-zero modes, and an alternative expansion scheme must be utilized [21]. The regime where the pion zero-modes must be integrated over explicitly while still treating the non-zero modes perturbatively is known as the ϵ-regime [21–33]. In the ϵ-regime, a new expansion parameter is introduced, $\epsilon \sim 2\pi/L\Lambda_\chi \sim 2\pi/\beta\Lambda_\chi$ and $\epsilon^2 \sim m_l/\Lambda_\chi$, where L and β are the spatial and temporal extents, respectively. At leading order, one may associate Δ_{Q}, the hyperfine splitting between the pseudoscalar meson P ($\{D^0, D^+, D_s^+\}$, $\{B^-, \bar{B}^0, \bar{B}_s^0\}$) and its respective vector meson P^* ($\{D^{*0}, D^{*+}, D_s^{*+}\}$, $\{B^{*-}, \bar{B}^{*0}, \bar{B}_s^{*0}\}$), with the physical values of $\sim$140 MeV and $\sim$50 MeV for the charm and bottom mesons, respectively. Therefore, it is reasonable to expect the hyperfine splitting to contribute at order ϵ^2 for charmed mesons ($\epsilon^2 \sim \Delta_c/\Lambda_\chi$) and approximately at order ϵ^3 for bottom mesons ($\epsilon^3 \sim \Delta_b/\Lambda_\chi$). For the sake of generality, both scenarios are considered.

This study presents the volume dependence of the heavy-meson masses at next-to-leading order (NLO), $\mathcal{O}(\epsilon^4)$, in the "mixed regime" of SU(2) and SU(3) HMχPT. In the mixed regime, the physical pion mass is small compared to the IR cutoff and therefore fall within the ϵ-regime, while the kaon and eta still satisfy the p-regime criteria [34,35]. Therefore in the mixed regime, the expansion in the π and $\{K, \eta\}$ masses is treated separately in order to satisfy $\epsilon^2 \sim m_\pi/\Lambda_\chi$ and $\epsilon \sim m_K/\Lambda_\chi \sim m_\eta/\Lambda_\chi$. The $\mathcal{O}(\epsilon^3)$ volume dependence of the heavy-meson mass for an $SU(2)_L \times SU(2)_R$ chiral theory with static heavy quarks has been previously calculated [35]. The results presented in this study are in agreement with those found in [35]. At $\mathcal{O}(\epsilon^3)$, the NLO couplings of HMχPT do not contribute and therefore extracting them requires going to the next order in the chiral and m_Q^{-1} expansions.

II. HEAVY MESON CHIRAL PERTURBATION THEORY

The field multiplet of the pseudoscalar P and the vector P^* can be conveniently represented as a single field operator [16–18],

$$H_a \equiv \frac{1+\not{v}}{2}[\not{P}^*_a + iP_a\gamma_5], \qquad \bar{H}_a \equiv \gamma^0 H_a^\dagger \gamma^0, \tag{1}$$

where v^μ is the velocity of the heavy meson. The representation of the composite field $H_a(x)$ assures it transforms as an SU(2) spinor under heavy-quark spin rotations, under the unbroken $SU(3)_V$ symmetry it transforms as an element of the $\bar{\mathbf{3}}$ fundamental representation (as denoted by the subscript "a"), and under both Lorentz and parity transformations it is a bilinear. In the rest frame, the LO HMχPT Lagrangian in m_Q and Λ_χ consistent with spontaneously broken $SU(3)_L \times SU(3)_R$ is [16–18]:

$$\mathcal{L}^0 = -i\,\mathrm{Tr}[\bar{H}_a(D^0)_{ba}H_b] - g\,\mathrm{Tr}[\bar{H}_a H_b \vec{\gamma}\cdot\vec{\mathcal{A}}_{ba}\gamma_5] + \frac{f^2}{8}\mathrm{Tr}[\partial^\mu\Sigma^\dagger\partial_\mu\Sigma] + \frac{f^2}{4}\mathrm{Tr}[\mathcal{M}\Sigma^\dagger + \mathrm{H.c.}], \tag{2}$$

where $\mathcal{M} = \frac{1}{2}\mathrm{diag}(m_\pi^2, m_\pi^2, 2m_K^2 - m_\pi^2)$ is the light meson mass matrix, $\vec{\gamma}$ is the spatial component of γ^μ, $D^\mu = \partial^\mu + \mathcal{V}^\mu$ is the covariant derivative, f is the pion decay constant, and the Goldstone bosons are encapsulated in the operators,

$$\begin{aligned}\Sigma &= \xi^2 = \exp\left(\frac{2iM}{f}\right)M\\ &= \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\sqrt{\frac{2}{3}}\eta \end{pmatrix}\\ \mathcal{A}^\mu &= \frac{i}{2}(\xi\partial^\mu\xi^\dagger - \xi^\dagger\partial^\mu\xi)\\ \mathcal{V}^\mu &= \frac{1}{2}(\xi\partial^\mu\xi^\dagger + \xi^\dagger\partial^\mu\xi).\end{aligned} \tag{3}$$

At NLO in HMχPT there are a large number of corrections to the Lagrangian that are consistent with velocity reparametrization invariance (VRI) [36], but the terms that will contribute to the volume dependence of the mass are the following:

$$\begin{aligned}\mathcal{L}^1 = &-\frac{g_1}{m_Q}\mathrm{Tr}[\bar{H}_a H_b\vec{\gamma}\cdot\vec{\mathcal{A}}_{ba}\gamma_5] - \frac{g_2}{m_Q}\mathrm{Tr}[\bar{H}_a\vec{\gamma}\cdot\vec{\mathcal{A}}_{ba}\gamma_5 H_b] + \frac{\lambda}{m_Q}\mathrm{Tr}[\bar{H}_a\sigma^{\mu\nu}H_a\sigma_{\mu\nu}] + \frac{\alpha_1}{\Lambda_\chi}\mathrm{Tr}[\bar{H}_a H_a](\mathcal{A}^0\mathcal{A}^0)_{bb}\\ &+ \frac{\sigma_1}{\Lambda_\chi}\mathrm{Tr}[\bar{H}_a H_b(\xi\mathcal{M}\xi + \mathrm{H.c.})_{ba}] + \frac{\sigma_2}{\Lambda_\chi}\mathrm{Tr}[\bar{H}_a H_a(\xi\mathcal{M}\xi + \mathrm{H.c.})_{bb}] + \frac{\alpha_2}{\Lambda_\chi}\mathrm{Tr}[\bar{H}_a H_c\mathcal{A}^0_{cb}\mathcal{A}^0_{ba}]\\ &+ \frac{\alpha_3}{\Lambda_\chi}\mathrm{Tr}[\bar{H}_a H_c\mathcal{A}_{cb}\cdot\mathcal{A}_{ba}] + \frac{\alpha_4}{\Lambda_\chi}\mathrm{Tr}[\bar{H}_a H_a](\mathcal{A}\cdot\mathcal{A})_{bb}.\end{aligned} \tag{4}$$

The $g's$, $\alpha's$, $\sigma's$, and λ are the relevant LECs of the theory. At leading order, the hyperfine splitting can be written in terms of λ, $\Delta_Q \equiv \frac{8\lambda}{m_Q}$.

III. ZERO-MODES INTEGRATION IN THE ϵ-REGIME

In the ϵ-regime, it is necessary to evaluate the pion zero-modes, $q^\mu = (0, \vec{0})$, contribution nonperturbatively. It is convenient to integrate zero-mode out of the theory, leaving an effective field theory in terms of the non-zero modes. In the mixed regime, only the pion zero-modes are removed [35], while the zero-modes of the kaon and eta are treated perturbatively. This can be done by rewriting the Σ operator as

$$\Sigma(x) = U\hat{\Sigma}(x)U, \qquad U = \exp\left[\frac{i}{f}\begin{pmatrix} \frac{\pi_z^0}{\sqrt{2}} & \pi_z^+ & 0 \\ \pi_z^- & -\frac{\pi_z^0}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \end{pmatrix}\right]. \tag{5}$$

The subscript z labels the zero-mode operators, while the operators with a hat are operators whose contribution can be treated perturbatively in the ϵ-expansion. When integrating over the zero-modes it is convenient to write the operator U in terms of hyperspherical coordinates,

$$U = \begin{pmatrix} \cos(\psi) + i\cos(\theta)\sin(\psi) & \sin(\theta)\sin(\psi)e^{i\phi} & 0 \\ \sin(\theta)\sin(\psi)e^{-i\phi} & \cos(\psi) - i\cos(\theta)\sin(\psi) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{6}$$

When constructing the Lagrangian that is invariant under chiral transformations, it is advantageous to define the operator $\xi \equiv \sqrt{\Sigma}$. Under the redefinition of Eq. (5), one finds,

$$\xi(x) = U\hat{\xi}(x)V^\dagger(x) = V(x)\hat{\xi}(x)U, \tag{7}$$

where the definitions $V^\dagger = \hat{\xi}U^\dagger\sqrt{U\hat{\Sigma}U}$ and $V = \sqrt{U\hat{\Sigma}U}U^\dagger\hat{\xi}^\dagger$ have been implicitly introduced. When integrating over U, one may substitute $\mathcal{A}_\mu = V\hat{\mathcal{A}}_\mu V^\dagger$ and $\mathcal{V}_\mu = V\hat{\mathcal{V}}_\mu V^\dagger + iV\partial_\mu V^\dagger$. The results presented here will be truncated at $\mathcal{O}(\epsilon^4)$, in which case one can safely make the following approximation:

$$\mathcal{A}^\mu \simeq \hat{\mathcal{A}}^\mu = \frac{i}{2}(\hat{\xi}\partial^\mu\hat{\xi}^\dagger - \hat{\xi}^\dagger\partial^\mu\hat{\xi}) = \frac{\partial^\mu\hat{M}}{f} + \mathcal{O}(\epsilon^3)$$
$$\mathcal{V}^\mu \simeq \hat{\mathcal{V}}^\mu = \frac{1}{2}(\hat{\xi}\partial^\mu\hat{\xi}^\dagger + \hat{\xi}^\dagger\partial^\mu\hat{\xi}) = \frac{\hat{M}\partial^\mu\hat{M}}{f^2} + \mathcal{O}(\epsilon^4). \tag{8}$$

The only contribution to the heavy-meson mass that originates from the zero-modes integration first appears at $\mathcal{O}(\epsilon^4)$, and comes from the second line in Eq. (4):

$$\begin{aligned}\delta\mathcal{L}_{\mathcal{M}} = {}& \frac{\sigma_1}{\Lambda_\chi}\mathrm{Tr}[\bar{H}_aH_b(\xi\mathcal{M}\xi + \mathrm{H.c.})_{ba}] \\ &+ \frac{\sigma_2}{\Lambda_\chi}\mathrm{Tr}[\bar{H}_aH_a(\xi\mathcal{M}\xi + \mathrm{H.c.})_{bb}] \\ \simeq{}& \frac{\sigma_1}{\Lambda_\chi}\mathrm{Tr}[\bar{H}_1H_1 + \bar{H}_2H_2](\cos(2\psi)m_\pi^2) \\ &+ \frac{\sigma_1}{\Lambda_\chi}\mathrm{Tr}[\bar{H}_3H_3](2m_k^2 - m_\pi^2) + \frac{\sigma_2}{\Lambda_\chi}\mathrm{Tr}[\bar{H}_aH_a] \\ &\times(2m_K^2 + (-1 + 2\cos(2\psi))m_\pi^2).\end{aligned} \tag{9}$$

In order to evaluate the contribution of this term to the heavy-meson mass, the ψ-dependence in this expression must be integrated out using the nonperturbative weight arising from the last term in Eq. (2). It is convenient to perform this integral by analytically continuing to Euclidean time $t \to -it$:

$$\begin{aligned}&\int\mathcal{D}U^2\exp\left[\int d^4x\left(\frac{f^2}{4}\mathrm{Tr}[\mathcal{M}U^2 + \mathrm{H.c.}] + \delta\mathcal{L}_{\mathcal{M}}\right)\right] \\ &= X(s)\exp\left[-\frac{X'(s)}{X(s)}m_\pi^2\int d^4x((\sigma_1 + 2\sigma_2) \\ &\quad\times(P_a^{*\dagger}P_a^* + \vec{P}_a^\dagger\cdot\vec{P}_a) - \sigma_1(P_3^{*\dagger}P_3^* + \vec{P}_3^\dagger\cdot\vec{P}_3))\right].\end{aligned} \tag{10}$$

where $s = \frac{1}{4}f^2m_\pi^2\beta L^3$, and X(s) can be expressed in terms of the modified Bessel function $I_1(2s)$ of the first kind,

$$X(s) \equiv \int\frac{8}{\pi}d\psi\cos^2(\psi)\sin^2(\psi)e^{2s\cos(2\psi)} = \frac{I_1(2s)}{s},$$
$$\int\frac{8}{\pi}d\psi\cos^2(\psi)\sin^2(\psi)e^{2s\cos(2\psi)}\cos(2\psi) = \frac{X'(s)}{2}. \tag{11}$$

Since $\mathcal{M}$ has no spin structure, it results in the same shift for both the masses of the pseudoscalar and vector fields, yet it explicitly breaks the $SU(3)_V$ symmetry,

$$\delta\mathrm{M}_{(P,P^*)} = \frac{m_\pi^2}{\Lambda_\chi}(\sigma_1 + 2\sigma_2)\frac{X'(s)}{X(s)} + \frac{\sigma_2}{\Lambda_\chi}2(2m_K^2 - m_\pi^2),$$
$$\delta\mathrm{M}_{(P_{\bar{s}},P^*_{\bar{s}})} = 2\frac{m_\pi^2}{\Lambda_\chi}\sigma_2\frac{X'(s)}{X(s)} + \frac{\sigma_1 + \sigma_2}{\Lambda_\chi}2(2m_K^2 - m_\pi^2). \tag{12}$$

This shifts the bare mass of the strange-pseudoscalar meson by δ_Q^s and the strange vector-meson mass by Δ_Q^s,

$$\delta_Q^s \equiv (\delta\mathrm{M}_{P_{\bar{s}}} - \delta\mathrm{M}_P)_{(L=\infty)} = 2\sigma_1\frac{(2m_K^2 - m_\pi^2)}{\Lambda_\chi},$$
$$\Delta_Q^s \equiv \Delta_Q + \delta_Q^s. \tag{13}$$

At leading order, one may associate δ_Q^s with physical value of the splitting between the isospin doublet P and strange-pseudoscalar $P_{\bar{s}}$, which is on the order of 100 MeV for both the charm and bottom mesons, respectively. Both δ_Q^s and Δ_Q^s will assume the same power counting as $\Delta_Q \sim \mathcal{O}(\epsilon^2)$.

This analysis introduces a volume dependence to the mass of the nonzero pion modes, as well as for the K's and η's,

$$m_\pi^2 \to m_\pi^2\frac{X'(s)}{2X(s)},$$
$$m_K^2 \to m_K^2 - \frac{m_\pi^2}{2} + m_\pi^2\frac{X'(s)}{4X(s)} = m_K^2 + \mathcal{O}(\epsilon^4) \tag{14}$$

$$m_\eta^2 \to \frac{4}{3}m_K^2 - 2\frac{m_\pi^2}{3} + m_\pi^2\frac{X'(s)}{6X(s)} = \frac{4}{3}m_K^2 + \mathcal{O}(\epsilon^4). \tag{15}$$

After performing the integration over the zero-modes, the finite-volume contribution from the remaining degrees of freedom can be evaluated perturbatively. The finite-volume Feynman diagrams can be evaluated in the standard way, where the integral is replaced by a sum over discretized four momenta and the zero mode is explicitly excluded in the pion loops [19]. An outline of the methods used in performing these sums is discussed in the appendix.

FIG. 1 (color online). ϵ^3 contribution to the pseudoscalar heavy-meson mass. The solid line corresponds to the heavy pseudoscalar, the double line denotes a vector meson, and the dashed line represents a Goldstone boson.

IV. RESULTS

In taking the isospin limit, the pseudoscalar pair $P = \{P_{\bar{u}}, P_{\bar{d}}\}$ will receive the same mass contribution. P will denote the isospin pair and $P_{\bar{s}}$ will denote the strange-heavy meson. In order to formally categorize the different terms contributing to the mass, it is important to consider the ratio $\Lambda_{\rm QCD}/m_{\rm Q} \sim \mathcal{O}(\epsilon^{\alpha})$. The most relevant cases are the following:

$$(i)\text{: } \Lambda_{\rm QCD}/m_{\rm Q} \sim \mathcal{O}(\epsilon^2) \quad (ii)\text{: } \Lambda_{\rm QCD}/m_{\rm Q} \sim \mathcal{O}(\epsilon), \tag{16}$$

corresponding to the static limit and LO heavy-quark mass corrections, respectively. For simplicity, the expressions below will include finite LO heavy-quark mass corrections. The static limit can easily be obtained by taking the $m_{\rm Q} \rightarrow \infty$ limit (note $\Delta_{\rm Q} \propto m_{\rm Q}^{-1} \rightarrow 0$). The individual diagrams contributing to M_P are written in the appendix. The notation $\delta {\rm M}_P \equiv M_P(L) - M_P(\infty)$ is used to denote the finite-volume dependence of the mass.

A. SU(3) HMχPT

The SU(3) volume dependence of the P and $P_{\bar{s}}$ masses up to and including $\mathcal{O}(\epsilon^4)$ is found by adding the finite-volume contributions from the self-energy diagrams depicted in Figs. 1 and 2, where the Goldstone bosons can be pions, kaons, and etas,

$$\begin{aligned}
\delta {\rm M}_P = {}& \left(g^2 + 2\frac{g(g_1 - g_2)}{m_{\rm Q}}\right)\left(\frac{3}{4f^2L^3} + \frac{1}{8\pi L f^2}\left(m_K^2 \mathcal{N}_1(m_K, L) + \frac{1}{6} m_\eta^2 \mathcal{N}_1(m_\eta, L)\right)\right) + \frac{g^2 2\sigma_1 m_K^2}{f^2\Lambda_\chi}\left(2\frac{\mathcal{N}_2(m_K, L)}{m_K^{1/2}L^{5/2}}\right) \\
& + \frac{g^2\Delta_{\rm Q}}{2f^2}\left(-\frac{3c_1}{8\pi L^2} + \frac{1}{3}\frac{\mathcal{N}_2(m_\eta, L)}{m_\eta^{1/2}L^{5/2}} + 2\frac{\mathcal{N}_2(m_K, L)}{m_K^{1/2}L^{5/2}}\right) - \frac{\alpha_1}{2f^2\Lambda_\chi}\left(6\frac{c_4}{2\pi^2L^4} + 8\frac{m_K^3}{16\pi^2 L}\mathcal{K}_1(m_K L)\right. \\
& \left. + 2\frac{m_\eta^3}{16\pi^2 L}\mathcal{K}_1(m_\eta L)\right) - \frac{\alpha_2}{2f^2\Lambda_\chi}\left(3\frac{c_4}{2\pi^2L^4} + 2\frac{m_K^3}{16\pi^2 L}\mathcal{K}_1(m_K L) + \frac{1}{3}\frac{m_\eta^3}{16\pi^2 L}\mathcal{K}_1(m_\eta L)\right) \\
& - \frac{\alpha_3}{2f^2\Lambda_\chi}\left(2\frac{m_K^3}{16\pi^2 L}\mathcal{K}_2(m_K L) + \frac{1}{3}\frac{m_\eta^3}{16\pi^2 L}\mathcal{K}_2(m_\eta L)\right) - \frac{\alpha_4}{2f^2\Lambda_\chi}\left(8\frac{m_K^3}{16\pi^2 L}\mathcal{K}_2(m_K L) + 2\frac{m_\eta^3}{16\pi^2 L}\mathcal{K}_2(m_\eta L)\right) \\
& + \frac{m_\pi^2}{2\Lambda_\chi}(\sigma_1 + 2\sigma_2)\frac{X'(s)}{X(s)},
\end{aligned} \tag{17}$$

$$\begin{aligned}
\delta {\rm M}_{P_{\bar{s}}} = {}& \frac{2}{f^2}\left(g^2 + 2\frac{g(g_1 - g_2)}{m_{\rm Q}}\right)\left(\frac{m_K^2}{8\pi L}\mathcal{N}_1(m_K, L) + \frac{1}{3}\frac{m_\eta^2}{8\pi L}\mathcal{N}_1(m_\eta, L)\right) - \frac{8g^2\sigma_1 m_K^2}{f^2L^{5/2}\Lambda_\chi}\frac{\mathcal{N}_2(m_K, L)}{m_K^{1/2}} \\
& + \frac{2g^2\Delta_{\rm Q}}{f^2L^{5/2}}\left(\frac{\mathcal{N}_2(m_\eta, L)}{3m_\eta^{1/2}} + \frac{\mathcal{N}_2(m_K, L)}{m_K^{1/2}}\right) - \frac{\alpha_1}{2f^2\Lambda_\chi}\left(3\frac{c_4}{2\pi^2L^4} + 8\frac{m_K^3}{16\pi^2 L}\mathcal{K}_1(m_K L) + 2\frac{m_\eta^3}{16\pi^2 L}\mathcal{K}_1(m_\eta L)\right) \\
& - \frac{\alpha_2}{2f^2\Lambda_\chi}\left(4\frac{m_K^3}{16\pi^2 L}\mathcal{K}_1(m_K L) + \frac{4}{3}\frac{m_\eta^3}{16\pi^2 L}\mathcal{K}_1(m_\eta L)\right) - \frac{\alpha_3}{2f^2\Lambda_\chi}\left(4\frac{m_K^3}{16\pi^2 L}\mathcal{K}_2(m_K L) + \frac{4}{3}\frac{m_\eta^3}{16\pi^2 L}\mathcal{K}_2(m_\eta L)\right) \\
& - \frac{\alpha_4}{2f^2\Lambda_\chi}\left(8\frac{m_K^3}{16\pi^2 L}\mathcal{K}_2(m_K L) + 2\frac{m_\eta^3}{16\pi^2 L}\mathcal{K}_2(m_\eta L)\right) + 2\sigma_2\frac{m_\pi^2}{2\Lambda_\chi}\frac{X'(s)}{X(s)}.
\end{aligned} \tag{18}$$

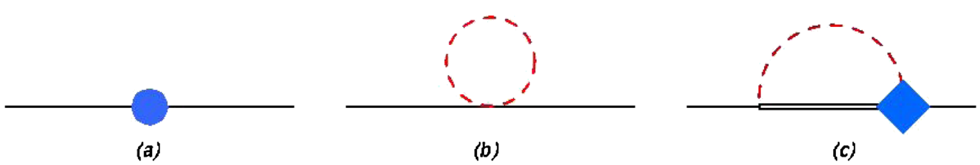

FIG. 2 (color online). ϵ^4 contribution to the pseudoscalar heavy-meson mass. (a) Denotes the zero-modes contribution. (b) Goldstone bosons loops originating from four-point vertices. (c) Incorporates operators that contribute the heavy flavor symmetry breaking corrections to the $PP^*\pi$ vertex.

where $m_\eta^2 \equiv 4m_K^2/3$, the discrete sums c_1, c_4, $\mathcal{N}_i$, and $\mathcal{K}_i$ are defined in Eqs. (A20), (A24), (A25), and (13) has used. Similarly, the $\mathcal{O}(\epsilon^3)$ and $\mathcal{O}(\epsilon^4)$ corrections to the vector-meson masses are depicted in Figs. 3 and 4, respectively. In the static limit, the pseudoscalar and the vector mesons are degenerate, therefore it is only necessary to evaluate the volume dependence of the hyperfine splitting:

$$\delta M_{P^*} - \delta M_P = 8\frac{gg_2}{3m_Q}\left(\frac{3}{4f^2L^3} + \frac{1}{8\pi L f^2}\left(m_K^2\mathcal{N}_1(m_K, L) + \frac{1}{6}m_\eta^2\mathcal{N}_1(m_\eta, L)\right)\right) + \frac{g^2\Delta_Q}{2f^2}\left(\frac{c_1}{2\pi L^2} - \frac{4}{3}\frac{\mathcal{N}_2(m_K, L)}{m_K^{1/2}L^{5/2}} + \frac{2}{9}\frac{\mathcal{N}_2(m_\eta, L)}{m_\eta^{1/2}L^{5/2}}\right) \tag{19}$$

$$\delta M_{P_{\bar{s}}^*} - \delta M_{P_{\bar{s}}} = \frac{16gg_2}{3m_Q f^2}\left(\frac{m_K^2}{8\pi L}\mathcal{N}_1(m_K, L) + \frac{1}{3}\frac{m_\eta^2}{8\pi L}\mathcal{N}_1(m_\eta, L)\right) - \frac{4}{3}\frac{g^2\Delta_Q}{f^2L^{5/2}}\left(2\frac{\mathcal{N}_2(m_K, L)}{m_K^{1/2}} + \frac{2}{3}\frac{2g^2\Delta_Q}{3f^2L^{5/2}}\frac{\mathcal{N}_2(m_\eta, L)}{m_\eta^{1/2}}\right). \tag{20}$$

B. SU(2) HMχPT

In SU(2) Chiral Perturbation Theory, the kaons and eta decouple from the theory. Only integrals including pions depicted in Figs. 1–4 contribute to the volume dependence of the masses,

$$\delta M_P^{(2)} = \left((g^{(2)})^2 + 2\frac{g^{(2)}(g_1^{(2)} - g_2^{(2)})}{m_Q}\right)\frac{3}{4f^2L^3} + \frac{m_\pi^2}{2\Lambda_\chi}(\sigma_1^{(2)} + 2\sigma_2^{(2)})\frac{X'(s)}{X(s)} - \frac{(g^{(2)})^2\Delta_Q^{(2)}}{2f^2}\frac{3c_1}{8\pi L^2} - \frac{3}{2f^2\Lambda_\chi}(2\alpha_1^{(2)} + \alpha_2^{(2)})\frac{c_4}{2\pi^2L^4}, \tag{21}$$

FIG. 3 (color online). ϵ^3 contribution to the heavy vector-meson mass.

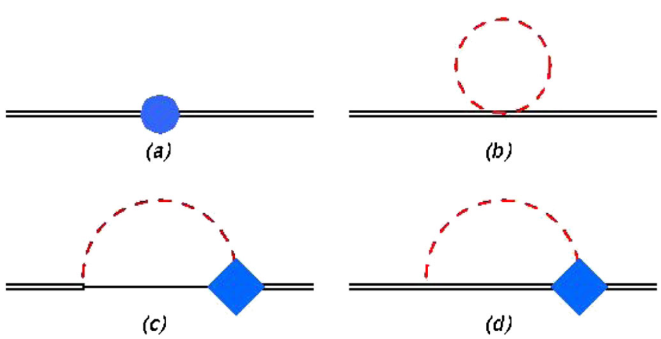

FIG. 4 (color online). ϵ^4 contribution to vector-meson mass.

$$\delta M_{P^*}^{(2)} - \delta M_P^{(2)} = 2\frac{g^{(2)}g_2^{(2)}}{m_Q f^2L^3} + \frac{(g^{(2)})^2\Delta_Q^{(2)}}{f^2}\frac{c_1}{4\pi L^2}, \tag{22}$$

where an additional superscript has been introduced in order to explicitly distinguish the SU(2) LECs from those contributing to the SU(3) theory. Note, these results have been derived assuming $\Delta_Q \sim \mathcal{O}(\epsilon^2)$, which should be expected to be the case for the charm mesons. For the bottom mesons one should expect $\Delta_b \sim \mathcal{O}(\epsilon^3)$. This would move finite volumes effects related to this coupling to $\mathcal{O}(\epsilon^5)$, displacing them outside the scope of this calculation.

V. ANALYSIS AND DISCUSSION

The results presented in the previous section allow determination of LECs that play an important role in the determination of heavy-light meson scattering phase shifts. In order to evaluate the LECs, one must fit the expressions $M_\infty + \delta M(L, m_\pi, m_Q)$ and $\Delta_\infty + \delta\Delta(L, m_\pi, m_Q)$ to LQCD results of the heavy-meson masses for different volumes and pion masses that fall within the ϵ-regime, where $(M_\infty, \Delta_\infty)$ are the physical mass and hyperfine splitting, and $(\delta M, \delta\Delta)$ denote the finite L, m_π and m_Q contribution described by Eqs. (12), (13), and (17)–(22). The corresponding LQCD calculation has not been performed yet. Nevertheless, the uncertainty of the LECs (δ_{l_i}) as a function of the standard deviation of the heavy-meson masses (δ_{M_h}) can be estimated. Because of the larger number of LECs and the larger expansion parameters for SU(3) HMχPT (m_K/Λ_χ, m_η/Λ_χ), the following discussion will focus on SU(2) HMχPT.

In order to determine δ_{l_i} as a function of δ_{M_h}, I have analyzed fake data for the heavy pseudoscalar and vector-meson masses. For both hadrons, a data set was generated that follows the trend predicted by $M_\infty + \delta M(L, m_\pi, m_Q)$; this required inputting randomly generated LECs. Additional L, m_π and m_Q dependent terms were added to $M_\infty + \delta M(L, m_\pi, m_Q)$ in order to simulate the $\mathcal{O}(\epsilon^5)$

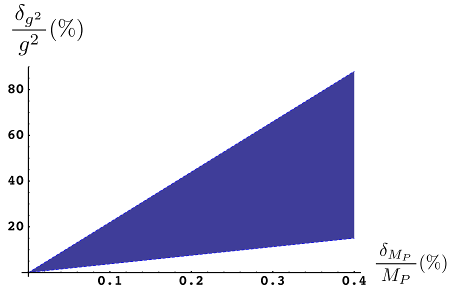

FIG. 5 (color online). Prediction for the level of precision for determining the LO LEC g^2 by fitting Eqs. (23) and (24) to 18 pseudoscalar and vector masses [corresponding to six different values of (m_π, L) that fall within the ϵ-regime and three different heavy-quark masses] with an uncertainty ranging from 0.5% to 0.01%.

corrections. The exact form of these terms is irrelevant for the discussion at hand. Each hadron mass has been given a corresponding uncertainty. Lastly, the set is fit to $\mathrm{M}_\infty + \delta\mathrm{M}(L, m_\pi, m_Q)$ in order to reproduce the randomly generated LECs. Since at leading order the heavy-meson mass depends linearly in the heavy-quark mass, the m_Q dependence can be rewritten in terms of physical heavy-meson mass, M_h. It is convenient to introduce variables that encapsulate the linear combination of LECs appearing in Eqs. (21) and (22): $g_3^2 \equiv 2\frac{g(g_1-g_2)}{\Lambda_\chi}$, $g_4^2 \equiv 2\frac{gg_1}{\Lambda_\chi}$, $\sigma \equiv (\sigma_1 + 2\sigma_2)$, $\tilde{\Delta} \equiv g^2\Delta_Q\frac{M_h}{\Lambda_\chi}$, $\alpha \equiv (2\alpha_1 + \alpha_2)$. Using these and Eqs. (12), (21), and (22), the m_π and L dependence of the pseudoscalar and vector masses can be written as follows,

$$\mathrm{M}_P(L, m_\pi, M_h) = M_h + g^2\frac{3}{4f^2L^3} + g_3^2\frac{\Lambda_\chi}{M_h}\frac{3}{4f^2L^3} + \sigma\frac{m_\pi^2}{2\Lambda_\chi}\frac{X'(s)}{X(s)} - \sigma_2\frac{2m_\pi^2}{\Lambda_\chi} - \tilde{\Delta}\frac{\Lambda_\chi}{M_h}\frac{3c_1}{16\pi f^2L^2} - \alpha\frac{3}{2f^2\Lambda_\chi}\frac{c_4}{2\pi^2L^4}, \tag{23}$$

$$M_{P^*}(L, m_\pi, M_{h^*}) = M_{h^*} + g^2\frac{3}{4f^2L^3} + g_4^2\frac{\Lambda_\chi}{M_{h^*}}\frac{3}{4f^2L^3} + \sigma\frac{m_\pi^2}{2\Lambda_\chi}\frac{X'(s)}{X(s)} - \sigma_2\frac{2m_\pi^2}{\Lambda_\chi} - \tilde{\Delta}\frac{\Lambda_\chi}{M_{h^*}}\frac{c_1}{16\pi f^2L^2} - \alpha\frac{3}{2f^2\Lambda_\chi}\frac{c_4}{2\pi^2L^4}, \tag{24}$$

where (M_h, M_{h^*}) are the bare pseudoscalar and vector masses.

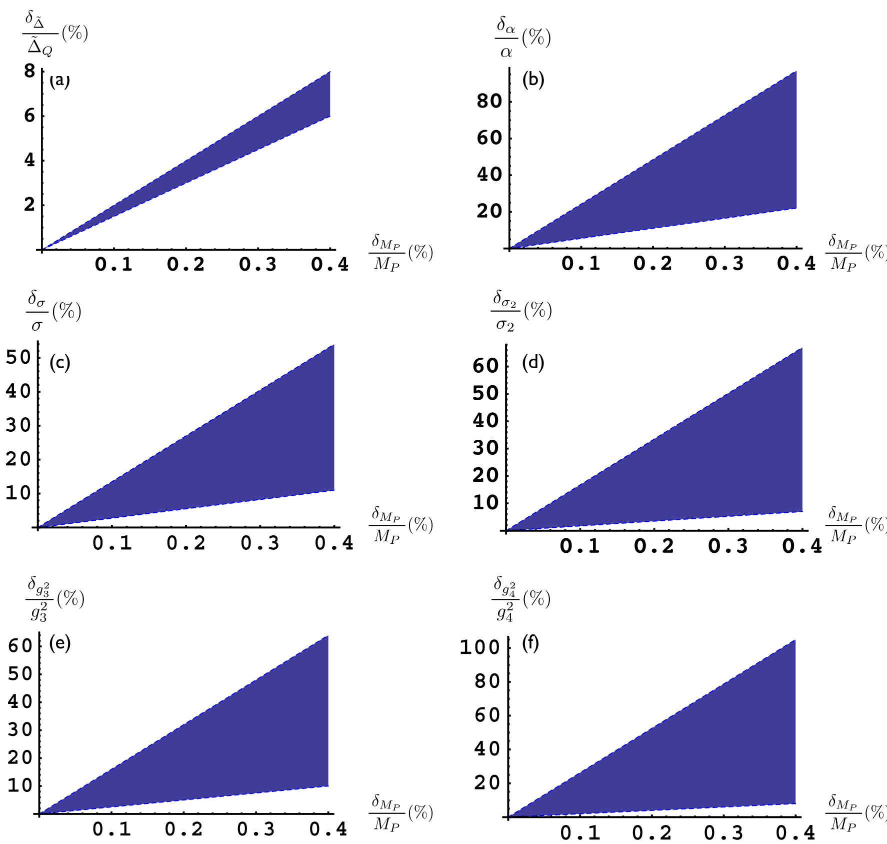

FIG. 6 (color online). Prediction for the level of precision for determining the NLO LECs $\{\tilde{\Delta}_Q, \alpha, \sigma, \sigma_2, g_3^2, g_4^2\}$ by fitting Eqs. (23) and (24) to 18 pseudoscalar and vector masses [corresponding to six different values of (m_π, L) that fall within the ϵ-regime and three different heavy-quark masses] with an uncertainty ranging from 0.5% to 0.01%.

Since the level of precision with which the LECs can be determined depends on their magnitude, the parameters were varied $\tilde{\Delta} = \{80\text{–}150\}$ MeV, $g^2 = \{0.5\text{–}1.5\}$, $|g_3^2| = \{0.5\text{–}1.5\}$, $|g_4^2| = \{0.5\text{–}1.5\}$, $|\sigma| = \{0.5\text{–}1.5\}$, $|\sigma_2| = \{0.5\text{–}1.5\}$, $|\alpha| = \{0.5\text{–}1.5\}$, while the pion decay constant was fixed at $f = 130$ MeV. For each value of the LECs, a set of 18 pseudoscalar and vector masses was generated, corresponding to six different (m_π, L) that fall within the ϵ-regime and three different heavy-quark masses. The three heavy-quark masses were chosen such that $M_h = \{1.8, 2.5, 3.0\}$ GeV. Each heavy-meson mass was given an uncertainty ranging from 0.5% to 0.01%. The randomly generated LECs were then obtain by simultaneously fitting the two data sets using Eqs. (23) and (24). The estimate of the expected fractional standard deviation of the LECs as a function of δ_{M_h}/M_h is plotted in Fig. 5 and 6 as the shaded region. The range of possible standard deviation for the LECs for a given uncertainty of the heavy-meson mass manifests the fact that the precision with which these LECs can be determined depend on their absolute value.

VI. CONCLUSION

HMχPT is the EFT for calculating strong-interaction quantities of heavy mesons. Currently, HMχPT is limited by the determination of the LECs of the theory. In particular, the LECs in the Lagrangian discussed in this paper, Eqs. (2)–(4), contribute to the evaluation of scattering lengths, and are currently known to within a factor of three [6,7].

These LECs can be evaluated from LQCD calculations. One way to achieve this is to extract the LECs from the volume dependence of the heavy-meson masses, since these finite-volume effects are parametrized by the LECs of the theory. With this in mind, the finite-volume dependence of the heavy pseudoscalar and vector-meson masses in the ϵ-regime of HMχPT have been calculated to $\mathcal{O}(\epsilon^4)$. In the ϵ-regime, LQCD calculations can be performed at the physical point of QCD ($m_\pi \approx 140$ MeV) if volumes are small ($L \leq 4$ fm).

Lastly, it was shown that by evaluating the pseudoscalar and vector-meson masses with a precision of 0.1% for six (m_π, L) and three m_Q values, six of the SU(2) NLO LECs can be determined within the 20% level of precision. In order to calculate the heavy-light scattering lengths, it is also necessary to determine the linear combination $2\alpha_4^{(2)} + \alpha_3^{(2)}$ [6,7], which could be determined from the volume dependence of the heavy-meson mass at $\mathcal{O}(\epsilon^6)$.

Because of the nature of the ϵ-regime, chiral corrections are suppressed, and finite m_π corrections to hadron masses come in at NLO in the expansion parameter. This is in contrast to the p-regime, where the finite m_π contributions are enhanced, contributing at LO in the expansion parameter. As a result, one would expect better determination of the σ couplings from studying the m_π dependence of the heavy-meson mass in the p-regime.

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APPENDIX A

Finite-volume Feynman diagrams can be performed by replacing integrals with sums over discretized four momenta [19]. The sums contributing to the calculation of the heavy-meson mass to $\mathcal{O}(\epsilon^4)$ are

$$\mathbb{A}(m_L, \Delta, L, \beta) = \frac{1}{\beta L^3} \sum_{n_\mu \neq 0} \frac{1}{i(\frac{2\pi n_0}{\beta} + \omega) - \Delta} \times \frac{(\frac{2\pi \vec{n}}{L})^2}{(\frac{2\pi n_0}{\beta})^2 + (\frac{2\pi \vec{n}}{L})^2 + m_L^2}, \tag{A1}$$

$$\mathbb{B}(m_L, L, \beta) = \frac{1}{\beta L^3} \sum_{n_\mu \neq 0} \frac{(\frac{2\pi n_0}{\beta})^2}{(\frac{2\pi n_0}{\beta})^2 + (\frac{2\pi \vec{n}}{L})^2 + m_L^2}, \tag{A2}$$

$$\mathbb{C}(m_L, L, \beta) = \frac{1}{\beta L^3} \sum_{n_\mu \neq 0} \frac{(\frac{2\pi \vec{n}}{L})^2}{(\frac{2\pi n_0}{\beta})^2 + (\frac{2\pi \vec{n}}{L})^2 + m_L^2}. \tag{A3}$$

where ω is the external energy, and m_L denotes the light meson mass (m_π, m_K, m_η). In the mixed regime, the π and $\{K, \eta\}$ loops must be treated separately. Because of the field convention, the corrections to the mass are defined as $\frac{i}{2}\Pi(\omega = 0) + \frac{i\delta_P}{2}\partial_\omega \Pi(\omega = 0)$, where Π is the sum of the amputated self-energy diagrams, and δ_P is the bare residual mass of the heavy meson. The superscripts of the terms below denote the order at which they contribute in the ϵ-expansion. The $\mathcal{O}(\epsilon^3)$ correction to the pseudoscalar mass, depicted in Fig. 1, is

$$\begin{aligned} M^{(3)}(m_\pi, \Delta, L, \beta) &= \frac{1}{2}\left(\frac{2\tilde{g}}{f}\right)^2 \frac{3}{2} \frac{1}{\beta L^3} \sum_{n_\mu \neq 0} \frac{1}{2(i(\omega + \frac{2\pi n_0}{\beta}) - \Delta)} \\ &\quad \times \frac{(\frac{2\pi \vec{n}}{L})^2}{(\frac{2\pi n_0}{\beta})^2 + (\frac{2\pi \vec{n}}{L})^2 + m_\pi^2} \xrightarrow{\omega \to 0} \frac{3\tilde{g}^2}{2f^2} \mathbb{A}(m_\pi, \Delta, L, \beta) \\ &= \frac{3\tilde{g}^2}{2f^2} \mathbb{A}(0, \Delta, L, \beta) + \mathcal{O}(\epsilon^5), \end{aligned} \tag{A4}$$

$$M^{(3)}(m_K, m_\eta, \Delta^s, L, \beta) = \frac{\tilde{g}^2}{f^2}\mathbb{A}(m_K, \Delta^s, L, \beta) + \frac{\tilde{g}^2}{6f^2}\mathbb{A}(m_\eta, \Delta^s, L, \beta) \quad \text{(A5)}$$

where $\tilde{g} = g + \frac{g_1}{m_Q}$. The first $\mathcal{O}(\epsilon^4)$ contribution comes from integrating out the zero-modes using Eq. (12), depicted by Fig. 2(a)],

$$M_a^{(4)}(m_\pi, L, \beta) = \frac{m_\pi^2}{2\Lambda_\chi}(\sigma_1 + 2\sigma_2)\frac{X'(s)}{X(s)}. \quad \text{(A6)}$$

The second graph, Fig. 2(b)], corresponds to the four-point vertex contribution to the mass,

$$M_b^{(4)}(m_\pi, L, \beta) = -\frac{3\alpha_1 + 6\alpha_2}{2f^2\Lambda_\chi}\mathbb{B}(0, L, \beta) + \mathcal{O}(\epsilon^6),$$

$$M_b^{(4)}(m_K, m_\eta, L, \beta) = -\frac{4\alpha_1 + \alpha_2}{f^2\Lambda_\chi}\mathbb{B}(m_K, L, \beta) - \frac{\alpha_3 + 4\alpha_4}{f^2\Lambda_\chi}(\mathbb{B}(m_K, L, \beta) + \mathbb{C}(m_K, L, \beta)) - \frac{6\alpha_1 + \alpha_2}{6f^2\Lambda_\chi}\mathbb{B}(m_\eta, L, \beta) - \frac{\alpha_3 + 6\alpha_4}{6f^2\Lambda_\chi}(\mathbb{B}(m_\eta, L, \beta) + \mathbb{C}(m_\eta, L, \beta)). \quad \text{(A7)}$$

The third diagram, Fig. 2(c)], comes from the three-point vertex corrections in the Lagrangian, and it results in the following contribution to the P meson mass:

$$M_c^{(4)}(m_\pi, \Delta, L, \beta) = -6\frac{gg_2}{2f^2 m_Q}\mathbb{A}(0, \Delta, L, \beta) + \mathcal{O}(\epsilon^5),$$

$$M_c^{(4)}(m_K, m_\eta \Delta^s, L, \beta) = -\frac{4gg_2}{2f^2 m_Q}\mathbb{A}(m_K, \Delta^s, L, \beta) - \frac{2gg_2}{6f^2 m_Q}\mathbb{A}\left(\frac{2}{\sqrt{3}}m_K, \Delta^s, L, \beta\right). \quad \text{(A8)}$$

In order to evaluate the temporal sum, the Abel-Plana formula will be used:

$$\frac{1}{\beta}\sum_n f\left(\frac{2\pi n}{\beta}\right) = \int_{-\infty}^{\infty}\frac{dz}{2\pi}f(z) - i\,\mathrm{Res}\left(\frac{f(z)}{e^{i\beta z} - 1}\right)\Bigg|_{\mathrm{lowerplane}} + i\,\mathrm{Res}\left(\frac{f(z)}{e^{-i\beta z} - 1}\right)\Bigg|_{\mathrm{upperplane}}. \quad \text{(A9)}$$

The spatial sum can be performed using Poisson's Resummation formula,

$$\frac{1}{L^3}\sum_{\vec{n}}\frac{(\frac{2\pi\vec{n}}{L})^{2m}}{(\frac{2\pi\vec{n}}{L})^2 + x^2} = \frac{1}{L^3}\int d^3k\frac{k^{2m}}{k^2 + x^2}\sum_{\vec{n}}\delta\left(\vec{k} - \frac{2\pi\vec{n}}{L}\right) = \int\frac{d^3k}{(2\pi)^3}\frac{k^{2m}}{k^2 + x^2}\underbrace{\sum_{\vec{n}}\delta\left(\frac{\vec{k}L}{2\pi} - \vec{n}\right)}_{\sum_{\vec{n}} e^{iL\vec{k}\cdot\vec{n}}} = \int\frac{d^3k}{(2\pi)^3}\frac{k^{2m}}{k^2 + x^2} + \frac{(-x^2)^m}{4\pi L}\times\sum_{\vec{n}\neq 0}\frac{e^{-nxL}}{n}. \quad \text{(A10)}$$

As carefully discussed in Ref. [37], the finite temperature contributions in the ϵ-regime are usually heavily suppressed, and such is the case in all the integrals considered here. This allows one to safely neglect finite temperature terms. With this, one can extract the volume dependence of the sum in Eq. (A1) as

$$\delta\mathbb{A}(m_L, \Delta, L, \beta) = \mathbb{A}(m_L, \Delta, L, \beta) - \mathbb{A}(m_L, \Delta, L \to \infty, \beta \to \infty) \overset{\omega\to 0}{\to} \Delta \times \int_{-\infty}^{\infty}\frac{dk_0}{2\pi}\frac{1}{k_0^2 + \Delta^2}\frac{k_0^2 + m_l^2}{4\pi L}\sum_{\vec{n}\neq 0}\frac{e^{-n\sqrt{k_0^2 + m_l^2}L}}{n} + \mathcal{O}(\epsilon^6). \quad \text{(A11)}$$

1. ϵ-Regime Integrals: $m_L = m_\pi$

In the case that the sum arises from a pion loop, one can take the chiral limit and substitute $\sqrt{k_0^2 + m_l^2} \to k_0$ in the above expression. Corrections to this approximation will result in $\mathcal{O}(\epsilon^6)$ contributions to the heavy-meson masses. All the integrals can be preformed using the following generating formula:

$$\mathcal{I}(\alpha, \Delta) \equiv \int_0^{\infty}dk_0\frac{1}{k_0^2 + \Delta^2}e^{-k_0\alpha} = \frac{\mathrm{Ci}(\alpha\Delta)\sin(\alpha\Delta)}{\Delta} + \frac{\cos(\alpha\Delta)(\frac{\pi}{2} - \mathrm{Si}(\alpha\Delta))}{\Delta}, \quad \text{(A12)}$$

$$\mathcal{I}''(\alpha, \Delta) \equiv \int_0^{\infty}dk_0\frac{k_0^{2m}}{k_0^2 + \Delta^2}e^{-k_0\alpha} = \frac{\partial^{2m}}{\partial\alpha^{2m}}\mathcal{I}(\alpha, \Delta). \quad \text{(A13)}$$

Where $\mathrm{Ci}(x) = \gamma + \log(x) + \int_0^x\frac{\cos(t) - 1}{t}dt$ and $\mathrm{Si}(x) = \int_0^x\frac{\sin(t)}{t}dt$ are the geometric integrals. From Eq. (A13), it follows:

$$\delta\mathbb{A}(0,\Delta,L,\beta)=\frac{\Delta^2}{4\pi^2}\sum_{\vec{n}\neq 0}\frac{1}{nL\Delta}\left(\frac{1}{nL\Delta}-\mathrm{Ci}(nL\Delta)\times\sin(nL\Delta)+\cos(nL\Delta)\left(\mathrm{Si}(nL\Delta)-\frac{\pi}{2}\right)\right). \tag{A14}$$

Assuming $\Delta L \sim \epsilon$, it is possible to expand about $\Delta L = 0$. In this limit the sum may be approximated as an integral over the variable $z \equiv nL\Delta$, [38],

$$L^2\Delta^2\sum_{\vec{n}\neq 0}\frac{1}{nL\Delta}\left(\frac{1}{nL\Delta}-\mathrm{Ci}(nL\Delta)\sin(nL\Delta)+\cos(nL\Delta)\times\left(\mathrm{Si}(nL\Delta)-\frac{\pi}{2}\right)\right)\overset{L\Delta\to 0}{\to}4\pi\int_0^\infty\frac{z^2dz}{z}\times\left(\frac{1}{z}-\mathrm{Ci}(z)\sin(z)+\cos(z)\left(\mathrm{Si}(z)-\frac{\pi}{2}\right)\right)=2\pi^2. \tag{A15}$$

At leading order this matches to the approximation made in Ref. [38] for the same integral. The next term in the expansion comes from taking a derivative with respect to $\alpha' \equiv L\Delta$

$$\frac{\partial}{\partial\alpha'}\alpha'^2\sum_{\vec{n}\neq 0}\frac{1}{n\alpha'}\left(\frac{1}{n\alpha'}-\mathrm{Ci}(n\alpha')\sin(n\alpha')+\cos(n\alpha')\times\left(\mathrm{Si}(n\alpha')-\frac{\pi}{2}\right)\right)\bigg|_{\alpha'\to 0}=-\sum_{\vec{n}\neq 0}\frac{\pi}{2n} \tag{A16}$$

Expanding about $L\Delta = 0$ leads to a $\mathcal{O}(\epsilon^4)$ approximation of Eq. (A14)

$$\delta\mathbb{A}(0,\Delta,L,\beta)=\frac{1}{2L^3}-\sum_{\vec{n}\neq 0}\frac{\Delta}{8n\pi L^2}+\mathcal{O}(\epsilon^5)=\underbrace{\frac{1}{2L^3}}_{\mathcal{O}(\epsilon^3)}-\underbrace{\frac{\Delta c_1}{8\pi L^2}}_{\mathcal{O}(\epsilon^4)}+\mathcal{O}(\epsilon^5), \tag{A17}$$

where Eq. (A20) has been used. To $\mathcal{O}(\epsilon^4)$ the remaining integrals are

$$\delta\mathbb{B}(m_\pi,L,\beta)=\frac{1}{4\pi L}\sum_{\vec{n}\neq 0}\int_{-\infty}^{\infty}\frac{dk_0}{2\pi}k_0^2\frac{e^{-nL\sqrt{k_0^2+m_\pi^2}}}{n}=\frac{1}{2\pi^2L^4}\sum_{\vec{n}\neq 0}\frac{1}{n^4}+\mathcal{O}(\epsilon^6)=\frac{c_4}{2\pi^2L^4}+\mathcal{O}(\epsilon^6), \tag{A18}$$

$$\delta\mathbb{C}(m_\pi,L,\beta)=-\frac{1}{2\pi^2L^4}\sum_{\vec{n}\neq 0}\frac{1}{n^4}+\mathcal{O}(\epsilon^6)=-\frac{c_4}{2\pi^2L^4}+\mathcal{O}(\epsilon^6). \tag{A19}$$

In writing out the full expression of the masses, it is important to note that $\delta\mathbb{B}(0,L,\beta)+\delta\mathbb{C}(0,L,\beta)=0+\mathcal{O}(\epsilon^6)$. Two previously calculated sums have been used [39–42]:

$$c_1=\sum_{\vec{n}\neq 0}\frac{1}{|n|}=-2.8372974\qquad c_4=\sum_{\vec{n}\neq 0}\frac{1}{|n|^4}=16.532315. \tag{A20}$$

2. p-Regime Integrals: $m_L=\{m_K, m_\eta\}$

In the p-regime, the light meson mass is comparable to the lowest nonzero momentum $m_L/\Lambda_\chi \sim 2\pi/L\Lambda_\chi \sim \epsilon$. In this regime, the small mass approximations used in the previous sections are no longer valid. One must perform the integral in Eq. (A11) without taking the chiral limit. Although this integral cannot be evaluated exactly, in the $\Delta \to 0$ limit the integral is dominated by small values of k_0. In this case, the argument in the exponential can be approximated as $\sqrt{k_0^2+m_l^2}=m_l+\frac{k_0^2}{2m_L}-\frac{k_0^4}{8m_L^3}+\cdots$,

$$\Rightarrow\delta\mathbb{A}(m_L,\Delta,L,\beta)=\Delta\sum_{\vec{n}\neq 0}\int_{-\infty}^{\infty}\frac{dk_0}{2\pi}\frac{1}{k_0^2+\Delta^2}\frac{k_0^2+m_l^2}{4\pi L}\frac{e^{-nm_lL}e^{-(nLk_0^2/2m_l)}}{n}\left(1+\frac{nLk_0^4}{8m_L^3}\right)+\cdots$$
$$=\frac{m_L^2}{8\pi L}\sum_{\vec{n}\neq 0}\frac{e^{-nm_lL}}{n}+\sum_{\vec{n}\neq 0}\left(\frac{3+9m_LnL-(m_LnL)^2}{64\pi^2m_L^{1/2}L^{5/2}n^{5/2}}\right)\sqrt{2\pi}\Delta e^{-nm_lL}+\mathcal{O}(\epsilon^5)$$
$$\equiv\frac{m_L^2}{8\pi L}\mathcal{N}_1(m_L,L)+\frac{\Delta}{m_L^{1/2}L^{5/2}}\mathcal{N}_2(m_L,L)+\mathcal{O}(\epsilon^5), \tag{A21}$$

where the definition in Eq. (A24) have been used. The remaining integrals can be performed exactly,

$$\delta\mathbb{B}(m_L, L, \beta) = \frac{1}{4\pi L}\sum_{\vec{n}\neq 0}\int_{-\infty}^{\infty}\frac{dk_0}{2\pi}k_0^2\frac{e^{-nL\sqrt{k_0^2+m_l^2}}}{n} = \frac{m_L^3}{16\pi^2 L}\sum_{\vec{n}\neq 0}(K_3(m_L nL) - K_1(m_L nL)) + \mathcal{O}(\epsilon^5),$$
$$\equiv \frac{m_L^3}{16\pi^2 L}\mathcal{K}_1(m_L L) + \mathcal{O}(\epsilon^5) \qquad \text{(A22)}$$

$$\delta\mathbb{C}(m_L, L, \beta) = -\frac{m_L^3}{16\pi^2 L}\sum_{\vec{n}\neq 0}(K_3(m_L nL) + 3K_1(m_L nL)) + \mathcal{O}(\epsilon^5) \equiv -\frac{m_L^3}{16\pi^2 L}(\mathcal{K}_1(m_L L) - \mathcal{K}_2(m_L L)). \qquad \text{(A23)}$$

In writing Eqs. (A21)–(A23) the following dimensionless functions were used,

$$\mathcal{N}_1(m_L, L) = \sum_{\vec{n}\neq 0}\frac{e^{-nm_lL}}{n} \qquad \mathcal{N}_2(m_L, L) = \sum_{\vec{n}\neq 0}\left(\frac{3 + 9m_L nL - (m_L nL)^2}{64\pi^2 n^{5/2}}\right)\sqrt{2\pi}e^{-nm_lL} \qquad \text{(A24)}$$

$$\mathcal{K}_1(m_L L) = \sum_{\vec{n}\neq 0}(K_3(m_L nL) - K_1(m_L nL)) \qquad \mathcal{K}_2(m_L L) = -4\sum_{\vec{n}\neq 0}K_1(m_L nL), \qquad \text{(A25)}$$

where K_α are the modified Bessel functions of the second kind. Finally, by adding the contributions from Eqs. (A4)–(A8) and substituting the expressions of the respective sums, one arrives at Eq. (17).

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