Adiabatic and nonadiabatic perturbation theory for coherence vector description of neutrino oscillations

Sebastian Hollenberg* and Heinrich Päs†

Fakultät für Physik, Technische Universität Dortmund, D-44221 Dortmund, Germany

(Received 30 March 2011; published 17 January 2012)

The standard wave function approach for the treatment of neutrino oscillations fails in situations where quantum ensembles at a finite temperature with or without an interacting background plasma are encountered. As a first step to treat such phenomena in a novel way, we propose a unified approach to both adiabatic and nonadiabatic two-flavor oscillations in neutrino ensembles with finite temperature and generic (e.g., matter) potentials. Neglecting effects of ensemble decoherence for now, we study the evolution of a neutrino ensemble governed by the associated quantum kinetic equations, which apply to systems with finite temperature. The quantum kinetic equations are solved formally using the Magnus expansion and it is shown that a convenient choice of the quantum mechanical picture (e.g., the interaction picture) reveals suitable parameters to characterize the physics of the underlying system (e.g., an effective oscillation length). It is understood that this method also provides a promising starting point for the treatment of the more general case in which decoherence is taken into account.

DOI: 10.1103/PhysRevD.85.013013 PACS numbers: 13.15.+g, 14.60.Pq, 14.60.St

I. INTRODUCTION

Ever since the conjecture that neutrinos have mass and thus might be subject to flavor oscillations [1], there has been a thriving interest in this very phenomenon which clearly indicates and gives rise to speculations as to how to describe physics beyond the electroweak standard model.

A convenient and established way to deal with neutrino flavor oscillations is to encode this effect in a Hamiltonian formulation in which the oscillatory behavior is captured in a Schrödinger-like equation for a wave function in neutrino flavor space. This formalism, in principle, applies to an arbitrary number of neutrino generations and is also capable of incorporating medium effects on neutrino propagation such as coherent elastic forward scattering in, e.g., stellar matter [2]. It was soon realized that the Hamiltonian formalism for neutrino oscillations can be given a geometrical interpretation in $N^2 - 1$-dimensional flavor space for N neutrino flavors [3]. This approach to neutrino oscillations sees equations of motion for a coherence vector [4] in that the Schrödinger-like equation of motion can be rephrased as a gyroscope equation, i.e., a formal equivalent to, e.g., the precession of a magnetic moment in an external magnetic field. Besides its apparent usefulness when it comes to picture neutrino oscillations, there is also a purely formal merit to the gyroscope-type equations in that they are introduced by means of decomposing the Hamiltonian in terms of the generators of the associated $SU(N)$, e.g., the Pauli matrices for a two-flavor system.

This decomposition procedure is also most convenient when the notion of a wave function is not suitable anymore to describe the physics of neutrino oscillations. A typical situation in which the breakdown of the wave function formalism is expected is quantum ensembles with a finite temperature or neutrino ensembles with a finite temperature and an interacting background plasma. The latter situation is encountered in the early Universe prior to big bang nucleosythesis. The crucial point in such an environment is the breaking of coherence due to the small mean free path of neutrinos at high temperatures. The other important modification results from the fact that neutrino oscillations in the early Universe can alter the lepton asymmetry, which in turn contributes to the refraction index of the primeval plasma rendering the equations of motion nonlinear. It is essentially for those two reasons that the wave function formalism must fail in describing neutrino oscillation phenomena now. The appropriate description is then given by the density matrix formalism. The density matrix of the neutrino ensemble obeys a von Neumann equation and the different contributions to the effective Hamiltonian are given by collisions (non-forward scattering), with particles from the background medium, which introduce decoherence. On the other hand, coherent oscillations are governed by a matter-dependent effective Hamiltonian which comes about via coherent forward-scattering processes of neutrinos off the background particles [5,6].

The dynamics of the neutrino ensemble are determined by the quantum kinetic equations (QKE) which present a generalization of the Pauli-Boltzmann equations. The former evolve quantum amplitudes as is indispensable if a consistent description of particle oscillation phenomena, which are inherently nonclassical, is sought. The Pauli-Boltzmann approach on the other hand evolves probabilities rather than amplitudes; this procedure is essentially classical since quantum mechanics only enters the problem when it comes to calculating cross sections for the various

*sebastian.hollenberg@tu-dortmund.de
†heinrich.paes@tu-dortmund.de

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possible reaction channels. The resultant quantum rate equations are inappropriate when neutrino oscillations occur. Thus, in order to obtain the QKE the full density matrix for all particles in the plasma is evolved forward in time by means of the S matrix and tracing over all degrees of freedom other than the neutrinos under consideration yields the equation of motion for the system's density matrix, the QKE, which do reduce to quantum rate equations in the appropriate limit [5]. The variable of interest in the QKE is the one-body reduced momentum-dependent density operator, which is conveniently decomposed in terms of the generators of the associated $SU(N)$.

In the forthcoming analysis, we account for a brief motivation on how to obtain the QKE from the density matrix formalism in the case of coherent forward scattering (which also dominates the bulk of the studies to follow) and how to relate the solutions to the QKE, i.e., the coherence vector, to physical observables in Sec. II. Moreover, we introduce the Magnus expansion, which allows for an analytic, yet approximative, solution to the underlying QKE. A short discussion of the convergence properties of the Magnus expansion proves useful to single out suitable quantities to describe the physics at hand. In Sec. III, we unfold a systematic way to develop an adiabatic perturbation theory starting from the QKE for a two-flavor neutrino ensemble with generic potentials. We show that the Magnus expansion not only allows to analytically solve the QKE in a perturbative way, but can also serve to motivate the definition of physical quantities, such as an effective mixing angle or an adiabaticity parameter. In order to isolate a convenient perturbation parameter, we perform different changes of the quantum mechanical picture (e.g., into the interaction picture) for the QKE. Hence, a suitable succession of bases changes can improve the convergence properties of the expansion. We solve the QKE for the coherence vector to first order in the Magnus expansion. The perturbation parameter in the adiabatic perturbation theory can then be used to identify a suitable expansion parameter for the nonadiabatic case. Once this parameter is identified, the nonadiabatic perturbation theory can be treated on the same ground as the adiabatic one, i.e., singling out an appropriate expansion parameter by changing the quantum mechanical picture and solving the QKE in the resulting representation by means of the Magnus expansion. This nonadiabatic perturbation theory is developed in Sec. IV. In Sec. V, we comment on the integrals appearing in both the adiabatic and nonadiabatic perturbation theories and show how the latter lead to the correct limits such as vacuum neutrino oscillations and the slab model approximation to nonadiabatic transitions in the Mikheev-Smirnov-Wolfenstein (MSW) framework. We sketch how to extend the developed perturbation theory to higher orders in the Magnus expansion. Moreover, we elaborate on possible extensions of the theory introduced to scenarios including decoherence in the ensemble due to collision processes with particles from a background plasma. It turns out that the continuation of the perturbation theory is somewhat nontrivial and deserves a more careful study.

II. NOTATION AND MATHEMATICAL TOOL BOX

The purpose of our analysis is to describe the evolution of a two-flavor neutrino ensemble with generic potentials at a finite temperature. To this end, we commence our considerations by a close inspection of the underlying QKE, which in this context can be readily derived from the density matrix formalism for neutrino oscillations. We remark that the latter has to be used to correctly deal with quantum systems at finite temperature as well as in situations where loss of coherence becomes important. The wave function formalism ceases to provide an appropriate handle for such systems. However, it is not until Sec. V C that we discuss an extension of the formalism to be discussed shortly to systems with a background plasma, in which ensemble decoherence due to collisions is encountered.

The density matrix $\rho(p,t)$ obeys a von Neumann equation

$$\dot{\rho}(p,t) = -i[H(p,t), \rho(p,t)], \tag{1}$$

where p is the neutrino four-momentum, H a generic Hamiltonian for the system. Since we are dealing with a two-flavor system, we can now decompose both the density matrix and the Hamiltonian in terms of the generators of the associated $SU(2)$, namely, the Pauli matrices σ^i. This yields

$$\rho = \tfrac{1}{2}\mathrm{Tr}\rho[1 + \mathrm{Tr}(\rho\sigma^i)\sigma^i] \equiv \tfrac{1}{2}P_0[1 + \vec{P}\,\vec{\sigma}], \tag{2}$$

$$H = \tfrac{1}{2}\mathrm{Tr}H[1 + \mathrm{Tr}(H\sigma^i)\sigma^i] \equiv \tfrac{1}{2}V_0[1 + \vec{V}\,\vec{\sigma}], \tag{3}$$

where repeated indices are to be summed over and the obvious identifications $P_0 = \mathrm{Tr}\rho$, $V_0 = \mathrm{Tr}H$, $\vec{P} = \mathrm{Tr}(\rho\vec{\sigma})$, $\vec{V} = \mathrm{Tr}(H\vec{\sigma})$, and $\vec{\sigma} = (\sigma^x, \sigma^y, \sigma^z)$ have been made. Here, $\vec{P}$ is the so-called coherence vector [7]. Making use of this notation, it is straightforward to recast the von Neumann equations according to

$$\dot{P}_i = [-V_0\varepsilon_{ilk}V^k]P^l. \tag{4}$$

Hence, we can identify

$$S_{il} \equiv -V_0\varepsilon_{ilk}V^k \quad \text{or} \quad S = V_0\begin{pmatrix} 0 & -V_z & V_y \\ V_z & 0 & -V_x \\ -V_y & V_x & 0 \end{pmatrix} \tag{5}$$

as the evolution matrix of the neutrino ensemble [8]. Put another way, the QKE can now be written [9] as

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{P}(t) = S(t)\vec{P}(t). \tag{6}$$

The entries of the effective potential vector $\vec{V} = (V_x, V_y, V_z)$ can also be readily obtained via

$$V_0 V_x \equiv \beta = 2\mathrm{Re}H_{12} = \frac{\Delta m^2}{2p}\sin 2\theta_0, \tag{7}$$

$$V_0 V_y = 2\mathrm{Im}H_{12} = 0, \tag{8}$$

$$V_0 V_z \equiv \lambda = (H_{11} - H_{22}) = -\frac{\Delta m^2}{2p}\cos 2\theta_0 + V_\alpha, \tag{9}$$

where H_{ij} are the elements of the Hamiltonian H, Δm^2 is the mass-squared difference of the two neutrino states, θ_0 is the associated vacuum mixing angle between the two flavors, which we shall denote as ν_a and ν_b for definiteness. V_α is the difference of potential terms affecting ν_a and ν_b, respectively. The last equality has been obtained using the 2×2 neutrino oscillation Hamiltonian in flavor space. Note, also, that all possible time dependences for the effective potential vector have been suppressed for reasons of notational convenience. It is, however, understood that all components of $\vec{V}$ depend on time in general and we shall in fact use $V_x = V_x(t)$ and $V_z = V_z(t)$ for the upcoming analysis.

In the coherence vector description, the expectation values of the generators of the associated $SU(2)$ are promoted to observables of interest. All information about the system can thus, in principle, be extracted from a solution to Eq. (6) for $\vec{P}$. Some comments related to this issue are in order: In the one-particle interpretation, the diagonal entries of the density matrix simply give the probability to find the system in one or the other state, i.e.,

$$\mathrm{prob}\,(\nu_a \to \nu_a) = \tfrac{1}{2}P_0[1 + P_z], \tag{10}$$

$$\mathrm{prob}\,(\nu_a \to \nu_b) = \tfrac{1}{2}P_0[1 - P_z]. \tag{11}$$

In the ensemble interpretation of the density matrix, the diagonal entries give the relative number densities for the different neutrino flavors normalized to the equilibrium Fermi-Dirac number distribution at zero chemical potential μ according to

$$N_a(p) = \tfrac{1}{2}P_0[1 + P_z]N^{\mathrm{EQ}}(p, 0), \tag{12}$$

$$N_b(p) = \tfrac{1}{2}P_0[1 - P_z]N^{\mathrm{EQ}}(p, 0), \tag{13}$$

$$N^{\mathrm{EQ}}(p, \mu) = \frac{1}{2\pi^2}\frac{p^2}{1 + e^{(p-\mu)/T}}, \tag{14}$$

where T is the temperature of the ensemble. Those relations are also easily inverted to yield a physical meaning of both

$$P_0 = \frac{N_a + N_b}{N^{\mathrm{EQ}}}, \tag{15}$$

$$P_z = \frac{N_a - N_b}{N_a + N_b}. \tag{16}$$

Hence, P_0 is connected to conservation of probability and, in a broader context, also lepton number. It is important to note that oscillations merely swap neutrinos from one flavor to another so that P_0 does not evolve in time, unless repopulation effects from some background plasma have to be taken into account as is the case, e.g., in the early Universe. On the other hand, P_z parametrizes the asymmetry of the system that is the excess of ν_a over ν_b. The latter fact also motivates the way of speaking in which P_x, P_y are called coherences encoding the amount of decoherence in the system. Therefore, the evolution of P_z is of special interest in most applications.

Our next concern is to provide mathematical means to approximately solve the differential equations (6). In order to do so, we rewrite the QKE supplied with an initial condition as

$$\frac{\partial}{\partial t}\vec{P}(t) = S(t)\vec{P}(t), \qquad \vec{P}^i = \vec{P}(t_0) \tag{17}$$

and notice that we are dealing with a nonautonomous set of linear differential equations. If the matrix S did not depend on time, the corresponding differential equation could be readily solved by taking the matrix exponential $\exp[S(t - t_0)]$ and writing $\vec{P}(t) = \exp[S(t - t_0)]\vec{P}^i$. It is then fair to ask whether the solution to Eq. (17) can always be written as an exponential via

$$\vec{P}(t) = \exp[\Omega(t, t_0)]\vec{P}^i. \tag{18}$$

A method for finding such a true exponential solution [10] has indeed been established under the label Magnus expansion [11,12]. The Magnus operator $\Omega = \ln S$ satisfies its own differential equation which in turn is solved by a series expansion $\Omega = \sum_{n=1}^{\infty}\Omega_n(t)$, where each Magnus approximant $\Omega_n(t)$ is small in an appropriate sense. The smallness of each Magnus approximant then determines the convergence properties of the expansion. A general theorem states that for a differential equation (17) defined in a Hilbert space with a bounded operator S the series converges in the interval $t \in [t_0, t_c[$ such that

$$\int_{t_0}^{t_c} \mathrm{d}\tau \|S(\tau)\| < \pi, \tag{19}$$

where $\|.\|$ is a matrix norm. As can be inferred from this condition, the convergence properties of the Magnus expansion can be improved by means of a change of the quantum mechanical picture and thus improving the convergence properties of the expansion can be rephrased as minimizing the matrix norm of the evolution matrix.

The first two Magnus approximants assume a form

$$\Omega_1(t) = \int_{t_0}^{t} \mathrm{d}\tau S(\tau), \tag{20}$$

$$\Omega_2(t) = \frac{1}{2}\int_{t_0}^{t}\mathrm{d}t_1\int_{t_0}^{t_1}\mathrm{d}t_2[S(t_1), S(t_2)] \tag{21}$$

and various methods have been worked out to calculate higher-order terms [13]. All such terms contain nested

commutators of S evaluated at different times from which it is readily seen that the Magnus expansion gives the exact result (the matrix exponential for a time-independent matrix S) already in first order, if the matrix S commutes with the matrix obtained by integrating S over a certain time interval. Note that each approximant Ω_n adopts the same properties as the matrix S. If S was (anti-) Hermitian, so would be the exponentials of all approximants; this means that unitarity of the solution will be preserved in each order of the perturbation expansion separately.

The convenience of the Magnus expansion is now apparent: In lieu of a calculation of eigenvalues and eigenvectors of a matrix, it poses the problem of calculating a matrix exponential. The latter problem is usually solvable for 3×3 matrices. Note, however, that the dimension of the evolution matrix corresponds to the number of generators of the associated $SU(N)$, i.e., for an N-dimensional system the dimension of the evolution matrix is $N^2 - 1$. So, for the case of three interacting neutrinos one already has to choose between diagonalizing an 8×8 matrix or calculating its exponential. Both problems are intricate.

Another nice feature of the Magnus expansion is that it provides a clear description of how to improve the approximation by going to higher-order terms. It also reproduces exact solutions to the QKE in the appropriate limits as will be seen later, but is also apparent at this stage of our analysis by inspecting the form of the Magnus approximants. Moreover, the condition for its convergence appears as an integral condition on the norm of the evolution matrix. This condition can be used as a cross-check for whether changing from one quantum mechanical picture to another improves convergence properties of the expansion or not. We will invoke this criterion in the upcoming analysis and as it turns out, it can even supply some physical insight.

III. ADIABATIC PERTURBATION THEORY

Having established QKE of the form

$$\frac{\partial}{\partial t}\vec{P}(t) = S(t)\vec{P}(t), \quad S(t) = \begin{pmatrix} 0 & -\lambda(t) & 0 \\ \lambda(t) & 0 & -\beta(t) \\ 0 & \beta(t) & 0 \end{pmatrix}, \tag{22}$$

it is easy to calculate the matrix norm [14] according to

$$\|S\|_{\mathrm{F}}^2 = \mathrm{Tr}(S^\dagger S) = 2\omega_{\mathrm{eff}}^2. \tag{23}$$

It is straightforward to show that the effective oscillation length of the system is indeed given by

$$\frac{2\pi}{l_{\mathrm{osc}}^{\mathrm{eff}}} \equiv \omega_{\mathrm{eff}} = \sqrt{\lambda^2 + \beta^2}. \tag{24}$$

At this point of our analysis, it might seem academic to calculate the matrix norm of the evolution equation under consideration, but it will be seen shortly that a comparison between matrix norms in different quantum mechanical pictures can provide physical insight into the nature of the neutrino ensemble at hand. Note, moreover, that we treat both β and λ as time-dependent quantities. This generic assumption provides greater freedom when it comes to adapting the QKE to early universe applications. To this end, we notice that β scales as p^{-1} in momentum and, in an expanding universe, this momentum is redshifted and thus depends on time [15]. We shall sketch how to treat such situations later on.

Furthermore, in order to get a grasp on how this oscillation length can be understood physically we transform the QKE to a basis which resembles the commonly encountered mass eigenbasis in the MSW framework. To do so, we notice that there are only nonvanishing contributions to the effective potential vector's x and z component, namely, V_x and V_z. Thus, it is only sensible to consider a generic time-dependent rotation in the xz plane by an angle $\Theta(t)$ as

$$\vec{P}(t) = R(t)\vec{Q}(t) \quad \text{with} \quad R[\Theta(t)] = \begin{pmatrix} \cos\Theta(t) & 0 & \sin\Theta(t) \\ 0 & 1 & 0 \\ -\sin\Theta(t) & 0 & \cos\Theta(t) \end{pmatrix}, \tag{25}$$

where $\vec{Q}$ is the coherence vector in the new *corotating frame* and $R(t)$ is the time-dependent rotation matrix. The QKE in the new basis appear as

$$\frac{\partial}{\partial t}\vec{Q}(t) = S_Q(t)\vec{Q}(t), \qquad S_Q = \begin{pmatrix} 0 & -\lambda\cos\Theta - \beta\sin\Theta & -\frac{\mathrm{d}\Theta}{\mathrm{d}t} \\ \lambda\cos\Theta + \beta\sin\Theta & 0 & -\beta\cos\Theta + \lambda\sin\Theta \\ \frac{\mathrm{d}\Theta}{\mathrm{d}t} & \beta\cos\Theta - \lambda\sin\Theta & 0 \end{pmatrix}. \tag{26}$$

Since we introduced Θ as a generic time-dependent mixing angle, we are endued with its explicit definition according to our needs. It is readily seen that the $(S_Q)_{23}$ and $(S_Q)_{32}$ elements of the evolution matrix can be eliminated by an appropriate choice of the mixing angle $\Theta(t)$. The advantage of this choice is the geometrical interpretation: in the Q picture, the motion of the coherence vector is confined to the xy plane, if there was not the additional perturbation by the time derivative of the effective angle, which introduces a nonzero z component to the problem and forces the motion to exit the xy plane as the ensemble evolves. The smaller the change of the effective mixing with time,

the smaller the urge of the coherence vector to exit the xy plane. Therefore, we fix the effective mixing angle to be

$$\cos\Theta(t) = \frac{\lambda(t)}{\sqrt{\lambda^2(t) + \beta^2(t)}}, \quad \sin\Theta(t) = \frac{\beta(t)}{\sqrt{\lambda^2(t) + \beta^2(t)}}. \tag{27}$$

The effective mixing angle reveals that mixing becomes maximal ($\Theta = \pi/2$) if the condition

$$\lambda(t_{\rm res}) = 0 \tag{28}$$

is satisfied for the so-introduced resonant time $t_{\rm res}$. A vanishing $\lambda(t)$, i.e., maximal effective mixing, hence coincides with the existence of a resonance in neutrino conversions, which can also be equivalently rephrased for a resonant temperature $T_{\rm res}$, depending on the application one has in mind.

We now recast the evolution matrix in the Q picture as

$$S_Q = \begin{pmatrix} 0 & -\omega_{\rm eff} & -\frac{{\rm d}\Theta}{{\rm d}t} \\ \omega_{\rm eff} & 0 & 0 \\ \frac{{\rm d}\Theta}{{\rm d}t} & 0 & 0 \end{pmatrix}. \tag{29}$$

Consider the matrix norm of this evolution matrix in the new quantum mechanical picture

$$\|S_Q\|_{\rm F}^2 = 2\omega_{\rm eff}^2\left[1 + \left(\frac{1}{\omega_{\rm eff}}\frac{{\rm d}\Theta}{{\rm d}t}\right)^2\right]. \tag{30}$$

At first glance, the above transformation seems to worsen the convergence properties due to the appearance of the additional

$$\gamma \equiv \frac{1}{\omega_{\rm eff}}\frac{{\rm d}\Theta}{{\rm d}t} \tag{31}$$

term. However, if this very term is sufficiently small, $\gamma \ll 1$, the convergence will only be marginally altered. Moreover, the smallness condition can be understood physically as well: the characteristic time scale of the system under study is $\tau_{\rm sys} = 1/\omega_{\rm eff}$, whereas the characteristic time scale of the interaction can be identified as $\tau_{\rm int} = ({\rm d}\Theta/{\rm d}t)^{-1}$. Hence, the parameter γ simply compares the characteristic time scale of the system to the characteristic time scale of the interaction, stating that a small γ can be paraphrased as the system's time scale being much smaller than the interaction's time scale. Put another way, the interaction is adiabatic. The parameter γ is thus readily interpreted and henceforth referred to as the adiabaticity parameter for the system [16].

The evolution matrix of the system thus reads

$$S_Q = \begin{pmatrix} 0 & -\omega_{\rm eff} & -\gamma\omega_{\rm eff} \\ \omega_{\rm eff} & 0 & 0 \\ \gamma\omega_{\rm eff} & 0 & 0 \end{pmatrix}. \tag{32}$$

Before we move on with our analysis, it is just to briefly comment on the definition of the adiabaticity parameter and the effective mixing angle. The effective mixing angle defined above is essentially the expression encountered when it comes to the usual MSW framework of matter-affected neutrino oscillations. Note, however, that the latter typically features a $\sin 2\Theta$ instead of $\sin\Theta$ as defined here. In order to streamline notation, we will nonetheless still omit this factor of 2. Moreover, we alert the reader that defining $1/\gamma$ as the adibaticity parameter is also quite common in the literature. However, the physics is not altered by this convention. Also, when comparing our analysis to other work it is important to notice that in the Θ instead of 2Θ convention the adiabaticity parameter lacks a factor of 2 as well. We will analyze the adiabaticity parameter further in Sec. IV.

Note that the concept of a Hamiltonian, in general, ceases to exist when QKE of thermal neutrino ensembles are considered. It is important to keep in mind this point in order to fully appreciate our paradigm. What is done here is the following: a rotation of the coherence vector into the xy plane for the adiabatic limit of taking $\gamma \to 0$. This looks similar to the diagonalization of the underlying Hamiltonian for single neutrino states in the sense that it allows to define mixing angles and adiabaticity parameters for the ensemble. The benefit of this procedure, however, is that it does not explicitly rely on the form of the time evolution matrix. It gives the prescription that an adiabaticity parameter for the system under consideration can be found by employing a suitable change of basis for the QKE without making any reference to an underlying Hamiltonian of the system. This point can prove especially useful when collision-affected neutrino conversions are considered; a system in which the Hamiltonian formulation definitely ceases to be applicable.

It is by now established that γ can serve as a small perturbation parameter in the adiabatic regime of neutrino conversions. It thus feels harmonious to struggle through just another transformation which is introduced to isolate the perturbation parameter γ in a convenient way and such that fast convergence of the expansion to come is assured. The prescription is as follows,

$$\vec{Q}(t) = U(t)\vec{X}(t), \text{ where } \frac{\partial}{\partial t}U(t) = S_Q^{\omega}(t)U(t), \quad U(t_0) = \mathbf{1}, \tag{33}$$

having decomposed the evolution matrix according to

$$S_Q = S_Q^{\omega} + S_Q^{\gamma} = \begin{pmatrix} 0 & -\omega_{\rm eff} & 0 \\ \omega_{\rm eff} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & -\gamma\omega_{\rm eff} \\ 0 & 0 & 0 \\ \gamma\omega_{\rm eff} & 0 & 0 \end{pmatrix} \tag{34}$$

in self-obvious notation. The subsidiary evolution equation for $U(t)$ is also readily solved to give

$$U(t) = \begin{pmatrix} \cos\tilde{\omega}_{\rm eff} & -\sin\tilde{\omega}_{\rm eff} & 0 \\ \sin\tilde{\omega}_{\rm eff} & \cos\tilde{\omega}_{\rm eff} & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \tilde{\omega}_{\rm eff}(t) = \int_{t_0}^{t} {\rm d}\tau\,\omega_{\rm eff}(\tau). \tag{35}$$

The QKE in disguise are recognized to be

$$\frac{\partial}{\partial t}\vec{X}(t) = S_X(t)\vec{X}(t),$$
$$S_X(t) = \begin{pmatrix} 0 & 0 & -\gamma\omega_{\mathrm{eff}}\cos\tilde{\omega}_{\mathrm{eff}} \\ 0 & 0 & \gamma\omega_{\mathrm{eff}}\sin\tilde{\omega}_{\mathrm{eff}} \\ \gamma\omega_{\mathrm{eff}}\cos\tilde{\omega}_{\mathrm{eff}} & -\gamma\omega_{\mathrm{eff}}\sin\tilde{\omega}_{\mathrm{eff}} & 0 \end{pmatrix} \tag{36}$$

and calculating the matrix norm yields

$$\|S_X\|_{\mathrm{F}}^2 = 2\gamma^2\omega_{\mathrm{eff}}^2. \tag{37}$$

It is evident now that the small parameter in the adiabatic regime, namely, γ, has been isolated and hence good convergence of the sought-after perturbation theory can be expected. This comforts us to seek a perturbative expansion in this basis (which is but an interaction picture for the $\vec{Q}$ basis).

The considerations unfolded in this section have seen two linear transformations $R(t)$, $U(t)$ from the original $\vec{P}$ basis to the $\vec{Q}$ and $\vec{X}$ basis. The reason for those transformations is twofold: On the one hand, changing the basis for the QKE discloses the physics of the system we are dealing with and on the other hand it seems advisable to find a basis for the QKE in which an approximate solution gives accurate results. For convenience, we shall now recapitulate the meaning of the transformations introduced so far.

The first transformation ($\vec{P} \xrightarrow{R} \vec{Q}$) is inherently physical. It gives a recipe on how to establish the concept of a *mass eigenbasis* in the coherence vector description of neutrino oscillations. The effective mixing angle defined in this way differs from the effective mixing angle encountered in the common MSW formalism by a conventional factor of 2. In this quantum mechanical picture, a clear path of approaching the resonance in neutrino oscillations is unveiled. A resonant conversion of neutrino flavors is encountered for $\lambda(t) = 0$. Moreover, the transformation to the matter eigenbasis sees the introduction of an effective mixing angle, which is a harbinger for the adiabaticity parameter γ subsequently laid open. Adiabatic neutrino conversion occurs for $\gamma \ll 1$, when the time scale of the system is much smaller than the time scale of the interaction. The mathematical benefit of this transformation is that we get a grasp on the convergence properties of the approximation we want to employ and we can furnish it with physical meaning. The convergence properties of the expansion get worse as the amount of adiabaticity violation increases, which later will be useful to construct a nonadiabatic perturbation theory. Also, the change to the matter eigenbasis suggests that the adiabaticity parameter γ should be the appropriate small quantity to expand in.

The second transformation ($\vec{Q} \xrightarrow{U} \vec{X}$) is convenient from a mathematical point of view. It removes an exactly integrable part of the evolution matrix and thus the matrix norm for the latter is directly proportional to the small expansion parameter γ. This truly renders γ into the sought-after perturbation parameter and we take comfort that the envisaged expansion converges fast. Put another way, already the first approximant should provide good guidance for the exact solution.

However, one final comment regarding the terminology is in order: The terms *quantum mechanical picture* or rather *change of the quantum mechanical picture* are used to denote a distinct basis for the time evolution matrix of the QKE or a basis change from one basis of the QKE to another, respectively. This is to be understood as a manner of speaking motivated by standard Schrödinger quantum mechanics.

Note eventually that no attempt for solving the QKE has been made so far. We have merely changed the quantum mechanical pictures to unfold the underlying physics. The paradigm of our analysis is that a careful treatment of the QKE, i.e., a succession of different changes of the quantum mechanical pictures already allows to extract important information about the system under consideration without ever explicitly solving the QKE.

In the $\vec{X}$ basis, the QKE are solved to first order in the Magnus expansion by

$$\vec{X}^{(1)}(t) = \exp\left[\int_{t_0}^{t} \mathrm{d}\tau S_X(\tau)\right]\vec{X}(t_0). \tag{38}$$

The formal solution for the coherence vector $\vec{P}(t)$ to first order in the Magnus expansion is thus obtained as

$$\vec{P}^{(1)}(t) = R(t)U(t)\exp\left[\int_{t_0}^{t} \mathrm{d}\tau S_X(\tau)\right]R^{-1}(t_0)\vec{P}(t_0). \tag{39}$$

In order to streamline notation, we write the terms contained in the matrix exponential as

$$J_s(t) = \int_{t_0}^{t} \mathrm{d}\tau\gamma\omega_{\mathrm{eff}}\sin\tilde{\omega}_{\mathrm{eff}}, \tag{40}$$

$$J_c(t) = \int_{t_0}^{t} \mathrm{d}\tau\gamma\omega_{\mathrm{eff}}\cos\tilde{\omega}_{\mathrm{eff}}, \tag{41}$$

as well as

$$|J| \equiv \sqrt{J_c^2 + J_s^2} \tag{42}$$

and the resultant expression for the coherence vector assumes a form

$$\vec{P}^{(1)}(t) = \begin{pmatrix} \cos\Theta(t) & 0 & \sin\Theta(t) \\ 0 & 1 & 0 \\ -\sin\Theta(t) & 0 & \cos\Theta \end{pmatrix} \begin{pmatrix} \cos\tilde{\omega}_{\rm eff} & -\sin\tilde{\omega}_{\rm eff} & 0 \\ \sin\tilde{\omega}_{\rm eff} & \cos\tilde{\omega}_{\rm eff} & 0 \\ 0 & 0 & 1 \end{pmatrix} \times \frac{1}{|J|^2} \begin{pmatrix} J_s^2 + J_c^2\cos|J| & -J_cJ_s(-1+\cos|J|) & -J_c|J|^2\mathrm{sinc}|J| \\ -J_cJ_s(-1+\cos|J|) & J_c^2 + J_s^2\cos|J| & J_s|J|^2\mathrm{sinc}|J| \\ J_c|J|^2\mathrm{sinc}|J| & -J_s|J|^2\mathrm{sinc}|J| & |J|^2\cos|J| \end{pmatrix} \times \begin{pmatrix} \cos\Theta_0 & 0 & -\sin\Theta_0 \\ 0 & 1 & 0 \\ \sin\Theta_0 & 0 & \cos\Theta_0 \end{pmatrix} \vec{P}^i, \quad (43)$$

where additionally $\Theta(t_0) \equiv \Theta_0$ and $\mathrm{sinc}x \equiv \frac{\sin x}{x}$ was defined. This expression, as cumbersome as it may look at a first glance, presents an analytic, yet perturbative, solution to the QKE as given in Eq. (22) for a generic potential, i.e., for a generic time (or equivalently temperature in early universe applications) dependence of both β and λ, as long as the transition can be considered adiabatic ($\gamma \ll 1$). Note also that oscillation probabilities in the one-particle interpretation can be extracted from this formal solution. Moreover, oscillating contributions to this very probability can be studied since there is no inherent averaging over rapidly oscillating contributions as is usually considered in the derivation of the oscillation probability in the MSW framework. Still, to fully appreciate this result a thorough discussion of various limiting cases, such as the adiabatic limit, is called for [17]. We postpone this endeavor until Sec. V.

IV. NONADIABATIC PERTURBATION THEORY

Before we proceed by developing a nonadiabatic perturbation theory on the same grounds as the foregoing adiabatic perturbation theory, it is instructive to briefly reconsider the adiabaticity parameter as defined in Eq. (31) and rephrase it in a way that allows for an understanding of the notion of adiabaticity in terms of the parameters β and λ. It is straightforward to show that

$$\gamma(t) = \frac{\beta\lambda}{\omega_{\rm eff}^3}\frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{\beta}{\lambda}. \quad (44)$$

However, physically the adiabaticity parameter at the neutrino conversion resonance ($\lambda = 0$) is of foremost interest. We find

$$\gamma_{\rm res} \equiv \gamma(t_{\rm res}) = -\frac{1}{\beta^2(t_{\rm res})}\frac{\mathrm{d}\lambda}{\mathrm{d}t}\bigg|_{t=t_{\rm res}}. \quad (45)$$

The adiabaticity parameter depends on the shape of the matter profile $\mathrm{d}\lambda/\mathrm{d}t$ as is expected from the MSW framework. Large variations of the matter profile at resonance are clearly disfavored for the sought-after perturbation expansion to work. Besides this contribution, also the term $1/\beta^2$ is familiar. It simply states that the larger β, the better the expansion works. Put another way, if the notion of adiabaticity as put forward in our analysis is adopted, a small β at resonance is incompatible with an adiabatic perturbation expansion to some extent. A regime with small β thus indicates nonadiabatic transitions; we will also refer to this regime as the sudden regime henceforth.

Recalling the QKE according to Eq. (22), it is obvious that β itself can be adopted as a small perturbation parameter if a nonadiabatic perturbation theory is desired. We split the evolution matrix S as

$$S(t) = S_\lambda(t) + S_\beta(t) = \begin{pmatrix} 0 & -\lambda(t) & 0 \\ \lambda(t) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\beta(t) \\ 0 & \beta(t) & 0 \end{pmatrix} \quad (46)$$

in obvious notation. The S_λ subsystem of this evolution equation can be integrated exactly and hence we impose the following change of the quantum mechanical picture,

$$\vec{P}(t) = V(t)\vec{Y}(t) \quad \text{with} \quad \frac{\partial}{\partial t}V(t) = S_\lambda(t)V(t). \quad (47)$$

The subsidiary evolution equation again is solved to give

$$V(t) = \begin{pmatrix} \cos\tilde{\lambda} & -\sin\tilde{\lambda} & 0 \\ \sin\tilde{\lambda} & \cos\tilde{\lambda} & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \tilde{\lambda}(t) = \int_{t_0}^{t}\mathrm{d}\tau\lambda(\tau), \quad (48)$$

rephrasing the QKE as

$$\frac{\partial}{\partial t}\vec{Y}(t) = S_Y(t)\vec{Y}(t), \qquad S_Y(t) = \begin{pmatrix} 0 & 0 & -\beta\sin\tilde{\lambda} \\ 0 & 0 & -\beta\cos\tilde{\lambda} \\ \beta\sin\tilde{\lambda} & \beta\cos\tilde{\lambda} & 0 \end{pmatrix}. \quad (49)$$

Note also that the new quantum mechanical picture is just the interaction picture. The matrix norm reveals isolation of the small perturbation parameter:

$$\|S_Y\|_{\rm F}^2 = 2\beta^2. \quad (50)$$

Solving the QKE for $\vec{Y}(t)$ will see the time-integrated evolution matrix S_Y since the first-order Magnus expansion gives the coherence vector as

$$\vec{P}^{(1)}(t) = V(t)\exp\left[\int_{t_0}^{t}\mathrm{d}\tau S_Y(\tau)\right]\vec{P}(t_0) \quad (51)$$

and it is thus sensible to define the following integrals to streamline notation,

$$K_s(t) = \int_{t_0}^{t} \mathrm{d}\tau \beta \sin\tilde{\lambda}, \tag{52}$$

$$K_c(t) = \int_{t_0}^{t} \mathrm{d}\tau \beta \cos\tilde{\lambda}, \tag{53}$$

as well as

$$|K| \equiv \sqrt{K_c^2 + K_s^2} \tag{54}$$

in order to mimic the notation introduced above for the adiabatic perturbation theory. The coherence vector to first order in the Magnus expansion is calculated to be

$$\vec{P}^{(1)}(t) = \begin{pmatrix} \cos\tilde{\lambda} & -\sin\tilde{\lambda} & 0 \\ \sin\tilde{\lambda} & \cos\tilde{\lambda} & 0 \\ 0 & 0 & 1 \end{pmatrix} \times \frac{1}{|K|^2} \begin{pmatrix} K_c^2 + K_s^2\cos|K| & K_cK_s(-1+\cos|K|) & -K_s|K|^2\mathrm{sinc}|K| \\ K_cK_s(-1+\cos|K|) & K_s^2 + K_c^2\cos|K| & -K_c|K|^2\mathrm{sinc}|K| \\ K_s|K|^2\mathrm{sinc}|K| & K_c|K|^2\mathrm{sinc}|K| & |K|^2\cos|K| \end{pmatrix} \vec{P}^i. \tag{55}$$

This is the analytic perturbative solution to the QKE in Eq. (22) with generic potential and time dependence for β and λ as long as the evolution can be considered nonadiabatic, which is equivalent to saying that β is a small quantity one can expand in. Again, this result can only be fully appreciated once the associated limiting cases are recovered. We will see to this in the next section [18].

V. PERTURBATION THEORY INGREDIENTS AND LIMITING CASES

We understand that our approach is a generic solution to the QKE in Eq. (22) without making any explicit reference to the physics it can be applied to. Hence, the integrals $J_{c/s}(t)$, $K_{c/s}(t)$ explicitly depend on the time dependence of both λ, i.e., the potential term V_α, as well as β and have to be evaluated in each application separately. However, certain general statements can be made already due to the fact that neutrino conversions reveal a resonance at $\lambda = 0$.

In any case, however, it is still necessary to demonstrate that the Magnus expansion does give exact results in the various physical limits.

A. The integrals: $J_{c/s}(t)$ and $K_{c/s}(t)$

The structure of the integrals $J_{c/s}(t)$ and $K_{c/s}(t)$, for the adiabatic and nonadiabatic case, respectively, at a first glance, is similar: the integrand is the expansion parameter ($\gamma\omega_{\mathrm{eff}}$ for the adiabatic case; β for the nonadiabatic case) multiplied by an oscillatory function. On second thought, however, there is a crucial difference; the oscillatory term in $K_{c/s}$ has a stationary phase ($\mathrm{d}\tilde{\lambda}/\mathrm{d}t = 0$ at resonance), whereas $J_{c/s}$ does not.

Let us therefore evaluate $K_{c/s}$ by means of the stationary phase method. Two main assumptions are needed to apply the stationary phase method: The oscillatory behavior of the integrand must be rapid enough to suppress all large contributions to the integral which might come from β so that the latter can simply be evaluated at resonance. The other requirement is that the resonance happens in a *small* region around t_{res}; put another way, the smallness of the aforementioned region is determined by whether the substitution $t_0 \to -\infty$ and $t \to \infty$ is justified in this region or not. Supposing that these two requirements are met, we find

$$|K_s| = 0, \tag{56}$$

$$|K_c| \simeq \left[\frac{2\pi}{\gamma_{\mathrm{res}}}\right]^{1/2}, \tag{57}$$

$$\text{hence } |K| \simeq \left[\frac{2\pi}{\gamma_{\mathrm{res}}}\right]^{1/2}. \tag{58}$$

Note that in the nonadiabatic perturbation theory the reciprocal value of the adiabaticity parameter at resonance is a small quantity.

Consider the integrals $J_{c/s}$ now. The first step that comes to mind here is integration by parts. We get

$$J_s(t) = -\gamma\cos\tilde{\omega}_{\mathrm{eff}}|_{t_0}^{t} + \int_{t_0}^{t} \mathrm{d}\tau \frac{\mathrm{d}\gamma}{\mathrm{d}\tau}\cos\tilde{\omega}_{\mathrm{eff}}, \tag{59}$$

$$J_c(t) = \gamma\sin\tilde{\omega}_{\mathrm{eff}}|_{t_0}^{t} - \int_{t_0}^{t} \mathrm{d}\tau \frac{\mathrm{d}\gamma}{\mathrm{d}\tau}\sin\tilde{\omega}_{\mathrm{eff}}. \tag{60}$$

If the variation of γ in the interval $[t_0, t]$ is sufficiently mild, the main contribution to the integrals is expected to come from the first term on the right-hand side.

Given these arguments, the integrals $K_{c/s}$, $J_{c/s}$ reveal a common trademark. Both integrals turn out to be small in the following sense: $K_{c/s}$ is proportional to the inverse of γ_{res} which, in a nonadiabatic perturbation theory, is a large quantity; likewise, in the adiabatic perturbation theory γ is the small quantity to expand in and again the integrals $J_{c/s}$ turn out to be proportional to γ.

B. Limiting cases

We begin our consideration by studying appropriate limits for the adiabatic perturbation theory first.

(1) *Adiabatic perturbation theory: The vacuum limit.* This limit to Eq. (43) is probably the most intuitive one, if we confine ourselves to the one-particle interpretation. We understand that for an exactly solvable system the first-order Magnus term should

already give the exact result, hence implying $\vec{P}^{(1)}(t) \equiv \vec{P}(t)$. Let us examine how this works out here: Firstly, we discard the potential term V_α. This gives $\lambda \to -\frac{\Delta m^2}{2p}\cos 2\theta_0$ and $\beta = \frac{\Delta m^2}{2p}\sin 2\theta_0$. Hence, for consistency we must take the adiabatic limit of $\gamma \to 0$, which immediately implies $J_{c/s} \to 0$. Moreover, $\tilde{\omega}_{\rm eff} \to \omega t$ if we define the common oscillation frequency $\omega = \frac{\Delta m^2}{2p}$ and set $t_0 = 0$ (as no resonance time exists, the choice of t_0 is arbitrary). Finally, it follows directly from the definition of the effective mixing angles that $\Theta(t) \to 2\theta_0$ for all times. All this reduces the coherence vector to

$$\vec{P}^{(1)}(t) = \begin{pmatrix} \cos^2 2\theta_0 \cos\omega t + \sin^2 2\theta_0 & \cos 2\theta_0 \sin\omega t & \sin 4\theta_0 \sin^2\frac{\omega t}{2} \\ \cos 2\theta_0 \sin\omega t & \cos\omega t & -\sin 2\theta_0 \sin\omega t \\ \sin 4\theta_0 \sin^2\frac{\omega t}{2} & -\sin 2\theta_0 \sin\omega t & \cos^2 2\theta_0 + \sin^2 2\theta_0 \cos\omega t \end{pmatrix} \vec{P}^i. \tag{61}$$

Suppose we start with a ν_a flavor such that $\vec{P}^i = (0, 0, 1)$. The probability to find the neutrino in the same/the other state after time t is then using Eqs. (10) and (11) as well as Eq. (61) given by

$$\text{prob}\,(\nu_a \to \nu_a) = 1 - \sin^2 2\theta_0 \sin^2\frac{\omega t}{2}, \tag{62}$$

$$\text{prob}\,(\nu_a \to \nu_b) = \sin^2 2\theta_0 \sin^2\frac{\omega t}{2}, \tag{63}$$

which is just the common oscillation probability. Note, moreover that this result was obtained solely using the truncated Magnus expansion as given above and that it accounts for probability conservation. Put another way, unitarity is guaranteed by means of the expansion itself and does not have to be imposed by hand.

We next turn our attention to the nonadiabatic case. There exist (at least) two interesting limits.

(1) *Nonadiabatic perturbation theory: The sudden limit*. The sudden limit of taking $\beta \to 0$ renders the QKE (22) into formally exactly solvable differential equations such that the Magnus expansion should give an exact result. The limit $\beta \to 0$ enforces $\omega_{\rm eff} = |\lambda|$ and hence $\cos\Theta = 1$, $\sin\Theta = 0$, which in turn implies $\gamma \to 0$. The coherence vector assumes a form

$$\vec{P}(t) = \begin{pmatrix} \cos\tilde{\lambda} & -\sin\tilde{\lambda} & 0 \\ \sin\tilde{\lambda} & \cos\tilde{\lambda} & 0 \\ 0 & 0 & 1 \end{pmatrix} \vec{P}^i. \tag{64}$$

The coherences of the ensemble are oscillating as a function of time (the ensemble is incoherent) and the flavor is frozen to its initial value. Put another way, in physical situations in which the evolution of the ensemble happens in a way that with increasing time also β increases, the unfreezing of the ensemble can be studied using nonadiabatic perturbation theory since it treats β as a small perturbation. We will point out in Sec. V C that this is typically the case in early universe applications. There is, however, a twist when it comes to early universe applications in that such systems are typically collision-dominated at high temperatures and thus the notion of adiabaticity is expected to be modified due to the presence of collisions. Put another way, a small β in early universe environments augmented by the presence of decohering collisions might as well allow for an adiabatic perturbation theory (see Sec. V C for some more details).

(2) *Nonadiabatic perturbation theory: The slab model limit*. Suppose that we start the evolution of the neutrino ensemble from a purely ν_a state $\vec{P}^i = (0, 0, 1)$. We obtain for the coherence vector

$$\vec{P}^{(1)}(t) = \begin{pmatrix} (-K_s\cos\tilde{\lambda} + K_c\sin\tilde{\lambda})\text{sinc}|K| \\ -(K_c\cos\tilde{\lambda} + K_s\sin\tilde{\lambda})\text{sinc}|K| \\ \cos|K| \end{pmatrix}. \tag{65}$$

The flavor oscillation probability is then written as

$$\text{prob}\,(\nu_a \to \nu_a) = \cos^2\tfrac{1}{2}|K|. \tag{66}$$

Suppose furthermore that the situation as described above to estimate the K-type integrals holds, i.e., the resonance in neutrino conversions happens in a narrow time interval centered around $t_{\rm res}$. Applying the stationary phase approximation, we then obtain

$$\text{prob}\,(\nu_a \to \nu_a) = \cos^2\sqrt{\frac{\pi}{2}\frac{1}{\gamma_{\rm res}}}. \tag{67}$$

This result is the oscillation probability for the slab model as outlined in Ref. [19]. So, the relevant limit is respected in this situation.

C. Higher-order corrections and applications

As has been seen in the previous sections, the Magnus expansion can be easily extended to higher orders by summing the associated approximants according to $\Omega(t) = \Omega_1(t) + \Omega_2(t) + \ldots$. In order to get a grasp on how this prescription unfolds, we calculate the 2nd-order Magnus approximant to be

$$\Omega_2(t) = \begin{pmatrix} 0 & \mathcal{J} & 0 \\ -\mathcal{J} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{68}$$

where $\mathcal{J}$ is given by

$$\mathcal{J}_{\rm ad} = \frac{1}{2}\int_{t_0}^{t} {\rm d}t_1 \int_{t_0}^{t_1} {\rm d}t_2 \gamma(t_1)\gamma(t_2)\omega_{\rm eff}(t_1)\omega_{\rm eff}(t_2) \times \sin[\tilde{\omega}_{\rm eff}(t_2) - \tilde{\omega}_{\rm eff}(t_1)] \tag{69}$$

for the adiabatic case and

$$\mathcal{J}_{\rm nad} = \frac{1}{2}\int_{t_0}^{t} {\rm d}t_1 \int_{t_0}^{t_1} {\rm d}t_2 \beta(t_1)\beta(t_2)\sin[\tilde{\lambda}(t_2) - \tilde{\lambda}(t_1)] \tag{70}$$

for the nonadiabatic case, respectively. The calculations are performed in the X picture for adiabatic transitions and in the Y picture for nonadiabatic transitions. Two things are readily inferred: the second-order approximant is indeed $\mathcal{O}(\gamma^2)$ and $\mathcal{O}(\beta^2)$ for adiabatic and nonadiabatic corrections, respectively, as is expected. Moreover, it is seen that the 2nd order populates the (23) and (32) entries of the Magnus operator Ω.

We shall now shortly elaborate on the complications which arise when ensemble decoherence is to be taken into account. This typically happens in early universe applications in which the time evolution of neutrinos is governed by three distinct physical processes: Firstly, the expansion of the Universe. Secondly, coherent oscillations governed by a matter-dependent effective Hamiltonian which results from coherent forward-scattering processes of neutrinos off the background particles. Thirdly, scattering processes with the background plasma of elementary particles. These collisions, or non-forward-scattering processes, with particles from the background medium typically introduce decoherence effects into the neutrino ensemble. In our analysis in this paper, we have neglected the ensemble decoherence due to non-forward scattering.

The epoch of foremost interest in studying neutrino oscillations in the early Universe is the one between muon decoupling at $T \sim m_\mu \sim 100$ MeV and neutrino decoupling, i.e., prior to big bang nucleosynthesis (BBN), at about $T \sim 1$ MeV, since during this time the initial conditions for nucleosynthesis, the electron neutrino abundance, are set [20], which then directly influence the neutron-to-proton ratio at the onset of BBN via β processes $p + e^- \rightleftharpoons n + \nu_e$. The primordial plasma during this epoch thus consists of electrons, positrons, neutrinos, and antineutrinos.

The density matrix ρ for the system of interacting and oscillating neutrinos encodes ratios of number density distributions and hence the expansion of the Universe does not directly contribute to the time evolution of the density matrix. However, the momenta of the particles are redshifted and the equilibrium number distributions $N^{\rm EQ}(p, 0)$ thus depend on time through this redshifting.

An interesting application of the approach developed here exists in scenarios as discussed in, e.g., Ref. [21], namely, active-sterile flavor oscillations. The latter are interesting since active-sterile oscillations would populate the additional sterile species and thus contribute significant additional energy density, which in turn would trigger an accelerated expansion of the Universe and hence lead to a higher weak freeze-out temperature. This again would alter the neutron-to-proton ratio at the onset of BBN. The coherent part of matter-affected active-sterile oscillations splits in two contributions. One is just the leading-order density-dependent contribution (the MSW [2] part). This part is only temperature dependent indirectly via the cosmological redshifting of fermion number density. The second contribution comes from leading-order finite temperature gauge boson effects [22], which cannot be neglected at the temperatures considered here.

The loss of coherence is due to neutrino collisions with the background medium. The decoherence (or damping) function for this process in thermal equilibrium turns out to be proportional to the total collision rate for the neutrino with momentum p under consideration.

The epoch of interest can now be decomposed into three distinct domains: At high temperatures, finite temperature gauge boson effects dominate. Repopulation effects from the background plasma can be neglected since at high temperatures thermal equilibrium for all relevant species is rapidly established. At intermediate temperatures, lepton number production starts and the forward-scattering contribution comes into play as a small perturbation. Finally, prior to the onset of BBN, at low temperatures, collisional effects and finite temperature gauge boson contributions cease to be important and coherent neutrino oscillations are the dominant process.

In each of the aforementioned temperature domains, the QKE should be solved to determine the evolution of the neutrino ensemble. It is obvious that decoherence and repopulation effects modify the underlying QKE and complicate their analysis by introducing new physical scales in the system. The early Universe framework thus deserves a more careful treatment which is beyond the scope of this work.

However, we do understand that our paradigms for obtaining the oscillation frequency, the effective mixing, as well as the adiabaticity parameter of the system do not necessitate a full solution for the coherence vector. Hence, we are at liberty to take a first glance at such collision-affected neutrino conversions invoking our formalism.

The time evolution matrix for a collision-affected two-flavor active-sterile neutrino ensemble in the early Universe is given by

$$S(t) = \begin{pmatrix} -D(t) & -\lambda(t) & 0 \\ \lambda(t) & -D(t) & -\beta(t) \\ 0 & \beta(t) & 0 \end{pmatrix} \tag{71}$$

in the framework described above. The potential term $\lambda(t)$ now contains the MSW potential from coherent elastic forward scattering of neutrinos on the background plasma as well as a finite temperature W-boson contribution. Moreover, $D(t)$ gives the decoherence (damping) function, which is proportional to the total collision rate for the active neutrino flavor. Applying our paradigm to this system, we find for the effective oscillation frequency

$$\omega_{\rm eff} \equiv \sqrt{\lambda^2 + \beta^2 + D^2}, \tag{72}$$

which readily reduces to the well-known expression in the collisionless limit of taking $D \to 0$. Moreover, we demand that effective mixing still have the property of being maximal at the resonance ($\lambda(t_{\rm res}) = 0$) and give way to the standard coherent oscillations results as discussed in our analysis above. We thus find

$$\cos\Theta(t) = \frac{\lambda(t)}{\sqrt{\lambda^2 + \beta^2 + D^2}}, \tag{73}$$

$$\sin\Theta(t) = \frac{\sqrt{\beta^2 + D^2}}{\sqrt{\lambda^2 + \beta^2 + D^2}}. \tag{74}$$

Since now we have an effective mixing angle at our disposal, we can identify an adiabaticity parameter via

$$\gamma \equiv \frac{1}{\omega_{\rm eff}} \frac{{\rm d}\Theta}{{\rm d}t}, \tag{75}$$

which can be calculated to yield

$$\gamma = \frac{1}{\omega_{\rm eff}^3}\left[\frac{\beta\lambda}{\sqrt{\beta^2 + D^2}}\frac{{\rm d}\beta}{{\rm d}t} + \frac{D\lambda}{\sqrt{\beta^2 + D^2}}\frac{{\rm d}D}{{\rm d}t} - \sqrt{\beta^2 + D^2}\frac{{\rm d}\lambda}{{\rm d}t}\right] \tag{76}$$

or rather at the resonance

$$\gamma_{\rm res} = -\frac{1}{\beta^2 + D^2}\frac{{\rm d}\lambda}{{\rm d}t}\bigg|_{t=t_{\rm res}}. \tag{77}$$

Two things are readily observed: Firstly, this result reduces to the standard paradigm of an adiabaticity parameter in the collision-unaffected regime ($D \to 0$) and secondly, it entails an intriguing modification of adiabaticity in the presence of collisions. The latter means that a small β does not necessarily coincide with nonadiabatic neutrino conversions any-more.

As has been mentioned before, an explicit solution for the coherence vector in a collision-affected regime demands good care and is beyond the scope of this work.

VI. CONCLUSIONS

The standard wave function approach for the treatment of neutrino oscillations fails in situations, where quantum ensembles at a finite temperature with or without an interacting background plasma are encountered. A first step to treat such phenomena consists in a thorough analysis of two-flavor neutrino oscillations. The most interesting (and general) system of collision-unaffected neutrino oscillations, which offers some insight into the physics at hand, is an oscillatory neutrino ensemble with generic matter potentials—as would be realized in a MSW-like scenario in which the matter potential due to coherent elastic forward scattering is augmented by nonstandard interactions. We study the time evolution, and hence the flavor oscillations, of such systems in a novel way, which also sheds light on how to extend the formalism on applications beyond collision-unaffected neutrino systems, e.g., collision-affected neutrino conversions in the early Universe.

In our analysis, we focus on providing novel and model-independent methods for solving the underlying quantum kinetic equations for a two-flavor neutrino ensemble at finite temperature subject to generic potentials rather than delving into explicit model building for specific neutrino configurations and hence certain physics models. To this end, we use the Magnus expansion and argue that it has virtues when it comes to perturbatively solve coupled nonautonomous differential equations such as the QKE considered here: the expansion is unitary to each order of the approximation, which is a most convenient feature if oscillation probabilities are calculated; moreover, we get a handle on the convergence properties of the expansion by studying the matrix norm of the underlying evolution matrix. The perturbation ansatz proposed maps the problem of diagonalizing a time evolution matrix with generic potentials (and hence possible time dependences) onto calculating the exponential of a matrix and the associated integrals as its emergent ingredients.

We understand that calculating the matrix norm of the evolution equation (via the Frobenius norm) also reveals new physics insight into the nature of the neutrino ensemble at hand. It is found that the matrix norm coincides with the effective oscillation frequency of the system (up to a proportionality factor). This feature can now be easily generalized to more complicated physics models by *interpreting* the matrix norm of the evolution matrix as the effective oscillation frequency. This can be done without making any reference to the Hamiltonian of the underlying system, which turns out to be particularly useful when a Hamiltonian description of the system at hand ceases to be applicable anyway.

We continue our analysis by performing several changes of basis for the QKE. The benefit of this procedure is twofold: On the one hand, it is of purely mathematical merit in that it demonstrates how successive transformations of the time evolution equation of the system can result in a shape of the latter most suitable for a perturbative solution of the QKE; the exactly solvable part of the differential equation is removed and hence the *small* perturbation to the exactly solvable system *isolated*. On the

other hand, we understand that the successive change of basis for the QKE also provides further physical insight.

A rotation to a *comoving* frame for the QKE reveals that the definition of an effective mixing angle becomes possible, which closely resembles, but, however, is *not* identical to the one encountered in the Hamiltonian formulation of the system at hand. It is found that the cosine of such a mixing angle is simply given by the expression containing the potential term normalized to the effective oscillation frequency. Again, this notion of an effective mixing angle is (largely) independent of the explicit shape of the time evolution matrix and could hence be adopted to seek a similar definition of effective mixing in more elaborate systems.

Moreover, with the oscillation frequency (a characteristic time scale of the system) and the effective mixing (a characteristic time scale of the interaction) at our disposal, it is straightforward to define an adiabaticity parameter of the system. Note, that all this physics insight is basically model independent and can easily be generalized to more complicated systems without the necessity of solving for the coherence vector explicitly.

Applying our novel paradigm for solving the QKE to the system of two-flavor neutrino conversions at a finite temperature subject to generic potentials, we obtain a perturbative solution for the coherence vector in the adiabatic regime. Moreover, keeping in mind the strategy to tackle QKE promulgated in our analysis, we understand that with the aid of the adiabaticity parameter as constructed during our analysis it is straightforward to develop a nonadiabatic perturbation theory on the same theoretical considerations as applied for the adiabatic case. This result is yet another advantage of our approach in that it treats both adiabatic and nonadiabatic neutrino conversions on the same ground; the two distinct physical situations do not have to be studied with two distinct perturbation theories. We take comfort that our approach yields accurate results since we demonstrate that it respects several limiting cases of the physics under considerations.

In order to illustrate the usefulness of our approach, we also eventually give an outlook how the ideas put forward in this article can be used to describe collision-affected neutrino conversions in the early Universe. Therefore, we understand that our analysis presents a promising basis for further investigations concerning the inclusion of decoherence in the ensemble and the resulting applications in early universe scenarios, e.g., the generation of lepton number in active-sterile oscillations prior to BBN.

ACKNOWLEDGMENTS

We are grateful to Danny van Dyk and Thomas J. Weiler for discussions on the topic during the course of this work.

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[10] Note that this is in contrast to the Dyson series solution which is frequently encountered in quantum field theory applications. The latter relies on time-ordered products of operators which sees some disadvantages if the series has to be truncated, e.g., that unitarity is only restored up to the order considered, i.e., by explicitly neglecting higher-order contributions.
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[14] We are dealing with a finite dimensional vector space and hence all matrix norms are equivalent. We choose to work with the unitarily invariant Frobenius norm $\|S\|_{\mathrm{F}}^2 = \mathrm{Tr}(S^\dagger S)$.
[15] A standard paradigm from cosmology derives a time-temperature relation using conservation of comoving entropy and the Friedmann equations to obtain $\mathrm{d}t(T) = -2\bar{m}_{\mathrm{Pl}} \frac{\mathrm{d}T}{T^3}$ with $\bar{m}_{\mathrm{Pl}} = \sqrt{\frac{90}{32\pi^3 g_*}} m_{\mathrm{Pl}}$, where m_{Pl} is the Planck mass and g_* gives the effective degrees of freedom

at the epoch under consideration. Keeping this in mind, all time dependences occurring throughout our analysis can easily be converted into temperature dependences, which might be more convenient in, for instance, early universe applications.

[16] Note that for the perturbation theory to be developed, we adopt here the notion of adiabaticity versus nonadiabaticity of Refs. [17,18].

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[20] Note, however, that an additional restriction in the derivation of the QKE emerges. If a treatment using the S matrix is employed, it is understood that the initial and final scattering states evolve as *free states*. This is only the case if the plasma is sufficiently dilute and the quanta do not spend most of their time interacting. For the case of a weakly interacting gas of relativistic particles, a rough estimate yields that the approach using the S matrix should be valid for temperatures $T \ll 100$ GeV, which does not interfere with the temperature regime under consideration here.

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