Hadronic contribution to the muon $g - 2$ factor: A theoretical determination

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The leading-order hadronic contribution to the muon $g - 2$, a_μ^{HAD}, is determined entirely from theory using an approach based on Cauchy's theorem in the complex squared energy s-plane. This is possible after fitting the integration kernel in a_μ^{HAD} with a simpler function of s. The integral determining a_μ^{HAD} in the light-quark region is then split into a low-energy and a high-energy part, the latter given by perturbative QCD (PQCD). The low energy integral involving the fit function to the integration kernel is determined by derivatives of the vector correlator at the origin, plus a contour integral around a circle calculable in PQCD. These derivatives are calculated using hadronic models in the light-quark sector. A similar procedure is used in the heavy-quark sector, except that now everything is calculable in PQCD, thus becoming the first entirely theoretical calculation of this contribution. Using the dual-resonance model realization of large N_c QCD to compute the derivatives of the correlator leads to agreement with the experimental value of a_μ. Accuracy, though, is currently limited by the model-dependent calculation of derivatives of the vector correlator at the origin. Future improvements should come from more accurate chiral perturbation theory and/or lattice QCD information on these derivatives, allowing for this method to be used to determine a_μ^{HAD} accurately entirely from theory, independently of any hadronic model.

The value of the muon $g - 2$ is well-known as a test of the standard model (SM) of particle physics. [1]. The SM result for the anomalous magnetic moment of the muon is conveniently separated into the contributions due to QED, the hadronic sector, and the electroweak sector. A sizable theoretical uncertainty arises from the (leading-order) hadronic vacuum polarization term, the second largest contribution after that of QED. A substantial effort has been made to determine this contribution from experimental data on $e^+ e^- \rightarrow$ hadrons and $\tau \rightarrow$ hadrons [2,3]. Currently, there is some yet unresolved discrepancy between both results, although a recent reanalysis appears to resolve this discrepancy [4]. Writing the muon anomaly in the SM as

$$a_\mu^{\text{SM}} = a_\mu^{\text{QED}} + a_\mu^{\text{HAD}} + a_\mu^{\text{EW}}, \qquad (1)$$

the leading contribution is that from QED, followed by the hadronic and the electroweak terms. In this paper, we concentrate on the leading-order hadronic contribution and discuss a new approach to its calculation entirely from theory. The method relies on Cauchy's theorem in the complex squared energy s-plane, after fitting the integration kernel entering a_μ^{HAD} with a simple function of s. In the region of the light-quark sector, the method requires knowledge of some of the derivatives of the (electromagnetic) vector correlator at zero momentum, as well as its perturbative QCD (PQCD) behavior. Currently, these derivatives will be obtained here from hadronic models, thus being affected by systematic uncertainties. Hence, at this stage, the method cannot rival in accuracy with the standard approach of using experimental data on the vector

correlator at low/intermediate energies. However, future precision determinations of these derivatives from chiral perturbation theory (CHPT) or lattice QCD would render this calculation of a_μ^{HAD} independent of experimental data on $e^+ e^- \rightarrow$ hadrons and $\tau \rightarrow$ hadrons. It must be emphasized that as a consequence of Cauchy's theorem, this method only requires knowledge of the derivatives of the vector correlator at the origin, rather than its full expression over an extended energy region. In addition, the method allows for a straightforward incorporation of the charm- and bottom-quark contributions to a_μ^{HAD} calculable exclusively from PQCD, i.e. without the need for data on the vector correlator. This leads to the first entirely theoretical calculation of this contribution. We begin with the standard expression of the (lowest-order) muon anomaly [1]

$$a_\mu^{\text{HAD}} = \frac{\alpha_{\text{EM}}^2}{3\pi^2} \int_{s_{th}}^\infty \frac{ds}{s} K(s) R(s), \qquad (2)$$

where α_{EM} is the electromagnetic coupling and the standard R-ratio is $R(s) = 3 \sum_f Q_f^2 [8\pi \text{Im}\Pi(s)]$, where Q_f are the quark charges and $\Pi(s)$ is the vector current correlator normalized to $8\pi \text{Im}\Pi(s) = 1 + \alpha_s/\pi + \dots$. The integration kernel $K(s)$ in Eq. (2) is given by [5]

$$K(s) = \int_0^1 dx \frac{x^2(1 - x)}{x^2 + \frac{s}{m_\mu^2}(1 - x)}, \qquad (3)$$

where m_μ is the muon mass. A popular approach to compute a_μ^{HAD} has been to split the integral in Eq. (2) into a low-energy region from threshold up to $s = s_0 \simeq (1.8 \text{ GeV})^2$, followed by a high-energy region from

$s = s_0$ to infinity. The integral in the former region was calculated using data of $e^+e^- \rightarrow$ hadrons or (isospin-rotated) data on $\tau \rightarrow$ hadrons. The integration in the high-energy region was performed assuming PQCD. In this paper, we discuss a new approach based entirely on theoretical input. For later convenience, we split the contributions to the leading-order a_μ^{HAD} into three pieces,

$$a_\mu^{HAD} = a_\mu^{HAD}|_{uds} + a_\mu^{HAD}|_c + a_\mu^{HAD}|_b, \qquad (4)$$

where the first term on the right-hand side corresponds to the contribution of the three light quarks, and the second and third term refer to the charm- and bottom-quark contributions. In the light-quark sector, the first step is to fit the integration kernel $K(s)$ in an interval $s_{th} \leq s \leq s_0$ with a function $K_1(s)$:

$$K_1(s) = a_0 s + \sum_{n=1} \frac{a_n}{s^n}, \qquad (5)$$

with coefficients determined by minimizing the chi-squared. The upper limit s_0 is below the charm threshold. Next, the integration range in Eq. (2) is split into a low-energy ($s \leq s_0$) and a high-energy ($s > s_0$) region where PQCD would be valid. In the former region, Cauchy's theorem is used to obtain

$$\int_{s_{th}}^{s_0} \frac{ds}{s} K_1(s) \frac{1}{\pi} \mathrm{Im}\Pi_{uds}(s)$$
$$= \mathrm{Res}\left[\Pi_{uds}(s) \frac{K_1(s)}{s} \right]_{s=0} - \frac{1}{2\pi i} \oint_{|s|=s_0} \frac{ds}{s} K_1(s) \Pi_{uds}(s), \qquad (6)$$

where the integral on the right-hand side, around the circle of radius $s_0 \simeq (1.8 \text{ GeV})^2$, is computed using PQCD in the light-quark sector. This is known up to the five-loop level [6]. The contour integration can be performed using fixed-order perturbation theory (FOPT) or, alternatively, contour-improved perturbation theory. There is no clear *a priori* criterion to decide which is best in a given application. However, in the present case, the difference between the two methods turns out to be negligible, as discussed later. The residues are given in terms of derivatives of the correlator at zero momentum, for which one can use hadronic models, CHPT or lattice QCD. Hence, Eq. (2) becomes

$$a_\mu^{HAD}|_{uds} = 8\alpha_{EM}^2 \sum_{i=u,d,s} Q_i^2 \Bigg\{ \mathrm{Res}\left[\Pi_{uds}(s) \frac{K_1(s)}{s} \right]_{s=0}$$
$$- \frac{1}{2\pi i} \oint_{|s|=s_0} \frac{ds}{s} K_1(s) \Pi_{uds}(s)|_{PQCD}$$
$$+ \int_{s_0}^\infty \frac{ds}{s} K(s) \frac{1}{\pi} \mathrm{Im}\Pi_{uds}(s)|_{PQCD} \Bigg\}, \qquad (7)$$

where the last integral above involves the exact integration kernel $K(s)$ and PQCD is used for the spectral function. It is

important to stress that this contribution to the anomaly only requires knowledge of a few derivatives of the vector correlator at the origin (to compute the residue). It does not require knowledge of the correlator itself in the extended energy region from threshold up to $s_0 \simeq (1.8 \text{ GeV})^2$. The choice of this particular value for the onset of PQCD will allow for a fair comparison with determinations based entirely on data [2]. It is also supported by experimental results from the BES Collaboration [7], which show the onset of PQCD at $s_0 \simeq 4.0 \text{ GeV}^2$. The stability of results against changes in this threshold value will be analyzed later.

In order to incorporate charm-quark information, we add an extra contribution determined as follows. A new fit to the integration kernel $K(s)$ is performed in a region $s_1 \leq s \leq s_2$, where $s_1 \simeq M_{J/\psi}^2$, and $s_2 \simeq (5.0 \text{ GeV})^2$. Using this kernel and Cauchy's theorem, the charm contribution is given by an expression similar to Eq. (7), except that s_0 is replaced by s_2 and $K_1(s)$ by the new fit function $K_2(s)$. The residues can now be computed directly from PQCD using the low-energy expansion of the heavy-quark correlator, known up to four-loop order. No hadronic model nor data is needed here. A similar procedure can be followed to incorporate the contribution of the bottom quark.

We proceed to fit the integration kernel, $K_1(s)$ in the region $s_{th} \leq s \leq s_0$. If one were to choose a polynomial fit of the form $K_1(s) = \sum_{i=1} c_i s^i$, then the residues in Eq. (7) would all vanish, and the anomaly would be determined entirely from QCD [8]. There are two drawbacks to such a fit. First, even taking many terms in the series expansion of $K_1(s)$, the fit is not accurate enough. Second, the higher powers of s bring in higher-dimensional condensates in the operator product expansion, thus reducing further the accuracy of this approach. An inspection of the s-behavior of the kernel $K(s)$ suggests that a series expansion involving inverse powers of s should be a better option. In fact, this turns out to be the case, e.g. the fit function, Eq. (5), up to s^{-3} becomes

$$K_1(s) = 2.257 \times 10^{-5} s + 3.482 \times 10^{-3} s^{-1}$$
$$- 1.467 \times 10^{-4} s^{-2} + 4.722 \times 10^{-6} s^{-3}, \quad (8)$$

where s is expressed in GeV^2, and the numerical coefficients have the appropriate units to render $K_1(s)$ dimensionless. Figure 1 shows the exact kernel $K(s)$ in Eq. (2) (solid curve) together with the fit $K_1(s)$ as in Eq. (8) (solid dots). The relative difference between the two curves lies in the range 0–1% in the low-energy region, where it contributes the most. A further estimate of the accuracy of the fit function, Eq. (8), can be obtained by using all available experimental data on $R(s)$ in Eq. (2), together with (a) the exact kernel Eq. (3), and (b) the fit kernel Eq. (8). We find $a_\mu^{HAD}|_{uds} = 641.69$ for procedure (a) and $a_\mu^{HAD}|_{uds} = 641.16$ for procedure (b), i.e. a difference of 0.08%. Using additional inverse powers of s terms in the
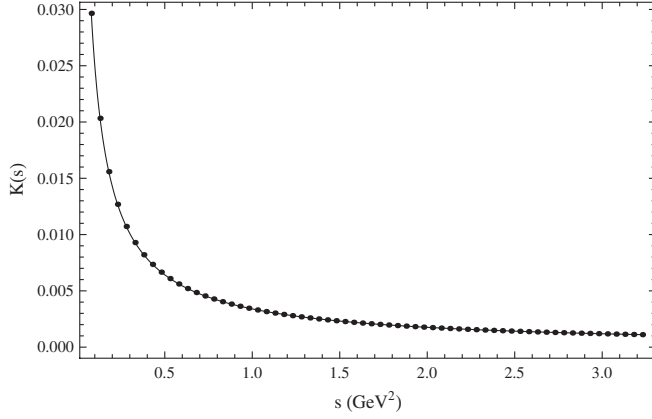
FIG. 1. The exact kernel $K(s)$, Eq. (3), (solid line) together with the fit in the light-quark region, Eq. (8), (solid circles).

fit, while improving it slightly, does not lead to any appreciable difference in the final result for a_μ^{HAD}. For instance, the difference in a_μ^{HAD} from adding two additional inverse powers in Eq. (8) is less than 0.16%.

We discuss now the incorporation of the heavy-quark sector contribution to the anomaly, starting with the charm-quark piece $a_\mu^{\text{HAD}}|_c$. The fit to the integration kernel, which we name $K_2(s)$, is performed in the region $s_1 \simeq M_{J/\psi}^2 \le s \le s_2 \simeq (5.0 \text{ GeV})^2$. The very simple function

$$K_2(s) = \frac{a_1}{s} + \frac{a_2}{s^2}, \qquad (9)$$

where $a_1 = 0.003712 \text{ GeV}^2$ and $a_2 = -0.0005122 \text{ GeV}^4$, provides an excellent fit with $K_2(s)$ differing from the exact kernel $K(s)$ by less than 0.02%. In this case, $a_\mu^{\text{HAD}}|_c$ is given by a similar expression as $a_\mu^{\text{HAD}}|_{uds}$, Eq. (7), with obvious replacements. An important difference is that now $a_\mu^{\text{HAD}}|_c$ only involves the correlator and its imaginary part calculable in PQCD, i.e. no hadronic model is needed for the residue. In fact, the Taylor expansion of the correlator around the origin is given by

$$\Pi_c(s)|_{\text{PQCD}} = \frac{3}{32\pi^2} Q_c^2 \sum_{n \ge 0} \bar{C}_n z^n, \qquad (10)$$

where $z = s/(4\bar{m}_c^2)$. Here, $\bar{m}_c \equiv \bar{m}_c(\mu)$ is the charm-quark mass in the $\overline{\text{MS}}$-scheme at a renormalization scale μ. The coefficients $\bar{C}_n$ up to $n = 30$ are known at the three-loop level [9,10]. At the four-loop level $\bar{C}_0$ and $\bar{C}_1$ were determined in Refs. [9–11], $\bar{C}_2$ in Ref. [10], and $\bar{C}_3$ in Ref. [12]. Because of the s-dependence of $K_2(s)$, no coefficients $\bar{C}_4$ or higher contribute to $\text{Res}[\Pi_c(s)p(s), s = 0]$. Using as input $\mu = 3 \text{ GeV}$, $\alpha_s^{(4)}(3 \text{ GeV}) = 0.2145(22)$ [13], and $\bar{m}_c(3 \text{ GeV}) = 0.986(10) \text{ GeV}$ [14], we find

$$\Pi_c(s) = 0.03604 + 0.001833s + 0.00012335s^2$$
$$+ 0.000012472s^3 + \mathcal{O}(s^4), \qquad (11)$$

where s is expressed in GeV^2, and the numerical coefficients have the appropriate units to render $\Pi_c(s)$ dimensionless. The residue in the charm-quark sector is

$$\text{Res}\left[\Pi_c(s)|_{\text{PQCD}} \frac{K_2(s)}{s}\right]_{s=0} = 76.1(5) \times 10^{-7}, \qquad (12)$$

where the error is due to the uncertainty in α_s and to the truncation of PQCD. For the bottom-quark case the fit function

$$K_3(s) = 0.003719 \text{ GeV}^2 s^{-1} - 0.0007637 \text{ GeV}^4 s^{-2} \qquad (13)$$

differs from the exact kernel by less than 0.0005% in the range $M_\Upsilon^2 \le s \le (12 \text{ GeV})^2$. The residue is now

$$\text{Res}\left[\Pi_b(s)|_{\text{PQCD}} \frac{K_3(s)}{s}\right]_{s=0} = 6.3 \times 10^{-7}, \qquad (14)$$

where the error is negligible. Next, in order to calculate the contour integral around the circle of radius s_2, we make use of PQCD, i.e.

$$\Pi_{\text{PQCD}}(s) = \sum_{n=0}^{\infty} \left(\frac{\alpha_s(\mu^2)}{\pi}\right)^n \Pi^{(n)}(s), \qquad (15)$$

where

$$\Pi^{(n)}(s) = \sum_{i=0}^{\infty} \left(\frac{\bar{m}^2}{s}\right)^i \Pi_i^{(n)}. \qquad (16)$$

The complete analytical result in PQCD up to $\mathcal{O}(\alpha_s^2, (\bar{m}^2/s)^6)$ is given in Ref. [15], with new results up to order $\mathcal{O}(\alpha_s^2(\bar{m}^2/s)^{30})$ obtained recently [16]. There are also exact results for $\Pi_0^{(3)}$ and $\Pi_1^{(3)}$ from Ref. [17], while $\Pi_2^{(3)}$ is known up to a constant term [18]. This constant term does not contribute to the contour integral due to the s-dependence of $K_2(s)$. Finally, at the five-loop level, the full logarithmic terms in $\Pi_0^{(4)}$ and $\Pi_1^{(4)}$ are known from Refs. [19,20], respectively. The contour integrals in FOPT are

$$\frac{1}{2\pi i} \oint \frac{ds}{s} K_n(s) \Pi_q(s)|_{\text{PQCD}} = \begin{cases} 135.3(6) \times 10^{-7} \\ 20.3(1) \times 10^{-7} \\ 3.6(2) \times 10^{-7} \end{cases}, \qquad (17)$$

for $n = 1, 2, 3$ and $q = uds, c, b$, respectively. For $n = 1$, the result in contour-improved perturbation theory is $135.6(6) \times 10^{-7}$, i.e. a 0.2% difference with FOPT. Also for $n = 1$, changing the PQCD threshold in the interval $s_0 = (1.8 - 2.0)^2 \text{ GeV}^2$ leads to a change of only 0.15% in the final value of a_μ^{HAD}. The BES Collaboration data [7] agrees well with PQCD in this region and beyond. The results for the third integral in Eq. (7) and their equivalent for the charm- and bottom-quark sectors are

$$\int_{s_j}^{\infty} \frac{ds}{s} K(s) \frac{1}{\pi} \mathrm{Im}\Pi_q(s)|_{\mathrm{PQCD}} = \begin{cases} 151.8(1) \times 10^{-7} \\ 20.0(4) \times 10^{-7} \\ 3.4(2) \times 10^{-7} \end{cases} \quad (18)$$

with $j = 0, 2, 4$ for $q = uds, c, b$, respectively. Substituting the results from Eqs. (17) and (18) into Eq. (7) and the corresponding expressions for the charm- and bottom-quark contributions, the leading order a_μ^{HAD} is

$$a_\mu^{\mathrm{HAD}} = \frac{16}{3} \alpha_{\mathrm{EM}}^2 \mathrm{Res}\left[\Pi_{uds}(s) \frac{K_1(s)}{s} \right]_{s=0} + 19.4(2) \times 10^{-10}. \quad (19)$$

The contributions to a_μ^{HAD} from the charm- and bottom-quark sectors obtained from PQCD are

$$a_\mu^{\mathrm{HAD}}|_c = 14.4(1) \times 10^{-10}, \quad (20)$$

and

$$a_\mu^{\mathrm{HAD}}|_b = 0.29(1) \times 10^{-10}. \quad (21)$$

Finally, we discuss the calculation of the first term on the right-hand side of Eq. (6). Given the parametrization in Eq. (8), this term can be conveniently written as

$$\mathrm{Res}\left[\Pi_{uds}(s) \frac{K_1(s)}{s} \right]_{s=0} = \lim_{s \to 0} \sum_{n=1}^{3} \frac{a_n}{n!} \frac{d^n}{ds^n} \Pi_{uds}(s), \quad (22)$$

where the a_n are the coefficients of the s^{-1}, s^{-2} and s^{-3} terms in Eq. (8), respectively. To demonstrate the effectiveness of the method, we consider three hadronic models for the vector correlator, (single ρ) vector meson dominance (VMD), the Kroll-Lee-Zumino (KLZ) quantum field theory model [21,22], and the dual-resonance model realization of QCD in the large N_c limit (Dual-QCD$_\infty$)[23–25]. We reiterate that the use of hadronic models to compute the derivatives of the vector correlator at the origin is only provisional. In the future, these derivatives will be provided with increased accuracy by CHPT and/or lattice QCD. The error on VMD can be estimated to be of order $\mathcal{O}(10$–$20\%)$, judging from its predictions of the pion radius and form factor. The KLZ model is a renormalizable theory of pions and a neutral ρ-meson which provides the necessary quantum field theory platform for VMD and leads to loop corrections to VMD. The loop corrections in the KLZ model bring the pion radius and form factor into better agreement with experiment. In VMD-type models, the vector correlator is related to the pion form factor through $\Pi_{uds}(s) = F_\pi(s)/f_\rho^2$, where $f_\rho = 4.96 \pm 0.02$ [13] is the $\gamma - \rho$ coupling and $F_\pi(s)$ is the pion form factor. The ρ-VMD expression for the correlator is

$$\Pi_{uds}(s)|_{\mathrm{VMD}} = \frac{1}{f_\rho^2} \frac{M_\rho^2}{(M_\rho^2 - s)}, \quad (23)$$

which involves the underlying standard VMD universality relation $g_{\rho\pi\pi}/f_\rho = 1$. With $g_{\rho\pi\pi} = 5.92 \pm 0.01$ from experiment [13], this relation is off by roughly 20%. The result for the residue in Eq. (19) is

$$\mathrm{Res}\left[\Pi_{uds}(s) K_1(s)/s \right]_{s=0}^{\mathrm{VMD}} = 2.20(2) \times 10^{-4}, \quad (24)$$

leading to

$$a_\mu^{\mathrm{HAD}}|_{\mathrm{VMD}} = 644(6) \times 10^{-10}. \quad (25)$$

For the correlator in the KLZ model, we use the result from Ref. [26] (see also Ref. [22]) and obtain

$$\mathrm{Res}\left[\Pi_{uds}(s) K_1(s)/s \right]_{s=0}^{\mathrm{KLZ}} = 2.22(2) \times 10^{-4}, \quad (26)$$

and

$$a_\mu^{\mathrm{HAD}}|_{\mathrm{KLZ}} = 650(6) \times 10^{-10}. \quad (27)$$

The errors for VMD and KLZ are only those due to the uncertainty in f_ρ and do not include possible (systematic) model errors. The latter can be gauged from the deviation from universality $g_{\rho\pi\pi}/f_\rho = 1$, off by some 20%, as well as from the pion charge radius in VMD $\langle r_\pi^2 \rangle = 0.394$ fm^2, to be compared with the experimental value [27] $\langle r_\pi^2 \rangle = 0.439 \pm 0.008$ fm^2. While QCD in the limit of an infinite number of colors leads to a hadronic spectrum consisting of an infinite number of zero-width resonances, it does not specify the mass spectrum nor the couplings. Dual-QCD$_\infty$ [23–25] provides this information leading to hadronic form factors in excellent overall agreement with data in the spacelike region. The vector correlator in this framework is given by

$$\Pi_{uds}(s)|_{\mathrm{QCD}_\infty} = \frac{1}{f_\rho^2} \frac{1}{\sqrt{\pi}} \frac{\Gamma(\beta - 1/2)}{\Gamma(\beta - 1)} B(\beta - 1, 1/2 - s/2M_\rho^2), \quad (28)$$

where β is a free parameter and $B(x, y)$ is the Euler beta function. From the power series expansion of $B(x, y)$, it is easy to see that Eq. (28) represents an infinite number of (zero-width) resonances corresponding to the ρ-meson and its radial excitations. The latter account for the deviation from the VMD result $g_{\rho\pi\pi}/f_\rho = 1$ leading to [23] $g_{\rho\pi\pi}/f_\rho = 1.2$ in agreement with the experiment. For $\beta = 2$, Eq. (28) reduces to single ρ-VMD. The value $\beta = 2.30(3)$ results in an excellent fit to all data on the pion form factor $F_\pi(s)$ in the spacelike region up to $s = -10$ GeV^2 with a chi-squared per degree of freedom $\chi_F \simeq 1.5$ [23]. In contrast, the VMD fit yields $\chi_F \simeq 11$. In addition, the Dual-QCD$_\infty$ model gives a value of the pion charge radius $\langle r_\pi^2 \rangle = 0.436 \pm 0.004$ fm^2 [23] to be compared with the most recent experimental value [27] $\langle r_\pi^2 \rangle = 0.439 \pm 0.008$ fm^2. Since the first derivative of the vector correlator dominates in Eq. (22), it is very important for a hadronic model to reproduce the pion radius. The result for the residue in Eq. (19) is

$$\mathrm{Res}\left[\Pi_{uds}(s) \frac{K_1(s)}{s} \right]_{s=0}^{\mathrm{QCD}_\infty} = 2.47(3) \times 10^{-4}, \quad (29)$$

and using Eq. (19), the hadronic $a_\mu^{\mathrm{QCD}_\infty}$ is

$$a_\mu^{\mathrm{HAD}}|_{\mathrm{QCD}_\infty} = 722(9) \times 10^{-10}, \qquad (30)$$

where the error is mostly due to that in β. The result, Eq. (30), can be compared with the value $a_\mu^{\mathrm{HAD}} = 692.3(4.2) \times 10^{-10}$ from Refs. [2,3] using e^+e^- data, or $a_\mu^{\mathrm{HAD}} = 701.5(4.7) \times 10^{-10}$ using τ data. However, a more recent reanalysis based on τ data [4] finds $a_\mu^{\mathrm{HAD}} = 690.96(4.65) \times 10^{-10}$. In the QCD_∞ framework, the $1/N_c$ corrections arise in the timelike region from finite width resonance effects. These corrections to the form factor are of order $\mathcal{O}(\Gamma^2/M^2)$ near the origin. While small, they might have an impact on the residues, Eq. (29). Shifting the poles in Eq. (28) to the second Riemann sheet in the complex s-plane, while preserving the normalization at the origin and the vanishing of the imaginary part of $F_\pi(s)$ at threshold [23,28], leads to a reasonable finite width model. The first derivative of the form factor at the origin, which is the main contribution to Eq. (29), receives no width correction. The second derivative is reduced with respect to the zero-width result by less than 2%, and the third derivative by some 3%. This translates into an increase in the value given in Eq. (30) of 0.1%.

Adding to Eq. (30) the QED contribution [29] $a_\mu^{\mathrm{QED}} = 11658471.809 \pm 0.015$, the electroweak [30] $a_\mu^{\mathrm{EW}} = 15.4 \pm 0.2$, the higher-order hadronic [3] $a_\mu^{\mathrm{HAD}}|_{\mathrm{HO}} = -9.79 \pm 0.09$, and the light-by-light contribution [31] $a_\mu^{\mathrm{LbL}}|_{\mathrm{HO}} = 11.6 \pm 4.0$, all in units of 10^{-10}, we find it intriguing that the Dual-QCD_∞ prediction, Eq. (30), leads to

$$a_\mu|_{\mathrm{QCD}_\infty} = 11659210.6 \pm 9.8 \times 10^{-10}, \qquad (31)$$

to be compared with the experimental value

$$a_\mu^{\mathrm{EXP}} = 11659208.9 \pm 6.3 \times 10^{-10}. \qquad (32)$$

Equation (31) does suggest that it might be possible to understand the muon anomaly entirely within the SM. Our approach to determine a_μ^{HAD} appears to be optimally designed for use in CHPT, as the main input is the power series of the correlator around the origin. The $\mathcal{O}(p^6)$ vector correlator was determined in Refs. [32,33]. The derivative at zero momentum, in terms of the usual chiral constants, is

$$\frac{d}{ds}\Pi_{uds}|_{\chi\mathrm{pT}}(0) = 0.0105557 - 4C_{93}^r - 0.77725L_{10}^r$$
$$+ 1.0346L_9^r. \qquad (33)$$

Two of these constants have been calculated on the lattice, $L_9^r = 3.08(23)(51) \times 10^{-3}$ [34] and $L_{10}^r = -5.2(2)(^{+5}_{-3}) \times 10^{-3}$ [35]. The constant C_{93}^r has not been determined on

the lattice yet. We have to rely on a very rough estimate of this constant from [32] using VMD, $C_{93}^r \approx -17 \times 10^{-3}\ \mathrm{GeV}^{-2}$. With these values, we obtain $\frac{d}{ds}\Pi_{uds}(0)|_{\chi\mathrm{pT}} \approx 0.0857$. Given the very large uncertainty in C_{93}^r and the fact that the contribution of the second and the third derivative is very small, we find

$$\mathrm{Res}\left[\Pi_{uds}(s)\frac{K_1(s)}{s}\right]_{s=0} \simeq 2.8, \qquad (34)$$

leading to

$$a_\mu^{\mathrm{HAD}} \simeq 815 \times 10^{-10}. \qquad (35)$$

This is a great deal larger than the value expected from the experiment. The reason for this is that the constant C_{93}^r dominates this result. Furthermore, in Ref. [32], it is argued that the estimates of the $\mathcal{O}(p^6)$ constants could be larger than the physical constants. Therefore, it makes sense that this result for a_μ^{HAD} represents an overestimate, rather than an underestimate. One can reverse this argument and give the first model independent determination of C_{93}^r. Making use of $a_\mu^{\mathrm{HAD}} = 692.3(4.2)$ from Ref. [2], we find that $C_{93}^r = -13.9(2) \times 10^{-3}\ \mathrm{GeV}^{-2}$.

In summary, we have discussed a new approach to the determination of the leading a_μ^{HAD} entirely from theory, i.e. without the use of experimental data on the vector correlator in an extended energy region. This can be achieved by fitting the integration kernel, Eq. (3), in the light-quark sector with the simple function Eq. (8) and subsequently invoking Cauchy's theorem in the complex s-plane. This leads to the result, Eq. (7), which only requires knowledge of a few derivatives of the vector correlator at the origin. This must be contrasted with the standard approach which requires the complete correlator in the wide energy region from threshold up to $s_0 \simeq (1.8\ \mathrm{GeV})^2$. Such detailed information can only be reliably and accurately obtained from data. Currently, these derivatives can be estimated using hadronic models, examples of which have been presented here. In the future, though, more accurate determinations of the derivatives from CHPT and/or lattice QCD should become available, thus allowing for a model-independent calculation of this contribution. In the heavy-quark sector, this problem does not arise, as it is possible to calculate the anomaly entirely from PQCD, with the results given in Eqs. (20) and (21).

[1] J. P. Miller, E. de Rafael, and B. Lee Roberts, Rep. Prog. Phys. **70**, 795 (2007); F. Jegerlehner, and A. Nyffeler, Phys. Rep. **477**, 1 (2009), and references therein.

[2] M. Davier *et al.*, Eur. Phys. J. C **71**, 1515 (2011).

[3] K. Hagiwara *et al.*, J. Phys. G **38**, 085003 (2011).

[4] F. Jegerlehner and R. Szafron, Eur. Phys. J. C **71**, 1632 (2011).

[5] S. J. Brodsky and E. de Rafael, Phys. Rev. **168**, 1620 (1968).

[6] P. A. Baikov, K. G. Chetyrkin, and J. H. Kühn, Phys. Rev. Lett. **96**, 012003 (2006).

[7] J. Z. Bai *et al.* (BES Coll.) Phys. Rev. Lett. **88**, 101802 (2002); M. Ablikim *et al.* (BES Coll.) Phys. Lett. B **677**, 239 (2009).

[8] N. Nasrallah, N. A. Papadopoulos, and K. Schilcher, Phys. Lett. **126B**, 379 (1983).

[9] R. Boughezal, M. Czakon, and T. Schutzmeier, Phys. Rev. D **74**, 074006 (2006).

[10] A. Maier, P. Maieröfer, and P. Marquard, Nucl. Phys. **B797**, 218 (2008); Phys. Lett. B **669**, 88 (2008).

[11] K. G. Chetyrkin, J. H. Kühn, and C. Sturm, Eur. Phys. J. C **48**, 107 (2006).

[12] A. Maier *et al.*, Nucl. Phys. **B824**, 1 (2010).

[13] K. Nakamura *et al.* (Particle Data Group), J. Phys. G **37**, 075021 (2010).

[14] K. G. Chetyrkin *et al.*, Phys. Rev. D **80**, 074010 (2009); S. Bodenstein *et al.*, Phys. Rev. D **82**, 114013 (2010); **83**, 074014 (2011).

[15] K. G. Chetyrkin *et al.*, Nucl. Phys. **B503**, 339 (1997).

[16] A. Maier, and P. Marquard, arXiv:1110.558.

[17] P. A. Baikov, K. G. Chetyrkin, and J. H. Kühn, Nucl. Phys. B, Proc. Suppl. **189**, 49 (2009).

[18] K. G. Chetyrkin, R. Harlander, and J. H. Kühn, Nucl. Phys. **B586**, 56 (2000).

[19] P. A. Baikov, K. G. Chetyrkin, and J. H. Kühn, Phys. Rev. Lett. **101**, 012002 (2008).

[20] P. A. Baikov, K. G. Chetyrkin, and J. H. Kühn, Nucl. Phys. B, Proc. Suppl. **135**, 243 (2004).

[21] N. M. Kroll, T. D. Lee, and B. Zumino, Phys. Rev. **157**, 1376 (1967).

[22] C. A. Dominguez *et al.*, Phys. Rev. D **76**, 095002 (2007); C. A. Dominguez, M. Loewe, and B. Willers, Phys. Rev. D **78**, 057901 (2008).

[23] C. A. Dominguez, Phys. Lett. B **512**, 331 (2001); C. A. Dominguez, and T. Thapedi, J. High Energy Phys. 10 (2004) 003; C. A. Dominguez, and R. Röntsch, *ibid.*10 (2007) 085.

[24] E. Ruiz Arriola and W. Broniowski, Phys. Rev. D **78**, 034031 (2008); **81**, 094021 (2010); W. Broniowski and E. Ruiz Arriola, Proc. Sci., LC 2010 (2010) 062.

[25] C. Bruch, A. Khodjamirian, and J. H. Kühn, Eur. Phys. J. C **39**, 41 (2005).

[26] C. Gale and J. I. Kapusta, Nucl. Phys. **B357**, 65 (1991).

[27] S. R. Amendolia *et al.*, Nucl. Phys. **B277**, 168 (1986).

[28] M. Gourdin, Phys. Rep. **11C**, 29 (1974).

[29] T. Kinoshita and M. Nio, Phys. Rev. D **73**, 013003 (2006).

[30] A. Czarnecki, W. J. Marciano, and A. Vainshtein, Phys. Rev. D **67**, 073006 (2003); M. Knecht *et al.*, J. High Energy Phys. 11 (2002) 003.

[31] A. Nyffeler, Phys. Rev. D **79**, 073012 (2009).

[32] J. Bijnens and P. Talavera, Nucl. Phys. **B568**, 319 (**2000**).

[33] E. Golowich and J. Kambor, Nucl. Phys. **B447**, 373 (1995).

[34] P. A. Boyle *et al.*, J. High Energy Phys. 07 (2008) 112.

[35] E. Shintani *et al.*, Phys. Rev. Lett. **101**, 242001 (2008).