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Supercurrent coupling destabilizes knot solitons

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In an influential paper of 2002, Babaev, Faddeev, and Niemi conjectured that two-component Ginzburg-Landau (TCGL) theory in three dimensions should support knot solitons, where the projective equivalence class of the pair of complex condensate fields $[\psi_1, \psi_2]$: $\mathbb{R}^3 \to \mathbb{C}P^1$ has nonzero Hopf degree. The conjecture was motivated by a certain truncation of the TCGL model which reduced it to the Faddeev-Skyrme model, long known to support knot solitons. Physically, the truncation amounts to ignoring the coupling between $[\psi_1, \psi_2]$ and the supercurrent of the condensates. The current paper presents a direct test of the validity of this truncation by numerically tracking the knot solitons as the supercurrent coupling is turned back on. It is found that the knot solitons shrink and disappear as the true TCGL model is reached. This undermines the reasoning underlying the conjecture and, when combined with other negative numerical studies, suggests the conjecture, in its original form, is very unlikely to be true.

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I. INTRODUCTION

Two-component Ginzburg-Landau (TCGL) theory is a phenomenological field theory which is enjoying increasing prominence in several areas of condensed matter theory. It consists of two complex scalar fields ψ_1, ψ_2: $\mathbb{R}^3 \to \mathbb{C}$ minimally coupled to a $U(1)$ gauge field $A \in \Omega^1(\mathbb{R}^3)$ and (perhaps) each other via an energy density of the form

$$\mathcal{E} = \tfrac{1}{2}|d_A\psi_1|^2 + \tfrac{1}{2}|d_A\psi_2|^2 + \tfrac{1}{2}|dA|^2 + U(\psi_1, \psi_2), \quad (1)$$

where $d_A\psi_a = d\psi_a - iA\psi_a$ and U is a potential function whose details depend strongly on the precise physical context. In an influential and much-cited paper of 2002 [1], Babaev, Faddeev, and Niemi conjectured that models of this type should possess "knot solitons" in which the projective equivalence class of the pair of complex fields ψ_1, ψ_2,

$$\varphi = [\psi_1, \psi_2]: \mathbb{R}^3 \to \mathbb{C}P^1 \cong S^2, \quad (2)$$

has nonzero Hopf degree. This conjecture is based on the observation that, when expressed in terms of the gauge-invariant fields φ, $\rho = \sqrt{|\psi_1|^2 + |\psi_2|^2}$ and

$$C = \frac{i}{2\rho^2}\sum_{a=1}^{2}(\bar{\psi}_a d_A\psi_a - \psi_a\overline{d_A\psi_a}), \quad (3)$$

the TCGL energy density is

$$\mathcal{E} = \tfrac{1}{8}\rho^2|d\varphi|^2 + \tfrac{1}{2}|dC + \tfrac{1}{2}\varphi^*\omega|^2 + \tfrac{1}{2}|d\rho|^2 + \tfrac{1}{2}\rho^2|C|^2 + U(\rho, \varphi), \quad (4)$$

where ω is the area form of the usual metric on the unit two-sphere. In superconductivity contexts, $\rho^2 C$ is the total supercurrent associated with condensates ψ_1, ψ_2 and so, in a slight abuse of terminology, we shall refer to C itself as the supercurrent. If we impose that $\rho = \rho_0 > 0$, a constant, and $C \equiv 0$, $\mathcal{E}$ coincides with the energy density of the Faddeev-Skyrme (FS) model, which is known to possess knot solitons of every Hopf degree Q. These are smooth, spatially localized, global energy minimizers in their homotopy class. Topologically, maps φ: $\mathbb{R}^3 \cup \{\infty\} \to S^2$ are classified by the framed cobordism class of $\varphi^{-1}(p)$, where $p \in S^2$ is any regular value of φ. In general, $\varphi^{-1}(p)$ is a union of closed curves in $\mathbb{R}^3$, that is, a union of knots. In the case of FS solitons these knots are simple unknots for low Q, but become more complicated as Q grows. For a survey of FS knot solitons, see [2].

So the conjecture of [1] results from identifying the FS model as a "submodel" of the original TCGL model. In replacing the TCGL model by its FS submodel, one is making a two-step truncation. The first truncation, imposing $\rho = \rho_0 > 0$ constant, is reasonable for suitable choices of U, for example,

$$U = \lambda(\rho_0^2 - |\psi_1|^2 - |\psi_2|^2)^2 \quad (5)$$

in the limit of large λ. Indeed, the analogous truncation in *ungauged* two-condensate systems has been found to introduce no drastic qualitative changes [3]. However, the second truncation, imposing $C = 0$, is harder to justify since, even for ρ constant, C is coupled to φ. Indeed, Babaev, Faddeev, and Niemi do not directly claim that setting $C = 0$ is a good approximation for the TCGL model [1]. Nonetheless, their claim that TCGL theory should support knot solitons assumes that the coupling between φ and C does not destroy the solitons and, hence, that the second truncation $C = 0$ is, if not a "good approximation," valid at least at a qualitative level. This is an assumption, not an established fact, a distinction missed by the authors of many of the papers citing [1], who treat the

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existence of knot solitons in TCGL models likewise as established fact rather than conjecture.

There have been many attempts to construct knot solitons in TCGL models numerically, by choosing ψ_1, ψ_2, A with $\varphi = [\psi_1, \psi_2]$ having $Q > 0$, usually in some toroidally symmetric ansatz, and minimizing $E = \int_{\mathbb{R}^3} \mathcal{E}$ using some gradient descent method. In almost all cases the initial field configurations have shrunk and fallen through the lattice mesh [4–6]. The only exception we know of is [7], which uses direct gradient flow to evolve the field configuration towards a critical point of E, and claims to have found ''definite convergence towards torus-shaped configurations.'' Given that this finding has not been reproduced by any other researchers, it seems likely that this is an artifact of having used too lax a convergence criterion (unfortunately, the precise convergence criterion used is not described in [7], but gradient flow is a notoriously slow method, and it is easy to mistake a slowly evolving field for a truly static one).

The failure of the direct approach to find knot solitons is not surprising given the results of [8]. Even if one imposes that ρ is nonvanishing, so that Q is well defined, in every degree class the infimum of $E(\psi_1, \psi_2, A)$ is 0 because, for suitably chosen A, any spatially localized configuration ψ_1, ψ_2 is unstable against Derrick scaling. This shows that, for all $Q \neq 0$, there is no *global* energy minimizer. Note that this contradicts the common misconception, repeated in [7], that the existence of a nontrivial topological charge protects a field configuration from collapse. So knot solitons, if they exist in TCGL theory, can only be *local* minimizers of E, and finding them is like looking for a hollow dip in a hilltop, rather than the floor of a valley: unless one's initial guess is very close to the dip, one is very unlikely to find it. Indeed this point was understood, from a more qualitative, physical viewpoint, by Babaev [9], some years after making the original conjecture. For this reason, the failure of previous numerical studies to find knot solitons, while discouraging, does not systematically undermine the conjecture of [1]. To do so using the direct approach one would have to systematically search the infinite dimensional space of all possible initial data.

In this paper we pursue a different approach. We study the TCGL model in the case of a hard-confining potential, (5) with $\lambda \to \infty$, or, equivalently, in the sigma model limit, where $|\psi_1|^2 + |\psi_2|^2 = \rho_0^2$. We may, without loss of generality, take $\rho_0 = 1$. This allows us to concentrate on the dynamically crucial issue of the coupling between φ and C. Rather than minimizing E starting with some plausible initial guess, we consider the one-parameter family of models with energies

$$E^{(\alpha)} = \frac{1}{8}\|d\varphi\|^2 + \frac{1}{8}\|\varphi^*\omega\|^2 + \frac{\alpha}{2}\langle dC, \varphi^*\omega\rangle + \frac{1}{2}\|dC\|^2 + \frac{1}{2}\|C\|^2 \tag{6}$$

parametrized by $\alpha \in [0, 1]$, where $\langle \cdot, \cdot \rangle$ denotes the L^2 inner product and $\| \cdot \|$ denotes the L^2 norm. Note that at $\alpha = 1$ this is the TCGL model (in the sigma model limit), while at $\alpha = 0$ it decouples into the FS model and an uncoupled Proca model for C. So the $\alpha = 0$ model clearly supports knot solitons with $C \equiv 0$—they are simply the usual FS knot solitons. The question is then, what is the fate of these solitons as the supercurrent coupling is ''turned on,'' that is, α is increased to 1? We address this problem numerically by a continuation method. Starting at $\alpha = 0$, we numerically minimize $E^{(\alpha)}$ within a given homotopy class using the limited memory quasi-Newton algorithm (also called a variable metric algorithm) with the BFGS formula for Hessian approximations [10]. Having found a minimizer for this value of α, we increase α slightly and minimize E again, starting from the energy minimizer just obtained. In this way we construct a curve of energy minimizers parametrized by α, and the crucial question is whether this curve continues all the way to $\alpha = 1$.

We will see that, for all $\alpha \in [0, 1)$, we have a lower energy bound

$$E^{(\alpha)} \geq c_0\sqrt{1 - \alpha}|Q|^{3/4}, \tag{7}$$

where $c_0 > 0$ is an absolute constant. This leads one to expect the knot solitons to persist whilever $\alpha < 1$, albeit with nonzero supercurrent C, and indeed this is what we find. We shall produce convincing numerical evidence, however, that as $\alpha \to 1$, these knot solitons shrink to zero size, because C deforms precisely to the form for which $E^{(1)}$ is not stable against Derrick scaling.

Note that our results are not just another negative finding in the same vein as [5], since our approach tests not just the conjecture of [1], but also the reasoning underlying it. That is, our results conclusively demonstrate that, notwithstanding the formal similarity between the TCGL model (in suitable variables) and the FS model, the knot solitons of the latter definitely do *not* persist in the former, even in the most favorable case where a hard-confining potential enforces $|\psi_1|^2 + |\psi_2|^2 = 1$. This also elucidates the mechanism by which knot solitons are destabilized. It has nothing, in general, to do with ρ shrinking and acquiring isolated zeroes (thus allowing Q to discontinuously drop). Even in the sigma model limit, where Q is a rigorously defined topological invariant, the coupling to the supercurrent C alone suffices to destabilize the knot solitons. Of course, this does not show for certain that E has no local minima with $Q \neq 0$, but it does invalidate the reasoning used to motivate the conjecture in the first place. The present results complement the work of Ward [6] (extended by one of us [4]) which also embeds TCGL theory into a one-parameter family of models, one end of which supports solitons. Again, these solitons disappear as the true TCGL theory is approached. It also complements the exact results of [8], where it was shown analytically that

supercurrent coupling destabilizes the $Q = 1$ Hopf soliton on physical space S^3. In the absence of evidence to the contrary, one should wield Occam's razor and conclude that, in all likelihood, the basic TCGL theory does not possess knot solitons. There remains the possibility that more elaborate, but physically relevant, versions of TCGL theory, involving direct current-current interactions, may have knot solitons [9], but until direct evidence for this is found, we prefer to remain sceptical.

II. ENERGY BOUND AND DERRICK SCALING

We seek local minimizers of the energy functional $E^{(\alpha)}$ defined in (6) with φ: $\mathbb{R}^3 \to S^2$ and $C \in \Omega^1(\mathbb{R}^3)$ having the boundary behavior

$$\lim_{|x|\to\infty} \varphi(x) = \varphi_\infty, \qquad \lim_{|x|\to\infty} C(x) = 0 \tag{8}$$

and $\alpha \in [0, 1]$. Without loss of generality, we choose $\varphi_\infty = (0, 0, 1)$. Configurations with this boundary behavior fall into disjoint homotopy classes labeled by the integer Hopf invariant

$$Q = \frac{1}{16\pi^2} \int_{\mathbb{R}^3} a \wedge da, \tag{9}$$

where a is any one-form on $\mathbb{R}^3$ such that $da = \varphi^*\omega$. It is well known that the Faddeev-Skyrme energy, which coincides (up to a factor of $\frac{1}{4}$ in the usual normalization) with $E^{(0)}(\varphi, 0)$, satisfies a topological lower energy bound

$$E^{(0)}(\varphi, 0) \geq c_0 |Q(\varphi)|^{3/4}, \tag{10}$$

where $c_0 > 0$ is an absolute constant [11]. Now, for all $\alpha \in [0, 1]$,

$$\begin{aligned} E^{(\alpha)} &= \frac{1}{8}\|d\varphi\|^2 + \frac{1}{8}(1-\alpha)\|\varphi^*\omega\|^2 + \frac{1}{2}(1-\alpha)\|dC\|^2 \\ &\quad + \frac{\alpha}{2}\|dC + \frac{1}{2}\varphi^*\omega\|^2 + \frac{1}{2}\|C\|^2 \\ &\geq \frac{1}{8}\|d\varphi\|^2 + \frac{1}{8}(1-\alpha)\|\varphi^*\omega\|^2 \\ &= \frac{1}{8}\sqrt{1-\alpha}(\|d\hat{\varphi}\|^2 + \|\hat{\varphi}^*\omega\|^2), \end{aligned} \tag{11}$$

where $\hat{\varphi}(x) = \varphi(\sqrt{1-\alpha}x)$. Hence, by the usual Vakulenko-Kapitanski bound (10),

$$E^{(\alpha)}(\varphi, C) \geq c_0\sqrt{1-\alpha}|Q(\varphi)|^{3/4}. \tag{12}$$

This leads us to expect that $E^{(\alpha)}$ will have a global energy minimizer in each homotopy class whilever $\alpha \in [0, 1)$.

It is instructive to subject $E^{(\alpha)}$ to the Derrick scaling test for all $\alpha \in [0, 1]$ since this gives an integral constraint on (φ, C) which we can use as a consistency check on our numerical scheme [12]. Assume that (φ, C) is a critical point of $E^{(\alpha)}$. Then $E^{(\alpha)}$ is stationary with respect to all variations of (φ, C) and so, in particular, with respect to the variation

$$\varphi_\lambda = \varphi \circ \mathcal{D}_\lambda, \qquad C_\lambda = \mathcal{D}_\lambda^* C, \tag{13}$$

where $\lambda \in (0, \infty)$ and $\mathcal{D}_\lambda$: $\mathbb{R}^3 \to \mathbb{R}^3$ is the dilation map $\mathcal{D}_\lambda(x) = \lambda x$. Note that $\varphi_1 = \varphi$ and $C_1 = C$. A short calculation shows that

$$\begin{aligned} E(\lambda)\text{: } & E^{(\alpha)}(\varphi_\lambda, C_\lambda) \\ &= \frac{1}{8\lambda}\|d\varphi\|^2 + \frac{\lambda}{8}\|\varphi^*\omega\|^2 + \frac{\lambda\alpha}{2}\langle\varphi^*\omega, dC\rangle + \frac{\lambda}{2}\|dC\|^2 \\ &\quad + \frac{1}{2\lambda}\|C\|^2. \end{aligned} \tag{14}$$

Hence, since (φ, C) is a critical point,

$$\begin{aligned} E'(1) &= -\frac{1}{8}\|d\varphi\|^2 + \frac{1}{8}\|\varphi^*\omega\|^2 + \frac{\alpha}{2}\langle\varphi^*\omega, dC\rangle \\ &\quad + \frac{1}{2}\|dC\|^2 - \frac{1}{2}\|C\|^2 \\ &= 0. \end{aligned} \tag{15}$$

We shall refer to this as the Derrick constraint.

Note that, in the case $\alpha = 1$ (the TCGL model), if C is chosen so that

$$dC + \tfrac{1}{2}\varphi^*\omega = 0, \tag{16}$$

and $(\varphi_\lambda, C_\lambda)$ defined as in (13), we have

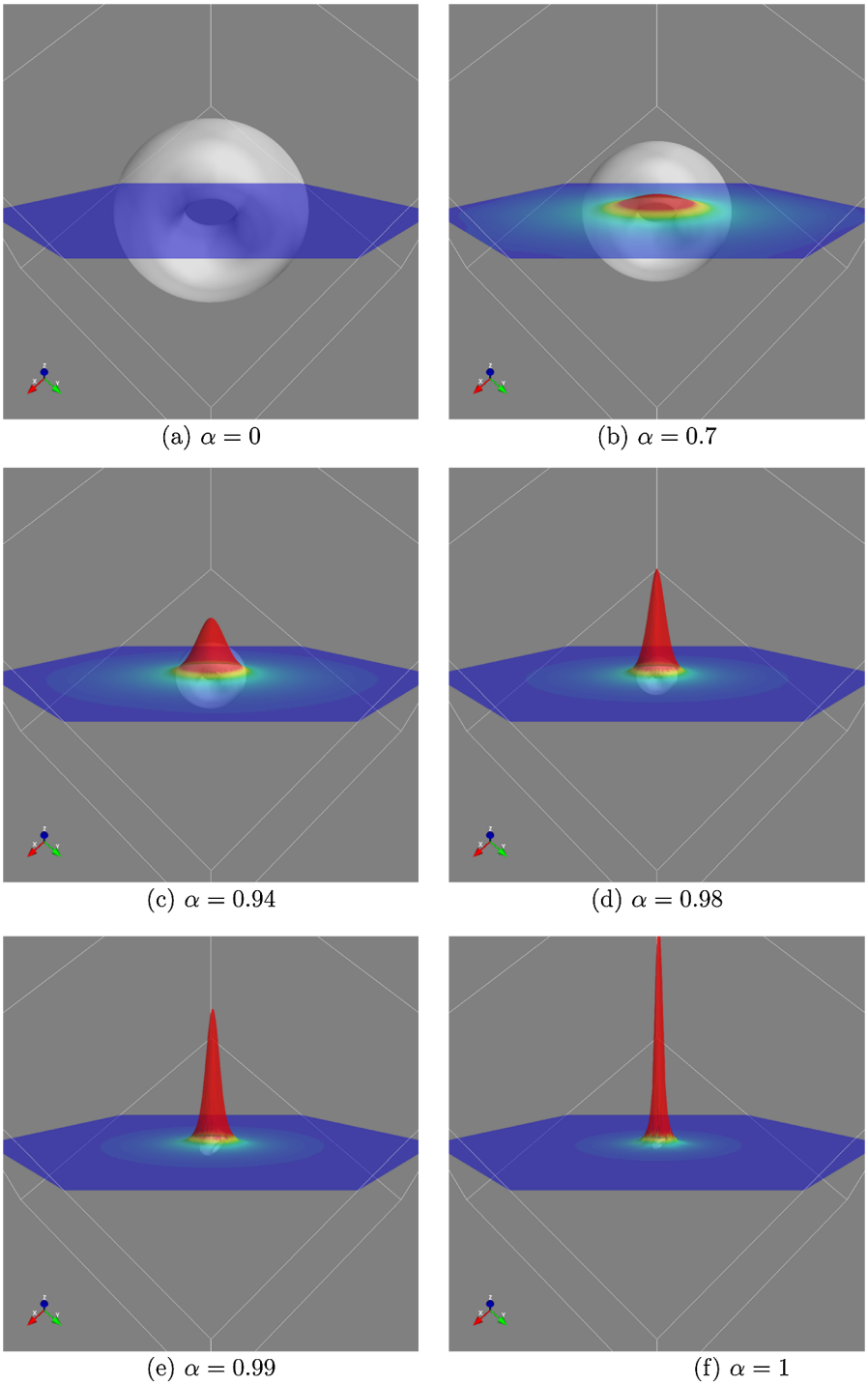
$$E^{(1)}(\varphi_\lambda, C_\lambda) = \frac{1}{8\lambda}\|d\varphi\|^2 + \frac{1}{2\lambda}\|C\|^2 \to 0 \tag{17}$$

as $\lambda \to \infty$. For all φ satisfying (8) there exists C satisfying (8) and (16), which shows that $E^{(1)}(\varphi, C)$ has infimum 0 in every homotopy class [8]. This argument, which is essentially equivalent to the one described in [2] (whose authors attribute it to unpublished work of Forgacs and Volkov), explains why the energy bound (12) becomes trivial at $\alpha = 1$. Of course, it does *not* follow that $E^{(1)}$ can have no critical points, because there is no reason why critical points of $E^{(1)}$ should satisfy (16).

For the purposes of numerics, we will study $E^{(\alpha)}$ not on $\mathbb{R}^3$, but in a large box $B = [-L, L] \times [-L, L] \times [-L, L]$ with Dirichlet boundary conditions $\varphi(x) = \varphi_\infty$, $C(x) = 0$, for all $x \in \partial B$. In this case, the Derrick scaling argument must be modified, as follows. First, we note that the variation $(\varphi_\lambda, C_\lambda)$ is only well defined for $\lambda \in [1, \infty)$, since for $\lambda \in (0, 1)$ $(\varphi_\lambda, C_\lambda)$ does not satisfy the boundary conditions. For $\lambda \geq 1$ we first notionally extend φ, C to the whole of $\mathbb{R}^3$ by their boundary values, then define $(\varphi_\lambda, C_\lambda)$ as in (13). Equation (14) for $E(\lambda)$ still holds, but now on $[1, \infty)$ rather than $(0, \infty)$. We require that $E(\lambda)$ has a local minimum at $\lambda = 1$, but now, since this is at an endpoint of the domain, it does *not* follow that $E'(1) = 0$. Rather, we know only that

$$E'(1) = -\frac{1}{8}\|d\varphi\|^2 + \frac{1}{8}\|\varphi^*\omega\|^2 + \frac{\alpha}{2}\langle\varphi^*\omega, dC\rangle + \frac{1}{2}\|dC\|^2 - \frac{1}{2}\|C\|^2 \geq 0, \tag{18}$$

so the Derrick scaling constraint becomes an integral inequality once we place the theory in a finite box. This argument, which could be called "Derrick scaling in a finite box" is of wide applicability in classical field theory. Note that it does not really require $B \subset \mathbb{R}^n$ to be a box: any star-shaped domain will do ($B \subset \mathbb{R}^n$ is star-shaped if there



(a) $\alpha = 0$ (b) $\alpha = 0.7$ (c) $\alpha = 0.94$ (d) $\alpha = 0.98$ (e) $\alpha = 0.99$ (f) $\alpha = 1$

FIG. 1 (color online). Minimizers of $E^{(\alpha)}$ for an increasing sequence of values of α, with Hopf charge $Q = 1$. The translucent white surface is the isosurface $\varphi_3 = 0$, which can be interpreted as bounding the soliton core. The colored surface is a plot of $|C|$ on a transverse slice through the soliton core. Note that as $\alpha \to 1$, the core shrinks and the C field accumulates in a singular spike.

exists $x_o \in B$ such that for all $x \in B$ the line segment from x_o to x lies in B). Of course, for the purposes of numerics, one would hope to choose the box size L to be so large (compared with the soliton core radius) that the numerical solution satisfies (15) to a good approximation. We will discuss this issue in more detail in Sec. III.

III. NUMERICAL RESULTS

We used a simple forward differencing scheme to discretize the energy functional $E^{(\alpha)}$ on a cubic lattice of spacing h with N^3 points, to give a lattice approximant $E_0^{(\alpha)}$: $(S^2 \times \mathbb{R}^3)^{N^3} \to \mathbb{R}$. Given an initial configuration (φ, C), we used the quasi-Newton BFGS method, as implemented by the TAO and PETSc parallel numerical libraries [13] and PETSc [14–16], to find a local minimum of $E_0^{(\alpha)}$. We considered the scheme to have converged to a minimum if the sup norm of the gradient of $E_0^{(\alpha)}$ was less than $0.01h^3$. We tested the accuracy of this scheme by comparing its results at $\alpha = 0$, for Hopf charges $Q = 1$, 2, 3, with previous studies of the pure FS model [17,18] and found good agreement (the energy minimizers had the same shape, energy, and core length to within a few, typically 2, percent). By experimenting in the $\alpha = 0$ case we found that a good balance of accuracy and computational speed was obtained with the choices $N = 480$ and $h = 0.0125$, and these values were used for the remaining calculations.

As described in Sec. I, having found a minimizer of $E_0^{(\alpha)}$, we incremented α slightly, $\alpha' = \alpha + \delta\alpha$, and minimized $E_0^{(\alpha')}$ starting with the α minimizer as our initial guess. In this way, we constructed a curve of minimizers of $E_0^{(\alpha)}$, parametrized by $\alpha \in [0, 1]$, starting at $\alpha = 0$ and working towards $\alpha = 1$, in each of the homotopy classes $Q = 1$, 2, 3. In every case, the knot soliton at $\alpha = 0$ shrinks rapidly as α approaches 1, so that, at $\alpha = 1$, its core size is comparable to the lattice spacing. This extremely coarse configuration can confidently be identified as a discretization artifact, rather than a genuine minimizer of $E^{(1)}$, as we shall argue below. The shrinking process is illustrated in Fig. 1, which shows a sequence of minimizers for increasing α in the case $Q = 1$. The results for $Q = 2$, 3 look very similar. A quantitative measure of the shrinking is given in Fig. 2, which shows the core length of the minimizer as a function of α. Recall that the core is, by definition, the preimage curve of $-\varphi_\infty$ which, in the cases $Q = 1$, 2, 3, is a single closed curve. Numerically, we construct this curve using interpolation tools of the MayaVi Data Visualizer [19]. This works well until α is very close to 1, when the core structure becomes too small for the lattice to properly resolve, and the interpolation tool produces a disconnected collection of segments instead of a closed curve. From this point, the core length calculation is unreliable, which is why the curves in Fig. 2 stop slightly short of $\alpha = 1$. Clearly, the data up to this point are consistent with the core length shrinking rapidly to 0 as $\alpha \to 1$.

Recall we have an energy bound (12) which vanishes at $\alpha = 1$, and that $E^{(1)}$ has infimum 0 in every homotopy class, meaning that every class contains fields of arbitrarily low $E^{(1)}$. Such fields are constructed by Derrick shrinking among the set of fields satisfying (16). Conversely, if (16) is not satisfied, the field is protected against shrinking by the presence of a quartic term in $E^{(1)}$. So if our claim is correct, that the minimizers shrink and vanish as $\alpha \to 1$, then they should satisfy (16) to a closer and closer approximation as $\alpha \to 1$. In Fig. 3 we present a graph of $\|dC + \frac{1}{2}\varphi^*\omega\|^2$ as a function of α which confirms that this is indeed what happens: the minimizer loses stability against Derrick scaling as $\alpha \to 1$, and consequently, its energy

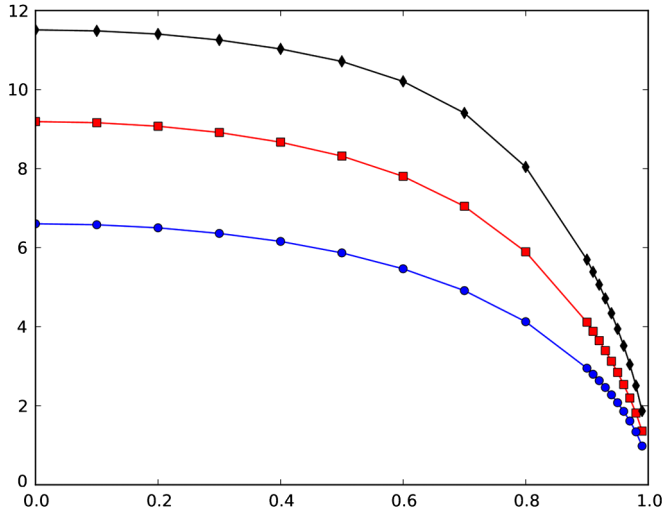


FIG. 2 (color online). Core lengths of the $Q = 1$ (blue disks), $Q = 2$ (red squares), and $Q = 3$ (black diamonds) minimizers as a function of α.

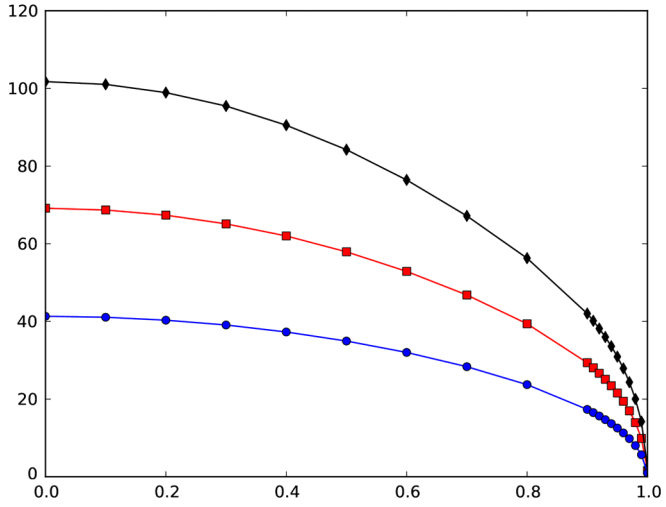


FIG. 3 (color online). The squared L^2 norm of $dC + \frac{1}{2}\varphi^*\omega$ of $Q = 1$ (blue disks), $Q = 2$ (red squares), and $Q = 3$ (black diamonds) minimizers as a function of α.

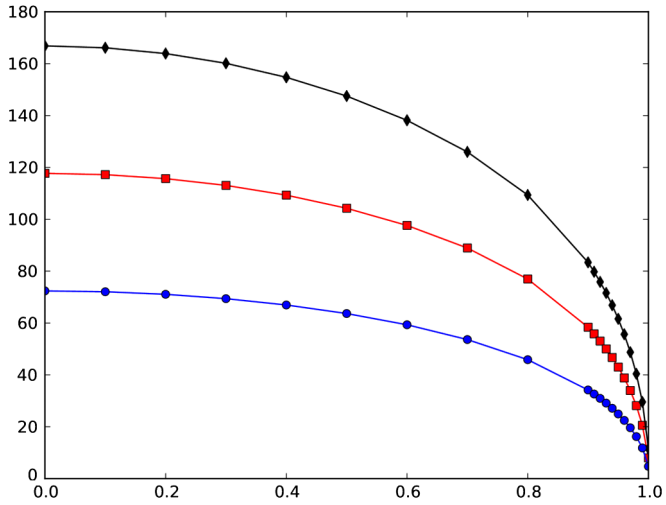


FIG. 4 (color online). The energy $E^{(\alpha)}$ of the $Q = 1$ (blue disks), $Q = 2$ (red squares), and $Q = 3$ (black diamonds) minimizers as a function of α.

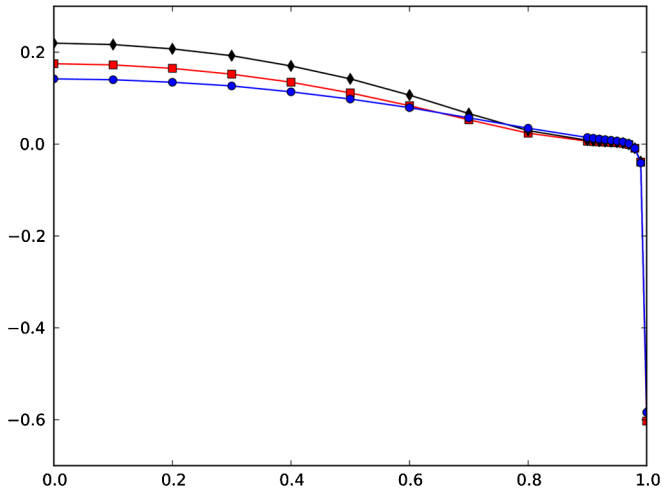


FIG. 5 (color online). The Derrick constraint function $D(\alpha)$ of $Q = 1$ (blue disks), $Q = 2$ (red squares), and $Q = 3$ (black diamonds) minimizers. This quantity should be non-negative for a genuine minimizer in a finite box.

vanishes rapidly as $\alpha \to 1$, as shown in Fig. 4. Note that these curves are consistent with the bound (12).

Clearly the energy does not vanish exactly at $\alpha = 1$, but this is a discretization artifact. To see this, we define for each $\alpha \in [0, 1]$ the quantity

$$D(\alpha) = \frac{1}{E^{(\alpha)}(\varphi, C)}\left\{-\frac{1}{8}\|d\varphi\|^2 + \frac{1}{8}\|\varphi^*\omega\|^2 + \frac{\alpha}{2}\langle\varphi^*\omega, dC\rangle + \frac{1}{2}\|dC\|^2 - \frac{1}{2}\|C\|^2\right\}, \quad (19)$$

where (φ, C) is the minimizer of $E^{(\alpha)}$. This is the left-hand side of the Derrick constraint (15), normalized by the total energy of the field. Hence, as shown in Sec. II, for a minimizer in $\mathbb{R}^3$, one should have $D(\alpha) = 0$, while for a minimizer in a finite box $D(\alpha) \geq 0$. Figure 5 presents a plot of $D(\alpha)$ for our numerical minimizers. One sees that $D(\alpha)$ is moderately small for small α, indicating that the boundary of the finite box exerts a moderate but appreciable influence on the numerical solution. As α increases $D(\alpha)$ decreases to very close to zero, because the minimizers shrink, so the finite box size becomes numerically irrelevant. However, as α gets very close to 1, $D(\alpha)$ becomes negative, indicating that from this point on the minimizers are strongly affected by discretization effects: if this were a continuum system, any field with $D(\alpha) < 0$ would be unstable against shrinking. In particular, $D(1)$ is large and negative, suggesting that the $\alpha = 1$ "minimizer" is a lattice artifact, and not representative of a genuine local minimizer of $E^{(1)}$.

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