Model building in AdS/CMT: DC conductivity and Hall angle

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Using the bottom-up approach in a holographic setting, we attempt to study both the transport and thermodynamic properties of a generic system in 3 + 1-dimensional bulk spacetime. We show the exact $1/T$ and T^2 dependence of the longitudinal conductivity and Hall angle, as seen experimentally in most copper-oxide systems, which are believed to be close to quantum critical point. This particular temperature dependence of the conductivities are possible in two different cases: (1) background solutions with scale invariant and broken rotational symmetry and (2) solutions with pseudoscaling and unbroken rotational symmetry, but only at low density limit. Generically, the study of the transport properties in a scale-invariant background solution, using the probe brane approach, at high density and at low temperature limit suggests we consider only metrics with two exponents. More precisely, the spatial part of the metric components should not be same, i.e. $g_{xx} \neq g_{yy}$. In doing so, we have generalized the above-mentioned behavior of conductivity with a very special behavior of specific heat which at low temperature goes as: $C_V \sim T^3$. However, if we break the scaling symmetry of the background solution by including a nontrivial dilaton, axion, or both and keep the rotational symmetry, then also we can generate such a behavior of conductivity, but only in the low density regime. As far as we are aware, this particular temperature dependence of both the conductivity and Hall angle is being shown for the first time using holography.

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I. INTRODUCTION

There are interesting model-building calculations that are being put forward using gauge/gravity duality, which suggests capturing the experimental results close to quantum criticality and the associated quantum phase transitions. In particular, for the copper-oxide systems at low temperature, the resistivity, which is the inverse of the conductivity, goes as $\sigma \sim T^{-1}$ [1–4]. This interesting behavior has been reported in a controllable yet unrealistic setting for a very special kind of gravitational system that displays the Lifshitz like property and is possible only when the Lifshitz exponent takes a special value, namely, $z = 2$,[1] see Ref. [6]. However, it is also suggested in Refs. [1–4] that for the copper-oxide systems, the Hall angle, $\cot\theta_H = \sigma^{xx}/\sigma^{xy}$, should have a quadratic dependence of temperature, $\cot\theta_H \sim T^2$. But, unfortunately, use of the gravitational solutions showing the Lifshitz-like scaling does not reproduce this behavior of the Hall angle; rather, it gives at low temperature a linear dependence of temperature and is not in complete agreement with the experimental results.

The experimental results for the transport properties of the copper-oxide systems near optimum doping at low temperature can be summarized as follows [1,3,4]:

$$\sigma^{xx} \sim 1/T, \qquad \cot\theta_H = \sigma^{xx}/\sigma^{xy} \sim T^2 \Rightarrow \sigma^{xy} \sim T^{-3}. \tag{1}$$

The basic reason of not getting the desired experimental behavior is due to the presence of a rotational symmetry in the x, y plane of the metric while having the scaling symmetry of the background solution, where x and y are the only two spatial directions available in field theory—even though this symmetry is broken explicitly in the presence of constant electric and magnetic field.

In this paper, we shall show that Eq. (1) can only be reproduced in two different cases: (1) background solutions respecting the scaling symmetry with broken rotational symmetry in the x, y plane and (2) pseudoscaling background solutions with unbroken rotational symmetry in the low density limit. Here, the pseudoscaling solutions mean the background geometry respects the scaling symmetry, but not the scalar fields like dilaton and axion. Furthermore, the background solutions having the scaling symmetry, time translation, spatial translation, and the rotational symmetry are completely ruled out by Eq. (1), e.g. pure anti-de Sitter (AdS) and pure Lifshitz solutions. It is worth it to emphasize that case (1) is the only choice that is permissible at high density, but at low density, we can have either of the choices. We are discussing both the limits of densities because it is not *a priori* clear the scale of optimum doping in Eq. (1).

The basic philosophy of Ref. [6] is to introduce charge carriers via Dirichlet p-branes in the probe brane approximation. The charge carriers are in thermal contact with a heat bath, which is taken as the Lifshitz black hole.

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[1]There is another paper [5], which does not require gravitational solution with Lifshitz scaling (rather with $z = 1$) in order to generate such a behavior of conductivity. More interestingly, it is shown that such a behavior follows at one loop.

Translating it into the language of Ref. [7], it is the bi-fundamental degrees of freedom that are charged, interacting among themselves and with the adjoint degrees of freedom giving us the desired feature of conductivity(in contrast to Ref. [5], where the authors have considered only the charged adjoint degrees of freedom to replicate the above mentioned experimental result at one loop).[2] In this paper, we have adopted the former approach (in the massless limit) and replace the heat bath of the Lifshitz kind by another, more general heat bath. The reason of such a replacement is that: (1) a Lifshitz-type heat bath is a special type compared to this more general heat bath and (2) it is Eq. (1), which was not possible to reproduce fully with the Lifshitz-type heat bath. Recall, the heat baths are essentially the source of studying physics around the quantum critical point at low temperature [8]. The consequences of replacing such a heat bath is addressed, thermodynamically.

In the holographic setting [9], the authors of Refs. [10–12] have proposed a beautiful algorithm to calculate the conductivities. Here, we have modified it slightly and obtain an equivalent way to calculate the conductivities. The result of the calculation matches precisely, as is done in Ref. [11], when the charge carriers move in a constant electric and magnetic field. Use of this equivalent prescription leads to the following dependence of conductivities on the metric components evaluated at some holographic energy scale, $r_\star$. At high densities compared to temperature,

$$\sigma^{xx} \sim \frac{c_\phi e^{-2\Phi(r_\star)}}{g_{xx}(r_\star)}, \qquad \sigma^{xy} \simeq \frac{B c_\phi e^{-4\Phi(r_\star)}}{g_{xx}(r_\star) g_{yy}(r_\star)}, \tag{2}$$

where c_ϕ is the charge density, B the magnetic field, and Φ the dilaton. In Eq. (2), the spatial parts of the metric components along x and y directions are denoted as g_{xx} and g_{yy}, respectively. Note, this result follows when the probe-brane action admits only the Dirac-Born-Infeld (DBI) type of action. If we do include the Chern-Simon part of the action to the probe brane as well, then the result of the conductivities gets slightly modified in the high-density limit compared to temperature:

$$\sigma^{xx} \sim \frac{c_\phi e^{-2\Phi(r_\star)}}{g_{xx}(r_\star)}, \qquad \sigma^{xy} \simeq \frac{B c_\phi e^{-4\Phi(r_\star)}}{g_{xx}(r_\star) g_{yy}(r_\star)} - \mu C_0(r_\star), \tag{3}$$

where μ is the coupling of the Chern-Simon action and C_0 is the axion field. Note that the scaling symmetry is broken for a nonconstant dilaton and axion field.

[2]There arises a natural question (which we are not going to address): Does this behavior of charged bi-fundamental degrees of freedom in a heat bath equal one-loop adjoint degrees of freedom in a different heat bath, generically?

Now, if we restrict ourselves to background solutions which possess the scaling symmetry and exhibit the rotational symmetry at the level of metric (not the full system), then the off diagonal part of the conductivity in the high density limit goes as $\sigma^{xy} \sim (\sigma^{xx})^2$, which is not in accordance with the experimental result, see Eq. (1). This means to reproduce Eq. (1) in the high density limit, we are forced to consider metric components for which $g_{xx} \neq g_{yy}$. This is one of the basic criteria that must be imposed in choosing the background metric, i.e. the heat bath, in order to study the physics associated with the transport properties around the quantum critical point.

In getting the results of the conductivity and the Hall angle as in Eq. (1), we have assumed the background metric respects the following scaling symmetry:

$$t \to \lambda^z t, \qquad x \to \lambda^w x, \qquad y \to \lambda y, \qquad r \to \frac{r}{\lambda}, \tag{4}$$

and also, we have assumed that there is not any nontrivial scalar field like dilaton or axion in the entire setup—as the presence of such a nontrivial background field would give rise to some kind of pseudoscaling theory. Of course, the charge density of the bi-fundamental degrees of freedom, i.e. two-form field strength (F_2) that appear in the DBI action, breaks the scaling symmetry. More exactly, the gravitational solution without any nonvanishing scalar field that gives us the desired result of the longitudinal conductivity and Hall angle should have the exponents $z = 1$ and $w = 1/2$. For this choice of exponents, the zero-temperature limit of the black hole solution, i.e. the solution without the thermal factor, has a boost symmetry along the t, y plane with a form

$$ds^2 = L^2\left[-r^2 dt^2 + r dx^2 + r^2 dy^2 + \frac{dr^2}{r^2}\right], \tag{5}$$

where L is the size of the $3+1$-dimensional bulk system, which we shall set to unity in our calculations later. The background geometry with two exponents z and w was proposed in Ref. [13] using a combination of Einstein-Hilbert action and several-form field strengths. Since the analytic nonextremal version of that solution is very difficult to obtain, we have adopted here a different path to generate such a solution by using only gravitons.

Let us do a little bit of dimensional analysis of various physical quantities. If the $d+1$-dimensional field theory spacetime coordinates (i.e. the bulk is $d+2$-dimensional spacetime) behaves under scaling as

$$t \to \lambda^z t, \qquad x \to \lambda^w x, \qquad y_i \to \lambda y_i, \qquad (i = 1, \cdots, d-1), \tag{6}$$

then the physical quantities possess the following length dimension:

$$[t] = z, \quad [x] = w, \quad [y_i] = 1, \quad [J^t] = -d,$$
$$[J^x] = w - z - d, \quad [J^i] = 1 - z - d, \quad [A_t] = -z,$$
$$[A_x] = -w, \quad [A_i] = -1, \quad [E_x] = -w - z,$$
$$[E_i] = -1 - z, \quad [B_x] = -2, \quad [B_i] = -1 - w,$$
$$[T] = -z = [\omega], \quad [F] = -z, \quad [\sigma^{xx}] = 2w - d,$$
$$[\sigma^{xy}] = 1 + w - d, \quad [\sigma^{yy}] = 2 - d, \tag{7}$$

where J^t, J_i, A_t, A_x, A_i, E, B, T, ω, F, and σ are charge density, current density, time component of the gauge potential, x-component of the gauge potential, y_i-component of the gauge potential, electric field, magnetic field, temperature, frequency, free energy, and conductivity, respectively. The two-form field strength has the following form, i.e. $F_2 = -E_x dt \wedge dx - E_{y_i} dt \wedge dy_i + B_{y_j} dx \wedge dy_i + B_x dy_i \wedge dy_j + \cdots$.

In the small magnetic field and at low density limit, $c_\phi^2 \ll \mathcal{N}^2 e^{2\Phi} g_{xx} g_{yy}$ but with $c_\phi \gg B\mu C_0$, the conductivities are

$$\sigma^{xx} \sim \mathcal{N} e^{-\Phi(r_\star)} \sqrt{\frac{g_{yy}(r_\star)}{g_{xx}(r_\star)}},$$
$$\sigma^{xy} \sim \frac{B c_\phi e^{-4\Phi(r_\star)}}{g_{xx}(r_\star) g_{yy}(r_\star)} - \mu C_0(r_\star), \tag{8}$$

where $\mathcal{N}$ is the effective tension of the brane. Upon comparing with Eq. (1), we can generate the desired experimental behavior of transport quantities for background solutions showing the pseudoscaling symmetry and unbroken rotational symmetry in the x, y plane. So, only in the low density limit, we need not consider a two-exponent solution as in Eq. (4). However, if we do, then we can also generate Eq. (1).In summary, the different possibilities with time translation and spatial translation symmetries are shown in Table I.

The holographic study of the transport properties using the approach of Refs. [10–12] gives us the nonlinear behavior at the critical point and help us understand the universal features, if any, in different limits of the parameter space, especially the quantity $dI/dV = 1/\mathcal{R} = \sigma$, where $\mathcal{R}$ is the resistance to the flow of current I with an applied voltage V. In this paper, we have generated successfully Eq. (1) and focused more on the model building than trying to find the universal features.

In the calculation of the conductivity, it is not *a priori* clear at what scale one should evaluate, i.e. how to choose the scale, $r_\star$, so as to capture the nonlinear effect—especially, for the system that is described by the Maxwell action. Of course, the gauge/gravity duality suggests we should do the calculations at the UV boundary. But the result of this calculation produces only the linearized effect. However, for the system whose action is described by the DBI type, there exists a very natural way to find the scale $r_\star$. This basically follows from the argument of Refs. [10–12], which says that either the integrand of the action or the solution, which is in the form of $\sqrt{\frac{A}{B}}$, needed to be real. At a special value of the radial coordinate, $r = r_\star$, both A and B vanish and there the action and the solution take an indeterminate $\frac{0}{0}$ form. Above or below this special scale, $r_\star$, both A and B become positive or negative together. In this paper, we give a physical argument to determine the scale $r_\star$ and show that it agrees precisely with the calculations done using the arguments of Refs. [10–12]. We use the fact that the Legendre-transformed action is the same as the energy density, H_L, evaluated on the static solution, which comes as the square root of one term; importantly, there is not any term in the denominator. On this energy, we use the argument of Ref. [10],[11] or [12] to find the scale $r_\star$. So, the scale, $r_\star$, is the point on the holographic direction, r, for which the Legendre-transformed action or the energy density vanishes and stays real:

$$(H_L)_{r_\star} = 0. \tag{9}$$

For the systems that are described by the DBI kind of actions, there exists another argument that precisely gives the same result for $r_\star$ as suggested in the previous paragraph, even though the precise physical reason is not that clear. The argument is to find the on-shell value of the norm of the field strength for which it takes a constant value, more precisely

TABLE I. Different possibilities that give the desired behavior of longitudinal and Hall conductivity.

Symmetries	Density	Limit	Eq. (1)
Scaling and rotation	Any density	$\cdots$	Not possible
Pseudoscaling and rotation	Medium to low density	$c_\phi \gg \mu B C_0 c_\phi \ll \mathcal{N} e^\Phi g_{xx}^2$	Possible
Pseudoscaling and rotation	High density	$\cdots$	Not possible
Pseudoscaling and rotation	Medium to low density	$\frac{B c_\phi e^{-4\Phi}}{g_{xx}^2} \ll \mu C_0 c_\phi \gg \mu B C_0$	Possible
Scaling with broken rotation	Low density	$\cdots$	Possible
Scaling with broken rotation	High density	$\cdots$	Possible

$$(F_{MN}F^{MN})_{r_\star} = -2. \tag{10}$$

There exists yet another way to determine the scale, $r_\star$; that is to find a scale where the determinant of $\det(g+F)_{ab}$ vanishes [11]. Here, the indices a and b run, only, over the field-theory directions. The equation for the condition is

$$(\det(g+F)_{ab})_{r_\star} = 0. \tag{11}$$

This can very easily be seen following the argument of vanishing of H_L at the scale $r_\star$. Generically, the Legendre-transformed action can be written as

$$H_L = \int \sqrt{\mathcal{A}(r)[\mathcal{A}_\in(r)\mathcal{A}_\ni(r) - (\mathcal{A}_\Delta(r))^2]}, \tag{12}$$

where $\mathcal{A}(r) = \sqrt{\frac{g_{rr}}{g_{tt}g_{xx}g_{yy}}}$, and the expression to $\mathcal{A}_\in(r)$, $\mathcal{A}_\ni(r)$, and $\mathcal{A}_\Delta(r)$ are given in Eq. (58). Generically, the term $(\mathcal{A}_\Delta(r))^2$ is nonzero when there exists more than one spatial current; more importantly, this term is always positive. Whereas, the term $\mathcal{A}_\in(r)$ and $\mathcal{A}_\ni(r)$ can change sign close to the horizon. Hence, we can use the arguments of Refs. [10–12] so as to have a real Legendre-transformed action or the energy. Moreover, one of the terms is nothing but $(-\det(g+F)_{ab})$. Hence, the condition, Eq. (12), follows from H_L.

The prescription of holography in Ref. [9] or that of Ref. [10] has been used to calculate the conductivity of several systems both in the top-down and bottom-up approaches. They include Refs. [14–29] as a partial list.

This paper is organized as follows. In Sec. II, we shall review the calculation of the conductivity following Ref. [10] and compare it with that using Eq. (13) for systems that are described by the DBI type of actions, but in the absence of the charge density. In Sec. III, we study the system in the presence of charge density and with the Chern-Simon type of actions. Studies of Sec. II and III are done in generic background solutions. Based on the calculations given in Sec. III, we give a toy example which is modeled in such a way that it gives us the desired behavior of conductivity and Hall angle in Sec. IV. In Sec. V, we study the thermodynamics of the charge carriers in the presence of a constant magnetic field. Finally, we conclude in Sec. VI. Several details of the calculations are relegated to the Appendices.

II. FROM NON-LINEAR DBI ACTION

In this section, we shall evaluate the on-shell value of the current using the definition $J^\mu = \frac{\delta S}{\delta A_\mu}$. In arbitrary spacetime dimensions, it is very difficult to solve the equations of motion that result from the DBI action, even in the massless and zero-condensate limit, i.e. for trivial embedding functions. Here, for simplicity, we shall restrict ourselves to $3+1$-dimensional bulk spacetime.

The DBI action is

$$S_{\rm DBI} = -T\int e^{-\phi}\sqrt{-\det([g]_{ab}+F_{ab})} \equiv -T\int e^{-\phi}\sqrt{-\det(M_{ab})}, \tag{13}$$

where [] is used to denote the pull-back of the bulk metric onto the world volume of the brane and T is the tension of the brane. For simplicity, we have dropped the Chern-Simon part of the action.

Looking at the existence of an exact solution to the Maxwell system in $3+1$ dimensions, as shown in Appendix A, suggests there could be an exact solution to the nonlinearly generalized Maxwell system that is the DBI action.

Let us assume the following structure to the metric and U(1) gauge field strength:

$$\begin{aligned} ds_4^2 &= -h(r)d\tau^2 + 2d\tau dr + e^{2s(r)}(dx^2+dy^2),\\ F_2 &= F_{r\tau}dr\wedge d\tau + F_{x\tau}dx\wedge d\tau + F_{y\tau}dy\wedge d\tau\\ &\quad + F_{xy}dx\wedge dy + F_{xr}dx\wedge dr + F_{yr}dy\wedge dr. \end{aligned} \tag{14}$$

The equation of motion to gauge field and the current associated with it are

$$\begin{aligned} &\partial_K[Te^{-\phi}\sqrt{-\det(M_{AB})}\theta^{KL}] = 0,\\ &J^\mu = -Te^{-\phi}\sqrt{-\det(M_{AB})}\theta^{r\mu}, \end{aligned} \tag{15}$$

where the indices N, K, L, etc., run over the entire bulk spacetime, whereas μ, ν, ρ, etc., run only over the field-theory directions, τ, x, and y. The function $\theta^{KL} = \frac{M^{KL}-M^{LK}}{2}$ and the inverse of matrix, M_{KL}, are defined as, $M^{KL}M_{LP} = \delta^K_P$. The explicit form of the spatial components of the current are

$$\begin{aligned} \sqrt{-\det(M_{AB})}J^x &= -Te^{-\phi}[F_{y\tau}(F_{r\tau}F_{xy}+F_{x\tau}F_{yr}-F_{xr}F_{y\tau})\\ &\quad + e^{2s}(F_{x\tau}+hF_{xr})],\\ \sqrt{-\det(M_{AB})}J^y &= -Te^{-\phi}[F_{x\tau}(-F_{r\tau}F_{xy}+F_{y\tau}F_{xr}-F_{yr}F_{x\tau})\\ &\quad + e^{2s}(F_{y\tau}+hF_{yr})]. \end{aligned} \tag{16}$$

Let us assume that the nonvanishing components of the field strengths are $F_{x\tau}$, $F_{y\tau}$, and F_{xy}, in which case there occurs a lot of simplification to both the currents and equations of motion:

$$J^x = -Te^{-\phi}\frac{F_{x\tau}}{\sqrt{1+e^{-4s}F_{xy}^2}},$$
$$J^y = -Te^{-\phi}\frac{F_{y\tau}}{\sqrt{1+e^{-4s}F_{xy}^2}},$$
$$\partial_y\left[T\frac{e^{-\phi}F_{xy}}{\sqrt{e^{4s}+F_{xy}^2}}\right]+\partial_r\left[T\frac{e^{-\phi+2s}F_{x\tau}}{\sqrt{e^{4s}+F_{xy}^2}}\right]=0,$$
$$\partial_x\left[-T\frac{e^{-\phi}F_{xy}}{\sqrt{e^{4s}+F_{xy}^2}}\right]+\partial_r\left[T\frac{e^{-\phi+2s}F_{y\tau}}{\sqrt{e^{4s}+F_{xy}^2}}\right]=0,$$
$$\partial_x\left[Te^{-\phi+2s}\frac{F_{x\tau}}{\sqrt{e^{4s}+F_{xy}^2}}\right]+\partial_y\left[Te^{-\phi+2s}\frac{F_{y\tau}}{\sqrt{e^{4s}+F_{xy}^2}}\right]=0. \tag{17}$$

The solution for $\phi = \text{constant} = \phi_0$ becomes

$$F_{y\tau} = E_y(\tau, r), \qquad F_{xy} = \text{constant} \equiv B, \qquad F_{x\tau} = E_x(\tau, r), \tag{18}$$

for some functions $E_x(\tau, r)$ and $E_y(\tau, r)$, whose functional form is

$$E_x(\tau, r) = f_2(\tau)e^{-2s}\sqrt{e^{4s}+B^2}, \qquad E_y(\tau, r) = f_3(\tau)e^{-2s}\sqrt{e^{4s}+B^2}, \tag{19}$$

determined in terms of two unknown functions $f_2(\tau)$ and $f_3(\tau)$. The Bianchi identity sets the condition on $F_{x\tau}$ and $F_{y\tau}$ that these components should not depend on r and can happen only when $B = 0$. This solution indeed is an exact solution to the complete equation of motion, and the only nonvanishing components of field strength are $F_{x\tau}$ and $F_{y\tau}$ [12].

Using this explicit structure of the solution in the expression of currents, we ended up with

$$J^x = -Te^{-\phi}F_{x\tau}, \qquad J^y = -Te^{-\phi}F_{y\tau}, \tag{20}$$

from which there follows the DC conductivities at the scale $r = r_c$, upon using Ohm's law:

$$\sigma^{xx}(r_c) = \sigma^{yy}(r_c) = -Te^{-\phi_0} \equiv \sigma. \tag{21}$$

This indeed reproduces the result of Ref. [12]. There exists another exact solution, but, unfortunately, with zero electric field, and the nonvanishing components of the field strengths are

$$F_{xy} = B = \text{constant}, \qquad F_{r\tau} = \frac{f_1}{\sqrt{f_1^2+e^{4s}+B^2}}, \tag{22}$$

where f_1 is a constant.

A. Comparing with the approach of Ref. [10]

In this subsection, we shall try to derive the expression of the conductivity from the DBI action in the absence of density. Let us work in a $d+2$-dimensional spacetime with dynamical exponent z. The exact form of the metric that we shall be considering is

$$ds^2_{d+2} = -r^{2z}f(r)dt^2 + r^2\sum_{i=1}^{d}dx_i^2 + \frac{dr^2}{r^2 f(r)}, \tag{23}$$

where we shall take $f = 1-(r_0/r)^{d+z}$. This form of the metric gives us the Hawking temperature, $T_H = (\frac{d+z}{4\pi})r_0^z$, where r_0 is the horizon. In order to carry out the analysis for conductivity, we need to turn on a U(1) gauge potential, which will give us the desired electric field in the field theory, and for convenience, we shall consider it constant. Along with this, we shall turn on another component of the field strength, whose one leg is along the radial direction and the other along the spatial direction. For specificity, we shall turn on F_{xr}. So, the complete form of the U(1) gauge field is $F_2 = -Edt\wedge dx - H'(r)dr\wedge dx$.

Let us consider a probe brane which is extended along time (t), radial direction (r), and $d_s - 1$-number of directions of the d number of spatial directions. Hence, the probe brane is a d_s brane. For $d_s = d+1$, the probe brane is a space-filling brane. Also, for simplicity, we shall consider the massless limit scenario, and the action becomes

$$S = -N\int dtdrdxd^{d-1}y\sqrt{\prod_1^{d-1}g_{y_a y_a}} \times\sqrt{g_{tt}g_{rr}g_{xx}+H'^2 g_{tt}-E^2 g_{rr}}, \tag{24}$$

where we have considered the metric to be far more general than that written in Eq. (24), but assumed to be diagonal.[3] The explicit form of the metric that we have considered has the following structure:

$$ds^2_{d+2} = -g_{tt}(r)dr^2 + g_{rr}(r)dr^2 + g_{xx}(r)dx^2 + \sum_1^{d-1}g_{ab}(r)dy^a dy^b, \tag{25}$$

where $\sum_1^{d-1}g_{ab}(r)dy^a dy^b$ is assumed to be diagonal, too, i.e. $\sum_1^{d-1}g_{ab}(r)dy^a dy^b = g_{11}(dy^1)^2+\cdots+g_{d-1,d-1}(dy^{d-1})^2$. The normalization N includes the tension and the number of the probe branes. Since the action Eq. (25) does not depend on the function $H(r)$ means the "momentum" associated with it must be a constant, i.e. $\frac{\delta S}{\delta H'} \equiv c$; from which there follows the solution:

[3]Note, this form of the metric can very easily be rewritten like that written in Eq. (A2), i.e. by doing a coordinate transformation.

$$H' = \pm c\sqrt{\frac{g_{rr}g_{tt}g_{xx} - E^2 g_{rr}}{N^2(\prod_a g_{y_a y_a})g_{tt}^2 - c^2 g_{tt}}}. \quad (26)$$

It is very easy to convince oneself that the constant, c, is nothing but the current density, J^x. Now, using the arguments of Ref. [10], we obtain the necessary equations to fix c, which is J^x, as

$$E^2 = g_{tt}(r_\star)g_{xx}(r_\star), \qquad J_x^2 = N^2\left(\prod_a g_{y_a y_a}(r_\star)\right)g_{tt}(r_\star), \quad (27)$$

where $r_\star$ is the value of r and where both the numerator and denominator of H' change sign. It is interesting to note that at $r_\star$, the gradient of the solution takes the $H' = \frac{0}{0}$ form, which is an indeterminate structure. So, the better way to find $r_\star$ is to go over to an equivalent form and demand that the energy density (or the Legendre-transformed action) that follows is real as well as have a "minimum" at some energy scale, which we denote it as $r_\star$, too.

The action Eq. (25) can equivalently be expressed by doing the Legendre transformation as

$$\begin{aligned} S_L &= S - \int \frac{\delta S}{\delta H'} H' \\ &= -\int \sqrt{[g_{rr}g_{tt}g_{xx} - E^2 g_{rr}]\left[N^2\left(\prod_a g_{y_a y_a}\right) - \frac{c^2}{g_{tt}}\right]}. \end{aligned} \quad (28)$$

For static configuration, the energy of the system is $H_L = -S_L$, where

$$H_L = \int \sqrt{[g_{rr}(g_{tt}g_{xx} - E^2)]\left[N^2\left(\prod_a g_{y_a y_a}\right) - \frac{c^2}{g_{tt}}\right]}. \quad (29)$$

For an illustration, let us take an example of the asymptotically AdS black hole: the first term in the square bracket under the square root changes sign somewhere close to horizon, the same is true for the second term in the square bracket, and their product is positive. Since both the terms in the square bracket change sign somewhere close to the horizon, we assume that this happens at the same value of radial coordinate, $r = r_\star$, so as to have a real energy or real Legendre-transformed action. Asymptotically, the first term in the square bracket diverges; so also the second term (for $d \geq 2$). Now, the only place it can vanish (i.e. minimum) is close to the horizon. For a discussion on the condition of minimization to energy, see Appendix B.

Demanding these two restrictions again gives us the same two equations as written in Eq. (28), from which there follows the expression of current

$$J^x = \pm N \frac{\sqrt{(\prod_a g_{y_a y_a}(r_\star))}}{\sqrt{g_{xx}(r_\star)}} E. \quad (30)$$

The absence of singular behavior to observable J^x means the terms under the square root is regular. Upon choosing the positive sign, the conductivity is

$$\sigma = N \frac{\sqrt{(\prod_a g_{y_a y_a}(r_\star))}}{\sqrt{g_{xx}(r_\star)}}. \quad (31)$$

The solution to the first equation of Eq. (28) gives the desired solution to $r_\star$ as a function of electric field E and Hawking temperature T_H, as g_{tt} is a function of T_H. If we assume that the metric components along the spatial directions are all the same, then the above formula of conductivity reduces to

$$\sigma = N\sqrt{\left(\prod_1^{d-2} g_{y_a y_a}(r_\star)\right)}. \quad (32)$$

This form of the conductivity is also found in the Maxwell system in Ref. [30], except the choice of $r_\star$, which is $r_\star = r_0$.

Let us find the complete form of the conductivity associated with the Lifshitz metric, as written in Eq. (24),in which case, the relevant equation that gives $r_\star$ as a function of electric field E is

$$E^2 = r_\star^{2(1+z)}\left[1 - \left(\frac{r_0}{r_\star}\right)^{d+z}\right]. \quad (33)$$

This algebraic equation is very nonlinear in nature and, hence, very difficult to find the exact solution, analytically. However, there exist exact solutions for a few specific cases,in which case, the number of spatial directions are tied to the exponent z, as $d = (n-1)z + n$ with $n = 0$, 1, 2, 3 and 4,

$$r_\star = E^{1/2(1+z)}, \qquad n=0,$$

$$r_\star = \left[\frac{(\frac{4\pi}{1+z})^{((1+z)/z)}T_H^{((1+z)/z)} \pm \sqrt{4E^2 + (\frac{4\pi}{1+z})^{(2(1+z)/z)}T_H^{(2(1+z)/z)}}}{2}\right]^{(1/2(1+z))}, \qquad n=1,$$

$$r_\star = \left[E^2 + \left(\frac{2\pi}{1+z}\right)^{(2(1+z)/z)} T_H^{2(1+z)/z}\right]^{(1/2(1+z))} = \left[E^2 + \left(\frac{2\pi}{1+z}\right)^{(2(1+z)/z)} T_H^{(2(1+z)/z)}\right]^{(1/(d+z))}, \qquad n=2,$$

$$r_\star = \left[\frac{2.3^{1/3}E^2 + 2^{1/3}[9(\frac{4\pi}{3(1+z)})^{(3(1+z)/z)}T_H^{(3(1+z)/z)} + \sqrt{81(\frac{4\pi}{3(1+z)})^{(6(1+z)/z)}T_H^{(6(1+z)/z)} - 12E^6}]^{2/3}}{6^{2/3}[9(\frac{4\pi}{3(1+z)})^{(3(1+z)/z)}T_H^{(3(1+z)/z)} + \sqrt{81(\frac{4\pi}{3(1+z)})^{(6(1+z)/z)}T_H^{(6(1+z)/z)} - 12E^6}]^{1/3}}\right]^{(1/2(1+z))}, \qquad n=3,$$

$$r_\star = \left[\frac{E^2 \pm \sqrt{E^4 + 4(\frac{\pi}{1+z})^{(4(1+z)/z)}T_H^{(4(1+z)/z)}}}{2}\right]^{(1/2(1+z))}, \qquad n=4. \tag{34}$$

It is interesting to note that the choice $n=0$ gives the negative exponent $z=-d$, whereas $n=1$ gives $d=1$, which essentially gives about a $1+1$-dimensional field theory for any exponent. The choice $n=2$, 3 and 4 gives the exponent $z=d-2$, $\frac{d-3}{2}$, and $z=\frac{d-4}{3}$, respectively.

Now, using the spatial part of the metric components from Eq. (24) in the expression of current gives $J \equiv E^{(d-1+z/(1+z))}Y_1$, with the function

$$Y_1 = N\left[1 + \left(\frac{2\pi}{1+z}\right)^{(2(1+z)/z)}\left(\frac{T_H^{1+(1/z)}}{E}\right)^2\right]^{(d-2/2(1+z))}$$
for $n=2$,
$$Y_1 = N\left[\frac{1 \pm \sqrt{1 + 4(\frac{\pi}{1+z})^{(4(1+z)/z)}(\frac{T_H^{1+(1/z)}}{E})^4}}{2}\right]^{(d-2/2(1+z))}$$
for $n=4$, (35)

for a couple of cases, and the conductivity in these special cases is

$$\sigma = NT_H^{((d-2)/z)}\left[\left(\frac{2\pi}{1+z}\right)^{(2(1+z)/z)} + \left(\frac{E}{T_H^{1+(1/z)}}\right)^2\right]^{(d-2/2(1+z))}$$
for $n=2$,
$$\sigma = \frac{N}{2^{((d-2)/2(1+z))}} \times \left[E^2 \pm \sqrt{E^4 + 4\left(\frac{\pi}{1+z}\right)^{(4(1+z)/z)} T_H^{(4(1+z)/z)}}\right]^{(1/2(1+z))}$$
for $n=4$. (36)

Hence, for very small electric field and at high temperature limit, $E \ll T^{1+(1/z)}$, the conductivity follows the power law behavior, in particular, $T_H^{((d-2)/z)}$.

Let us go away from this special case of $d=(n-1)z+n$ and find the solution to $r_\star$ from Eq. (34). In the weak field limit, $E \ll T^{(1+(1/z))}$, the solution to $r_\star$ can be approximated as

$$r_\star \simeq r_0\left[1 + \left(\frac{E}{r_0^{1+z}}\right)^2\right]^{((d-2)/(d+z))} + \cdots = \left(\frac{4\pi T_H}{d+z}\right)^{1/z}\left[1 + \left(\frac{d+z}{4\pi}\right)^{(2(1+z)/z)}\left(\frac{E}{T_H^{(1+(1/z))}}\right)^2\right]^{(1/(d+z))} + \cdots, \tag{37}$$

which gives the current to leading order

$$J_x \simeq NE\left(\frac{4\pi T_H}{d+z}\right)^{((d-2)/z)} \times \left[1 + \left(\frac{d+z}{4\pi}\right)^{(2(1+z)/z)}\left(\frac{E}{T_H^{(1+(1/z))}}\right)^2\right]^{((d-2)/(d+z))} + \cdots, \tag{38}$$

whereas in the strong field limit, $E \gg T^{(1+(1/z))}$, the solution becomes

$$r_\star \simeq E^{(1/(1+z))}\left[1 + \left(\frac{r_0}{E^{(1/(1+z))}}\right)^{d+z}\right]^{(1/(d+z))} + \cdots = E^{(1/(1+z))}\left[1 + \left(\frac{4\pi}{d+z}\right)^{((d+z)/z)} \times \left(\frac{T_H^{(1+(1/z))}}{E}\right)^{((d+z)/(1+z))}\right]^{(1/(d+z))} + \cdots, \tag{39}$$

which gives the current to leading order

$$J_x \simeq NE^{((d+z-1)/(1+z))}\left[1 + \left(\frac{4\pi}{d+z}\right)^{((d+z)/z)} \times \left(\frac{T_H^{(1+(1/z))}}{E}\right)^{((d+z)/(1+z))}\right]^{((d-2)/2(1+z))} + \cdots. \tag{40}$$

This form of current essentially gives us the function

$$Y_1 = N\left[1 + \left(\frac{4\pi}{d+z}\right)^{((d+z)/z)} \times \left(\frac{T_H^{(1+(1/z))}}{E}\right)^{((d+z)/(1+z))}\right]^{((d-2)/2(1+z))} + \cdots. \quad (41)$$

On comparing this expression of Y_1 for the $n = 2$ case as in Eq. (36), it follows that the subleading terms to Y_1 in Eq. (42) vanish exactly for $d = z + 2$.

Multiple electric fields

Let us consider a situation where we have turned on more than one constant electric field. For simplicity, let us take the gauge potential as $A = -(E_1 t + H(r))dx - E_2 t dy$, which gives the field strength as

$$F_2 = -E_1 dt \wedge dx - E_2 dt \wedge dy - H'(r)dr \wedge dx. \quad (42)$$

Let us consider the previous brane configuration, again, but with this new from of the gauge-field strength in the background metric:

$$ds^2_{d+2} = -g_{tt}(r)dr^2 + g_{rr}(r)dr^2 + g_{xx}(r)dx^2 + g_{yy}(r)dy^2 + \sum_1^{d-2} g_{ab}(r)dz^a dz^b. \quad (43)$$

Going through the procedure as outlined above, we ended up with

$$J^x = \pm N E_1 \sqrt{\prod_1^{d-2} g_{z_a z_a}(r_\star)} \sqrt{\frac{g_{yy}(r_\star)}{g_{xx}(r_\star)}}, \quad (44)$$

where $r_\star$ is to be determined by solving

$$g_{tt}(r_\star)g_{xx}(r_\star) = E_1^2 + E_2^2 \frac{g_{xx}(r_\star)}{g_{yy}(r_\star)}. \quad (45)$$

Now, note that the functional expression of the current density remains the same as is found in the DBI action with one electric field, but the condition on $r_\star$ is different. For $g_{xx}(r_\star) = g_{yy}(r_\star)$, the condition almost remains the same as for one electric field except with the substitution $E_1^2 \to E_1^2 + E_2^2$, but for unequal $g_{xx}(r_\star)$ and $g_{yy}(r_\star)$, one has to find the choice of cutoff $r_\star$ by solving Eq. (46).

With a constant electric and magnetic field

Let us rerun the calculation with a constant electric and magnetic field, in which case the field strength is

$$F_2 = -E dt \wedge dx + B dx \wedge dy - H'(r)dr \wedge dx. \quad (46)$$

For our choice of constant electric and magnetic field, the formula of current density becomes

$$J^x = \pm N E \sqrt{\frac{\left(\prod_{a=1}^{d-2} g_{z_a z_a}(r_\star) g_{yy}(r_\star)\right)}{g_{xx}(r_\star)}} \sqrt{1 - \frac{B^2}{E^2}\frac{g_{tt}(r_\star)}{g_{yy}(r_\star)}}. \quad (47)$$

It is easy to note that the current is modified with an additional multiplicative factor $\sqrt{1 - \frac{B^2}{E^2}\frac{g_{tt}(r_\star)}{g_{yy}(r_\star)}}$ in comparison to the cases without any magnetic field. This is because the scale, $r_\star$, is also modified by the same multiplicative factor on the right-hand side of Eq. (28), but without the square root.

At first sight, it looks as if the results of the current in Eq. (21), after substituting the solution, are not compatible with Eq. (48), in 3 + 1 dimensions. Actually, to compare both the equations, we should go to a frame where both the calculations are done in one coordinate system. To do that, we can either do some change of coordinates or directly compute the current using the approach of Ref. [10].

In either way, note that the computation to Eq. (21) is done for which $F^{(\tau)}_{xr}$ vanishes in the frame of (τ, r, x_i). We have used a superscript (τ) in the expression of field strength to denote it, which in the frame (t, r, x_i), e.g., as in Eq. (47), suggests that they are related as $F^{(\tau)}_{xr} = H'(r) - E/h(r)$. The vanishing of $F^{(\tau)}_{xr}$ means $H'(r) = E/h(r)$, and equating this with the solution to $H'(r)$ that follows from the DBI action gives

$$J^x = \pm NE, \quad (48)$$

for zero magnetic field. This precisely matches with Eq. (21) at the scale, $r = r_\star$, up to an overall normalization.

B. Subsummary

To summarize, in this section, we have studied the expression of the current density in terms of one or more constant electric and magnetic fields. Essentially, use of the prescription of Ref. [10] or, equivalently, that of Eqs. (10) and (13), results in a recipe to calculate the current density in $d + 1$-dimensional field theory if the dual bulk geometry is of the form of Eq. (44). With a constant electric field, say, along x, one of the spatial directions, the ratio of the current density to electric field is the square root of the ratio of the product of the metric component (up to an overall factor) of $d - 1$ space, which is perpendicular to the t, x, r plane, to the metric component along x-axis, i.e. Eq. (32). This quantity should be evaluated at an energy scale, $r_\star$, for which the product of the metric components along t- and x-axis, i.e. $g_{tt}(r_\star)g_{xx}(r_\star)$ becomes same as the square of the electric field, which is the first equation of Eqs. (28). The condition that determines the point, $r_\star$, is generalized when there exist more than one constant electric field and a constant magnetic field in the theory.

It is worth it to emphasize that $r_\star$ should be close to the horizon, r_0, rather than to boundary in order for Eq. (48) to make sense. The factor $\sqrt{1-\frac{B^2}{E^2}\frac{g_{tt}(r_\star)}{g_{yy}(r_\star)}}$ should be a real quantity and can happen only when $r_\star$ is close to the horizon for any strength of the magnetic and electric fields. This can be seen as follows: close to the horizon, the ratio $\frac{g_{tt}(r_\star)}{g_{yy}(r_\star)}$ is very small, whereas close to the boundary, this ratio approaches unity. So, for $B > E$, the second factor in the square root can become greater than unity.

III. WITH CHARGE DENSITY

Let us discuss the effect of the nonvanishing charge density along with the Chern-Simon term on the conductivity. The inclusion of the Chern-Simon term makes an interesting change of the Hall conductivity; that is, it adds a piece and could potentially change the structure unless we take the axion to be constant. Moreover, the Chern-Simon term does not make any surprising changes in the Hall angle, $\cot\theta_H = \sigma^{xy}/\sigma^{xx}$, at the leading order in the large density and small magnetic field limit.

Charge density with the Chern-Simon term

It is not *a priori* clear whether the low-energy effective action of the probe brane admits a Chern-Simon type term or not. We assume that it does and takes the form similar to that in string theory except that the target space here is 3 + 1-dimensional. In this section, we have considered the following form of the field strength:

$$\begin{aligned}F_2 &= -E_1 dt\wedge dx - E_2 dt\wedge dy + B dx\wedge dy\\ &\quad - H'(r) dr\wedge dx + h'(r) dr\wedge dy + \phi'(r) dr\wedge dt.\end{aligned}\tag{49}$$

The inclusion of the Chern-Simon term in the probe-brane action adds the following term to the 3 + 1-dimensional action:

$$S_{\rm CS} = \mu\int\Big(\frac{C_0}{2}F\wedge F + [C_2]\wedge F + [C_4]\Big),\tag{50}$$

in the absence of the B_2 field from the Neveu-Schwarz–Neveu-Schwarz sector. The bulk fields $[C_n]$ are to be understood as the pullback onto the world volume of the probe brane. Let us also assume for simplicity C_4 vanishes, the C_2 has the the structure $[C_2] = -\tilde{C}_2(r) dt\wedge dy$, and $[C_0]$ depends only on the radial coordinate. Using the field strength as written in Eq. (50) results in

$$\begin{aligned}S_{\rm CS} &= \mu\int[C_0(E_1h' + E_2H' - B\phi')\\ &\quad + \tilde{C}_2H']dt\wedge dx\wedge dy\wedge dr.\end{aligned}\tag{51}$$

Let us redefine $\tilde{\mu} := \mu V_2$, where V_3 is the volume of $R^{1,2}$. Finally, the Chern-Simon action becomes

$$S_{\rm CS} = \tilde{\mu}\int[C_0E_1h' + (C_0E_2 + \tilde{C}_2)H' - C_0B\phi')]dr.\tag{52}$$

So, the full action of the probe brane is

$$\begin{aligned}S &= -\mathcal{N}\int dr[g_{rr}(B^2g_{tt} - E_1^2g_{yy} - E_2^2g_{xx} + g_{tt}g_{xx}g_{yy})\\ &\quad + (g_{tt}g_{xx} - E_1^2)h'^2 - 2E_1E_2h'H' + (g_{tt}g_{yy} - E_2^2)H'^2\\ &\quad + 2B(E_1h' + E_2H')\phi' - (g_{xx}g_{yy} + B^2)\phi'^2]^{1/2}\\ &\quad + \tilde{\mu}\int dr[C_0E_1h' + (C_0E_2 + \tilde{C}_2)H' - C_0B\phi')]\\ &\equiv -\mathcal{N}\int dr\bar{\mathcal{L}} + \tilde{\mu}\int dr[C_0E_1h'\\ &\quad + (C_0E_2 + \tilde{C}_2)H' - C_0B\phi')].\end{aligned}\tag{53}$$

Once again, the action does not depend on the fields ϕ, H, and h, so the corresponding momenta are constants. Let us denote the constant momentum for the field ϕ, h, and H as c_ϕ, c_h, and c_H, respectively:

$$\begin{aligned}\frac{\delta S}{\delta\phi'} &\equiv c_\phi\\ &= -\frac{\mathcal{N}\sqrt{\prod g_{z_az_a}}}{\bar{\mathcal{L}}}[B(E_1h' + E_2H')\\ &\quad - (g_{xx}g_{yy} + B^2)\phi'] - \tilde{\mu}BC_0,\\ \frac{\delta S}{\delta h'} &\equiv c_h\\ &= -\frac{\mathcal{N}\sqrt{\prod g_{z_az_a}}}{\bar{\mathcal{L}}}[(g_{tt}g_{xx} - E_1^2)h'\\ &\quad - E_1E_2H' + BE_1\phi'] + \tilde{\mu}E_1C_0,\\ \frac{\delta S}{\delta H'} &\equiv c_H\\ &= -\frac{\mathcal{N}\sqrt{\prod g_{z_az_a}}}{\bar{\mathcal{L}}}[(g_{tt}g_{yy} - E_2^2)H'\\ &\quad + BE_2\phi' - E_1E_2h'] + \tilde{\mu}(E_2C_0 + \tilde{C}_2].\end{aligned}\tag{54}$$

From now on, we shall drop the tildes from the field C_2 and the coupling μ so as to avoid cluttering of it. By taking the ratio of the momenta, we determine h' and ϕ' in terms of H':

$$h' \equiv \frac{h_1}{h_2}, \quad \text{where } h_1 = g_{yy}[E_2(c_H E_1 - c_h E_2 - E_1 \mu C_2) g_{xx} + g_{tt}(B(Bc_h + c_\phi E_1) + (c_h - E_1 \mu C_0) g_{xx} g_{yy})] H'$$
$$h_2 = g_{xx}[E_1(-c_H E_1 + c_h E_2 + E_1 \mu C_2) g_{yy} + g_{tt}(B(Bc_H + c_\phi E_2) + (c_H - E_2 \mu C_0) g_{xx} g_{yy} - \mu C_2(B^2 + g_{xx} g_{yy}))],$$
$$\phi' \equiv \frac{\phi_1}{\phi_2}, \quad \text{where } \phi_1 = g_{tt}[E_1(Bc_h + c_\phi E_1) g_{yy} + g_{xx}(E_2(Bc_H + c_\phi E_2) - BE_2 \mu C_2 - (c_\phi + B\mu C_0) g_{tt} g_{yy})] H'$$
$$\phi_2 = g_{xx}[E_1(-cHE_1 + c_h E_2 + E_1 \mu C_2) g_{yy} + g_{tt}(B(Bc_H + c_\phi E_2) + (c_H - E_2 \mu C_0) g_{xx} g_{yy} - \mu C_2(B^2 + g_{xx} g_{yy}))]. \tag{55}$$

The function H' can be evaluated by substituting Eq. (56) into the last equation of Eq. (55). We are not writing down the explicit form of H', as it is a very long expression. As is done previously, we perform the Legendre-transformed action:

$$\begin{aligned} S_L &= S - \int \frac{\delta S}{\delta \phi'} \phi' - \int \frac{\delta S}{\delta h'} h' - \int \frac{\delta S}{\delta H'} H' \\ &= -\mathcal{N} \int \frac{dr}{\bar{\mathcal{L}}} [g_{rr}(B^2 g_{tt} - E_1^2 g_{yy} - E_2^2 g_{xx} + g_{tt} g_{xx} g_{yy})] \\ &= -\int dr \sqrt{\frac{g_{rr}}{g_{tt} g_{xx} g_{yy}}} \times \sqrt{[\mathcal{A}_{\in}(r) \mathcal{A}_{\ni}(r) - (\mathcal{A}_{\triangle}(r))^2]}, \end{aligned} \tag{56}$$

where

$$\begin{aligned} \mathcal{A}_{\in}(r) &= B^2 g_{tt} - E_2^2 g_{xx} - E_1^2 g_{yy} + g_{tt} g_{xx} g_{yy}, \\ \mathcal{A}_{\ni}(r) &= g_{tt} g_{xx} g_{yy} - (c_H - \mu(E_2 C_0 + C_2))^2 g_{xx} \\ &\quad - (c_h - E_1 \mu C_0)^2 g_{yy} + (c_\phi + B\mu C_0)^2 g_{tt}, \\ \mathcal{A}_{\triangle}(r) &= (c_\phi + B\mu C_0) B g_{tt} + (c_h - E_1 \mu C_0) E_1 g_{yy} \\ &\quad + (c_H - \mu(E_2 C_0 + C_2)) E_2 g_{xx}. \end{aligned} \tag{57}$$

From now on, we shall set $E_2 = 0$, as it does not change much of the physics that we are going to do. The scale $r_\star$ is determined by solving

$$\mathcal{A}_{\in}(r_\star) = [B^2 g_{tt} - E_1^2 g_{yy} + g_{tt} g_{xx} g_{yy}]_{r_\star} = 0. \tag{58}$$

The form of the currents which follow are

$$\begin{aligned} J^x &= \mu C_2 \mp \frac{E_1 g_{yy}}{B^2 + g_{xx} g_{yy}} \\ &\quad \times \sqrt{(c_\phi + B\mu C_0)^2 + \mathcal{N}^2(B^2 + g_{xx} g_{yy})}, \\ J^y &= E_1 \left[\frac{B(c_\phi + B\mu C_0)}{B^2 + g_{xx} g_{yy}} - \mu C_0 \right] \\ &= \frac{E_1 [Bc_\phi - \mu C_0 g_{xx} g_{yy}]}{B^2 + g_{xx} g_{yy}}, \end{aligned} \tag{59}$$

where we have used Eq. (58). The conductivity that follows upone using Ohm's law is

$$\begin{aligned} \sigma^{xx} &= \mp \frac{g_{yy}}{B^2 + g_{xx} g_{yy}} \\ &\quad \times \sqrt{(c_\phi + B\mu C_0)^2 + \mathcal{N}^2(B^2 + g_{xx} g_{yy})}, \\ \sigma^{xy} &= \frac{Bc_\phi - \mu C_0 g_{xx} g_{yy}}{B^2 + g_{xx} g_{yy}}. \end{aligned} \tag{60}$$

The quantity $\mathcal{N}^2 \sim N_f^2 / g_s^2$, where N_f is the number of flavor branes and g_s is the string coupling. Once again, let us look at a special corner of the parameter space of charge density c_ϕ and the magnetic field B for which $\mathcal{N} B$ is very small in comparison to density. In this case, the conductivity reduces to

$$\begin{aligned} \sigma^{xx} &\simeq \mp \left[\frac{c_\phi}{g_{xx}} + \mathcal{N}^2 \frac{g_{yy}}{2c_\phi^2} + \cdots \right], \\ \sigma^{xy} &\simeq \frac{Bc_\phi}{g_{xx} g_{yy}} - \mu C_0 + \cdots, \end{aligned} \tag{61}$$

where the ellipses denote higher powers of magnetic field. Choosing the positive branch from σ^{xx} and dropping the second term in σ^{xx} gives us

$$\sigma^{xx} \sim \frac{c_\phi}{g_{xx}}, \qquad \sigma^{xy} \simeq \frac{Bc_\phi}{g_{xx} g_{yy}} - \mu C_0, \tag{62}$$

which in the small μC_0 limit, i.e. $\mu C_0 \ll \frac{Bc_\phi}{g_{xx} g_{yy}}$, the Hall angle reduces to

$$\frac{\sigma^{xx}}{\sigma^{xy}} \sim \frac{g_{yy}}{B}. \tag{63}$$

So, the presence of the Chern-Simon term in the action parametrically does not change much of the conductivity in the small magnetic field or large density limit but adds a piece to the off-diagonal part of the conductivity. The Hall angle in the high-density limit, $\mu C_0 \ll \frac{Bc_\phi}{g_{xx} g_{yy}}$, remains same as in the absence of the Chern-Simon term.

In the presence of a nontrivial dilaton, Φ, the form of the conductivities becomes

$$\sigma^{xx} = \mp \frac{e^{2\Phi} g_{yy}}{B^2 + e^{4\Phi} g_{xx} g_{yy}} \times \sqrt{(c_\phi + B\mu C_0)^2 + \mathcal{N}^2 e^{-2\Phi}(B^2 + e^{4\Phi} g_{xx} g_{yy})},$$
$$\sigma^{xy} = \frac{B c_\phi - \mu C_0 e^{4\Phi} g_{xx} g_{yy}}{B^2 + e^{4\Phi} g_{xx} g_{yy}}. \qquad (64)$$

Note that we are calculating the conductivities in the Einstein frame, i.e. we have changed the metric components as $g_{ab} \to e^{2\Phi} g_{ab}$. In the small magnetic field limit, $B \ll e^{2\Phi(r_\star)}\sqrt{g_{xx}(r_\star) g_{yy}(r_\star)}$, and at large-density limit, $c_\phi \gg B\mu C_0(r_\star)$, the conductivities reduce to

$$\sigma^{xx} \sim \frac{e^{-2\Phi}}{g_{xx}} \sqrt{c_\phi^2 + \mathcal{N}^2 e^{2\Phi} g_{xx} g_{yy}},$$
$$\sigma^{xy} \sim \frac{B c_\phi e^{-4\Phi}}{g_{xx} g_{yy}} - \mu C_0. \qquad (65)$$

If we, further, take the axion as constant, $C_0 = \theta$, with a rotationally invariant geometry in the very high-density limit, $c_\phi \gg \mathcal{N} e^{\Phi(r_\star)}\sqrt{g_{xx}(r_\star) g_{yy}(r_\star)}$, and assume that the first term in the Hall conductivity dominates over the axionic term, then there is no solution to dilaton or metric component that can give the result as written in Eq. (1). However, if we consider a limit for which, $B c_\phi e^{-4\Phi(r_\star)} \ll \mu C_0(r_\star) g_{xx}(r_\star) g_{yy}(r_\star)$, then the second term in the Hall conductivity dominates over the first term; in which case, it is possible to reproduce Eq. (1) up to a sign.

Let us consider another limit, $c_\phi \gg B\mu C_0$ but $c_\phi \ll \mathcal{N} e^{\Phi(r_\star)}\sqrt{g_{xx}(r_\star) g_{yy}(r_\star)}$, with small magnetic field in a rotationally invariant geometry with a nonconstant axion. Then the conductivities reduce to

$$\sigma^{xx} \sim \mathcal{N} e^{-\Phi}, \qquad \sigma^{xy} \sim \frac{B c_\phi e^{-4\Phi}}{g_{xx}^2} - \mu C_0. \qquad (66)$$

Upon comparing to Eq. (1), the dilaton should go as $e^\Phi \sim \mathcal{N} T$, and the combination of metric component and axion should be as

$$\frac{B c_\phi e^{-4\Phi}}{g_{xx}^2} - \mu C_0 \sim T^{-3}. \qquad (67)$$

It would be interesting to find such background solutions that have the property as is written in Eq. (68).

IV. GEOMETRY WITH TWO EXPONENTS: AN EXAMPLE

In this section, we shall write down a gravitational black hole solution for which the geometry exhibits the required two exponents, explicitly. The extremal solution is already found in Ref. [13] in a specific setting that is with several-form field strengths and metric. But to find the nonextremal solution in that setup is very cumbersome. Instead, here, we shall find such solutions by adopting a different form of the gravitational action than that which is considered in Ref. [13], but it comes up with a cost: that is, the entropy vanishes even though there is a finite size of the horizon. The on-shell action vanishes identically as a result of the vanishing of the free energy and the energy. A similar kind of behavior was seen previously in the context of generating a Lifshitz type of solutions in Refs. [31,32].

The action that we shall consider is a Ricci squared corrected term to Einstein-Hilbert action with a cosmological constant

$$S = \frac{1}{2\kappa_4^2} \int \sqrt{-g}[R - 2\Lambda + \beta R^2] \equiv \int L. \qquad (68)$$

The equation of motion that follows from it is

$$R_{MN} - \frac{1}{2} g_{MN} R + \Lambda g_{MN} + 2\beta g_{MN} \Box R - 2\beta \nabla_M \nabla_N R + 2\beta R R_{MN} - \frac{1}{2}\beta R^2 g_{MN} = 0. \qquad (69)$$

The solution to the equation of motion comes as

$$ds^2 = L^2\left[-r^{2z} f(r) dt^2 + r^{2w} dx^2 + r^2 dy^2 + \frac{dr^2}{r^2 f(r)}\right], \qquad (70)$$

which respect to the scaling symmetry

$$t \to \lambda^z t, \qquad x \to \lambda^w x, \qquad y \to \lambda y, \qquad r \to \frac{r}{\lambda}. \qquad (71)$$

The function

$$f(r) = 1 - \left(\frac{r_h}{r}\right)^{\alpha_\pm}, \qquad (72)$$

where

$$\alpha_\pm = 1 + w + \tfrac{3}{2} z \pm \tfrac{1}{2}\sqrt{-4(1+w^2) + 4z + 4wz + z^2}, \qquad (73)$$

from which there follows a restriction on the exponents, $4z + 4wz + z^2 \geq 4(1 + w^2)$ and the dimensionful objects β and Λ are

$$\Lambda = -\frac{1}{2L^2}[1 + w + z + w^2 + z^2 + wz],$$
$$\beta = \frac{L^2}{4[1 + w + z + w^2 + z^2 + wz]} \qquad (74)$$

with the Hawking temperature

$$T_H = \frac{\alpha_\pm}{4\pi} r_h^z. \qquad (75)$$

It follows trivially that for a solution with exponents for which $z = 1$ and $w = 1/2$ satisfy the restrictions, $\alpha_\pm$ is a real quantity, and hence the solution is real. For this choice of the exponents, the cosmological constant and the coupling are

$$\Lambda = -\frac{17}{8L^2}, \qquad \beta = \frac{L^2}{17}, \qquad \alpha_\pm = \frac{3\sqrt{2} \pm 1}{\sqrt{2}}. \qquad (76)$$

If we calculate the entropy of the system using Wald's formula [33],

$$S_{BH} = -2\pi \int_{r_h} \frac{\partial L}{\partial R_{abcd}} \epsilon_{ab}\epsilon_{cd}, \qquad (77)$$

where the quantity ϵ_{ab} is binormal to the bifurcation surface and is normalized in such a way that it obeys $\epsilon_{ab}\epsilon^{ab} = -2$. We use the convention of Ref. [34] to calculate it, which reads as

$$\epsilon_{ab} = \xi_a \eta_b - \xi_b \eta_a, \qquad (78)$$

where ξ and η are null vectors normal to the bifurcate killing horizon, with $\xi.\eta = 1$. In our choice of 3 + 1-dimensional metric, the nonvanishing components of the null vectors are

$$\xi_t = -g_{tt} = -L^2 r^{2z} f(r), \qquad \eta_t = 1, \qquad \eta_r = -\sqrt{\frac{g_{rr}}{g_{tt}}} = -\frac{1}{f(r) r^{1+z}}. \qquad (79)$$

In fact, for the action like Eq. (69), the entropy is

$$S_{BH} = \frac{2\pi}{\kappa_4^2}(\sqrt{-g}[1 + 2\beta R])_{r_h}, \qquad (80)$$

and using all the ingredients into this formula gives us zero entropy, which means the solution Eq. (71) has the constant curvature $R = -1/2\beta$. From the trace of the equation of motion to metric Eq. (70), it follows that the scalar curvature obeys

$$R = 4\Lambda + 6\beta \Box R. \qquad (81)$$

Now, combining these two facts, we obtain the curvature

$$R = -1/2\beta = 4\Lambda, \qquad (82)$$

which is precisely the behavior of the solution in Eq. (77).

A. Parameter space

In this subsection, we shall write down the exact form of both the conductivity and Hall angle that follow from Sec. III. Before doing the evaluation of the conductivity, we need to know the scale, $r_\star$. From Eq. (59), it follows that for small electric field and magnetic field, the scale

$$r_\star \sim r_h \sim T_H^{1/z}. \qquad (83)$$

The correction to this scale occurs in the dimensionless ratios of $E/T_H^{1+1/z}$ and $B/T_H^{(1+w)/z}$.

Now, substituting the explicit form of the metric components from Eq. (71) into Eq. (61) results in

$$\sigma^{xx} = \mp \frac{r_\star^2}{B^2 + r_\star^{2(1+w)}} \times \sqrt{(c_\phi + B\mu C_0)^2 + \mathcal{N}^2(B^2 + r_\star^{2(1+w)})}, \qquad \sigma^{xy} = \frac{Bc_\phi - \mu C_0 r_\star^{2(1+w)}}{B^2 + r_\star^{2(1+w)}}. \qquad (84)$$

In the small magnetic field, large density, and at low-temperature limit, the expression of the conductivities reduces to

$$\sigma^{xx} \sim c_\phi r_\star^{-2w} + \cdots \sim c_\phi T_H^{-2w/z}, \qquad \sigma^{xy} \sim Bc_\phi r_\star^{-2(1+w)} - \mu C_0 + \cdots \sim Bc_\phi T_H^{-2(1+w)/z} - \mu C_0, \qquad \Rightarrow \sigma^{xx}/\sigma^{xy} \sim r_\star^2/c_\phi + \cdots \sim (T_H^{2/z})/c_\phi + \cdots, \qquad (85)$$

where, in the last line, we have assumed $Bc_\phi > \mu C_0 T_H^{2(1+w)/z}$. Demanding that this temperature dependence of conductivities should match the experimental results, Eq. (1) gives us the following values of exponent: $z = 1$ and $w = 1/2$. So, the above form of the exponents gives us the strange-metal behavior of copper-oxide systems as seen in experiments[1,3,4]. If we consider the other regime of parameter space with a constant axion at low temperature for which the magnetic field is small in comparison to charge density such that $Bc_\phi < \mu C_0 T_H^{2(1+w)/z}$, then the off-diagonal conductivity does not depend on the temperature for constant axion. So, it is not of much interest as far as the experimental results are concerned. Hence, this regime of parameter space may not be that useful. However, if we consider the nonconstant axion field in the same limit, i.e. $Bc_\phi < \mu C_0 T_H^{2(1+w)/z}$, then by matching with Eq. (1), we get the exponents as $2w = z$, and the axion field should have the following behavior: $C_0 \sim T_H^{-3} \sim r_\star^{-3z}$. It would be interesting to find such background solutions.

B. Fermi liquid

In this subsection, we shall reproduce a well-known transport property of the Fermi-liquid theory. It is known, see for example Ref. [35], that the conductivity at low temperature goes as $\sigma_{\rm FL} \sim T^{-2}$. Now, upon using Eq. (86), we see that in order to reproduce this particular behavior of the temperature we are required to take the exponents as $w = z$. Here, the exponents are not fixed to a particular value. In the next section, we shall demand that the specific heat should have a linear dependence of temperature, parametrically. The result of this fixes the exponents to $z = w = -2$. Note, for this choice of exponents, the quantity $\alpha_\pm$ defined in Eq. (74) becomes pure imaginary, which is an artifact of the action we used to construct it. However, in what follows, we shall not be worried about the nature of $\alpha_\pm$, as we believe the above-mentioned constraint on the exponents can be removed by looking at better solutions.

V. PROBE BRANE THERMODYNAMICS

In this section, we shall study some thermodynamic properties of the probe brane but without the Chern-Simon term and nontrivial dilaton. Let us recall that the charge carriers are introduced via probe brane and the study of their thermodynamic behavior is very important so as to have a better understanding of the nature of quantum critical point. It is reported in Refs. [36,37] (for a review, see Ref. [35]) that at low temperature, the specific heat (for non-Fermi liquid [NFL]) goes as $C_V \sim T \,\mathrm{Log}\, T$. But, unfortunately, with our choice of exponents as demanded by the transport properties, $z=1$ and $w=1/2$ give us the specific heat to go instead as $C_V \sim T_H^3$. This kind of behavior of specific heat resembles that of the Debye theory.

Let us see this particular behavior of specific heat in detail. We shall proceed to calculate the free energy of the probe-brane system following Ref. [38]. The proper holographic treatment is also done in Refs. [21,39]. The Gibbs free energy, i.e. the thermodynamic potential, Ω, in the grand canonical ensemble is just the negative of the on-shell value of the action times temperature. Here, we have chosen to work in the canonical ensemble. The easiest way to include the effect of the charge density and magnetic field is by using the field strength, $F_2 = \phi'(r) dr \wedge dt + B dx \wedge dy$ in the DBI action. In $3+1$ dimensions, using the metric as in Eq. (44) gives us the thermodynamic potential and chemical potential $\mu = \int_{r_h}^\infty dr F_{rt} = \int_{r_h}^\infty dr \phi'$. The chemical potential, μ, should not be confused with the Chern-Simon coupling that appeared in Sec. III:

$$\Omega = \mathcal{N} V_2 \int_{r_h}^\infty dr \frac{(g_{xx}g_{yy} + B^2)\sqrt{g_{tt}g_{rr}}}{\sqrt{g_{xx}g_{yy} + B^2 + \rho^2}}, \qquad \mu = \rho \int_{r_h}^\infty dr \frac{\sqrt{g_{tt}g_{rr}}}{\sqrt{g_{xx}g_{yy} + B^2 + \rho^2}}, \tag{86}$$

where $\mathcal{N}\rho = c_\phi$ and c_ϕ is the charge density. V_2 is the flat-space volume of the x, y plane. Using the metric structure as written in Eq. (71) gives

$$\Omega = \mathcal{N} V_2 \int_{r_h}^\infty dr \frac{(r^{2+2w} + B^2) r^{z-1}}{\sqrt{r^{2+2w} + B^2 + \rho^2}}, \qquad \mu = \rho \int_{r_h}^\infty dr \frac{r^{z-1}}{\sqrt{r^{2+2w} + B^2 + \rho^2}}. \tag{87}$$

For generic choice of the exponents, the integral in the thermodynamic potential and in the chemical potential diverges at UV, so we need to regulate it. The way we shall do is to subtract an equivalent amount but without the charge density and magnetic field. It means

$$\begin{aligned}\Omega &= \mathcal{N} V_2 \int_0^\infty dr \left(\frac{r^{1+2w+z}}{\sqrt{r^{2+2w} + B^2 + \rho^2}} - r^{z+w} \right) - \mathcal{N} V_2 \int_0^{r_h} dr \frac{r^{1+2w+z}}{\sqrt{r^{2+2w} + B^2 + \rho^2}} \\ &\quad + \mathcal{N} V_2 B^2 \int_0^\infty dr \left(\frac{r^{z-1}}{\sqrt{r^{2+2w} + B^2 + \rho^2}} - r^{z-w-2} \right) - \mathcal{N} V_2 B^2 \int_0^{r_h} dr \frac{r^{z-1}}{\sqrt{r^{2+2w} + B^2 + \rho^2}}, \\ \mu &= \rho \int_0^\infty dr \left(\frac{r^{z-1}}{\sqrt{r^{2+2w} + B^2 + \rho^2}} - r^{z-w-2} \right) - \rho \int_0^{r_h} dr \frac{r^{z-1}}{\sqrt{r^{2+2w} + B^2 + \rho^2}}.\end{aligned} \tag{88}$$

The second term in the parentheses of the first equation should not be there when $z + w = 0$. Similarly, the second term in the square bracket of the first equation comes into picture only when $z > 2 + w$, so also for the second term in the second square bracket. Let us assume the case where $z \ngtr 2 + w$ and $z \neq -w$. It means we want to regulate it in the following way:

$$\begin{aligned}\Omega &= \mathcal{N} V_2 \int_0^\infty dr \left(\frac{r^{1+2w+z}}{\sqrt{r^{2+2w} + B^2 + \rho^2}} - r^{z+w} \right) - \mathcal{N} V_2 \int_0^{r_h} dr \frac{r^{1+2w+z}}{\sqrt{r^{2+2w} + B^2 + \rho^2}} \\ &\quad + \mathcal{N} V_2 B^2 \int_0^\infty dr \frac{r^{z-1}}{\sqrt{r^{2+2w} + B^2 + \rho^2}} - \mathcal{N} V_2 B^2 \int_0^{r_h} dr \frac{r^{z-1}}{\sqrt{r^{2+2w} + B^2 + \rho^2}}, \\ \mu &= \rho \int_0^\infty dr \frac{r^{z-1}}{\sqrt{r^{2+2w} + B^2 + \rho^2}} - \rho \int_0^{r_h} dr \frac{r^{z-1}}{\sqrt{r^{2+2w} + B^2 + \rho^2}}.\end{aligned} \tag{89}$$

After doing these integrals, we find

$$\frac{\Omega}{\mathcal{N}V_2} = \alpha(w,z)(B^2+\rho^2)^{((z+w+1)/(2+2w))} - \frac{1}{(2+2w+z)}\frac{r_h^{(2+2w+z)}}{\sqrt{B^2+\rho^2}}\,{}_2F_1\left[1+\frac{z}{2+2w},\frac{1}{2};2+\frac{z}{2+2w};-\frac{r_h^{2+2w}}{B^2+\rho^2}\right]$$
$$+\frac{B^2}{z\sqrt{\pi}}(B^2+\rho^2)^{((z-w-1)/(2+2w))}\Gamma\left(\frac{1+w-z}{2+2w}\right)\Gamma\left(\frac{2+2w+z}{2+2w}\right)$$
$$-\frac{r_h^z}{z}\frac{B^2\rho}{\sqrt{B^2+\rho^2}}\,{}_2F_1\left[\frac{z}{2+2w},\frac{1}{2};1+\frac{z}{2+2w};-\frac{r_h^{2+2w}}{B^2+\rho^2}\right],$$
$$\mu = \frac{1}{z\sqrt{\pi}}(B^2+\rho^2)^{((z-w-1)/(2+2w))}\Gamma\left(\frac{1+w-z}{2+2w}\right)\Gamma\left(\frac{2+2w+z}{2+2w}\right)$$
$$-\frac{r_h^z}{z}\frac{\rho}{\sqrt{B^2+\rho^2}}\,{}_2F_1\left[\frac{z}{2+2w},\frac{1}{2};1+\frac{z}{2+2w};-\frac{r_h^{2+2w}}{B^2+\rho^2}\right], \tag{90}$$

where $\alpha(w,z)$ is a function of the exponents and whose explicit structure is not that important for the understanding of thermodynamics. $\Gamma(x)$ and ${}_2F_1[a,b;c;x]$ are the gamma function and hypergeometric function, respectively. In the limit of high density, low magnetic field, and low temperature, i.e. $T^{1+w/z}/\sqrt{B^2+\rho^2} \ll 1$, Eq. (91) can be expanded in the series form. Let us recall the free energy in the canonical ensemble $F = \Omega + \mu J^t$, where $J^t = \mathcal{N}V_2\rho$ is the charge. From this, the entropy density goes as

$$s = -\frac{1}{V_2}\left(\frac{\partial F}{\partial T_H}\right) = s_0 + \frac{\mathcal{N}}{2z\sqrt{B^2+\rho^2}}(4\pi/\alpha_\pm)^{((2+2w+z)/z)}T_H^{((2+2w)/)z}, \tag{91}$$

where $s_0 = \frac{4\pi\mathcal{N}}{z\alpha_\pm}\sqrt{B^2+\rho^2}$ is the entropy density at zero temperature. The specific heat is defined as the heat capacity per unit volume, and at low temperature, it goes as

$$C_V = T_H\left(\frac{\partial s}{\partial T_H}\right) = \frac{\mathcal{N}}{\sqrt{B^2+\rho^2}}\left(\frac{1+w}{z^2}\right)(4\pi/\alpha_\pm)^{((2+2w+z)/z)}T_H^{((2+2w)/z)}. \tag{92}$$

The magnetic susceptibility, which we shall call "susceptibility," at low temperature is

$$\chi/V_2 = -\left(\frac{\partial^2 F}{\partial B^2}\right) = -\chi_0(B,\rho) + \frac{\mathcal{N}4\pi}{z\alpha_\pm}\frac{\rho^2}{(B^2+\rho^2)^{3/2}}T_H, \tag{93}$$

where χ_0 is some function of B and ρ and whose exact form is not that illuminating. The effect of the Chern-Simon term with the field strength, $F_2 = \phi'(r)dr \wedge dt + Bdx \wedge dy$, is to replace ρ in all of the above formulas by $\rho + \mu\theta B$, where we have considered the axion field to be a constant and identified it with $C_0 \equiv \theta$.

At high temperature, low magnetic field, and low density

In this subsection, we shall write down the behavior of thermodynamic quantities in the high-temperature but low-magnetic-field limit. One of the main reasons to study this regime of parameter space is to see the behavior of susceptibility. Probably, it is correct to say that when we are in the proximity of quantum critical point, the magnetization should not obey the Curie-Weiss–type behavior in the high-temperature limit.

The temperature dependence of free energy in this regime can be obtained very easily by looking at the following integrals:

$$\frac{\Omega}{\mathcal{N}V_2} \sim -\int_0^{r_h} dr\frac{r^{1+2w+z}}{\sqrt{r^{2+2w}+B^2+\rho^2}} - B^2\int_0^{r_h} dr\frac{r^{z-1}}{\sqrt{r^{2+2w}+B^2+\rho^2}},$$
$$= -\frac{r_h^{z+w+1}}{z+w+1} - \frac{B^2 r_h^{z-w-1}}{z-w-1} + \frac{(B^2+\rho^2)}{2(z-w-1)}r_h^{z-w-1} + \frac{B^2(B^2+\rho^2)}{2(z-3w-3)}r_h^{z-3w-3} + \cdots,$$
$$\mu \sim -\rho\int_0^{r_h} dr\frac{r^{z-1}}{\sqrt{r^{2+2w}+B^2+\rho^2}} = -\frac{\rho r_h^{z-w-1}}{z-w-1} + \frac{\rho(B^2+\rho^2)}{2(z-3w-3)}r_h^{z-3w-3} + \cdots. \tag{94}$$

So, the free energy in the canonical ensemble has the following behavior in the high-temperature limit:

$$\frac{F}{\mathcal{N}V_2} = -\frac{1}{z+w+1}(4\pi/\alpha_\pm)^{((z+w+1)/z)}T_H^{((z+w+1)/z)} - \frac{(B^2+\rho^2)}{2(z-w-1)}(4\pi/\alpha_\pm)^{((z-w-1)/z)}T_H^{((z-w-1)/z)} + \frac{(B^2+\rho^2)^2}{2(z-3w-3)}(4\pi/\alpha_\pm)^{((z-3w-3)/z)}T_H^{((z-3w-3)/z)} + \cdots, \quad (95)$$

from which the magnetization, $\frac{M}{\mathcal{N}V_2} = -(\frac{\partial F/(\mathcal{N}V_2)}{\partial B})$, and the susceptibility, $\frac{\chi}{\mathcal{N}V_2} = -(\frac{\partial^2 F/(\mathcal{N}V_2)}{\partial B^2})$, are

$$\frac{M}{\mathcal{N}V_2} = -\frac{B}{(z-w-1)}(4\pi/\alpha_\pm)^{((z-w-1)/z)}T_H^{((z-w-1)/z)} - \frac{2B(B^2+\rho^2)}{(z-3w-3)}(4\pi/\alpha_\pm)^{((z-3w-3)/z)}T_H^{((z-3w-3)/z)},$$

$$\frac{\chi}{\mathcal{N}V_2} = \frac{1}{(z-w-1)}(4\pi/\alpha_\pm)^{((z-w-1)/z)}T_H^{((z-w-1)/z)} - \frac{2(3B^2+\rho^2)}{(z-3w-3)}(4\pi/\alpha_\pm)^{((z-3w-3)/z)}T_H^{((z-3w-3)/z)} \equiv \tilde{\chi}_0 T_H^{((z-w-1)/z)} - \tilde{\chi}_1 T_H^{((z-3w-3)/z)}. \quad (96)$$

Now, if we demand that the magnetization, or, more precisely, the susceptibility, has the Curie-Weiss–type behavior, the above result forces us to put the following constraints on the exponents:

$$2z = 1 + w. \quad (97)$$

Recalling the results to the exponents that follow from the study of conductivity and the Hall angle in Sec. IV suggests that near to quantum critical point, the system does not show the Curie-Weiss–type behavior. In fact, the behavior of susceptibility using the exponents, $z = 1$ and $w = 1/2$, gives

$$\chi = \tilde{\chi}_0 T_H^{-1/2}, \quad (98)$$

and for Fermi liquid (FL), $z = -2 = w$, at high-temperature limit goes as

$$\chi = \tilde{\chi}_0 T_H^{1/2}. \quad (99)$$

The Curie-Weiss–type behavior is possible only when Eq. (98) is obeyed, from which it follows trivially that the asymptotically AdS spacetime possesses such a kind of behavior, as an example, for which $z = 1 = w$. Once again, the effect of the Chern-Simon term is to replace ρ in all of the above formulas by $\rho + \mu\theta B$ for constant axion as stated in the previous section.

VI. CONCLUSION

In this paper, we have shown that there exist two possible ways (see Table I) with different symmetries to find the precise temperature dependence of the longitudinal conductivity and Hall angle, $1/T$ and T^2, respectively, as seen in the non-Fermi liquid. The calculation is done similarly in spirit to the proposal of Ref. [10], where the charge density is introduced via flavor brane. It is done in a generic 3 + 1-dimensional (bulk) background solution possessing the symmetries, like scaling, time translation, and spatial translations. The result of the calculation suggests that in order to get the desired experimental result for the transport quantities that is mentioned above, in the high-density limit, we should not take the spatial part of the metric components same, i.e. $g_{xx} \neq g_{yy}$. It means the theory should have the symmetries like scaling, time translation, and spatial translation symmetry but without any rotational symmetry. For this purpose, we have considered a metric with two exponents, z and w, as defined in Eq. (4). The end result of this requirement is that the exponents take the values $z = 1$ and $w = 1/2$.

The study of the thermodynamic behavior of various physical quantities are equally important in the study of the quantum critical point or otherwise. For the above choice of the exponents, the specific heat at low temperature goes as $C_V \sim T_H^3$, which resembles that of the Debye type. The susceptibility at zero magnetic field and at low temperature goes as $\chi = -\chi_0 + (\text{constants}) \times T_H/\rho$, where χ_0 is a function of charge density.

However, if we consider a theory to have the symmetries like pseudoscaling (nontrivial scalar field), time translation, spatial translation, and rotation in the low density limit, we can reproduce Eq. (1) without the need to introduce two exponents. We leave the detailed study of the thermodynamic behavior of this class of solution for future research.

From this study, there follows an interesting outcome: we are completely ruling out those background solutions that possess the symmetries like scaling symmetry, time translation, spatial translation, and rotational symmetry. In other words, these symmetries are not consistent with Eq. (1).

The transport and thermodynamic behavior of various physical quantities at high density and at low temperature can be summarized in this two-exponent model as shown in Tables II and III.

The behavior of the thermodynamic and transport quantities at high temperature, low density, and low-magnetic-field limit is given in Table IV.

It follows from Table IV that it is only the asymptotically AdS spacetime that shows the Curie-Weiss type of behavior, fFrom which it is natural to think that the asymptotically AdS spacetime may be associated to metals (more specifically, to the paramagnets) even though the specific heat shows a quadratic dependence of temperature.

In this study, we have constructed a background black hole geometry with two exponents, for illustration. The future goal would be to construct other background

TABLE II. Behavior of transport quantities at low temperatures and at high density.

Type	Physical quantity	Experiment result	Reference	In this model	Experimental result forces the choice of exponents
NFL	Conductivity	T^{-1}	[1,3,4]	$T_H^{-2w/z}$	$z = 1,\ w = 1/2$
NFL	Hall Angle	T^2	[1,3,4]	$T_H^{2/z}$	$z = 1,\ w = 1/2$
FL	Conductivity	T^{-2}	[38]	$T_H^{-2w/z}$	$z = -2,\ w = -2$
FL	Hall Angle	Not known to author	$\cdots$	$T_H^{2/z} \sim T_H^{-1}$	$\cdots$

TABLE III. Behavior of thermodynamic quantities at low temperature and at high density.

Type	Physical quantity	Experiment result	Reference	In this model	Experimental result forces the choice of exponents
FL	Specific heat	T	[38]	$-T_H^{(2+2w)/z}$	$z = -2,\ w = -2$
FL	Susceptibility: $\chi(B = 0)$	independent of T	[38]	$-\chi_0 + \mathrm{const}/\alpha_\pm \times T_{H/\rho}$	$\cdots$
NFL	Specific heat	should not be as T	[38]	$T_H^{(2+2w)/z} \sim T_H^3$	$\cdots$
NFL	Susceptibility: $\chi(B = 0)$	Not known to author	$\cdots$	$-\chi_0 + \mathrm{const} \times T_{H/\rho}$	$\cdots$

TABLE IV. Behavior of thermodynamic quantities at high temperature, low density, and low magnetic field.

Type	Physical quantity	Exponents	In this model	Prediction
NFL	Specific heat	$z = 1,\ w = 1/2$	$T_H^{(1+w)/z}$	$T_H^{3/2}$
NFL	Susceptibility	$z = 1,\ w = 1/2$	$T_H^{(z-w-1)/z}$	$T_H^{-1/2}$
FL	Specific heat	$z = -2 = w$	$T_H^{(1+w)/z}$	$T_H^{1/2}$
FL	Susceptibility	$z = -2 = w$	$T_H^{(z-w-1)/z}$	$T_H^{1/2}$
AdS spacetime	Specific heat	$z = 1 = w$	$T_H^{(1+w)/z}$	T_H^2
AdS spacetime	Susceptibility	$z = 1 = w$	$T_H^{(z-w-1)z}$	T_H^{-1}

solutions having nontrivial spacetime thermodynamics, i.e. the thermodynamics of adjoint degrees of freedom, in the sense of having nonzero entropy for finite horizon size and maybe nonzero free energy, depending on the requirement of the model, which is *a priori* not clear at present. Moreover, the thermodynamic quantities in the Fermi-liquid phase need to be real.

There are several other checks that needed to be done. In particular, the AC conductivity, $\sigma(\omega)$, which in the interval $T_H < \omega < \tilde{\Omega}$, shows a very specific behavior [40] where ω and $\tilde{\Omega}$ are the frequency and some high-energy cutoff scale, respectively. This result of Ref. [40] for copper-oxide systems puts some serious restrictions on the form of the bulk geometry. In the study of superconductors [41] at low temperature (close to extremality), it was suggested that if the potential energy close to IR behaves as $V = V_0/r^2$, then the real part of AC conductivity goes as $\Re[\sigma(\omega)] \propto \omega^{\sqrt{4V_0+1}-1}$. Now, upon matching with the results of Ref. [40], we get $V_0 = -2/9$, i.e. there should be an attractive potential energy close to IR. This is an interesting prediction, but we leave this aspect of holographic model building for future research.

There is one further comment that deserves to be mentioned. In Ref. [6], it is shown that both for scaling and presudoscaling theories with unbroken rotational symmetry in the x, y plane, the resistivity and the AC conductivity have the following temperature and frequency dependence, at low temperature: $\rho \sim T^{\nu_1}$ and $\sigma(\omega) \sim \omega^{-\nu_1}$ for $\nu_1 \le 1$. Now, if we demand Eq. (1) on this result, then it fixes $\nu_1 = 1$; it means $\sigma(\omega) \sim \omega^{-1}$, which is not allowed by Ref. [40]. So, it is natural to think of some more exotic models that are either proposed in Ref. [6] or that are discussed in this paper in order to get as close as possible to the experimental results.

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APPENDIX A: SOLUTION TO MAXWELL SYSTEM

In this section, we shall write down the exact solution to the Maxwell's equation of motion in the notation of Ref. [42]. Let us start with a system whose dynamics is described by Maxwell's action:

$$S = -\frac{1}{4}\int d^{d+1}x \frac{\sqrt{-g}}{g_{\rm YM}^2} F_{\rm MN}F^{\rm MN}, \qquad (A1)$$

with coordinate-dependent coupling, $g_{\rm YM}$ and whose explicit dependence we do not specify. Also, we assume that the $d+1$-dimensional spacetime possesses the symmetries like rotation in $d-1$-dimensional space, along with time and spatial translations, and has a structure like

$$ds^2_{d+1} = -h(r)d\tau^2 + 2d\tau dr + e^{2s(r)}\delta_{ij}dx^i dx^j, \qquad (A2)$$

where the radial coordinate can have a range, $r_0 \le r \le r_c$, with r_0 denote the horizon of a black hole and r_c is the upper cutoff, which is the UV. The nonvanishing components of the field strength are $F_{\tau r}$, $F_{\tau i}$, F_{ij}, F_{ir}. The equations of motion are

$$\begin{aligned} &e^{2s}\partial_\tau F_{\tau r} + \partial^i F_{\tau i} = h\partial^i F_{ir},\\ &\partial_\tau F_{ir} + [(2-p)s' - \phi']F_{\tau i} - \partial_r F_{\tau i} + h' F_{ir}\\ &\quad = -h\partial_r F_{ir} + h[(2-p)s' - \phi']F_{ir} + e^{-2s}\partial^j F_{ji},\\ &\partial_r F_{\tau r} + (ps' + \phi')F_{\tau r} + e^{-2s}\partial^i F_{ir} = 0, \end{aligned} \qquad (A3)$$

where we have used $1/g^2_{\rm YM} = e^{\phi(r)}$, and in the derivatives, the indices $i, j, \cdots$ are raised using δ^{ij}, i.e. $\partial^i = \delta^{ij}\partial_j$. The normalization of the coupling is assumed to be $\phi(r_0) = 0$. We shall solve these equations of motion along with the Bianchi identities:

$$\begin{aligned} \partial_\tau F_{ri} + \partial_r F_{i\tau} + \partial_i F_{\tau r} = 0,\\ \partial_\tau F_{ij} + \partial_i F_{j\tau} + \partial_j F_{\tau i} = 0,\\ \partial_r F_{ij} + \partial_i F_{jr} + \partial_j F_{ri} = 0, \end{aligned} \qquad (A4)$$

with the infalling boundary condition at the horizon, which means the momentum flux tangent to the horizon vanishes, i.e. $T_{rr}(r_0) = 0$ [42], which suggests $F_{ir}(r_0) = 0$.

The current and charge density at the horizon are

$$\begin{aligned} J_i(\tau, x_i, r_0) = F_{i\tau}(\tau, x_i, r_0)\\ q(\tau, x_i, r_0) = F_{r\tau}(\tau, x_i, r_0), \end{aligned} \qquad (A5)$$

which obey the continuity equation $\partial_\tau q + \partial^i J_i = 0$, courtesy of the first equation of Eq. (A3) after setting the condition $s(r_0) = 0$.

Exact solutions

In this subsection, we shall find the exact solution to the Maxwell system, first for $3+1$-dimensional spacetime and then for any arbitrary spacetime dimension.

(i) $3+1$ dimension: Let us denote the spacetime coordinate as τ, x, y and r, the solution for which the coupling is constant, i.e. $\phi' = 0$ with a nontrivial electric field and a constant magnetic field,

$$\begin{aligned} &F_{xr} = 0, \quad F_{yr} = 0, \quad F_{r\tau} = 0,\\ &F_{y\tau} = E_y(\tau), \quad F_{xy} = \text{constant} \equiv B,\\ &F_{x\tau} = E_x(\tau), \end{aligned} \qquad (A6)$$

for some functions $E_x(\tau)$ and $E_y(\tau)$ whose functional form is not fixed by the equations of motion or the Bianchi identity.

For $p = 2$, there exists another exact solution for which the coupling is constant, i.e. $\phi' = 0$, and the rest of the components of field strength are

$$\begin{aligned} &F_{xr} = 0, \quad F_{yr} = 0, \quad F_{r\tau} = qe^{2s(r)},\\ &F_{y\tau} = E_y(\tau), \quad F_{xy} = \text{constant} \equiv B,\\ &F_{x\tau} = E_x(\tau), \end{aligned} \qquad (A7)$$

where q is a constant. The functional form of the functions $E_x(\tau)$ and $E_y(\tau)$ are fixed neither by the equations of motion nor by the Bianchi identity.

(ii) Any arbitrary dimension: There exists an exact solution to the Maxwell system in any arbitrary spacetime dimension but, unfortunately, with zero electric field. In fact, all the components of field strength vanish except

$$F_{r\tau} = qe^{-ps(r)-\phi(r)}, \qquad (A8)$$

where q is a constant. One can find another solution with a nontrivial electric field, provided the inverse coupling goes as

$$1/g^2_{\rm YM} = e^{-(p-2)s(r)+\text{constant}}, \qquad (A9)$$

with $F_{r\tau} = qe^{-2s(r)}$, and the other nonvanishing component of the field strength is

$$F_{i\tau} = E_i(\tau) \qquad (A10)$$

for some functions $E_i(\tau)$, again, whose functional form is not fixed by the equations of motion nor the Bianchi identity.

APPENDIX B: ENERGY MINIMIZATION

After extremizing Eq. (30), it follows that the extremum occurs when the following equation is satisfied at some r:

$$\begin{aligned} &[g'_{rr}(g_{tt}g_{xx} - E^2) + g_{rr}(g'_{tt}g_{xx} + g_{tt}g'_{xx})]\left[N^2\left(\prod g_{y_a y_a}\right) - \frac{c^2}{g_{tt}}\right]\\ &+ [g_{rr}(g_{tt}g_{xx} - E^2)]\left[N^2\left(\prod g_{y_a y_a}\right)' + \frac{c^2 g'_{tt}}{g^2_{tt}}\right] = 0. \end{aligned} \qquad (B1)$$

Let us denote it as r_m, which is a function of (T_H, E, J^x). But recall that vanishing and reality of energy, H_L, implies that it occurs at a scale $r_\star$, which is a function of T_H and E only. So one can ask the question: can $r_\star$ be same as r_m, i.e. $r_\star(T_H, E) = r_m(T_H, E, J^x)$? The answer to this question is: it can happen only if the current J^x is a function of T_H and E. If we take the case as in Eq. (28), then it gives us a solution to Eq. (B1). In fact, for this solution the energy extremization at $r_\star$ again is in the indeterminate form, i.e. $(\frac{dH_L}{dr})_{r_\star} = \frac{0}{0}$ because in this case, both H_L and the numerator of $\frac{dH_L}{dr}$ vanishes. So we shall take the physical reason of choosing a scale $r_\star$ as the condition of reality and vanishing of energy, H_L. In general, *a priori*, it is not clear what other value of r one should choose so as to find the current as a function of temperature and electric field that solves Eq. (B1) for which $r_\star = r_m$.

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