

Anomalous decays of η' and η into four pionsFeng-Kun Guo,^{1,*} Bastian Kubis,^{1,†} and Andreas Wirzba^{2,‡}¹*Helmholtz-Institut für Strahlen- und Kernphysik (Theorie) and Bethe Center for Theoretical Physics, Universität Bonn, D-53115 Bonn, Germany*²*Institute for Advanced Simulation and Jülich Center for Hadron Physics, Institut für Kernphysik, Forschungszentrum Jülich, D-52425 Jülich, Germany*

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We calculate the branching ratios of the yet unmeasured η' decays into four pions, based on a combination of chiral perturbation theory and vector-meson dominance. The decays $\eta' \rightarrow 2(\pi^+\pi^-)$ and $\eta' \rightarrow \pi^+\pi^-2\pi^0$ are P-wave dominated and can largely be thought to proceed via two ρ resonances; we predict branching fractions of $(1.0 \pm 0.3) \times 10^{-4}$ and $(2.4 \pm 0.7) \times 10^{-4}$, respectively, not much lower than the current experimental upper limits. The decays $\eta' \rightarrow 4\pi^0$ and $\eta \rightarrow 4\pi^0$, in contrast, are D-wave driven as long as conservation of CP symmetry is assumed, and are significantly further suppressed; any experimental evidence for the decay $\eta \rightarrow 4\pi^0$ could almost certainly be interpreted as a signal of CP violation. We also calculate the CP -violating amplitudes for $\eta' \rightarrow 4\pi^0$ and $\eta \rightarrow 4\pi^0$ induced by the QCD θ -term.

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I. INTRODUCTION

Processes in low-energy QCD that involve an odd number of (pseudo-)Goldstone bosons (and possibly photons), which are, therefore, of odd intrinsic parity, are thought to be governed by the Wess-Zumino-Witten (WZW) term [1] via chiral anomalies. While the so-called triangle anomaly is well tested in processes such as π^0 , $\eta \rightarrow \gamma\gamma$, and the box anomaly contributes e.g. to $\gamma\pi \rightarrow \pi\pi$ and $\eta \rightarrow \pi\pi\gamma$, the pentagon anomaly remains more elusive; the simplest possible process that is usually cited is $K^+K^- \rightarrow \pi^+\pi^-\pi^0$, which however has not been experimentally tested yet, and is likely to be subject to large corrections to the chiral-limit amplitude that is dictated by the WZW term.

A different set of processes involving five light pseudoscalars is the four-pion decays of η and η' . Experimental information about these is scarce: only upper limits on branching ratios exist [2]; however, this may change in the near future for at least some of the possible final states with the advent of high-statistics η' experiments such as BES-III, WASA-at-COSY, ELSA, CB-at-MAMI-C, CLAS at Jefferson Lab, etc. We are only aware of one previous theoretical calculation of these decays, performed in the framework of a quark model [3], whose partial width predictions, however, have in the meantime been ruled out by the experimental upper limits, at least for the channel $\eta' \rightarrow 2(\pi^+\pi^-)$.

In principle, the decays $\eta' \rightarrow 4\pi$, in contradistinction to many other η' decay channels, seem not terribly forbidden by approximate symmetries: they are neither isospin forbidden, nor required to proceed via electromagnetic inter-

actions. The reaction $\eta \rightarrow 4\pi$, in contrast, is essentially suppressed by tiny phase space: only the decay into $4\pi^0$ is kinematically allowed ($M_\eta - 4M_{\pi^0} = 7.9$ MeV, $M_\eta - 2(M_{\pi^\pm} + M_{\pi^0}) = -1.2$ MeV). Furthermore, the fact that anomalous amplitudes always involve the totally antisymmetric tensor $\epsilon_{\mu\nu\alpha\beta}$ can be used to show that no two pseudoscalars are allowed to be in a relative S wave: assuming they were, this would reduce the five-point function $PPPPP$ effectively to a four-point function $SPPP$ (where S stands for a scalar and P for a pseudoscalar), in which there are no four independent vectors left to contract the ϵ tensor with. The decays $\eta' \rightarrow 2(\pi^+\pi^-)$ and $\eta' \rightarrow \pi^+\pi^-2\pi^0$ can therefore be expected to be P-wave dominated. As furthermore Bose symmetry forbids two neutral pions to be in an odd partial wave, $\eta' \rightarrow 4\pi^0$ and $\eta \rightarrow 4\pi^0$ even require all π^0 to be at least in relative D waves [4]. This, combined with the tiny phase space available, leads to the notion of $\eta \rightarrow 4\pi^0$ being CP forbidden [2,5,6], although strictly speaking it is only S-wave CP forbidden.

The outline of the article is as follows. We begin by discussing the two decay channels with charged pions in the final state, $\eta' \rightarrow 2(\pi^+\pi^-)$ and $\eta' \rightarrow \pi^+\pi^-2\pi^0$, in Sec. II. There, we calculate the corresponding decay amplitudes at leading nonvanishing order in the chiral expansion, saturate the appearing low-energy constants by vector-meson contributions, and calculate the corresponding branching ratios. In Sec. III, we then construct a CP -conserving (D-wave) decay mechanism for η , $\eta' \rightarrow 4\pi^0$ and determine the resulting branching fractions, before discussing the CP -violating (S-wave) η , $\eta' \rightarrow 4\pi^0$ decay as induced by the QCD θ -term in Sec. IV. Finally, we summarize and conclude. The Appendices contain technical details on four-particle phase space integration as well as on a (suppressed) tensor-meson mechanism for η , $\eta' \rightarrow 4\pi^0$.

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II. $\eta' \rightarrow 2(\pi^+ \pi^-)$ AND $\eta' \rightarrow \pi^+ \pi^- 2\pi^0$

A. Chiral perturbation theory

We wish to calculate the leading (nontrivial) chiral contribution to the anomalous decays

$$\begin{aligned} \eta' &\rightarrow \pi^+(p_1)\pi^-(p_2)\pi^+(p_3)\pi^-(p_4), \\ \eta' &\rightarrow \pi^+(p_1)\pi^0(p_2)\pi^-(p_3)\pi^0(p_4). \end{aligned} \quad (1)$$

The amplitudes can be written in terms of the invariant variables $s_{ij} = (p_i + p_j)^2$, $i, j = 1, \dots, 4$, which are subject to the constraint

$$s_{12} + s_{13} + s_{14} + s_{23} + s_{24} + s_{34} = M_{\eta'}^2 + 8M_{\pi}^2 \quad (2)$$

(in the isospin limit of equal pion masses). The five-meson vertices of the WZW term can be deduced from the Lagrangian

$$\mathcal{L}_{p^5}^{\text{WZW}} = \frac{N_c \epsilon_{\mu\nu\alpha\beta}}{240\pi^2 F_{\pi}^5} \langle \varphi \partial^{\mu} \varphi \partial^{\nu} \varphi \partial^{\alpha} \varphi \partial^{\beta} \varphi \rangle + \dots, \quad (3)$$

where N_c is the number of colors and will be taken to be 3 in this paper, $F_{\pi} = 92.2$ MeV is the pion decay constant, and $\langle \dots \rangle$ denotes the trace in flavor space. For simplicity, we refrain from spelling out the WZW term in its full, chirally invariant form. Furthermore,

$$\frac{\varphi}{\sqrt{2}} = \begin{pmatrix} \frac{\eta_0}{\sqrt{3}} + \frac{\eta_8}{\sqrt{6}} + \frac{\pi^0}{\sqrt{2}} & \pi^+ & K^+ \\ \pi^- & \frac{\eta_0}{\sqrt{3}} + \frac{\eta_8}{\sqrt{6}} - \frac{\pi^0}{\sqrt{2}} & K^0 \\ K^- & \bar{K}^0 & \frac{\eta_0}{\sqrt{3}} - \frac{2\eta_8}{\sqrt{6}} \end{pmatrix}. \quad (4)$$

We assume a simple, one-angle $\eta\eta'$ mixing scheme,

$$\begin{aligned} |\eta\rangle &= \cos\theta_P |\eta_8\rangle - \sin\theta_P |\eta_0\rangle, \\ |\eta'\rangle &= \sin\theta_P |\eta_8\rangle + \cos\theta_P |\eta_0\rangle, \end{aligned} \quad (5)$$

and use the standard mixing angle $\theta_P = \arcsin(-1/3) \approx -19.5^\circ$. As we are going to present what in some sense corresponds to a leading-order calculation of the decay amplitudes, we regard the more elaborate two-angle mixing schemes [7] as beyond the scope of this study; we

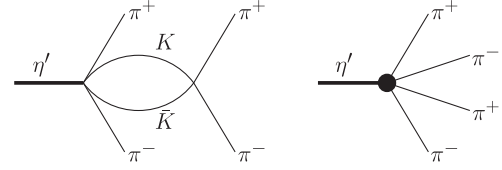


FIG. 1. Feynman diagrams contributing to $\eta' \rightarrow 2(\pi^+ \pi^-)$ (and similarly to $\eta' \rightarrow \pi^+ \pi^- 2\pi^0$) at $\mathcal{O}(p^6)$. The thick dot in the right diagram denotes a vertex from $\mathcal{L}_{\text{odd}}^{(6)}$.

expect the error made thereby to be covered by our generous final uncertainty estimate.

The flavor structure of Eq. (3) is such that there are no direct contributions to η , $\eta' \rightarrow 4\pi$, and the decay amplitudes vanish at leading order (in the anomalous sector) $\mathcal{O}(p^4)$. Nonvanishing contributions only occur at $\mathcal{O}(p^6)$, where the amplitudes are given by sums of (kaon) loops and counterterm contributions from the $\mathcal{O}(p^6)$ Lagrangian of odd intrinsic parity [8], see Fig. 1. Only two different structures ($\propto C_1^W, C_{12}^W$) remain when external currents are switched off. Ref. [8] only considers the Goldstone boson octet; we add terms $\propto \tilde{C}_1^W, \tilde{C}_{12}^W$ that only contribute for the singlet field η_0 :

$$\begin{aligned} \mathcal{L}_{\text{odd}}^{(6)} &= iC_1^W \epsilon_{\mu\nu\alpha\beta} \langle \chi_- u^\mu u^\nu u^\alpha u^\beta \rangle \\ &\quad - \frac{i\tilde{C}_1^W}{3} \epsilon_{\mu\nu\alpha\beta} \langle \chi_- \rangle \langle u^\mu u^\nu u^\alpha u^\beta \rangle \\ &\quad + C_{12}^W \epsilon_{\mu\nu\alpha\beta} \langle h^{\gamma\mu} [u_\gamma, u^\nu u^\alpha u^\beta] \rangle \\ &\quad - \frac{\tilde{C}_{12}^W}{3} \epsilon_{\mu\nu\alpha\beta} \langle h^{\gamma\mu} [u_\gamma, u^\nu u^\alpha] \rangle \langle u^\beta \rangle + \dots, \end{aligned} \quad (6)$$

with the usual chiral vielbein $u_\mu = i(u^\dagger \partial_\mu u - u \partial_\mu u^\dagger)$ (neglecting external currents), $u = \exp(i\varphi/2F_\pi)$, $h_{\mu\nu} = \nabla_\mu u_\nu + \nabla_\nu u_\mu$ with $\nabla_\mu X = \partial_\mu X + [\Gamma_\mu, X]$ and $\Gamma_\mu = \frac{1}{2}(u^\dagger \partial_\mu u + u \partial_\mu u^\dagger)$ (neglecting again external currents). Furthermore, we use $\chi_- = u^\dagger \chi u^\dagger - u \chi^\dagger u$, where $\chi = 2B \text{diag}(m_u, m_d, m_s) + \dots$ contains the quark mass matrix and B is related to the quark condensate according to $B = -\langle \bar{q}q \rangle / F_\pi^2$. The decay amplitudes at $\mathcal{O}(p^6)$ take the compact forms

$$\begin{aligned} \mathcal{A}(\eta_{0/8} \rightarrow \pi^+ \pi^- \pi^+ \pi^-) &= -\mathcal{A}(\eta_{0/8} \rightarrow \pi^+ \pi^0 \pi^- \pi^0) = \frac{N_c \epsilon_{\mu\nu\alpha\beta}}{3\sqrt{3}F_\pi^5} p_1^\mu p_2^\nu p_3^\alpha p_4^\beta [\mathcal{F}_{0/8}(s_{12}) + \mathcal{F}_{0/8}(s_{34}) \\ &\quad - \mathcal{F}_{0/8}(s_{14}) - \mathcal{F}_{0/8}(s_{23})], \\ \mathcal{F}_0(s) &= -16\sqrt{2}(C_{12}^{Wr}(\mu) - \tilde{C}_{12}^{Wr}(\mu))s, \\ \mathcal{F}_8(s) &= \frac{1}{8\pi^2 F_\pi^2} \left\{ (s - 4M_K^2) \bar{J}_{KK}(s) - \frac{s}{16\pi^2} \left(2 \log \frac{M_K}{\mu} + \frac{1}{3} \right) \right\} - 16C_{12}^{Wr}(\mu)s, \\ \bar{J}_{KK}(s) &= \frac{1}{8\pi^2} (1 - \sigma_K \text{arccot} \sigma_K), \\ \sigma_K &= \sqrt{\frac{4M_K^2}{s} - 1}, \end{aligned} \quad (7)$$

with the scale-dependent renormalized low-energy constants $C_{12}^{Wr}(\mu)$ and $\tilde{C}_{12}^{Wr}(\mu)$. There are no loop contributions to the η_0 amplitudes at this order, since at $\mathcal{O}(p^4)$ the anomalous five-pseudoscalar term (3) (the left vertex of the loop diagram in Fig. 1) contributes only to the octet case. Equation (7) is scale independent with the β function for $C_{12}^{Wr}(\mu)$ obtained in Ref. [8], if we demand \tilde{C}_{12}^{Wr} to have the same infinite part and scale dependence as C_{12}^{Wr} . A numerical estimate for the finite part $C_{12}^{Wr}(M_\rho)$ will be obtained by resonance saturation through vector-meson contributions.

B. Resonance saturation from hidden local symmetry

Resonance saturation for the $\mathcal{O}(p^6)$ chiral Lagrangian of odd intrinsic parity has been studied in great generality recently in Ref. [9]. Here, however, we opt for the simpler, but on the other hand more predictive hidden-local-symmetry scheme [10–13], which has the additional advantage of having been tested phenomenologically in great detail [14].

In the framework of hidden local symmetry (HLS), there are four additional terms involving vector-meson fields, with coefficients c_i ($i = 1, \dots, 4$), in addition to the WZW action for anomalous processes [10,12]; as already noted in Ref. [15], only three independent combinations of these contribute to low-energy amplitudes at $\mathcal{O}(p^6)$. HLS amplitudes for any given anomalous process contain two kinds of contributions: contact terms and resonance exchange terms. The contact terms have the same form as those derived from the WZW action, but with a modified coefficient (see below); the gauge-invariant construction of the HLS Lagrangian density guarantees that the additional, c_i -dependent contributions will be canceled by vector-meson exchange in the low-energy limit. In the following, we again for simplicity reasons refrain from properly defining all the HLS Lagrangian terms in their chirally invariant forms, but only quote the terms relevant for vertices of five pseudoscalars; the full Lagrangians can be retrieved e.g. from Refs. [12,13].

The contact terms for five-pseudoscalar vertices can be read off from the Lagrangian

$$\mathcal{L}_{P^5}^{\text{HLS}} = \frac{N_c \epsilon_{\mu\nu\alpha\beta}}{240\pi^2 F_\pi^5} \left[1 - \frac{15}{8}(c_1 - c_2) \right] \langle \varphi \partial^\mu \varphi \partial^\nu \varphi \partial^\alpha \varphi \partial^\beta \varphi \rangle. \quad (8)$$

The low-energy limit of the vector-meson-exchange contribution can be obtained by integrating out the heavy fields: substituting the leading-order equation of motion of the vector-meson fields

$$V_\mu = \frac{1}{8igF_\pi^2} [\partial_\mu \varphi, \varphi], \quad (9)$$

where g is the universal vector-meson coupling constant, into the HLS Lagrangians [13,14]

$$\begin{aligned} \mathcal{L}_{VVP} &= -\frac{N_c c_3 g^2}{8\pi^2 F_\pi} \epsilon_{\mu\nu\alpha\beta} \langle \partial^\mu V^\nu \partial^\alpha V^\beta \varphi \rangle, \\ \mathcal{L}_{VPPP} &= -\frac{iN_c (c_1 - c_2 - c_3) g}{32\pi^2 F_\pi^3} \epsilon_{\mu\nu\alpha\beta} \langle V^\mu \partial^\nu \varphi \partial^\alpha \varphi \partial^\beta \varphi \rangle, \end{aligned} \quad (10)$$

where the vector-meson nonet (with ideal mixing) is defined as

$$V_\mu = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\rho_\mu^0}{\sqrt{2}} + \frac{\omega_\mu}{\sqrt{2}} & \rho_\mu^+ & K_\mu^{*+} \\ \rho^- & -\frac{\rho_\mu^0}{\sqrt{2}} + \frac{\omega_\mu}{\sqrt{2}} & K_\mu^{*0} \\ K_\mu^{*-} & \bar{K}_\mu^{*0} & \phi_\mu \end{pmatrix}, \quad (11)$$

we find

$$\mathcal{L}_{P^5,V}^{(4)} = \frac{N_c (c_1 - c_2)}{128\pi^2 F_\pi^5} \epsilon_{\mu\nu\alpha\beta} \langle \varphi \partial^\mu \varphi \partial^\nu \varphi \partial^\alpha \varphi \partial^\beta \varphi \rangle, \quad (12)$$

which exactly cancels the second term inside the square brackets in Eq. (8).

If we extend the equation of motion Eq. (9) to next-to-leading order in the derivative expansion,

$$V_\mu = \frac{1}{8igF_\pi^2} \left(1 - \frac{\partial^2}{M_V^2} \right) [\partial_\mu \varphi, \varphi], \quad (13)$$

where M_V is the vector-meson mass, we can derive the vector-meson contribution to the five-meson vertices at $\mathcal{O}(p^6)$. Inserting Eq. (13) into Eq. (10), we find

$$\begin{aligned} \mathcal{L}_{P^5,V}^{(6)} &= \frac{N_c (c_1 - c_2 + c_3)}{128\pi^2 F_\pi^5 M_V^2} \epsilon_{\mu\nu\alpha\beta} \\ &\quad \times \langle \partial^\lambda \partial^\mu \varphi [\partial_\lambda \varphi, \partial^\nu \varphi \partial^\alpha \varphi \partial^\beta \varphi] \\ &\quad - 2\partial^2 \varphi \partial^\mu \varphi \partial^\nu \varphi \partial^\alpha \varphi \partial^\beta \varphi \rangle. \end{aligned} \quad (14)$$

The first term is exactly of the form of the Lagrangian term $\propto C_{12}^{Wr}$ in Eq. (6). For the second term, we use the equation of motion for the Goldstone bosons, which, neglecting higher orders in the fields, reads (compare e.g. Ref. [8])

$$\partial^2 \varphi = -\frac{1}{2} \langle \varphi, \chi \rangle + \frac{1}{3} \langle \varphi \chi \rangle, \quad (15)$$

so we also identify a vector-meson contribution to C_1^{Wr} and \tilde{C}_1^{Wr} . Our results read altogether

$$\begin{aligned} C_1^{Wr}(M_V) &= \tilde{C}_1^{Wr}(M_V) = -2C_{12}^{Wr}(M_V) \\ &= \frac{N_c (c_1 - c_2 + c_3)}{128\pi^2 M_V^2}, \end{aligned} \quad (16)$$

where we have indicated the conventional assumption of the resonance-saturation hypothesis to be valid roughly at the resonance scale, $\mu = M_V$ (which in the following we will identify with the mass of the ρ , $M_\rho = 775.5$ MeV). The numerical values of the HLS coupling constants are often taken to be given by $c_1 - c_2 \approx c_3 \approx 1$ [10], fairly consistent with more elaborate phenomenological fits that yield $c_1 - c_2 = 1.21$, $c_3 = 0.93$ [14].

In principle, this completes the task to provide the necessary input for an evaluation of the chiral representation of the decay amplitude, Eq. (7). We observe, however, the following. First, evaluating the slope of (the largely linear function) $\mathcal{F}_8(s)$ in Eq. (7) with this input (using $\bar{J}'_{KK}(0) = 1/(96\pi^2 M_K^2)$), we find

$$\begin{aligned} & 8\pi^2(4\pi F_\pi)^2 \times \mathcal{F}'_8(0) \\ &= 3(c_1 - c_2 + c_3) \frac{(4\pi F_\pi)^2}{2M_\rho^2} - \left(1 + 2 \log \frac{M_K}{M_\rho}\right). \end{aligned} \quad (17)$$

Numerically, the first term is about $6.7 \times (c_1 - c_2 + c_3)/2$, and the second is 0.1. Hence, at the scale $\mu = M_\rho$, the slope is entirely dominated by the vector-meson contribution, and the kaon loops are negligible.

Second, the maximal value for the kinematical invariants in $\eta' \rightarrow 4\pi$ allowed by phase space is $\sqrt{s_{ij}} \leq M_{\eta'} - 2M_\pi \approx 680$ MeV, therefore replacing the ρ propagator by its leading linear approximation is not phenomenologically reliable. Even deviations induced by the finite *width* of the ρ , $\Gamma_\rho = 149.1$ MeV, will be clearly visible. In the following, we will therefore use the full vector-meson-exchange amplitudes as derived from the HLS formalism, with the ρ -meson propagators including the width, which in addition is expected to be a very good estimate of the higher-order pairwise P-wave interaction of the pions in the final state (of course neglecting any crossed-channel effects). They are given by

$$\begin{aligned} \mathcal{A}_V(\eta_8 \rightarrow \pi^+ \pi^- \pi^+ \pi^-) &= \frac{1}{\sqrt{2}} \mathcal{A}_V(\eta_0 \rightarrow \pi^+ \pi^- \pi^+ \pi^-) = -\mathcal{A}_V(\eta_8 \rightarrow \pi^+ \pi^0 \pi^- \pi^0) = -\frac{1}{\sqrt{2}} \mathcal{A}_V(\eta_0 \rightarrow \pi^+ \pi^0 \pi^- \pi^0) \\ &= \frac{N_c \epsilon_{\mu\nu\alpha\beta}}{16\sqrt{3}\pi^2 F_\pi^5} p_1^\mu p_2^\nu p_3^\alpha p_4^\beta \left\{ (c_1 - c_2 - c_3) \left[\frac{M_\rho^2}{D_\rho(s_{12})} + \frac{M_\rho^2}{D_\rho(s_{34})} - \frac{M_\rho^2}{D_\rho(s_{14})} - \frac{M_\rho^2}{D_\rho(s_{23})} \right] \right. \\ &\quad \left. + 2c_3 \left[\frac{M_\rho^4}{D_\rho(s_{12})D_\rho(s_{34})} - \frac{M_\rho^4}{D_\rho(s_{14})D_\rho(s_{23})} \right] \right\} \end{aligned} \quad (18)$$

$$\begin{aligned} &\approx \frac{N_c \epsilon_{\mu\nu\alpha\beta}}{16\sqrt{3}\pi^2 F_\pi^5} p_1^\mu p_2^\nu p_3^\alpha p_4^\beta \left\{ (c_1 - c_2) \left[\frac{s_{12}}{D_\rho(s_{12})} + \frac{s_{34}}{D_\rho(s_{34})} - \frac{s_{14}}{D_\rho(s_{14})} - \frac{s_{23}}{D_\rho(s_{23})} \right] \right. \\ &\quad \left. + c_3 \left[\frac{M_\rho^2(s_{12} + s_{34})}{D_\rho(s_{12})D_\rho(s_{34})} - \frac{M_\rho^2(s_{14} + s_{23})}{D_\rho(s_{14})D_\rho(s_{23})} \right] \right\}, \end{aligned} \quad (19)$$

where

$$\begin{aligned} D_\rho(s) &= M_\rho^2 - s - iM_\rho\Gamma_\rho(s), \\ \Gamma_\rho(s) &= \frac{M_\rho}{\sqrt{s}} \left(\frac{s - 4M_\pi^2}{M_\rho^2 - 4M_\pi^2} \right)^{3/2} \Gamma_\rho \end{aligned} \quad (20)$$

is the inverse ρ propagator, and we have neglected the width term in the transformation from Eq. (18) to Eq. (19) in order to demonstrate the correct chiral dimension $\mathcal{O}(p^6)$ of the vector-meson contribution explicitly. Expanding the resonance propagators in Eq. (19) and comparing to Eq. (7) easily leads back to the coupling constant estimate for C_{12}^{Wr} found on the Lagrangian level in Eq. (16).

At this point, we can try to answer the introductory question on which parts of the WZW anomaly action—triangle, box, or pentagon—the decays $\eta' \rightarrow 2(\pi^+ \pi^-)$ and $\eta' \rightarrow \pi^+ \pi^- 2\pi^0$ yield information. As the pentagon anomaly only enters via the kaon-loop contributions, we have found above that its significance for the decays under investigation here is negligible; the vector-meson contributions are derived from the triangle and box-anomaly terms, see Eq. (10). As the phenomenological values of the HLS coupling constants suggest $c_1 - c_2 - c_3 \ll 2c_3$,

the box anomaly yields the lesser part of the two, and the decays are dominated by the triangle-anomaly term.

C. Branching ratios

We calculate the partial widths of the decays $\eta' \rightarrow \pi^+ \pi^- \pi^+ \pi^-$ and $\eta' \rightarrow \pi^+ \pi^0 \pi^- \pi^0$ using

$$\Gamma(\eta' \rightarrow 4\pi) = \frac{1}{2SM_{\eta'}} \int |\mathcal{A}(\eta' \rightarrow 4\pi)|^2 d\Phi_4, \quad (21)$$

where the evaluation of the four-particle phase space Φ_4 is discussed in detail in Appendix A. S is a symmetry factor— $S = 4$ for the $2(\pi^+ \pi^-)$ final state, and $S = 2$ for the $\pi^+ \pi^- 2\pi^0$ one. Note that with the relation $\mathcal{A}(\eta_0 \rightarrow 4\pi) = \sqrt{2}\mathcal{A}(\eta_8 \rightarrow 4\pi)$ and the standard mixing according to Eq. (5), we have $\mathcal{A}(\eta' \rightarrow 4\pi) = \mathcal{A}(\eta_8 \rightarrow 4\pi)$. To obtain branching ratios, we normalize the partial widths by the total width of the η' as quoted by the particle data group, $\Gamma_{\eta'} = (0.199 \pm 0.009)$ MeV [2]. Note that by using the most precise single measurement of this width alone, $\Gamma_{\eta'} = (0.226 \pm 0.017 \pm 0.014)$ MeV [16], our predictions for the branching fractions would be reduced by more than 10%. Given the observation of Eq. (17), we neglect the kaon-loop contributions altogether and evaluate the matrix

elements using Eq. (18). In order to account for trivial isospin-breaking effects due to phase space corrections, we calculate the branching ratio for $\eta' \rightarrow \pi^+ \pi^0 \pi^- \pi^0$ using an average pion mass $M_\pi = (M_{\pi^+} + M_{\pi^0})/2$, while we employ the charged pion mass for the decay into four charged pions. All results are first quoted as a function of the coupling constants $c_1 - c_2$ and c_3 , before inserting two sets of values: (i) $c_1 - c_2 = c_3 = 1$, and (ii) $c_1 - c_2 = 1.21$, $c_3 = 0.93$ [14]. We refrain from employing the errors given in the fits in Ref. [14]: the uncertainties in the HLS coupling constants are well below what we estimate to be the overall uncertainty of our prediction. The results are

$$\begin{aligned} \mathcal{B}(\eta' \rightarrow 2(\pi^+ \pi^-)) & \\ &= [0.15(c_1 - c_2)^2 + 0.47(c_1 - c_2)c_3 + 0.37c_3^2] \times 10^{-4} \\ &= \{1.0, 1.1\} \times 10^{-4}, \end{aligned} \quad (22)$$

$$\begin{aligned} \mathcal{B}(\eta' \rightarrow \pi^+ \pi^- 2\pi^0) & \\ &= [0.35(c_1 - c_2)^2 + 1.09(c_1 - c_2)c_3 + 0.87c_3^2] \times 10^{-4} \\ &= \{2.3, 2.5\} \times 10^{-4}. \end{aligned} \quad (23)$$

We therefore find that the uncertainties due to the HLS coupling constants are small. We wish to point out that although $\pi\pi$ P-wave dynamics are usually well approximated by the ρ resonance, and crossed-channel effects are expected to occur rather at the 10% level (as inferred from studies of decays such as $\omega \rightarrow 3\pi$, $\phi \rightarrow 3\pi$ [17]), the present study in some sense still amounts to a leading-order calculation: SU(3)-breaking effects of the order of $F_\eta/F_\pi \approx 1.3$ [18] may occur, and in the treatment of the η' (η_0), we have implicitly evoked the $1/N_c$ expansion. We therefore deem a generic uncertainty of 30% realistic, and quote our predictions accordingly as

$$\begin{aligned} \mathcal{B}(\eta' \rightarrow 2(\pi^+ \pi^-)) &= (1.0 \pm 0.3) \times 10^{-4}, \\ \mathcal{B}(\eta' \rightarrow \pi^+ \pi^- 2\pi^0) &= (2.4 \pm 0.7) \times 10^{-4}. \end{aligned} \quad (24)$$

These are to be compared to the current experimental upper limits [2,19]

$$\begin{aligned} \mathcal{B}_{\text{exp}}(\eta' \rightarrow 2(\pi^+ \pi^-)) &< 2.4 \times 10^{-4}, \\ \mathcal{B}_{\text{exp}}(\eta' \rightarrow \pi^+ \pi^- 2\pi^0) &< 2.6 \times 10^{-3}, \end{aligned} \quad (25)$$

hence signals of these decays ought to be within reach of modern high-statistics experiments soon.

III. $\eta, \eta' \rightarrow 4\pi^0$

As we have mentioned in the Introduction, the P-wave mechanism described in the previous section, proceeding

essentially via two ρ intermediate resonances, cannot contribute to the $4\pi^0$ final states. In fact, we can show that the D-wave characteristic of $\eta, \eta' \rightarrow 4\pi^0$ suppresses these decays to $\mathcal{O}(p^{10})$ in chiral power counting, that is to the level of three loops in the anomalous sector. This is, in particular, due to the flavor and isospin structure of the anomaly, which does not contain five-meson vertices including $2\pi^0$ at leading order ($\mathcal{O}(p^4)$), and to the chiral structure of meson-meson scattering amplitudes, which only allows for S and P waves at tree level ($\mathcal{O}(p^2)$). As a complete three-loop calculation would be a formidable task and is certainly beyond the scope of our exploratory study, we instead consider the decay mechanisms shown in Fig. 2. As shown in Appendix B, the contribution from two f_2 mesons is negligible in comparison to the pion loop. We therefore focus on the pion loop as shown in the left panel of Fig. 2. It represents a decay mechanism that, we believe, ought to capture at least the correct order of magnitude of the corresponding partial width.

A. Pion-loop contribution

Our decay mechanism for $\eta, \eta' \rightarrow 4\pi^0$ is built on the observation that there is a specific diagrammatic contribution that we can easily calculate, and that, in particular, comprises the complete leading contribution to the *imaginary part* of the decay amplitude. This is given by $\pi^+ \pi^-$ intermediate states, and hence harks back to the results of the previous section. As argued above, it appears at chiral $\mathcal{O}(p^{10})$: $\eta_{0/8} \rightarrow \pi^+ \pi^- 2\pi^0$ as calculated in Eq. (7) to $\mathcal{O}(p^6)$, followed by rescattering $\pi^+ \pi^- \rightarrow \pi^0 \pi^0$, where the S wave does not contribute, and D and higher partial waves start to appear at $\mathcal{O}(p^4)$ [20]; see Fig. 2 for illustration. We calculate this first in the following approximation: given the numerical dominance of the counterterm contribution in Eq. (7), the amplitudes $\mathcal{F}_{0/8}(s)$ are taken to be linear, $\mathcal{F}_{0/8}(s) \approx \mathcal{F}'_{0/8}(s)s$, neglecting tiny curvature effects from the kaon loops; and we approximate $\pi\pi$ rescattering by a phenomenological D wave, thus improving on the leading chiral representation, but neglecting G and higher partial waves. We find

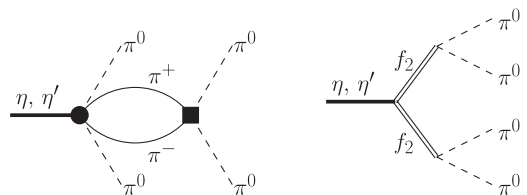


FIG. 2. Left: Pion-loop contribution to $\eta, \eta' \rightarrow 4\pi^0$. The black circle denotes an effective local $\eta, \eta' \rightarrow \pi^+ \pi^- 2\pi^0$ coupling at $\mathcal{O}(p^6)$, the black square an effective local D-wave $\pi\pi$ scattering vertex at $\mathcal{O}(p^4)$. Right: $\eta, \eta' \rightarrow 4\pi^0$ through two intermediate f_2 mesons.

$$\begin{aligned}
\mathcal{A}(\eta_8 \rightarrow 4\pi^0) &= \frac{1}{\sqrt{2}} \mathcal{A}(\eta_0 \rightarrow 4\pi^0) = -\frac{N_c(c_1 - c_2 + c_3)}{8\pi} \frac{\epsilon_{\mu\nu\alpha\beta}}{\sqrt{3}F_\pi^5} p_1^\mu p_2^\nu p_3^\alpha p_4^\beta \left\{ \mathcal{G}(s_{12}, s_{23}, s_{14}, s_{34}; s_{13}) + \mathcal{G}(s_{12}, s_{14}, s_{23}, s_{34}; s_{24}) \right. \\
&\quad \left. - \mathcal{G}(s_{13}, s_{23}, s_{14}, s_{24}; s_{12}) - \mathcal{G}(s_{13}, s_{14}, s_{23}, s_{24}; s_{34}) - \mathcal{G}(s_{12}, s_{24}, s_{13}, s_{34}; s_{14}) - \mathcal{G}(s_{12}, s_{13}, s_{24}, s_{34}; s_{23}) \right\}, \\
\mathcal{G}(v, w, x, y; s) &= \frac{v - w - x + y}{M_\rho^2} \frac{16(t_2^0(s) - t_2^2(s))}{3(s - 4M_\pi^2)^2} \left\{ (s - 4M_\pi^2)^2 \bar{J}_{\pi\pi}(s) - 2(s^2 - 10sM_\pi^2 + 30M_\pi^4) \left(L + \frac{1}{16\pi^2} \log \frac{M_\pi}{\mu} \right) \right. \\
&\quad \left. + \frac{1}{16\pi^2} \left(\frac{s^2}{15} - \frac{8}{3}sM_\pi^2 + 15M_\pi^4 \right) \right\}, \\
\bar{J}_{\pi\pi}(s) &= \frac{1}{8\pi^2} \left[1 - \frac{\sigma}{2} \left(\log \frac{1+\sigma}{1-\sigma} - i\pi \right) \right], \quad \sigma = \sqrt{1 - \frac{4M_\pi^2}{s}}, \quad L = \frac{\mu^{d-4}}{16\pi^2} \left\{ \frac{1}{d-4} + \frac{1}{2}(\gamma_E - 1 - \log 4\pi) \right\}. \quad (26)
\end{aligned}$$

$t_2^\ell(s)$ is the partial wave of angular momentum $\ell = 2$ for the appropriate isospin quantum number I ; the expression $16t_2^\ell(s)(s - 4M_\pi^2)^{-2} = a_2^\ell + \mathcal{O}(s - 4M_\pi^2)$, with the D-wave scattering length a_2^ℓ , is therefore finite at threshold. Note furthermore that $t_2^\ell = \mathcal{O}(p^4)$ in chiral counting, such that the chiral order of Eq. (26) is indeed $\mathcal{O}(p^{10})$. L contains the infinite part of the divergent loop diagram in the usual way, using dimensional regularization. Of course, this individual loop contribution is both divergent and scale dependent: only the imaginary part is complete (to this order) and in that sense well-defined and finite. We display the full expression here as we will use the scale dependence as a rough independent consistency check below.

Without the knowledge of counterterms of an order as high as $\mathcal{O}(p^{10})$, one cannot make a quantitative prediction using the loop amplitude derived in the above. Hence, we have to resort to a certain phenomenological representation. The imaginary part of Eq. (26), which is complete at $\mathcal{O}(p^{10})$ as mentioned, is used to establish a connection to a one- f_2 exchange in the s channel. Note that the $f_2(1270)$ exchange dominates the available $\pi\pi$ scattering phase shifts in the $I = 0$, $\ell = 2$ channel, see e.g. Ref. [21].

We will proceed to estimate the full D-wave $\pi\pi$ rescattering contribution as follows. Neglecting again any crossed-channel effects, rescattering of two pions can be summed by the Omnès factor,

$$\Omega_\ell^I(s) = \exp \left\{ \frac{s}{\pi} \int_{4M_\pi^2}^{\infty} \frac{\delta_\ell^I(z) dz}{z(z - s - i\epsilon)} \right\}, \quad (27)$$

where δ_ℓ^I is the $\pi\pi$ scattering phase shifts in the channel with isospin I and angular momentum ℓ . Near threshold, its imaginary part can be approximated as

$$\text{Im} \Omega_\ell^I(s) \approx \delta_\ell^I(s) \{1 + \mathcal{O}(\sigma^2)\} \approx \sigma t_\ell^I(s) \{1 + \mathcal{O}(\sigma^2)\} \quad (28)$$

(neglecting the shift from unity in $\Omega(4M_\pi^2)$, which is justified in the D wave for our intended accuracy), while in the approximation of a phase being dominated by a narrow resonance of mass M and width Γ , the Omnès factor is given by

$$\begin{aligned}
\Omega_\ell^I(s) &\approx \frac{M^2 \exp(i\delta_\ell^I(s))}{\sqrt{(M^2 - s)^2 + M^2\Gamma^2(s)}}, \\
\Gamma(s) &= \frac{M}{\sqrt{s}} \left(\frac{s - 4M_\pi^2}{M^2 - 4M_\pi^2} \right)^{\ell+1/2} \Gamma. \quad (29)
\end{aligned}$$

Despite D-wave scattering near threshold not being dominated by the $f_2(1270)$,¹ we still use Eq. (28) to invoke the f_2 . This is because at somewhat higher energies, the $I = 0$ $\pi\pi$ D wave dominates over the $I = 2$ component and will be well approximated by the $f_2(1270)$ resonance. One may wonder whether, in particular, for $\eta \rightarrow 4\pi^0$, which stays close to $\pi\pi$ threshold throughout the allowed phase space, this approximation may not lead to sizeable errors. We have checked for the numerical results for the branching fraction discussed below that, employing the full Omnès function according to Eq. (27) with the phase parameterization provided in Ref. [23], the branching ratio changes by about 10%, well below the accuracy we can aim for here. On the other hand, within the $\eta' \rightarrow 4\pi^0$ decay, we stay sufficiently far below the resonance energy that the phase of the D wave can still be neglected. With the correspondence between Eqs. (28) and (29), we conclude that the $f_2(1270)$ contribution to the amplitude can be estimated as

¹It is dominated by the low-energy constant $\bar{\ell}_2$ from the $\mathcal{O}(p^4)$ Lagrangian [20], or by t -channel vector-meson exchange in the spirit of resonance saturation [22].

$$\begin{aligned} \mathcal{A}_{f_2}(\eta_8 \rightarrow 4\pi^0) &= \frac{1}{\sqrt{2}} \mathcal{A}_{f_2}(\eta_0 \rightarrow 4\pi^0) = -\frac{N_c(c_1 - c_2 + c_3)}{24\pi^2} \frac{\epsilon_{\mu\nu\alpha\beta}}{\sqrt{3}F_\pi^5} p_1^\mu p_2^\nu p_3^\alpha p_4^\beta \{ \mathcal{G}_{f_2}(s_{12}, s_{23}, s_{14}, s_{34}; s_{13}, s_{24}) \\ &\quad - \mathcal{G}_{f_2}(s_{13}, s_{23}, s_{14}, s_{24}; s_{12}, s_{34}) - \mathcal{G}_{f_2}(s_{12}, s_{24}, s_{13}, s_{34}; s_{14}, s_{23}) \}, \\ \mathcal{G}_{f_2}(v, w, x, y; s, t) &= \frac{v - w - x + y}{M_\rho^2} \left[\frac{M_{f_2}^2}{M_{f_2}^2 - s} + \frac{M_{f_2}^2}{M_{f_2}^2 - t} \right], \end{aligned} \quad (30)$$

neglecting for simplicity the width of the f_2 , which is justified in the kinematic regime accessible in $\eta' \rightarrow 4\pi^0$. Note that, due to the special symmetry of the amplitude, Eq. (30) can be rewritten identically by employing a “twice-subtracted” version of the resonance term, i.e. replacing $\mathcal{G}_{f_2} \rightarrow \mathcal{G}_{f_2}''$,

$$\mathcal{G}_{f_2}''(v, w, x, y; s, t) = \frac{v - w - x + y}{M_\rho^2 M_{f_2}^2} \left[\frac{s^2}{M_{f_2}^2 - s} + \frac{t^2}{M_{f_2}^2 - t} \right], \quad (31)$$

which makes the correct chiral dimension of the resonance contribution manifest.

As a rough final consistency check, we compare the order of magnitude of a chiral counterterm induced by the f_2 exchange, see Eq. (31) in the low-energy limit $s, t \ll M_{f_2}^2$, with the scale running of such a counterterm as necessitated by the $\log\mu$ dependence in Eq. (26). If we only retain the scattering lengths in the D-wave partial waves, the relevant part to be compared to Eq. (31) (that does not cancel in the full amplitude) is

$$\begin{aligned} \mu \frac{d}{d\mu} [\mathcal{G}(v, w, x, y; s) + \mathcal{G}(v, w, x, y; t)] \\ = \frac{v - w - x + y}{3M_\rho^2} (a_2^0 - a_2^2) \frac{s^2 + t^2}{8\pi^2}. \end{aligned} \quad (32)$$

Comparing the numerical prefactors, we find that the scale dependence is suppressed versus the estimate for the finite counterterm by

$$\frac{a_2^0 - a_2^2}{16\pi} \times M_{f_2}^4 \approx 0.22, \quad (33)$$

where we have used $a_2^0 = 1.75 \times 10^{-3} M_\pi^{-4}$, $a_2^2 = 0.17 \times 10^{-3} M_\pi^{-4}$ [24]. In other words, the scale dependence suggests the order of magnitude of our counterterm estimate using f_2 saturation to be reasonable.

B. Pion-loop contribution improved: including vector propagators

We have seen in Sec. II on the P-wave dominated, (partially) charged four-pion final states that the leading approximation in an expansion of the ρ meson propagators is not a sufficient description of these decays, given the available phase space in η' decays. With the $\eta_{0/8} \rightarrow \pi^+ \pi^- 2\pi^0$ transitions entering the decay mechanism for $\eta_{0/8} \rightarrow 4\pi^0$ as described in the previous section, this deficit would be fully inherited in our estimate of the all-neutral final states. In fact, the imaginary part of the corresponding diagram including the full ρ propagators, see Fig. 3, can even be calculated exactly, using Cutkosky rules; however, the resulting expressions are extremely involved and not very illuminating. It turns out, though, that the main effects of the not-so-large vector-meson mass can be approximated by the following expression for the imaginary part:

$$\begin{aligned} \text{Im} \mathcal{A}(\eta_8 \rightarrow 4\pi^0) &= \frac{1}{\sqrt{2}} \text{Im} \mathcal{A}(\eta_0 \rightarrow 4\pi^0) = -\frac{N_c}{8\pi} \frac{\epsilon_{\mu\nu\alpha\beta}}{\sqrt{3}F_\pi^5} p_1^\mu p_2^\nu p_3^\alpha p_4^\beta \{ (c_1 - c_2 - c_3) [\text{Im} \mathcal{G}_1^\rho(s_{12}, s_{23}, s_{14}, s_{34}; s_{13}) \\ &\quad + \text{Im} \mathcal{G}_1^\rho(s_{12}, s_{14}, s_{23}, s_{34}; s_{24}) - \text{Im} \mathcal{G}_1^\rho(s_{13}, s_{23}, s_{14}, s_{24}; s_{12}) - \text{Im} \mathcal{G}_1^\rho(s_{13}, s_{14}, s_{23}, s_{24}; s_{34}) \\ &\quad - \text{Im} \mathcal{G}_1^\rho(s_{12}, s_{24}, s_{13}, s_{34}; s_{14}) - \text{Im} \mathcal{G}_1^\rho(s_{12}, s_{13}, s_{24}, s_{34}; s_{23})] + 2c_3 [\text{Im} \mathcal{G}_2^\rho(s_{12}, s_{23}, s_{14}, s_{34}; s_{13}) \\ &\quad + \text{Im} \mathcal{G}_2^\rho(s_{12}, s_{14}, s_{23}, s_{34}; s_{24}) - \text{Im} \mathcal{G}_2^\rho(s_{13}, s_{23}, s_{14}, s_{24}; s_{12}) \\ &\quad - \text{Im} \mathcal{G}_2^\rho(s_{13}, s_{14}, s_{23}, s_{24}; s_{34}) - \text{Im} \mathcal{G}_2^\rho(s_{12}, s_{24}, s_{13}, s_{34}; s_{14}) - \text{Im} \mathcal{G}_2^\rho(s_{12}, s_{13}, s_{24}, s_{34}; s_{23})] \}, \\ \text{Im} \mathcal{G}_1^\rho(v, w, x, y; s) &= \left[\frac{M_\rho^2(v - w)}{(M_\rho^2 - \frac{1}{2}(v + w))^2} - \frac{M_\rho^2(x - y)}{(M_\rho^2 - \frac{1}{2}(x + y))^2} \right] (t_2^0(s) - t_2^2(s)) \frac{\sigma}{3\pi} + \mathcal{O}(\sigma^7), \\ \text{Im} \mathcal{G}_2^\rho(v, w, x, y; s) &= \frac{M_\rho^4(M_\rho^2(v - w - x + y) - vx + wx)}{(M_\rho^2 - \frac{1}{2}(v + w))^2 (M_\rho^2 - \frac{1}{2}(x + y))^2} (t_2^0(s) - t_2^2(s)) \frac{\sigma}{3\pi} + \mathcal{O}(\sigma^7). \end{aligned} \quad (34)$$

We find, furthermore, that the neglected terms indicated as $\mathcal{O}(\sigma^7)$ are also suppressed in inverse powers of M_ρ , starting at $\mathcal{O}(M_\rho^{-6})$ compared to the leading terms of $\mathcal{O}(M_\rho^{-2})$ in the above. Numerically, the indicated higher-order corrections in σ^2 are found to be small, less than about 10% all over phase space. However, the corrections by the remnants of the ρ propagators are large compared to the limit $M_\rho \rightarrow \infty$, given the available phase space and the high power of these

propagators in the denominator. Using the same trick as in the previous section to transform the imaginary part into an estimate for the whole (resonance-dominated) partial wave via the Omnès function, we arrive at

$$\begin{aligned} \mathcal{A}(\eta_8 \rightarrow 4\pi^0) &= \frac{1}{\sqrt{2}} \mathcal{A}(\eta_0 \rightarrow 4\pi^0) = -\frac{N_c}{24\pi^2} \frac{\epsilon_{\mu\nu\alpha\beta}}{\sqrt{3}F_\pi^5} p_1^\mu p_2^\nu p_3^\alpha p_4^\beta \{(c_1 - c_2 - c_3)[\mathcal{G}_{f_2,1}^\rho(s_{12}, s_{23}, s_{14}, s_{34}; s_{13}) \\ &+ \mathcal{G}_{f_2,1}^\rho(s_{12}, s_{14}, s_{23}, s_{34}; s_{24}) - \mathcal{G}_{f_2,1}^\rho(s_{13}, s_{23}, s_{14}, s_{24}; s_{12}) - \mathcal{G}_{f_2,1}^\rho(s_{13}, s_{14}, s_{23}, s_{24}; s_{34}) \\ &- \mathcal{G}_{f_2,1}^\rho(s_{12}, s_{24}, s_{13}, s_{34}; s_{14}) - \mathcal{G}_{f_2,1}^\rho(s_{12}, s_{13}, s_{24}, s_{34}; s_{23})] + 2c_3[\mathcal{G}_{f_2,2}^\rho(s_{12}, s_{23}, s_{14}, s_{34}; s_{13}) \\ &+ \mathcal{G}_{f_2,2}^\rho(s_{12}, s_{14}, s_{23}, s_{34}; s_{24}) - \mathcal{G}_{f_2,2}^\rho(s_{13}, s_{23}, s_{14}, s_{24}; s_{12}) - \mathcal{G}_{f_2,2}^\rho(s_{13}, s_{14}, s_{23}, s_{24}; s_{34}) \\ &- \mathcal{G}_{f_2,2}^\rho(s_{12}, s_{24}, s_{13}, s_{34}; s_{14}) - \mathcal{G}_{f_2,2}^\rho(s_{12}, s_{13}, s_{24}, s_{34}; s_{23})]\}, \\ \mathcal{G}_{f_2,1}^\rho(v, w, x, y; s) &= \left[\frac{M_\rho^2(v-w)}{(M_\rho^2 - \frac{1}{2}(v+w))^2} - \frac{M_\rho^2(x-y)}{(M_\rho^2 - \frac{1}{2}(x+y))^2} \right] \frac{M_{f_2}^2}{M_{f_2}^2 - s}, \\ \mathcal{G}_{f_2,2}^\rho(v, w, x, y; s) &= \frac{M_\rho^4(M_\rho^2(v-w-x+y) - vy + wx)}{(M_\rho^2 - \frac{1}{2}(v+w))^2(M_\rho^2 - \frac{1}{2}(x+y))^2} \frac{M_{f_2}^2}{M_{f_2}^2 - s}. \end{aligned} \quad (35)$$

Note that this result is far from the one- f_2 dominance estimate, with a f_2 coupling constant $\propto M_\rho^{-2}$ as the previous section suggested. Expanding Eq. (35) simultaneously around the limits $M_\rho \rightarrow \infty$, $M_{f_2} \rightarrow \infty$, the leading term (corresponding to chiral dimension $\mathcal{O}(p^{10})$) is not dominated by terms of $\mathcal{O}(M_\rho^{-2}M_{f_2}^{-4})$, but also contains other terms of $\mathcal{O}(M_\rho^{-4}M_{f_2}^{-2})$ and $\mathcal{O}(M_\rho^{-6})$. In other words, Eq. (30) is numerically no reasonable approximation to Eq. (35) even for the decay $\eta \rightarrow 4\pi^0$, with its tiny phase space available.

C. Branching ratios

We calculate the partial width using Eq. (21) with the symmetry factor $S = 4!$. Note again that $\mathcal{A}(\eta' \rightarrow 4\pi^0) = \mathcal{A}(\eta_8 \rightarrow 4\pi^0)$, assuming standard mixing. We employ the amplitude as given in Eq. (35) as our “best guess” for an estimate of the branching fraction. With the same numerical input as in Sec. II C (except using the *neutral* pion mass everywhere), we find

$$\begin{aligned} \mathcal{B}(\eta' \rightarrow 4\pi^0) &= [0.4(c_1 - c_2)^2 + 1.6(c_1 - c_2)c_3 + 1.7c_3^2] \times 10^{-8} \\ &= \{3.7, 3.9\} \times 10^{-8}, \end{aligned} \quad (36)$$

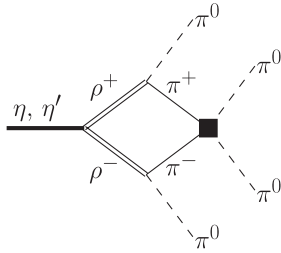


FIG. 3. Pion-loop contribution to $\eta, \eta' \rightarrow 4\pi^0$ via ρ^\pm intermediate states; see the vector-meson dominated amplitude discussed in Sec. II. The black square denotes an effective local D-wave $\pi\pi$ scattering vertex at $\mathcal{O}(p^4)$.

for the two sets of coupling constants c_i . Note that the use of the amplitude (30) leads to a branching fraction of the order of 4×10^{-11} , i.e. almost 3 orders of magnitude smaller.

We can trivially also calculate the branching fraction for $\eta \rightarrow 4\pi^0$, the only $\eta \rightarrow 4\pi$ decays that is kinematically allowed. We again employ the amplitude (35), and note that mixing according to Eq. (5) suggests $\mathcal{A}(\eta \rightarrow 4\pi^0) = \sqrt{2}\mathcal{A}(\eta_8 \rightarrow 4\pi^0)$. Normalized to the total width of the η , $\Gamma_\eta = (1.30 \pm 0.07)$ keV [2], we find

$$\begin{aligned} \mathcal{B}(\eta \rightarrow 4\pi^0) &= [0.4(c_1 - c_2)^2 + 1.1(c_1 - c_2)c_3 + 1.0c_3^2] \times 10^{-30} \\ &= \{2.4, 2.6\} \times 10^{-30}, \end{aligned} \quad (37)$$

in other words, the D-wave characteristic of the decay combined with tiny phase space leads to an enormous suppression of the CP -allowed $\eta \rightarrow 4\pi^0$ decay. We again compare these estimates to the available experimental upper limits [6,25],

$$\begin{aligned} \mathcal{B}_{\text{exp}}(\eta' \rightarrow 4\pi^0) &< 5 \times 10^{-4}, \\ \mathcal{B}_{\text{exp}}(\eta \rightarrow 4\pi^0) &< 6.9 \times 10^{-7}; \end{aligned} \quad (38)$$

further improvements of these experimental upper limits are planned (see e.g. Ref. [26] for $\eta \rightarrow 4\pi^0$). In this case, our predictions are smaller than those by several orders of magnitude.

The uncertainties of Eqs. (36) and (37) are hard to assess. The generic $SU(3)$ and $1/N_c$ error of about 30% assumed in Sec. II C is probably too small, as here, we do not even have a complete leading-order calculation at our disposal. We therefore rather assume these numbers to be the correct orders of magnitude, without quantifying the uncertainty of the prediction any further.

IV. CP -VIOLATING $\eta, \eta' \rightarrow 4\pi^0$ DECAYS

Given the smallness of the branching fractions predicted for $\eta' \rightarrow 4\pi^0$, $\eta \rightarrow 4\pi^0$ via a D-wave dominated, CP -conserving decay mechanism in the previous section, it is desirable to compare these numbers with possible CP -violating contributions that may, on the other hand, avoid the huge angular-momentum suppression. One such CP -violating mechanism that is expected to affect strong-interaction processes is induced by the so-called θ -term, an additional term in the QCD Lagrangian necessitated for the solution of the $U(1)_A$ problem. The θ -term violates P and CP symmetry and may induce observable symmetry-violating effects, in particular, in flavor-conserving processes. Its effective-Lagrangian treatment includes a term that can be rewritten as (see Ref. [27] and references therein)

$$\mathcal{L}_\theta = i\bar{\theta}_0 \frac{F_\pi^2 M_{\eta_0}^2}{12} \left\{ \langle U - U^\dagger \rangle - \log \left(\frac{\det U}{\det U^\dagger} \right) \right\},$$

$$U = u^2 = \exp \left(\frac{i\varphi}{F_\pi} \right), \quad (39)$$

which, in addition to the well-known $\eta \rightarrow 2\pi$ amplitude [27,28], also induces a CP -violating $\eta \rightarrow 4\pi$ amplitude,

$$\begin{aligned} \mathcal{A}_{CP}(\eta_8 \rightarrow 4\pi^0) &= \frac{1}{\sqrt{2}} \mathcal{A}_{CP}(\eta_0 \rightarrow 4\pi^0) \\ &= \mathcal{A}_{CP}(\eta' \rightarrow 4\pi^0) = \frac{1}{\sqrt{2}} \mathcal{A}_{CP}(\eta \rightarrow 4\pi^0) \\ &= -\frac{M_{\eta_0}^2 \bar{\theta}_0}{3\sqrt{3}F_\pi^3}. \end{aligned} \quad (40)$$

We will use $M_{\eta_0} \approx M_{\eta'}$ for numerical evaluation. The fact that this amplitude is a constant makes the phase space integration almost trivial, with the results for the branching fractions

$$\begin{aligned} \mathcal{B}(\eta \xrightarrow{CPV} 4\pi^0) &= 5 \times 10^{-5} \times \bar{\theta}_0^2, \\ \mathcal{B}(\eta' \xrightarrow{CPV} 4\pi^0) &= 9 \times 10^{-2} \times \bar{\theta}_0^2. \end{aligned} \quad (41)$$

We remark that we do not consider the branching ratio estimate for $\eta' \rightarrow 4\pi^0$ in Eq. (41) reliable in any sense: given the available phase space and the possibility of strong S-wave $\pi\pi$ final-state interactions, it could easily be enhanced by an order of magnitude. Were $\bar{\theta}_0$ a quantity of natural size, Eq. (41) would demonstrate the enhancement of the CP -violating S-wave mechanism compared to the CP -conserving D-wave one, see Eqs. (36) and (37). With current limits on the QCD vacuum angle derived from neutron electric dipole moment measurements, $\bar{\theta}_0 \leq 10^{-11}$ [29], these branching fractions are already bound beyond anything measurable; however, we note that for $\eta \rightarrow 4\pi^0$, the suppression of the CP -conserving D-wave mechanism, see Eq. (37), is so strong that it is even smaller than the CP -violating (S-wave) one in Eq. (41) if the current bounds are inserted for $\bar{\theta}_0$.

V. SUMMARY AND CONCLUSIONS

In this article, we have calculated the branching fractions of the η and η' decays into four pions. These processes of odd intrinsic parity are anomalous, and—as long as CP symmetry is assumed to be conserved—prohibit the pions to be in relative S-waves. We organize the amplitudes according to chiral power-counting rules, and find the leading contributions to the η' decay amplitudes with charged pions in the final state at $\mathcal{O}(p^6)$. Utilizing the framework of hidden local symmetry for vector mesons, we assume that vector-meson exchange saturates the $\mathcal{O}(p^6)$ low-energy constants, and find that the (P-wave) decay amplitude is entirely governed by ρ intermediate states. The dominant contribution is hence given by the triangle anomaly via $\eta' \rightarrow \rho\rho$ (with numerically subleading box terms), not by the pentagon anomaly. In this way, the branching fractions for $\eta' \rightarrow 2(\pi^+\pi^-)$ and $\eta' \rightarrow \pi^+\pi^-2\pi^0$ are predicted to be

$$\begin{aligned} \mathcal{B}(\eta' \rightarrow 2(\pi^+\pi^-)) &= (1.0 \pm 0.3) \times 10^{-4}, \\ \mathcal{B}(\eta' \rightarrow \pi^+\pi^-2\pi^0) &= (2.4 \pm 0.7) \times 10^{-4}, \end{aligned} \quad (42)$$

respectively. The former is only a factor of 2 smaller than the current experimental upper limit, so should be testable in the near future with the modern high-statistics facilities.

Predictions for the decays into four neutral pions are much more difficult, as Bose symmetry requires them to emerge in relative D waves (assuming CP conservation), suppressing the amplitudes to $\mathcal{O}(p^{10})$ in chiral power counting. We here do not even obtain the full leading-order amplitudes, as these would require a three-loop calculation. We estimate the decay via a charged-pion-loop contribution with D-wave pion-pion charge-exchange rescattering; an alternative mechanism through two f_2 mesons is found to be completely negligible in comparison, based on an estimate of the tensor-tensor-pseudoscalar coupling constant in the framework of QCD sum rules. Because of these phenomenological approximations, the CP -conserving branching ratios thus obtained,

$$\begin{aligned} \mathcal{B}(\eta' \rightarrow 4\pi^0) &\sim 4 \times 10^{-8}, \\ \mathcal{B}(\eta \rightarrow 4\pi^0) &\sim 3 \times 10^{-30}, \end{aligned} \quad (43)$$

should only be taken as order-of-magnitude estimates. It thus turns out that the CP -conserving decay width of $\eta \rightarrow 4\pi^0$ is so small that any signal to be observed would indicate CP -violating physics. For the latter, we calculate one specific example using the QCD θ -term.

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APPENDIX A: FOUR-BODY PHASE SPACE INTEGRATION

The n -body phase space is defined as

$$d\Phi_n(P; p_1, \dots, p_n) \equiv (2\pi)^4 \delta^4\left(P - \sum_{i=1}^n p_i\right) \prod_{i=1}^n \frac{d^3 p_i}{(2\pi)^3 2p_i^0}. \quad (\text{A1})$$

Using the recursive relation [2]

$$d\Phi_n(P; p_1, \dots, p_n) = d\Phi_j(q; p_1, \dots, p_j) \times d\Phi_{n-j+1}(P; q, p_{j+1}, \dots, p_n) \frac{dq^2}{2\pi}, \quad (\text{A2})$$

we have

$$\begin{aligned} d\Phi_4(P; p_1, \dots, p_4) &= d\Phi_2(q; p_1, p_2) d\Phi_2(k; p_3, p_4) d\Phi_2(P; q, k) \frac{dq^2}{2\pi} \frac{dk^2}{2\pi} \\ &= \frac{1}{(8\pi^2)^4 M} \int_{m_1+m_2}^{M-m_3-m_4} d\sqrt{s_{12}} \int_{m_3+m_4}^{M-\sqrt{s_{12}}} d\sqrt{s_{34}} \\ &\quad \times \int d\Omega_1^* d\Omega_3' d\Omega |p_1^*| |p_3'| |q|, \end{aligned} \quad (\text{A3})$$

where m_i , $i = 1 \dots 4$ are the masses associated with the final-state particles of momentum p_i , M is the mass of the decaying particle, $s_{12} = q^2$, $s_{34} = k^2$. $d\Omega_1^* = d\varphi_1^* d\cos\theta_1^*$ is the solid angle of particle 1 in the center-of-mass frame (cmf) of particles 1 and 2, $d\Omega_3'$ is the solid angle of particle 3 in the cmf of 3 and 4, and $d\Omega$ is the solid angle of the 1, 2 system in the rest frame of the decaying particle. The three-momenta are given by

$$|p_1^*| = \frac{\lambda^{1/2}(s_{12}, m_1^2, m_2^2)}{2\sqrt{s_{12}}}, \quad |p_3'| = \frac{\lambda^{1/2}(s_{34}, m_3^2, m_4^2)}{2\sqrt{s_{34}}}, \\ |q| = \frac{\lambda^{1/2}(M^2, s_{12}, s_{34})}{2M}, \quad (\text{A4})$$

with the usual Källén function $\lambda(x, y, z) \equiv x^2 + y^2 + z^2 - 2(xy + xz + yz)$.

Denoting quantities in the cmf of 1 and 2 (3 and 4) by $*$ ($'$), one can relate them with those in the rest frame of the decay particle by Lorentz transformation. Explicitly,

$$\begin{aligned} p_1^\mu &= \{\gamma_{12}(p_1^{0*} + \boldsymbol{\beta}_{12} \cdot \mathbf{p}_1^*), \gamma_{12}(\boldsymbol{\beta}_{12} p_1^{0*} + p_{1\parallel}^*), p_{1\perp}^*\}, \\ p_2^\mu &= \{\gamma_{12}(p_2^{0*} - \boldsymbol{\beta}_{12} \cdot \mathbf{p}_1^*), \gamma_{12}(\boldsymbol{\beta}_{12} p_2^{0*} - p_{1\parallel}^*), -p_{1\perp}^*\}, \\ p_3^\mu &= \{\gamma_{34}(p_3^{0'} + \boldsymbol{\beta}_{34} \cdot \mathbf{p}_3'), \gamma_{34}(\boldsymbol{\beta}_{34} p_3^{0'} + p_{3\parallel}'), p_{3\perp}'\}, \\ p_4^\mu &= \{\gamma_{34}(p_4^{0'} - \boldsymbol{\beta}_{34} \cdot \mathbf{p}_3'), \gamma_{34}(\boldsymbol{\beta}_{34} p_4^{0'} - p_{3\parallel}'), -p_{3\perp}'\}, \end{aligned} \quad (\text{A5})$$

where $\boldsymbol{\beta}_{12} = \mathbf{q}/q^0$ ($\boldsymbol{\beta}_{34} = \mathbf{k}/k^0$) is the velocity of the 1, 2 (3, 4) system in the rest frame of the decay particle, and $\gamma_{12(34)} = (1 - \boldsymbol{\beta}_{12(34)}^2)^{-1/2}$. Moreover, $p_{1\parallel(\perp)}^*$ are the components of \mathbf{p}_1^* parallel (perpendicular) to \mathbf{q} , and $p_{3\parallel(\perp)}'$ are the components of \mathbf{p}_3' parallel (perpendicular) to \mathbf{k} . One can define θ_1^* as the angle between the directions of \mathbf{q} and \mathbf{p}_1^* , and θ_3' as the one between $\mathbf{k} = -\mathbf{q}$ and \mathbf{p}_3' . The angle between \mathbf{p}_1^* and \mathbf{p}_3' , θ_{13} , is related to the solid angles Ω_1^* and Ω_3' by

$$\cos\theta_{13} = -\cos\theta_1^* \cos\theta_3' - \sin\theta_1^* \sin\theta_3' \cos(\varphi_3' + \varphi_1^*). \quad (\text{A6})$$

The angles are shown for illustration in Fig. 4. It is obvious that the integration $d\Omega$ as well as the one over either φ_1^* or φ_3' are trivial, such that Eq. (A3) simplifies to

$$\begin{aligned} d\Phi_4(P; p_1, \dots, p_4) &= \frac{1}{(8\pi^2)^3 M} \int_{m_1+m_2}^{M-m_3-m_4} d\sqrt{s_{12}} \int_{m_3+m_4}^{M-\sqrt{s_{12}}} d\sqrt{s_{34}} \\ &\quad \times \int d\cos\theta_1^* d\cos\theta_3' d\varphi_3' |p_1^*| |p_3'| |q|. \end{aligned} \quad (\text{A7})$$

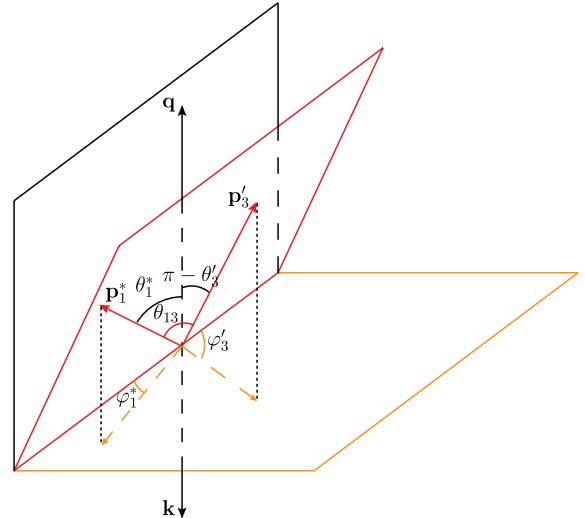


FIG. 4 (color online). The solid angles of particle 1 (3) in the cmf of 1 and 2 (3 and 4). θ_{13} is the angle between the momentum of particle 1 in the cmf of 1 and 2 and the momentum of particle 3 in the cmf of 3 and 4.

APPENDIX B: TENSOR-MESON CONTRIBUTIONS TO $\eta, \eta' \rightarrow 4\pi^0$

1. Amplitude, decay width

In this Appendix, we discuss an alternative, resonance-driven decay mechanism for the decays $\eta, \eta' \rightarrow 4\pi^0$, namely, via two $f_2(1270)$ tensor mesons, see the right panel of Fig. 2. The Lagrangian for the tensor-tensor-pseudoscalar interaction reads

$$\mathcal{L}_{TTP} = \frac{g_{TTP}}{\sqrt{2}} \epsilon_{\mu\nu\rho\sigma} g_{\alpha\beta} \langle \partial^\mu T^{\nu\alpha} T^{\rho\beta} \partial^\sigma \varphi \rangle, \quad (\text{B1})$$

where the tensor nonet is given by

$$T_{\mu\nu} = \begin{pmatrix} \frac{a_{2\mu\nu}^0}{\sqrt{2}} + \frac{f_{2\mu\nu}}{\sqrt{2}} & a_{2\mu\nu}^+ & K_{2\mu\nu}^+ \\ a_{2\mu\nu}^- & -\frac{a_{2\mu\nu}^0}{\sqrt{2}} + \frac{f_{2\mu\nu}}{\sqrt{2}} & K_{2\mu\nu}^0 \\ K_{2\mu\nu}^- & \bar{K}_{2\mu\nu}^0 & f_{2\mu\nu}' \end{pmatrix}. \quad (\text{B2})$$

The coupling constant g_{TTP} is not easily determined phenomenologically; we will first write the resulting decay width as a function of g_{TTP} , and then proceed to estimate it in Appendix B 2 using QCD sum rules.

The decay of the f_2 into two pseudoscalars is described by the Lagrangian [21,30]

$$\mathcal{L}_{f_2} = g_T f_{2\mu\nu} \langle u^\mu u^\nu \rangle = \frac{g_T}{F_\pi^2} f_{2\mu\nu} \langle \partial^\mu \varphi \partial^\nu \varphi \rangle + \dots \quad (\text{B3})$$

Using the polarization sum for a tensor meson [31]

$$\begin{aligned} & \sum_\lambda \phi_{\mu\nu}^{(\lambda)}(p) \phi_{\rho\sigma}^{(\lambda)}(p)^\dagger \\ &= \frac{1}{2} (X_{\mu\rho} X_{\nu\sigma} + X_{\mu\sigma} X_{\nu\rho}) - \frac{1}{3} X_{\mu\nu} X_{\rho\sigma}, \end{aligned} \quad (\text{B4})$$

with $X_{\mu\nu} \equiv g_{\mu\nu} - p_\mu p_\nu / M_{f_2}^2$, it is straightforward to derive the $f_2 \rightarrow \pi\pi$ decay width as [21]

$$\Gamma(f_2 \rightarrow \pi\pi) = \frac{g_T^2}{80\pi M_{f_2}^2 F_\pi^4} (M_{f_2}^2 - 4M_\pi^2)^{5/2}. \quad (\text{B5})$$

Inserting $M_{f_2} = 1275.1$ MeV, $\Gamma_{f_2} = 185.1$ MeV, and $\mathcal{B}(f_2 \rightarrow \pi\pi) = 84.8\%$ [2] (neglecting the corresponding uncertainties), the coupling constant can be obtained as $g_T = 39.4$ MeV.

Applying the polarization sum of a tensor field as given in Eq. (B4), the decay amplitude for the process $\eta' \rightarrow f_2 f_2 \rightarrow 4\pi^0$ takes the simple form

$$\begin{aligned} \mathcal{A}_{f_2 f_2} &= -\sqrt{\frac{2}{3}} \frac{2g_{TTP} g_T^2}{F_\pi^4} \epsilon_{\mu\nu\alpha\beta} p_1^\mu p_2^\nu p_3^\alpha p_4^\beta \\ &\times [\mathcal{H}(s_{12}, s_{23}, s_{14}, s_{34}; s_{13}, s_{24}) \\ &- \mathcal{H}(s_{13}, s_{23}, s_{14}, s_{24}; s_{12}, s_{34}) \\ &- \mathcal{H}(s_{12}, s_{24}, s_{13}, s_{34}; s_{14}, s_{23})]. \end{aligned} \quad (\text{B6})$$

The three terms in the square brackets are due to interchange of identical pions in the final state, and $\mathcal{H}(v, w, x, y; s, t)$ is given by

$$\mathcal{H}(v, w, x, y; s, t) = \frac{v - w - x + y}{(M_{f_2}^2 - s)(M_{f_2}^2 - t)}, \quad (\text{B7})$$

where we have again neglected the f_2 width in the propagators.

We calculate the partial width as in Sec. III C, and find as the result for the decay mechanism through two virtual f_2 states

$$\Gamma(\eta' \rightarrow f_2 f_2 \rightarrow 4\pi^0) \approx 1 \times 10^{-16} \frac{g_{TTP}^2}{\text{GeV}^{-2}} \text{ MeV}. \quad (\text{B8})$$

The value of g_{TTP} is estimated using the method of QCD sum rules in Appendix B 2 to be about 9 GeV^{-1} . Thus, the branching fraction is

$$\mathcal{B}(\eta' \rightarrow f_2 f_2 \rightarrow 4\pi^0) \approx 4 \times 10^{-14}. \quad (\text{B9})$$

It is orders of magnitude smaller than the value in Eq. (36), and hence can be safely neglected.

2. Estimate of g_{TTP} via QCD sum rules

In this Appendix, we estimate the unknown coupling constant g_{TTP} using QCD sum rules [32,33]. We choose to estimate it from the $a_2 f_2 \pi$ coupling. Since we are not aiming at a precise calculation, complications due to mixing with gluon operators and anomalous dimensions will be neglected. The interpolating fields for the a_2^+ [34] and π^+ are

$$\begin{aligned} j_{\mu\nu}^{(a_2^+)}(x) &= \frac{i}{2} \bar{d}(x) (\gamma_\mu \vec{D}_\nu + \gamma_\nu \vec{D}_\mu) u(x), \\ j_5^{(\pi^+)}(x) &= im_q \bar{d}(x) \gamma_5 u(x), \end{aligned} \quad (\text{B10})$$

where $\vec{D}_\mu \equiv (\vec{D}_\mu - \vec{D}_\mu)/2$, with D_μ the standard covariant derivative, and m_q is the light quark mass. The flavor wave function for the f_2 is $(\bar{u}u + \bar{d}d)/\sqrt{2}$, and the corresponding interpolating field follows from the above equation. We will study the three-point correlation function

$$\begin{aligned} \Pi_{\mu\nu\alpha\beta}(p', q) &= \int d^4x d^4y e^{i(p'x + qy)} \langle 0 | T \{ J_{\mu\nu}^{(f_2)}(x) j_5^{(\pi^+)}(y) \\ &\times [J_{\alpha\beta}^{(a_2^+)}(z)]^\dagger \} | 0 \rangle_{z \rightarrow 0} \\ &\equiv \Pi(p', q) p'^\rho q^\sigma (g_{\alpha\mu} \epsilon_{\beta\nu\rho\sigma} + g_{\alpha\nu} \epsilon_{\beta\mu\rho\sigma} \\ &+ g_{\beta\mu} \epsilon_{\alpha\nu\rho\sigma} + g_{\beta\nu} \epsilon_{\alpha\mu\rho\sigma}) + \dots \end{aligned} \quad (\text{B11})$$

In the last step, only one Lorentz structure is kept, which we will use for the estimate.

The operator product expansion for the correlation function can be calculated in the deep Euclidean region. Keeping operators of the lowest dimension only, we find

$$\Pi(p', q) = -\frac{m_q \langle \bar{q}q \rangle}{16\sqrt{2}} \frac{1}{q^2} [\log(-p^2) + \log(-p'^2)], \quad (\text{B12})$$

with $p = q + p'$, where we have neglected all polynomial terms since they will not contribute after the Borel transform. The correlation function can also be expressed in terms of hadronic quantities, which reads

$$\Pi(p', q) = -\frac{1}{4} F_\pi M_\pi^2 \frac{g_{TTP} f_T^2 M_T^6}{(q^2 - M_\pi^2)(p^2 - M_T^2)(p'^2 - M_T^2)}, \quad (\text{B13})$$

where SU(3) symmetry is assumed for the a_2 and f_2 by requiring their decay constants and masses to be the same. The hadronic quantities are defined as

$$\langle 0 | j_5^{(\pi^+)} | \pi^+ \rangle = \frac{1}{\sqrt{2}} F_\pi M_\pi^2, \quad \langle 0 | j_{\mu\nu}^{(f_2)} | f_2 \rangle = f_T M_T^3 \phi_{\mu\nu}, \quad (\text{B14})$$

with $\phi_{\mu\nu}$ as defined in Eq. (B4). Note that f_T defined in this way is dimensionless. Following Ref. [35], we take $p^2 = p'^2$, and perform the Borel transform only once. We obtain the sum rule

$$g_{TTP} = \frac{F_\pi M_B^4}{4\sqrt{2} f_T^2 M_T^6} e^{M_T^2/M_B^2}, \quad (\text{B15})$$

where M_B is the so-called Borel mass, and $m_q \langle \bar{q}q \rangle = -M_\pi^2 F_\pi^2/2$ has been used. In addition, f_T was already calculated in QCD sum rules [34]. Because of a cancellation between the gluon condensate and the four-quark condensate, the sum rule for f_T is dominated by a perturbative contribution, which reads [34]

$$M_T^6 f_T^2 e^{-M_T^2/M_B^2} \approx \frac{3}{160\pi^2} \int_0^{s_0} s^2 e^{-s/M_B^2} ds, \quad (\text{B16})$$

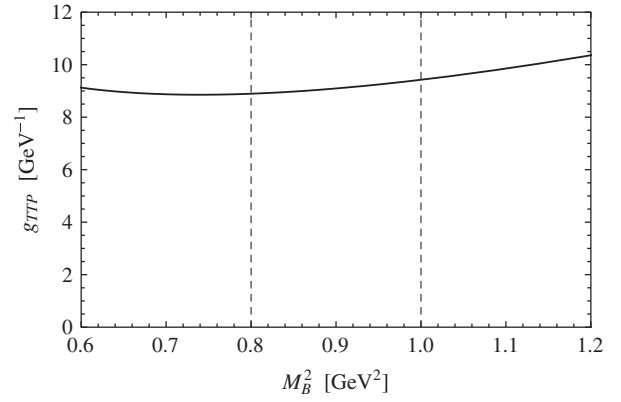


FIG. 5. QCD sum rule estimate of g_{TTP} . The interval $M_B^2 \in [0.8, 1.0]$ GeV², marked by the vertical dashed lines, is optimal to suppress both higher power corrections and continuum contribution in Ref. [34].

where s_0 is a threshold parameter introduced to mimic the spectral function in the region $q^2 > s_0$ by the one calculated using perturbative QCD. Finally, we obtain

$$g_{TTP} \approx \frac{20\sqrt{2}\pi^2}{3} F_\pi M_B^4 \left(\int_0^{s_0} s^2 e^{-s/M_B^2} ds \right)^{-1}. \quad (\text{B17})$$

This estimate is plotted in Fig. 5 as a function of M_B^2 . Taking the same interval of $M_B^2 \in [0.8, 1.0]$ GeV² and $s_0 = 2.5$ GeV² as in Ref. [34], the coupling constant is estimated as

$$g_{TTP} \approx 9 \text{ GeV}^{-1}. \quad (\text{B18})$$

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