

Full parametrization of the 6×6 flavor mixing matrix in the presence of three light or heavy sterile neutrinos

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In addition to three active neutrinos ν_e , ν_μ , and ν_τ , one or more light sterile neutrinos have been conjectured to account for the LSND, MiniBooNE, and reactor antineutrino anomalies (at the sub-eV mass scale) or for warm dark matter in the Universe (at the keV mass scale). Heavy Majorana neutrinos at or above the TeV scale have also been assumed in some seesaw models. Such hypothetical particles can weakly mix with active neutrinos, and thus their existence can be detected at low energies. In the $(3 + 3)$ scenario with three sterile neutrinos we present a full parametrization of the 6×6 flavor mixing matrix in terms of 15 rotation angles and 15 phase angles. We show that this standard parametrization allows us to clearly describe the salient features of some problems in neutrino phenomenology, such as (a) possible contributions of light sterile neutrinos to the tritium beta decay and neutrinoless double-beta decay, (b) leptonic CP violation and deformed unitarity triangles of the 3×3 flavor mixing matrix of three active neutrinos, (c) a reconstruction of the 6×6 neutrino mass matrix in the type-(I+II) seesaw mechanism, and (d) flavored and unflavored leptogenesis scenarios in the type-I seesaw mechanism with three heavy Majorana neutrinos.

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I. INTRODUCTION

One of the fundamental questions in neutrino physics and cosmology is whether there exist extra species of neutrinos which do not directly participate in the standard weak interactions. Such sterile neutrinos are certainly hypothetical, but their possible existence is either theoretically motivated or experimentally implied. For example,

- (i) heavy Majorana neutrinos at or above the TeV scale are expected in many seesaw models [1], which can not only interpret the small masses of three active neutrinos but can also account for the cosmological matter-antimatter asymmetry via the leptogenesis mechanism [2];
- (ii) the LSND antineutrino anomaly [3], the MiniBooNE antineutrino anomaly [4], and the reactor antineutrino anomaly [5] can all be explained as the active-sterile antineutrino oscillations in the assumption of two species of sterile antineutrinos whose masses are close to 1 eV [6];
- (iii) an analysis of the existing data on the CMB, galaxy clustering, and supernovae Ia favors some extra radiation content in the Universe and one or two species of sterile neutrinos at the sub-eV mass scale [7]¹;
- (iv) sufficiently long-lived sterile neutrinos in the keV mass range can serve as a good candidate for warm dark matter, whose presence may allow us to solve

or soften several problems that we have recently encountered in the dark matter simulations [9] (e.g., to damp the inhomogeneities on small scales by reducing the number of dwarf galaxies or to smooth the cusps in the dark matter halos).²

No matter how small or how large the mass scale of sterile neutrinos is, they are undetectable unless they mix with three active neutrinos to some extent. The strength of active-sterile neutrino mixing can be described in terms of some rotation angles and phase angles, just like the parametrization of the 3×3 quark flavor mixing in the standard model [13].

The main purpose of this paper is to present a full parametrization of the 6×6 flavor mixing matrix \mathcal{U} in the $(3 + 3)$ scenario with three sterile neutrinos denoted as ν_x , ν_y , and ν_z :

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \\ \nu_x \\ \nu_y \\ \nu_z \end{pmatrix} = \mathcal{U} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \\ \nu_4 \\ \nu_5 \\ \nu_6 \end{pmatrix}, \quad (1)$$

where ν_i (for $i = 1, \dots, 6$) stand for the mass eigenstates of active and sterile neutrinos. Such a complete parametriza-

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¹If the bound obtained from the big bang nucleosynthesis is taken into account, however, only one species of light sterile neutrinos and antineutrinos is allowed [8].

²There are some interesting models which can accommodate sterile neutrinos at either keV [10] or sub-eV [11] mass scales. A model-independent argument is also supporting the conjecture of warm dark matter particles hiding out in the “flavor desert” of the fermion mass spectrum [12].

tion, which has been lacking in the literature [14], is expected to be very useful for the study of neutrino phenomenology at both low and high energy scales. We propose a simple but novel way to establish the connection between active and sterile neutrinos in terms of 15 mixing angles and 15 CP -violating phases. It allows us to clearly describe the salient features of some interesting problems, such as (a) possible contributions of light sterile neutrinos to the tritium beta (β) decay and neutrinoless double-beta ($0\nu 2\beta$) decay, (b) leptonic CP violation and deformed unitarity triangles of the 3×3 flavor mixing matrix of three active neutrinos, (c) a reconstruction of the 6×6 neutrino mass matrix in the type-(I+II) seesaw mechanism, and (d) flavored and unflavored leptogenesis scenarios in the type-I seesaw mechanism with three heavy Majorana neutrinos.

II. THE STANDARD PARAMETRIZATION

The 6×6 unitary matrix \mathcal{U} defined in Eq. (1) can be decomposed as

$$\mathcal{U} = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & U_0 \end{pmatrix} \begin{pmatrix} A & R \\ S & B \end{pmatrix} \begin{pmatrix} V_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}, \quad (2)$$

in which $\mathbf{0}$ and $\mathbf{1}$ stand, respectively, for the 3×3 zero and identity matrices, U_0 and V_0 are the 3×3 unitary matrices, and A , B , R , and S are the 3×3 matrices which satisfy the conditions

$$\begin{aligned} AA^\dagger + RR^\dagger &= BB^\dagger + SS^\dagger = \mathbf{1}, \\ AS^\dagger + RB^\dagger &= A^\dagger R + S^\dagger B = \mathbf{0}, \\ A^\dagger A + S^\dagger S &= B^\dagger B + R^\dagger R = \mathbf{1}, \end{aligned} \quad (3)$$

as a result of the unitarity of \mathcal{U} . In the limit of $R = S = \mathbf{0}$, $A = B = \mathbf{1}$ holds, and thus there is no correlation between the active sector (described by V_0) and the sterile sector (characterized by U_0). In view of Eq. (A7) in Appendix A, we parametrize \mathcal{U} as follows:

$$\begin{aligned} A &= \begin{pmatrix} c_{14}c_{15}c_{16} & 0 & 0 \\ -c_{14}c_{15}\hat{\delta}_{16}\hat{\delta}_{26}^* - c_{14}\hat{\delta}_{15}\hat{\delta}_{25}^*c_{26} & c_{24}c_{25}c_{26} & 0 \\ -\hat{\delta}_{14}\hat{\delta}_{24}^*c_{25}c_{26} & -c_{24}c_{25}\hat{\delta}_{26}\hat{\delta}_{36}^* - c_{24}\hat{\delta}_{25}\hat{\delta}_{35}^*c_{36} & c_{34}c_{35}c_{36} \\ -c_{14}c_{15}\hat{\delta}_{16}c_{26}\hat{\delta}_{36}^* + c_{14}\hat{\delta}_{15}\hat{\delta}_{25}^*\hat{\delta}_{26}\hat{\delta}_{36}^* & -\hat{\delta}_{24}\hat{\delta}_{34}^*c_{35}c_{36} & \\ -c_{14}\hat{\delta}_{15}c_{25}\hat{\delta}_{35}^*c_{36} + \hat{\delta}_{14}\hat{\delta}_{24}^*c_{25}\hat{\delta}_{26}\hat{\delta}_{36}^* & & \\ +\hat{\delta}_{14}\hat{\delta}_{24}^*\hat{\delta}_{25}\hat{\delta}_{35}^*c_{36} - \hat{\delta}_{14}c_{24}\hat{\delta}_{34}^*c_{35}c_{36} & & \end{pmatrix}, \\ B &= \begin{pmatrix} c_{14}c_{24}c_{34} & 0 & 0 \\ -c_{14}c_{24}\hat{\delta}_{34}^*\hat{\delta}_{35} - c_{14}\hat{\delta}_{24}^*\hat{\delta}_{25}c_{35} & c_{15}c_{25}c_{35} & 0 \\ -\hat{\delta}_{14}^*\hat{\delta}_{15}c_{25}c_{35} & -c_{15}c_{25}\hat{\delta}_{35}^*\hat{\delta}_{36} - c_{15}\hat{\delta}_{25}^*\hat{\delta}_{26}c_{36} & c_{16}c_{26}c_{36} \\ -c_{14}c_{24}\hat{\delta}_{34}^*c_{35}\hat{\delta}_{36} + c_{14}\hat{\delta}_{24}^*\hat{\delta}_{25}\hat{\delta}_{35}^*\hat{\delta}_{36} & -\hat{\delta}_{15}^*\hat{\delta}_{16}c_{26}c_{36} & \\ -c_{14}\hat{\delta}_{24}^*c_{25}\hat{\delta}_{26}c_{36} + \hat{\delta}_{14}^*\hat{\delta}_{15}c_{25}\hat{\delta}_{35}^*\hat{\delta}_{36} & & \\ +\hat{\delta}_{14}^*\hat{\delta}_{15}\hat{\delta}_{25}^*\hat{\delta}_{26}c_{36} - \hat{\delta}_{14}^*c_{15}\hat{\delta}_{16}c_{26}c_{36} & & \end{pmatrix}, \end{aligned} \quad (6)$$

and

$$\begin{aligned} \begin{pmatrix} V_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} &= O_{23}O_{13}O_{12}, & \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & U_0 \end{pmatrix} &= O_{56}O_{46}O_{45}, \\ \begin{pmatrix} A & R \\ S & B \end{pmatrix} &= O_{36}O_{26}O_{16}O_{35}O_{25}O_{15}O_{34}O_{24}O_{14}, \end{aligned} \quad (4)$$

where 15 two-dimensional rotation matrices O_{ij} (for $1 \leq i < j \leq 6$) in a six-dimensional complex space have been given in Eqs. (A2)–(A6). To be explicit,

$$\begin{aligned} V_0 &= \begin{pmatrix} c_{12}c_{13} & \hat{\delta}_{12}^*c_{13} & \hat{\delta}_{13}^* \\ -\hat{\delta}_{12}c_{23} - c_{12}\hat{\delta}_{13}\hat{\delta}_{23}^* & c_{12}c_{23} - \hat{\delta}_{12}^*\hat{\delta}_{13}\hat{\delta}_{23}^* & c_{13}\hat{\delta}_{23}^* \\ \hat{\delta}_{12}\hat{\delta}_{23} - c_{12}\hat{\delta}_{13}c_{23} & -c_{12}\hat{\delta}_{23} - \hat{\delta}_{12}^*\hat{\delta}_{13}c_{23} & c_{13}c_{23} \end{pmatrix}, \\ U_0 &= \begin{pmatrix} c_{45}c_{46} & \hat{\delta}_{45}^*c_{46} & \hat{\delta}_{46}^* \\ -\hat{\delta}_{45}c_{56} - c_{45}\hat{\delta}_{46}\hat{\delta}_{56}^* & c_{45}c_{56} - \hat{\delta}_{45}^*\hat{\delta}_{46}\hat{\delta}_{56}^* & c_{46}\hat{\delta}_{56}^* \\ \hat{\delta}_{45}\hat{\delta}_{56} - c_{45}\hat{\delta}_{46}c_{56} & -c_{45}\hat{\delta}_{56} - \hat{\delta}_{45}^*\hat{\delta}_{46}c_{56} & c_{46}c_{56} \end{pmatrix}, \end{aligned} \quad (5)$$

in which $c_{ij} \equiv \cos\theta_{ij}$ and $\hat{\delta}_{ij} \equiv e^{i\delta_{ij}} \sin\theta_{ij}$, with θ_{ij} and δ_{ij} being the rotation angle and phase angle, respectively. Both V_0 and U_0 have the standard form as advocated in Ref. [13], and either of them consists of three mixing angles and three CP -violating phases. If the sterile sector is switched off, we are then left with the 3×3 unitary matrix V_0 which describes the flavor mixing of three active neutrinos. If the active sector is switched off, one will arrive at the 3×3 unitary matrix U_0 which purely describes the flavor mixing of three sterile neutrinos. In the type-I seesaw mechanism [1], for example, U_0 is essentially equivalent to the unitary transformation used to diagonalize the 3×3 heavy Majorana neutrino mass matrix M_R and therefore relevant to the leptogenesis mechanism [2].

With the help of Eqs. (4) and (A2)–(A6), a lengthy but straightforward calculation leads us to the explicit expressions of A , B , R , and S as follows:

$$\begin{aligned}
R &= \begin{pmatrix} \hat{s}_{14}^* c_{15} c_{16} & \hat{s}_{15}^* c_{16} & \hat{s}_{16}^* \\ -\hat{s}_{14}^* c_{15} \hat{s}_{16} \hat{s}_{26}^* - \hat{s}_{14}^* \hat{s}_{15} \hat{s}_{25}^* c_{26} & -\hat{s}_{15}^* \hat{s}_{16} \hat{s}_{26}^* + c_{15} \hat{s}_{25}^* c_{26} & c_{16} \hat{s}_{26}^* \\ +c_{14} \hat{s}_{24}^* c_{25} c_{26} & & \\ -\hat{s}_{14}^* c_{15} \hat{s}_{16} c_{26} \hat{s}_{36}^* + \hat{s}_{14}^* \hat{s}_{15} \hat{s}_{25}^* \hat{s}_{26} \hat{s}_{36}^* & -\hat{s}_{15}^* \hat{s}_{16} c_{26} \hat{s}_{36}^* - c_{15} \hat{s}_{25}^* \hat{s}_{26} \hat{s}_{36}^* & c_{16} c_{26} \hat{s}_{36}^* \\ -\hat{s}_{14}^* \hat{s}_{15} c_{25} \hat{s}_{35}^* c_{36} - c_{14} \hat{s}_{24}^* c_{25} \hat{s}_{26} \hat{s}_{36}^* & +c_{15} c_{25} \hat{s}_{35}^* c_{36} & \\ -c_{14} \hat{s}_{24}^* \hat{s}_{25} \hat{s}_{35}^* c_{36} + c_{14} c_{24} \hat{s}_{34}^* c_{35} c_{36} & & \end{pmatrix}, \\
S &= \begin{pmatrix} -\hat{s}_{14} c_{24} c_{34} & -\hat{s}_{24} c_{34} & -\hat{s}_{34} \\ \hat{s}_{14} c_{24} \hat{s}_{34}^* \hat{s}_{35} + \hat{s}_{14} \hat{s}_{24}^* \hat{s}_{25} c_{35} & \hat{s}_{24} \hat{s}_{34}^* \hat{s}_{35} - c_{24} \hat{s}_{25} c_{35} & -c_{34} \hat{s}_{35} \\ -c_{14} \hat{s}_{15} c_{25} c_{35} & & \\ \hat{s}_{14} c_{24} \hat{s}_{34}^* c_{35} \hat{s}_{36} - \hat{s}_{14} \hat{s}_{24}^* \hat{s}_{25} \hat{s}_{35}^* \hat{s}_{36} & \hat{s}_{24} \hat{s}_{34}^* c_{35} \hat{s}_{36} + c_{24} \hat{s}_{25} \hat{s}_{35}^* \hat{s}_{36} & -c_{34} c_{35} \hat{s}_{36} \\ +\hat{s}_{14} \hat{s}_{24}^* c_{25} \hat{s}_{26} c_{36} + c_{14} \hat{s}_{15} c_{25} \hat{s}_{35}^* \hat{s}_{36} & -c_{24} c_{25} \hat{s}_{26} c_{36} & \\ +c_{14} \hat{s}_{15} \hat{s}_{25}^* \hat{s}_{26} c_{36} - c_{14} c_{15} \hat{s}_{16} c_{26} c_{36} & & \end{pmatrix}. \tag{7}
\end{aligned}$$

We see that the textures of A and B are rather similar, and so are the textures of R and S . In fact, the expression of B can be obtained from that of A^* with the subscript replacements $15 \leftrightarrow 24$, $16 \leftrightarrow 34$, and $26 \leftrightarrow 35$; the expression of S can be obtained from that of $-R^*$ with the same subscript replacements. Note that the results of A and R have been obtained in Ref. [15], and here we present the results of B and S to complete a full parametrization of the 6×6 flavor mixing matrix \mathcal{U} .

It proves convenient to define $V \equiv AV_0$ and $U \equiv U_0B$, which describe the flavor mixing phenomena of three active neutrinos and three sterile neutrinos, respectively. Furthermore, $\hat{S} \equiv U_0SV_0$ links the mass eigenstates (ν_1, ν_2, ν_3) to the sterile flavor eigenstates (ν_x, ν_y, ν_z) in the chosen basis. We therefore have

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = V \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} + R \begin{pmatrix} \nu_4 \\ \nu_5 \\ \nu_6 \end{pmatrix} \tag{8}$$

and

$$\begin{pmatrix} \nu_x \\ \nu_y \\ \nu_z \end{pmatrix} = U \begin{pmatrix} \nu_4 \\ \nu_5 \\ \nu_6 \end{pmatrix} + \hat{S} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}. \tag{9}$$

Equation (8) directly leads us to the standard weak charged-current interactions of six neutrinos:

$$-\mathcal{L}_{cc} = \frac{g}{\sqrt{2}} (\overline{e\mu\tau})_L \gamma^\mu \left[V \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}_L + R \begin{pmatrix} \nu_4 \\ \nu_5 \\ \nu_6 \end{pmatrix}_L \right] W_\mu^- + \text{H.c.}, \tag{10}$$

where V is just the Maki-Nakagawa-Sakata-Pontecorvo matrix [16] responsible for the active neutrino mixing, and R measures the strength of charged-current interactions between (e, μ, τ) and (ν_4, ν_5, ν_6) . Because of

$$VV^\dagger = AA^\dagger = \mathbf{1} - RR^\dagger, \quad U^\dagger U = B^\dagger B = \mathbf{1} - R^\dagger R, \tag{11}$$

we find that neither V nor U is exactly unitary and their nonunitary effects are simply characterized by nonvanishing R and S .

In view of current observational constraints on sterile neutrinos, we expect that the mixing angles between active and sterile neutrinos are strongly suppressed [at most, at the $\mathcal{O}(0.1)$ level [6]³]. The smallness of θ_{ij} (for $i = 1, 2, 3$ and $j = 4, 5, 6$) allows us to make the following excellent approximations to Eqs. (7) and (8):

$$\begin{aligned}
A &\simeq \mathbf{1} - \begin{pmatrix} \frac{1}{2}(s_{14}^2 + s_{15}^2 + s_{16}^2) & 0 & 0 \\ \hat{s}_{14} \hat{s}_{24}^* + \hat{s}_{15} \hat{s}_{25}^* + \hat{s}_{16} \hat{s}_{26}^* & \frac{1}{2}(s_{24}^2 + s_{25}^2 + s_{26}^2) & 0 \\ \hat{s}_{14} \hat{s}_{34}^* + \hat{s}_{15} \hat{s}_{35}^* + \hat{s}_{16} \hat{s}_{36}^* & \hat{s}_{24} \hat{s}_{34}^* + \hat{s}_{25} \hat{s}_{35}^* + \hat{s}_{26} \hat{s}_{36}^* & \frac{1}{2}(s_{34}^2 + s_{35}^2 + s_{36}^2) \end{pmatrix}, \\
B &\simeq \mathbf{1} - \begin{pmatrix} \frac{1}{2}(s_{14}^2 + s_{24}^2 + s_{34}^2) & 0 & 0 \\ \hat{s}_{14}^* \hat{s}_{15} + \hat{s}_{24}^* \hat{s}_{25} + \hat{s}_{34}^* \hat{s}_{35} & \frac{1}{2}(s_{15}^2 + s_{25}^2 + s_{35}^2) & 0 \\ \hat{s}_{14}^* \hat{s}_{16} + \hat{s}_{24}^* \hat{s}_{26} + \hat{s}_{34}^* \hat{s}_{36} & \hat{s}_{15}^* \hat{s}_{16} + \hat{s}_{25}^* \hat{s}_{26} + \hat{s}_{35}^* \hat{s}_{36} & \frac{1}{2}(s_{16}^2 + s_{26}^2 + s_{36}^2) \end{pmatrix}, \tag{12}
\end{aligned}$$

³For example, the nonunitarity of $V = AV_0$ or the deviation of V from V_0 can at most be at the 1% level as constrained by current neutrino oscillation data and precision electroweak data [17].

where the terms of $\mathcal{O}(s_{ij}^4)$ have been omitted, and

$$R \simeq \mathbf{0} + \begin{pmatrix} \hat{s}_{14}^* & \hat{s}_{15}^* & \hat{s}_{16}^* \\ \hat{s}_{24}^* & \hat{s}_{25}^* & \hat{s}_{26}^* \\ \hat{s}_{34}^* & \hat{s}_{35}^* & \hat{s}_{36}^* \end{pmatrix},$$

$$S \simeq \mathbf{0} - \begin{pmatrix} \hat{s}_{14} & \hat{s}_{24} & \hat{s}_{34} \\ \hat{s}_{15} & \hat{s}_{25} & \hat{s}_{35} \\ \hat{s}_{16} & \hat{s}_{26} & \hat{s}_{36} \end{pmatrix}, \quad (13)$$

where the terms of $\mathcal{O}(s_{ij}^3)$ have been omitted. It turns out that $R \simeq -S^\dagger$ holds in the same approximation.

Note that the 6×6 unitary matrix \mathcal{U} can be used to describe not only the flavor mixing between active and sterile neutrinos, but also the flavor mixing between ordinary and extra quarks. Note also that it is straightforward to obtain the $(3 + 1)$ flavor mixing scenario from Eqs. (6) and (7) by switching off the mixing angles θ_{i5} and θ_{j6} (for $1 \leq i \leq 4$ and $1 \leq j \leq 5$), or the $(3 + 2)$ flavor mixing scenario by turning off the mixing angles θ_{j6} (for $1 \leq j \leq 5$).

III. SOME APPLICATIONS

To illustrate the usefulness of our parametrization of the 6×6 flavor mixing matrix \mathcal{U} , let us briefly discuss its four simple but instructive applications in neutrino phenomenology.

A. The effective masses of β and $0\nu 2\beta$ decays

One or two light sterile neutrinos at the sub-eV mass scale have been hypothesized for quite a long time to interpret the LSND antineutrino anomaly [3] and the subsequent MiniBooNE antineutrino puzzle [4]. In general, however, it seems more natural to assume the number of sterile neutrino species to be equal to that of active neutrino species [18],⁴ such that even possible warm dark matter in the form of one or two species of keV sterile neutrinos could be taken into account.

For simplicity and illustration, we are only concerned about the effective masses of the tritium beta (β) decay ${}^3_1\text{H} \rightarrow {}^3_2\text{He} + e^- + \bar{\nu}_e$ and the neutrinoless double-beta ($0\nu 2\beta$) decay $A(Z, A) \rightarrow A(Z + 2, N - 2) + 2e^-$ in the $(3 + 3)$ neutrino mixing scenario. The former is

$$\langle m \rangle'_e \equiv \left[\sum_{i=1}^6 m_i^2 |V_{ei}|^2 \right]^{1/2}$$

$$= \sqrt{\langle m \rangle_e^2 c_{14}^2 c_{15}^2 c_{16}^2 + m_4^2 s_{14}^2 c_{15}^2 c_{16}^2 + m_5^2 s_{15}^2 c_{16}^2 + m_6^2 s_{16}^2}, \quad (14)$$

⁴In order to avoid any severe conflict between such a $(3 + 3)$ scenario and the standard Λ CDM cosmology, it is perhaps necessary to either loosen the mass hierarchy of three sterile neutrinos [i.e., not all of them are of $\mathcal{O}(0.1)$ eV] or refer to some nonstandard models of cosmology [7].

where $\langle m \rangle_e = \sqrt{m_1^2 c_{12}^2 c_{13}^2 + m_2^2 s_{12}^2 c_{13}^2 + m_3^2 s_{13}^2}$ is the standard contribution from three active neutrinos. We see that $\langle m \rangle'_e \geq \langle m \rangle_e$ always holds. The effective mass of the $0\nu 2\beta$ decay is

$$\langle m \rangle'_{ee} \equiv \sum_{i=1}^6 m_i V_{ei}^2$$

$$= \langle m \rangle_{ee} (c_{14} c_{15} c_{16})^2 + m_4 (\hat{s}_{14}^* c_{15} c_{16})^2$$

$$+ m_5 (\hat{s}_{15}^* c_{16})^2 + m_6 (\hat{s}_{16}^*)^2 \quad (15)$$

with $\langle m \rangle_{ee} = m_1 (c_{12} c_{13})^2 + m_2 (\hat{s}_{12}^* c_{13})^2 + m_3 (\hat{s}_{13}^*)^2$ being the standard contribution from three active neutrinos. It is difficult to compare between the magnitudes of $\langle m \rangle_{ee}$ and $\langle m \rangle'_{ee}$, because the CP -violating phases may give rise to more or less cancellations of different terms in them. In particular, even $\langle m \rangle_{ee} = 0$ [19] or $\langle m \rangle'_{ee} = 0$ [20] is not impossible. If both $\langle m \rangle'_e$ and $\langle m \rangle'_{ee}$ can be determined or constrained in future experiments, a comparison between them might be able to probe the existence of light sterile neutrinos [21].

B. Deformed unitarity triangles and CP violation

Switching off three sterile neutrinos, one may describe flavor mixing and CP violation of three active neutrinos in terms of six unitarity triangles in the complex plane [22]. Three of them, defined by the orthogonality conditions

$$\Delta_e: V_{\mu 1} V_{\tau 1}^* + V_{\mu 2} V_{\tau 2}^* + V_{\mu 3} V_{\tau 3}^* = 0,$$

$$\Delta_\mu: V_{\tau 1} V_{e 1}^* + V_{\tau 2} V_{e 2}^* + V_{\tau 3} V_{e 3}^* = 0,$$

$$\Delta_\tau: V_{e 1} V_{\mu 1}^* + V_{e 2} V_{\mu 2}^* + V_{e 3} V_{\mu 3}^* = 0, \quad (16)$$

are illustrated in Fig. 1 (left panel). The area of each triangle is equal to $J_0/2$, where J_0 is the Jarlskog parameter given in Eq. (B2) and measures the strength of leptonic CP -violating effects in $\nu_\mu \rightarrow \nu_\tau$, $\nu_\tau \rightarrow \nu_e$, and $\nu_e \rightarrow \nu_\mu$ oscillations. Now let us turn on the contributions of three sterile neutrinos to flavor mixing and CP violation. Then Eq. (16) approximates to

$$\Delta'_e: V_{\mu 1} V_{\tau 1}^* + V_{\mu 2} V_{\tau 2}^* + V_{\mu 3} V_{\tau 3}^* \simeq -Z^*,$$

$$\Delta'_\mu: V_{\tau 1} V_{e 1}^* + V_{\tau 2} V_{e 2}^* + V_{\tau 3} V_{e 3}^* \simeq -\mathcal{Y},$$

$$\Delta'_\tau: V_{e 1} V_{\mu 1}^* + V_{e 2} V_{\mu 2}^* + V_{e 3} V_{\mu 3}^* \simeq -\mathcal{X}^*, \quad (17)$$

where $\mathcal{X} \equiv \hat{s}_{14} \hat{s}_{24}^* + \hat{s}_{15} \hat{s}_{25}^* + \hat{s}_{16} \hat{s}_{26}^*$, $\mathcal{Y} \equiv \hat{s}_{14} \hat{s}_{34}^* + \hat{s}_{15} \hat{s}_{35}^* + \hat{s}_{16} \hat{s}_{36}^*$, and $Z \equiv \hat{s}_{24} \hat{s}_{34}^* + \hat{s}_{25} \hat{s}_{35}^* + \hat{s}_{26} \hat{s}_{36}^*$. These deformed unitarity triangles are also illustrated in Fig. 1 (right panel). The small differences of their areas from $J_0/2$ just signify the new CP -violating effects.

Let us take a look at the CP -violating asymmetries between $\nu_\alpha \rightarrow \nu_\beta$ and $\bar{\nu}_\alpha \rightarrow \bar{\nu}_\beta$ oscillations, defined as $\mathcal{A}_{\alpha\beta} \equiv P(\nu_\alpha \rightarrow \nu_\beta) - P(\bar{\nu}_\alpha \rightarrow \bar{\nu}_\beta)$. With the help of Eq. (B5), we explicitly obtain

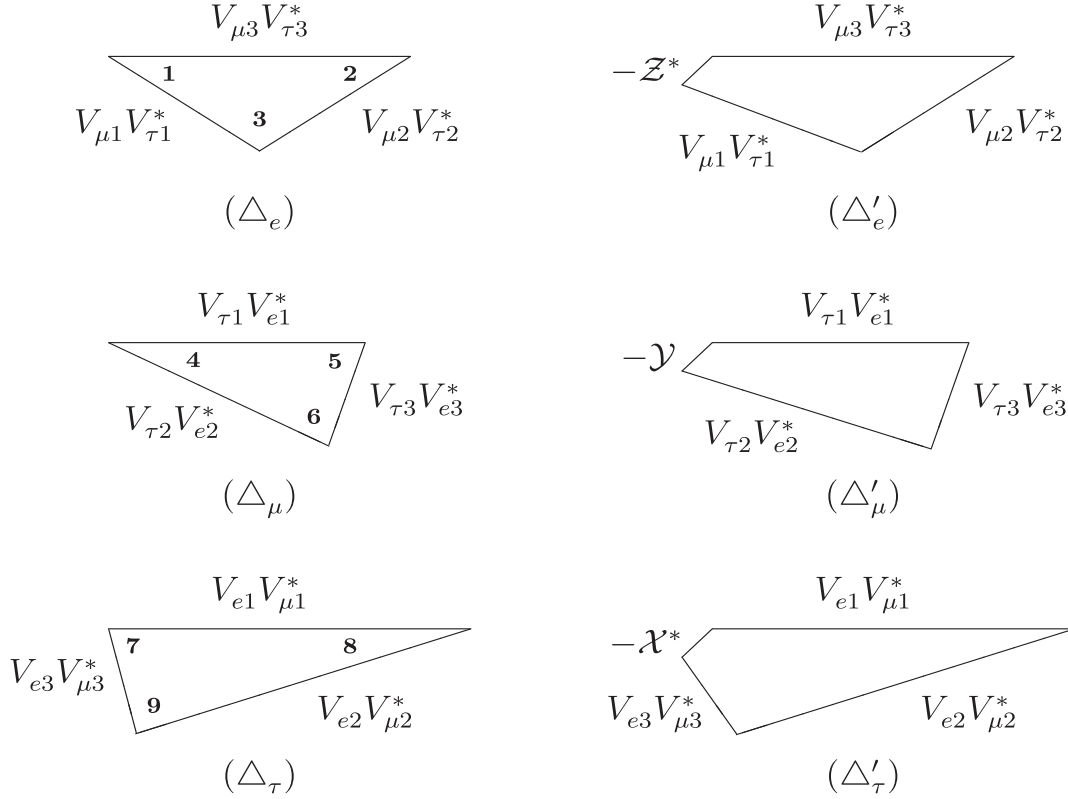


FIG. 1. Left panel: Three unitarity triangles of $V \equiv AV_0$ in the absence of three sterile neutrinos (i.e., $A = \mathbf{1}$ and $V = V_0$). Right panel: Three deformed unitarity triangles of $V \equiv AV_0$ in the presence of three sterile neutrinos, where \mathcal{X} , \mathcal{Y} , and \mathcal{Z} are defined below Eq. (17).

$$\begin{aligned}
 \mathcal{A}_{\mu e} &\simeq -4(J_0 + c_{12}s_{12}c_{23} \text{Im}X) \sin \frac{\Delta m_{21}^2 L}{2E}, \\
 \mathcal{A}_{e\tau} &\simeq -4(J_0 + c_{12}s_{12}s_{23} \text{Im}Y) \sin \frac{\Delta m_{21}^2 L}{2E}, \\
 \mathcal{A}_{\mu\tau} &\simeq +4[J_0 + c_{12}s_{12}c_{23}s_{23}(s_{23} \text{Im}X + c_{23} \text{Im}Y)] \\
 &\quad \times \sin \frac{\Delta m_{21}^2 L}{2E} + 4c_{23}s_{23} \text{Im}Z \sin \frac{\Delta m_{32}^2 L}{2E}, \quad (18)
 \end{aligned}$$

where $X \equiv \mathcal{X}e^{-i\delta_{12}}$, $Y \equiv \mathcal{Y}e^{-i(\delta_{12} + \delta_{23})}$, and $Z \equiv \mathcal{Z}e^{-i\delta_{23}}$. We see that $\mathcal{A}_{\mu e}$ and $\mathcal{A}_{e\tau}$ are related to the deformed unitarity triangles Δ'_e and Δ'_μ , respectively. In comparison, Δ'_τ has something to do with $\mathcal{A}_{\mu\tau}$. It is therefore possible to determine three new CP -violating terms $\text{Im}X$, $\text{Im}Y$, and $\text{Im}Z$ by measuring the CP -violating effects in neutrino oscillations. Note that three CP -violating asymmetries in Eq. (18) satisfy the correlation

$$\mathcal{A}_{\mu\tau} + (s_{23}^2 \mathcal{A}_{\mu e} + c_{23}^2 \mathcal{A}_{e\tau}) \simeq 4c_{23}s_{23} \text{Im}Z \sin \frac{\Delta m_{32}^2 L}{2E}. \quad (19)$$

When $\Delta m_{32}^2 L/E \sim \pi$ holds, both $\mathcal{A}_{\mu e}$ and $\mathcal{A}_{e\tau}$ are suppressed such that $\mathcal{A}_{\mu\tau}$ becomes a pure measure of the nonunitary CP -violating parameter $\text{Im}Z$. This interesting possibility, together with terrestrial matter effects, has been discussed before (e.g., Refs. [15,23]).

C. Reconstruction of the 6×6 neutrino mass matrix

The type-(I+II) seesaw mechanism [24] is a good example to illustrate the flavor mixing between three active neutrinos and three heavy Majorana neutrinos. In this mechanism the mass term of six neutrinos is usually written as

$$-\mathcal{L}_{\text{mass}} = \frac{1}{2} \overline{(\nu_L N_R^c)} \begin{pmatrix} M_L & M_D \\ M_D^T & M_R \end{pmatrix} \begin{pmatrix} \nu_L^c \\ N_R \end{pmatrix} + \text{H.c.}, \quad (20)$$

where ν_L and N_R represent the column vectors of three left-handed neutrinos and three right-handed neutrinos, respectively. The overall 6×6 neutrino mass matrix in Eq. (20) can be diagonalized by a unitary transformation:

$$\mathcal{U}^\dagger \begin{pmatrix} M_L & M_D \\ M_D^T & M_R \end{pmatrix} \mathcal{U}^* = \begin{pmatrix} \hat{M}_\nu & \mathbf{0} \\ \mathbf{0} & \hat{M}_N \end{pmatrix}, \quad (21)$$

where \mathcal{U} is already given in Eq. (2); $\hat{M}_\nu \equiv \text{Diag}\{m_1, m_2, m_3\}$ and $\hat{M}_N \equiv \text{Diag}\{M_1, M_2, M_3\}$, with m_i or M_i (for $i = 1, 2, 3$) being the physical masses of light or heavy Majorana neutrinos. The standard weak charged-current interactions of six neutrinos are given by Eq. (10) with $\nu_4 = M_1$, $\nu_5 = N_2$, and $\nu_6 = N_3$ in the basis of mass eigenstates. With the help of Eq. (2),

$$\begin{aligned}
M_L &= V\hat{M}_\nu V^T + R\hat{M}_N R^T \simeq V_0\hat{M}_\nu V_0^T + R\hat{M}_N R^T, \\
M_D &= V\hat{M}_\nu \hat{S}^T + R\hat{M}_N U^T \simeq R\hat{M}_N U_0^T, \\
M_R &= \hat{S}\hat{M}_\nu \hat{S}^T + U\hat{M}_N U^T \simeq U_0\hat{M}_N U_0^T,
\end{aligned} \tag{22}$$

where $V \equiv AV_0$, $U \equiv U_0B$, and $\hat{S} \equiv U_0SV_0$ have been defined before. The approximations made on the right-hand side of Eq. (22) follow the spirit that only the leading terms of M_L , M_D , and M_R are kept. It is then possible to reconstruct these 3×3 neutrino mass matrices, at least in principle, in terms of neutrino masses and flavor mixing parameters [25].

Given the basis where M_R is diagonal, real, and positive, Eq. (22) implies that $M_R \simeq \hat{M}_N$ together with $U_0 \simeq \mathbf{1}$ is a

good approximation. Note that such a flavor basis is often chosen in the study of leptogenesis, because the decays of N_i (for $i = 1, 2, 3$) need to be calculated. It is also much easier to reconstruct M_D and M_L in this special basis. For example,

$$M_D \simeq R\hat{M}_N \simeq \begin{pmatrix} M_1\hat{s}_{14}^* & M_2\hat{s}_{15}^* & M_3\hat{s}_{16}^* \\ M_1\hat{s}_{24}^* & M_2\hat{s}_{25}^* & M_3\hat{s}_{26}^* \\ M_1\hat{s}_{34}^* & M_2\hat{s}_{35}^* & M_3\hat{s}_{36}^* \end{pmatrix}, \tag{23}$$

and six independent matrix elements of $M_L \simeq V_0\hat{M}_\nu V_0^T + M_D R^T$ can similarly be obtained:

$$\begin{aligned}
(M_L)_{ee} &\simeq m_1(c_{12}c_{13})^2 + m_2(\hat{s}_{12}^*c_{13})^2 + m_3(\hat{s}_{13}^*)^2 + M_1(\hat{s}_{14}^*)^2 + M_2(\hat{s}_{15}^*)^2 + M_3(\hat{s}_{16}^*)^2, \\
(M_L)_{e\mu} &\simeq -m_1c_{12}c_{13}(\hat{s}_{12}c_{23} + c_{12}\hat{s}_{13}\hat{s}_{23}^*) + m_2\hat{s}_{12}^*c_{13}(c_{12}c_{23} - \hat{s}_{12}^*\hat{s}_{13}\hat{s}_{23}^*) + m_3c_{13}\hat{s}_{13}^*\hat{s}_{23}^* + M_1\hat{s}_{14}^*\hat{s}_{24}^* + M_2\hat{s}_{15}^*\hat{s}_{25}^* + M_3\hat{s}_{16}^*\hat{s}_{26}^*, \\
(M_L)_{e\tau} &\simeq m_1c_{12}c_{13}(\hat{s}_{12}\hat{s}_{23} - c_{12}\hat{s}_{13}c_{23}) - m_2\hat{s}_{12}^*c_{13}(c_{12}\hat{s}_{23} + \hat{s}_{12}^*\hat{s}_{13}c_{23}) + m_3c_{13}\hat{s}_{13}^*c_{23} + M_1\hat{s}_{14}^*\hat{s}_{34}^* + M_2\hat{s}_{15}^*\hat{s}_{35}^* + M_3\hat{s}_{16}^*\hat{s}_{36}^*, \\
(M_L)_{\mu\mu} &\simeq m_1(\hat{s}_{12}c_{23} + c_{12}\hat{s}_{13}\hat{s}_{23}^*)^2 + m_2(c_{12}c_{23} - \hat{s}_{12}^*\hat{s}_{13}\hat{s}_{23}^*)^2 + m_3(c_{13}\hat{s}_{23}^*)^2 + M_1(\hat{s}_{24}^*)^2 + M_2(\hat{s}_{25}^*)^2 + M_3(\hat{s}_{26}^*)^2, \\
(M_L)_{\mu\tau} &\simeq -m_1(\hat{s}_{12}c_{23} + c_{12}\hat{s}_{13}\hat{s}_{23}^*)(\hat{s}_{12}\hat{s}_{23} - c_{12}\hat{s}_{13}c_{23}) - m_2(c_{12}c_{23} - \hat{s}_{12}^*\hat{s}_{13}\hat{s}_{23}^*)(c_{12}\hat{s}_{23} + \hat{s}_{12}^*\hat{s}_{13}c_{23}) \\
&\quad + m_3c_{13}^2c_{23}\hat{s}_{23}^* + M_1\hat{s}_{24}^*\hat{s}_{34}^* + M_2\hat{s}_{25}^*\hat{s}_{35}^* + M_3\hat{s}_{26}^*\hat{s}_{36}^*, \\
(M_L)_{\tau\tau} &\simeq m_1(\hat{s}_{12}\hat{s}_{23} - c_{12}\hat{s}_{13}c_{23})^2 + m_2(c_{12}\hat{s}_{23} + \hat{s}_{12}^*\hat{s}_{13}c_{23})^2 + m_3(c_{13}c_{23})^2 + M_1(\hat{s}_{34}^*)^2 + M_2(\hat{s}_{35}^*)^2 + M_3(\hat{s}_{36}^*)^2.
\end{aligned} \tag{24}$$

So any specific textures of M_D and M_L predicted in a specific type-(I+II) seesaw model must have direct and important impacts on the magnitudes of neutrino masses, flavor mixing angles, and CP -violating phases.

D. Flavored and unflavored leptogenesis scenarios

It is straightforward to obtain the type-I seesaw mechanism from the type-(I+II) seesaw mechanism by switching off the M_L term. In this special case one arrives at the exact type-I seesaw relation $V\hat{M}_\nu V^T + R\hat{M}_N R^T = \mathbf{0}$, where V and R satisfy the unitarity condition $VV^\dagger + RR^\dagger = \mathbf{1}$. Given $M_i \gg m_i$, it is more popular to write the 3×3 light Majorana neutrino mass matrix as

$$M_\nu \equiv V\hat{M}_\nu V^T = -R\hat{M}_N R^T \simeq -M_D M_R^{-1} M_D^T, \tag{25}$$

where $V \equiv AV_0 \simeq V_0$ holds in the same approximation [26]. Associated with this seesaw picture, the leptogenesis mechanism [2] provides a natural possibility of accounting for the observed matter-antimatter asymmetry of the Universe via the lepton-number-violating, CP -violating, and out-of-equilibrium decays of N_i and the $(B - L)$ -conserving sphaleron processes. The CP -violating asymmetry between $N_i \rightarrow \ell_\alpha + H$ and $N_i \rightarrow \bar{\ell}_\alpha + \bar{H}$ decays, denoted as $\varepsilon_{i\alpha}$ (for $i = 1, 2, 3$ and $\alpha = e, \mu, \tau$), is given by [27]

$$\begin{aligned}
\varepsilon_{i\alpha} &\equiv \frac{\Gamma(N_i \rightarrow \ell_\alpha + H) - \Gamma(N_i \rightarrow \bar{\ell}_\alpha + \bar{H})}{\Gamma(N_i \rightarrow \ell_\alpha + H) + \Gamma(N_i \rightarrow \bar{\ell}_\alpha + \bar{H})} \\
&= \frac{1}{8\pi v^2 (M_D^\dagger M_D)_{ii}} \sum_{j \neq i} \left\{ \text{Im}[(M_D^*)_{\alpha i} (M_D)_{\alpha j} (M_D^\dagger M_D)_{ij}] \right. \\
&\quad \left. \times \mathcal{F}\left(\frac{M_j^2}{M_i^2}\right) + \text{Im}[(M_D^*)_{\alpha i} (M_D)_{\alpha j} (M_D^\dagger M_D)_{ij}^*] \mathcal{G}\left(\frac{M_j^2}{M_i^2}\right) \right\},
\end{aligned} \tag{26}$$

where the loop functions $\mathcal{F}(x) = \sqrt{x}\{2 - x\}/(1 - x) + (1 + x)\ln[x/(1 + x)]$ and $\mathcal{G}(x) = 1/(1 - x)$ have been introduced. If all the interactions in the era of leptogenesis were blind to lepton flavors, then only the total CP -violating asymmetry ε_i should be relevant:

$$\varepsilon_i = \sum_\alpha \varepsilon_{i\alpha} = \frac{1}{8\pi (M_D^\dagger M_D)_{ii}} \sum_{j \neq i} \text{Im}[(M_D^\dagger M_D)_{ij}^2] \mathcal{F}\left(\frac{M_j^2}{M_i^2}\right). \tag{27}$$

The leptogenesis mechanisms associated with Eqs. (26) and (27) are usually referred to as flavored and unflavored leptogenesis scenarios, respectively.

In the flavor basis where $M_R \simeq \hat{M}_N$ and $U_0 \simeq \mathbf{1}$ hold, we have obtained the explicit expression of $M_D \simeq R\hat{M}_N$ in Eq. (23). It is then straightforward to arrive at

$$\begin{aligned}
(M_D^\dagger M_D)_{11} &\simeq M_1^2 (R^\dagger R)_{11} \simeq M_1^2 (s_{14}^2 + s_{24}^2 + s_{34}^2), \\
(M_D^\dagger M_D)_{22} &\simeq M_2^2 (R^\dagger R)_{22} \simeq M_2^2 (s_{15}^2 + s_{25}^2 + s_{35}^2), \\
(M_D^\dagger M_D)_{33} &\simeq M_3^2 (R^\dagger R)_{33} \simeq M_3^2 (s_{16}^2 + s_{26}^2 + s_{36}^2)
\end{aligned} \tag{28}$$

and

$$\begin{aligned}
(M_D^\dagger M_D)_{12} &\simeq M_1 M_2 (R^\dagger R)_{12} \\
&\simeq M_1 M_2 (\hat{s}_{14} \hat{s}_{15}^* + \hat{s}_{24} \hat{s}_{25}^* + \hat{s}_{34} \hat{s}_{35}^*), \\
(M_D^\dagger M_D)_{13} &\simeq M_1 M_3 (R^\dagger R)_{13} \\
&\simeq M_1 M_3 (\hat{s}_{14} \hat{s}_{16}^* + \hat{s}_{24} \hat{s}_{26}^* + \hat{s}_{34} \hat{s}_{36}^*), \\
(M_D^\dagger M_D)_{23} &\simeq M_2 M_3 (R^\dagger R)_{23} \\
&\simeq M_2 M_3 (\hat{s}_{15} \hat{s}_{16}^* + \hat{s}_{25} \hat{s}_{26}^* + \hat{s}_{35} \hat{s}_{36}^*).
\end{aligned} \tag{29}$$

So the unflavored CP -violating asymmetry ε_i depends on nine phase differences $\delta_{i4} - \delta_{i5}$, $\delta_{i4} - \delta_{i6}$, and $\delta_{i5} - \delta_{i6}$ (for $i = 1, 2, 3$), but only six of them are independent. Because $V \simeq V_0$ holds in the same approximations as made above, we conclude that there is *not* any direct relationship between the CP violation at low energies (governed by δ_{12} , δ_{13} , and δ_{23}) and the unflavored leptogenesis at high energies. As for the flavored leptogenesis, the relevant CP -violating asymmetries $\varepsilon_{i\alpha}$ also rely on the aforementioned nine phase differences. This point can be clearly seen from

$$\begin{aligned}
\text{Im}[(M_D^*)_{\alpha i} (M_D)_{\alpha j} (M_D^\dagger M_D)_{ij}] &\simeq M_i^2 M_j^2 \text{Im}[R_{\alpha i}^* R_{\alpha j} (R^\dagger R)_{ij}], \\
\text{Im}[(M_D^*)_{\alpha i} (M_D)_{\alpha j} (M_D^\dagger M_D)_{ij}^*] &\simeq M_i^2 M_j^2 \text{Im}[R_{\alpha i}^* R_{\alpha j} (R^\dagger R)_{ij}^*].
\end{aligned} \tag{30}$$

Given the masses of three heavy Majorana neutrinos and the flavor mixing parameters between light and heavy Majorana neutrinos, it is then possible to determine $\varepsilon_{i\alpha}$ and ε_i (for $i = 1, 2, 3$ and $\alpha = e, \mu, \tau$) so as to realize the flavored or unflavored leptogenesis mechanism.

IV. SUMMARY AND REMARKS

An appealing and puzzling feature of the standard model is that it happens to have three species of leptons and quarks. If extra species of matter particles exist, no matter whether they are sequential in or exotic to the standard model itself, they definitely signify new physics. In this paper we have conjectured the presence of three species of sterile neutrinos and presented a full parametrization of the 6×6 flavor mixing matrix for active and sterile neutrinos in terms of 15 rotation angles and 15 phase angles. Such an exercise makes sense because we *do* have some preliminary observational hints on light sterile neutrinos, and the theoretical motivation for the existence of heavy Majorana neutrinos in the seesaw and leptogenesis mechanisms is also very strong.

We have shown that this standard parametrization of the 6×6 flavor mixing matrix in the $(3 + 3)$ scenario of active and sterile neutrino mixing allows us to clearly describe the salient features of some problems in neutrino phenomenology. Four examples have been briefly discussed for this connection: (a) possible contributions of light sterile neutrinos to the tritium beta decay and neutrinoless double-beta decay, (b) leptonic CP violation and deformed unitarity triangles of the 3×3 flavor mixing matrix V of three active neutrinos, (c) a reconstruction of the 6×6 neutrino mass matrix in the type-(I+II) seesaw mechanism, and (d) flavored and unflavored leptogenesis scenarios in the type-I seesaw mechanism with three heavy Majorana neutrinos. Our results are expected to be useful to understand the impacts of extra neutrino species on the standard weak interactions and neutrino oscillations in a better and more straightforward way. Furthermore, the parametrization itself can also be applied to the quark sector if extra species of quarks are conjectured.

Let us stress that the presence of new degrees of freedom in the neutrino sector may apparently violate the unitarity of the 3×3 flavor mixing matrix V of three active neutrinos in the weak charged-current interactions. Hence testing the unitarity of V in neutrino oscillations and searching for the signatures of new neutrinos at TeV-scale colliders can be complementary to each other, both qualitatively and quantitatively, in order to deeply understand the intrinsic properties of Majorana particles. On the other hand, light or heavy sterile neutrinos can have significant consequences in cosmology. The latter provides us with a good playground to study hot dark matter in the presence of sub-eV sterile neutrinos and warm dark matter in the form of keV sterile neutrinos, together with the cosmological matter-antimatter asymmetry via leptogenesis in which heavy Majorana neutrinos and their decays play the key role. We hope that any experimental breakthrough in these aspects will pave the way towards the true theory of massive neutrinos.

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APPENDIX A

The 6×6 unitary matrix \mathcal{U} in Eq. (1) can be expressed as a product of 15 two-dimensional rotation matrices in a six-dimensional complex space [15]:

$$\begin{aligned}
\mathcal{U} &= (O_{56} O_{46} O_{36} O_{26} O_{16}) (O_{45} O_{35} O_{25} O_{15}) \\
&\quad \times (O_{34} O_{24} O_{14}) (O_{23} O_{13}) O_{12},
\end{aligned} \tag{A1}$$

where O_{ij} (for $1 \leq i < j \leq 6$) are unitary and read as follows:

$$O_{12} = \begin{pmatrix} c_{12} & \hat{s}_{12}^* & 0 & 0 & 0 & 0 \\ -\hat{s}_{12} & c_{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (\text{A2})$$

and

$$O_{13} = \begin{pmatrix} c_{13} & 0 & \hat{s}_{13}^* & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -\hat{s}_{13} & 0 & c_{13} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad O_{23} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & c_{23} & \hat{s}_{23}^* & 0 & 0 & 0 \\ 0 & -\hat{s}_{23} & c_{23} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (\text{A3})$$

and

$$O_{14} = \begin{pmatrix} c_{14} & 0 & 0 & \hat{s}_{14}^* & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -\hat{s}_{14} & 0 & 0 & c_{14} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad O_{24} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & c_{24} & 0 & \hat{s}_{24}^* & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -\hat{s}_{24} & 0 & c_{24} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad O_{34} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_{34} & \hat{s}_{34}^* & 0 & 0 \\ 0 & 0 & -\hat{s}_{34} & c_{34} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (\text{A4})$$

and

$$O_{15} = \begin{pmatrix} c_{15} & 0 & 0 & 0 & \hat{s}_{15}^* & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -\hat{s}_{15} & 0 & 0 & 0 & c_{15} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad O_{25} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & c_{25} & 0 & 0 & \hat{s}_{25}^* & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -\hat{s}_{25} & 0 & 0 & c_{25} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$O_{35} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_{35} & 0 & \hat{s}_{35}^* & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\hat{s}_{35} & 0 & c_{35} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad O_{45} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{45} & \hat{s}_{45}^* & 0 \\ 0 & 0 & 0 & -\hat{s}_{45} & c_{45} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (\text{A5})$$

and

$$\begin{aligned}
O_{16} &= \begin{pmatrix} c_{16} & 0 & 0 & 0 & 0 & \hat{s}_{16}^* \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -\hat{s}_{16} & 0 & 0 & 0 & 0 & c_{16} \end{pmatrix}, & O_{26} &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & c_{26} & 0 & 0 & 0 & \hat{s}_{26}^* \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -\hat{s}_{26} & 0 & 0 & 0 & c_{26} \end{pmatrix}, & O_{36} &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_{36} & 0 & 0 & \hat{s}_{36}^* \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -\hat{s}_{36} & 0 & 0 & c_{36} \end{pmatrix}, \\
O_{46} &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{46} & 0 & \hat{s}_{46}^* \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\hat{s}_{46} & 0 & c_{46} \end{pmatrix}, & O_{56} &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{56} & \hat{s}_{56}^* \\ 0 & 0 & 0 & 0 & -\hat{s}_{56} & c_{56} \end{pmatrix}. & & (A6)
\end{aligned}$$

In the above equations we have defined $c_{ij} \equiv \cos\theta_{ij}$ and $\hat{s}_{ij} \equiv e^{i\delta_{ij}} \sin\theta_{ij}$, with θ_{ij} and δ_{ij} being the rotation angle and phase angle, respectively. Because some of the two-dimensional rotation matrices can commute with each other, it is possible to rewrite Eq. (A1) as

$$\begin{aligned}
\mathcal{U} &= (O_{56}O_{46}O_{45})(O_{36}O_{26}O_{16}O_{35}O_{25}O_{15}O_{34}O_{24}O_{14}) \\
&\quad \times (O_{23}O_{13}O_{12}). & (A7)
\end{aligned}$$

On the right-hand side of Eq. (A7), the combination $(O_{23}O_{13}O_{12})$ purely describes the flavor mixing among three active neutrinos, while $(O_{56}O_{46}O_{45})$ purely describes the flavor mixing among three sterile neutrinos. It is the combination $(O_{36} \cdots O_{14})$ in Eq. (A7) that allows the active and sterile sectors to “talk” to each other, as discussed in Sec. II.

APPENDIX B

Given the $n \times n$ flavor mixing matrix, one may always calculate its $(n-1)^2(n-2)^2/4$ rephasing invariants of CP violation, the so-called Jarlskog parameters [28]. As for the 6×6 unitary matrix \mathcal{U} under discussion, we totally have 100 invariants of this nature. But we are mainly concerned about the Jarlskog parameters of $V \equiv AV_0$ defined in Sec. II:

$$J_{\alpha\beta}^{ij} \equiv \text{Im}(V_{\alpha i}V_{\beta j}V_{\alpha j}^*V_{\beta i}^*), \quad (B1)$$

in which the Greek indices run over (e, μ, τ) and the Latin indices run over $(1, 2, 3)$. Since V describes the flavor mixing of three active neutrinos, $J_{\alpha\beta}^{ij}$ measure the corresponding CP -violating effects in their oscillations. In the absence of three sterile neutrinos (i.e., $A = \mathbf{1}$ and $V = V_0$), one arrives at a universal Jarlskog parameter

$$\begin{aligned}
J_0 &\equiv J_{e\mu}^{12} = J_{e\mu}^{23} = J_{e\mu}^{31} = J_{\mu\tau}^{12} = J_{\mu\tau}^{23} = J_{\mu\tau}^{31} = J_{\tau e}^{12} \\
&= J_{\tau e}^{23} = J_{\tau e}^{31} = c_{12}s_{12}c_{13}^2s_{13}c_{23}s_{23} \sin\delta, & (B2)
\end{aligned}$$

with $\delta \equiv \delta_{13} - \delta_{12} - \delta_{23}$, as guaranteed by the unitarity of V_0 . When the contributions of three sterile neutrinos are switched on, $J_{\alpha\beta}^{ij}$ can be calculated with the help of Eqs. (5) and (12) in a good approximation. One finds

$$\begin{aligned}
J_{e\mu}^{12} &\simeq J_0 + c_{12}s_{12}c_{23} \text{Im}X, & J_{\tau e}^{12} &\simeq J_0 + c_{12}s_{12}s_{23} \text{Im}Y, \\
J_{\mu\tau}^{12} &\simeq J_0 + c_{12}s_{12}c_{23}s_{23}(s_{23} \text{Im}X + c_{23} \text{Im}Y), \\
J_{\mu\tau}^{23} &\simeq J_0 + c_{12}c_{23}s_{23}(s_{12}s_{23} \text{Im}X + s_{12}c_{23} \text{Im}Y + c_{12} \text{Im}Z), \\
J_{\mu\tau}^{31} &\simeq J_0 + s_{12}c_{23}s_{23}(c_{12}s_{23} \text{Im}X + c_{12}c_{23} \text{Im}Y - s_{12} \text{Im}Z), & (B3)
\end{aligned}$$

and $J_{e\mu}^{23} \simeq J_{e\mu}^{31} \simeq J_{\tau e}^{23} \simeq J_{\tau e}^{31} \simeq J_0$ [29], where $X \equiv \mathcal{X}e^{-i\delta_{12}}$, $Y \equiv \mathcal{Y}e^{-i(\delta_{12}+\delta_{23})}$, and $Z \equiv \mathcal{Z}e^{-i\delta_{23}}$, with \mathcal{X} , \mathcal{Y} , and \mathcal{Z} being defined below Eq. (17). Note that we have assumed θ_{13} , θ_{i4} , θ_{i5} , and θ_{i6} (for $i = 1, 2, 3$) to be small in our calculations, and thus the terms of $\mathcal{O}(s_{13}|X|)$, $\mathcal{O}(s_{13}|Y|)$, and $\mathcal{O}(s_{13}|Z|)$ together with those higher-order terms have been omitted from the above results. The fact that $J_{e\mu}^{23} \simeq J_{e\mu}^{31} \simeq J_{\tau e}^{23} \simeq J_{\tau e}^{31} \simeq J_0$ holds in the above approximation is simply because they all involve the smallest matrix element of V (i.e., $V_{e3} \simeq \hat{s}_{13}^*$) [30].

It is well known that the maximal value of J_0 is $J_0^{\text{max}} = 1/(6\sqrt{3}) \simeq 9.6\%$ [28]. In comparison, the magnitudes of $\text{Im}X$, $\text{Im}Y$, and $\text{Im}Z$ are likely to reach the percent level if those active-sterile mixing angles are of $\mathcal{O}(0.1)$ and the relevant CP -violating phases are of $\mathcal{O}(1)$. So the five Jarlskog parameters in Eq. (B3) might deviate from the standard one J_0 in a significant way, depending on the constructive or destructive contributions from three sterile neutrinos. It is therefore important to observe all the nine Jarlskog parameters in neutrino oscillations. Taking account of the nonunitarity of V , one may easily derive the probabilities of $\nu_\alpha \rightarrow \nu_\beta$ and $\bar{\nu}_\alpha \rightarrow \bar{\nu}_\beta$ oscillations [15,17]:

$$\begin{aligned}
P(\nu_\alpha \rightarrow \nu_\beta) &= \frac{\sum_i |V_{\alpha i}|^2 |V_{\beta i}|^2 + 2 \sum_{i < j} \text{Re}(V_{\alpha i} V_{\beta j} V_{\alpha j}^* V_{\beta i}^*) \cos \Delta_{ij} - 2 \sum_{i < j} J_{\alpha\beta}^{ij} \sin \Delta_{ij}}{(VV^\dagger)_{\alpha\alpha} (VV^\dagger)_{\beta\beta}}, \\
P(\bar{\nu}_\alpha \rightarrow \bar{\nu}_\beta) &= \frac{\sum_i |V_{\alpha i}|^2 |V_{\beta i}|^2 + 2 \sum_{i < j} \text{Re}(V_{\alpha i} V_{\beta j} V_{\alpha j}^* V_{\beta i}^*) \cos \Delta_{ij} + 2 \sum_{i < j} J_{\alpha\beta}^{ij} \sin \Delta_{ij}}{(VV^\dagger)_{\alpha\alpha} (VV^\dagger)_{\beta\beta}}, \tag{B4}
\end{aligned}$$

where $\Delta_{ij} \equiv \Delta m_{ij}^2 L / (2E)$, with $\Delta m_{ij}^2 \equiv m_i^2 - m_j^2$, E being the neutrino beam energy, and L being the baseline length. As a consequence,

$$\begin{aligned}
\mathcal{A}_{\alpha\beta} &\equiv P(\nu_\alpha \rightarrow \nu_\beta) - P(\bar{\nu}_\alpha \rightarrow \bar{\nu}_\beta) \\
&= \frac{4}{(AA^\dagger)_{\alpha\alpha} (AA^\dagger)_{\beta\beta}} \sum_{i < j} J_{\alpha\beta}^{ij} \sin \Delta_{ij} \\
&\simeq 4[J_{\alpha\beta}^{12} \sin \Delta_{21} + (J_{\alpha\beta}^{13} + J_{\alpha\beta}^{23}) \sin \Delta_{32}], \tag{B5}
\end{aligned}$$

where $AA^\dagger \simeq \mathbf{1}$ and $\Delta_{31} \simeq \Delta_{32}$ (i.e., $\Delta m_{31}^2 \simeq \Delta m_{32}^2$ [13]) have been taken into account. This result implies that both $J_{\alpha\beta}^{12}$ and $J_{\alpha\beta}^{13} + J_{\alpha\beta}^{23}$ can, in principle, be determined if the baseline of neutrino oscillations is sufficiently long. Of course, terrestrial matter effects may more or less contaminate the genuine CP -violating effects in such long-baseline neutrino oscillation experiments and should be properly treated [23].

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