Nonadiabatic quantum Vlasov equation for Schwinger pair production

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Using Lewis-Riesenfeld theory, we derive an exact nonadiabatic master equation describing the time evolution of the QED Schwinger pair-production rate for a general time-varying electric field. This equation can be written equivalently as a first-order matrix equation, as a Vlasov-type integral equation, or as a third-order differential equation. In the last version it relates to the Korteweg-de Vries equation, which allows us to construct an exact solution using the well-known one-soliton solution to that equation. The case of timelike delta function pulse fields is also briefly considered.

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I. INTRODUCTION

Vacuum pair production by a strong electric field, predicted by Schwinger in 1951 [1], may now finally be seen due to the construction of ultrastrong laser systems [2]. However, the corresponding fields are very different from the few special configurations for which an exact calculation of the pair creation rate is possible. Thus, recently there has been increased interest in the development of approximation schemes, such as semiclassical methods [3–5] and Monte Carlo simulations [6].

A case that is relatively amenable to an exact treatment is the one of a purely time-dependent electric field. Here the spatial momentum is a good quantum number, which allows one to reduce the time evolution of the system to a collection of mode equations labeled by the fixed momentum k. The pair-production calculation can then be further reduced to a one-dimensional scattering problem, suitable for standard numerical or WKB methods [7–9]. Alternatively, the mode equation can be transformed to the quantum Vlasov equation, an integral equation for $\mathcal{N}_k(t)$, the total expected number of created pairs in the mode k [10–14] (see Ref. [15] for a comparison of the two approaches).

In this paper, we reconsider the time evolution of the QED Hamiltonian in a time-varying field using Lewis-Riesenfeld invariant theory [16] and a suitable operator basis forming a spectrum generating algebra $SU(1, 1)$. We derive an exact nonadiabatic master equation for the time evolution of the Schwinger pair-production rate. This equation can be written equivalently as a first-order matrix equation, as a quantum Vlasov equation, or as a third-order differential equation. For a specific solution ansatz this third-order equation relates to the Korteweg-de Vries (KdV) equation, which allows us to construct an exact solution using the well-known one-soliton solution to that equation. We also consider the case of an alternating timelike delta function pulse field, a type of field which is of relevance for a recent proposal to apply Ramsey interferometry to the Schwinger effect [17].

II. DERIVATION OF THE MASTER EQUATION

We will give the derivation of the master equation for the scalar QED case; the derivation for the spinor QED case is similar, and will be included in a forthcoming, more detailed publication [18]. A scalar particle with charge q and mass m in a homogeneous time-dependent electric field with the gauge potential $A_{\|}(t)$ has the Fourier decomposed Hamiltonian of time-dependent oscillators (in units of $\hbar = c = 1$),

$$\hat{H}(t) = \int \frac{d^3k}{(2\pi)^3}[\pi_k^\dagger \pi_k + \omega_k^2(t)\phi_k^\dagger \phi_k], \tag{1}$$

where

$$\omega_k^2(t) = (k_{\|} - qA_{\|}(t))^2 + \mathbf{k}_\perp^2 + m^2. \tag{2}$$

We will quantize the theory in the Schrödinger picture, where the time-dependent quantum state obeys the functional Schrödinger equation

$$i\frac{\partial \Psi(t)}{\partial t} = \hat{H}(t)\Psi(t). \tag{3}$$

In this picture the field operators $\hat{\phi}(x)$ and $\hat{\pi}(x) = \dot{\hat{\phi}}^\dagger$ are time independent with the momentum space commutation relations

$$[\hat{\phi}_k, \hat{\pi}_{k'}] = [\hat{\phi}_k^\dagger, \hat{\pi}_{k'}^\dagger] = i(2\pi)^3\delta_{kk'}, \tag{4}$$

but the corresponding creation and annihilation operators with the equal-time commutators

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$$[\hat{a}_k(t), \hat{a}^\dagger_{k'}(t)] = [\hat{b}_{-k}(t), \hat{b}^\dagger_{-k'}(t)] = (2\pi)^3 \delta_{kk'} \quad (5)$$

are generally time dependent [19–22],

$$\hat{\phi}_k = \hat{a}_k(t)\varphi_k(t) + \hat{b}^\dagger_{-k}(t)\varphi^*_k(t), \\ \hat{\pi}_k = \hat{a}^\dagger_k(t)\dot{\varphi}^*_k(t) + \hat{b}_{-k}(t)\dot{\varphi}_k(t). \quad (6)$$

Here φ_k is an auxiliary field satisfying the classical mode equation

$$\ddot{\varphi}_k(t) + \omega_k^2(t)\varphi_k(t) = 0, \quad (7)$$

as well as the Wronskian constraint

$$\mathrm{Wr}[\varphi_k, \varphi^*_k] \equiv \varphi_k(t)\dot{\varphi}^*_k(t) - \varphi^*_k(t)\dot{\varphi}_k(t) = i. \quad (8)$$

Equation (7) and the Wronskian determine $\varphi_k(t)$ up to a phase factor, which we fix by requiring that $\varphi_k(t)$ be real at the initial time t_0. Thus if t_0 is finite, then for $t \le t_0$ one has

$$\varphi_k(t) = \frac{e^{-i\omega_k(0)(t-t_0)}}{\sqrt{2\omega_k(0)}} \quad (9)$$

(for $t_0 = -\infty$ this should hold in the asymptotic sense). We note that the operators $\hat{a}_k(t)$, $\hat{b}_{-k}(t)$ and their Hermitian conjugates are Lewis-Riesenfeld invariants; that is, they fulfill the Liouville-von Neumann equation

$$i\frac{\partial \hat{I}_k(t)}{\partial t} + [\hat{I}_k(t), \hat{H}_k(t)] = 0, \quad (10)$$

as can be easily checked.

The ground state $|0_k, t\rangle$ for the kth mode is annihilated by both $\hat{a}_k(t)$ and $\hat{b}_{-k}(t)$, and the nth excited state is

$$|n_k, t\rangle := \frac{[\hat{a}^\dagger_k(t)\hat{b}^\dagger_{-k}(t)]^{n_k}}{n_k!}|0_k, t\rangle. \quad (11)$$

Thus the total time-dependent vacuum state is given by

$$|0, t\rangle = \prod_k |0_k, t\rangle. \quad (12)$$

In the free theory, the time-dependent vacuum state reduces to the Minkowski vacuum, as expected. The scalar product for the quantized fields and their Hermitian conjugates allows us to find the Bogoliubov transformation between the past time t_0 and the present time t, which is given by

$$\hat{a}_k(t_0) = \mu_k(t_0, t)\hat{a}_k(t) + \nu_k(t_0, t)\hat{b}^\dagger_{-k}(t), \\ \hat{b}^\dagger_{-k}(t_0) = \mu^*_k(t_0, t)\hat{b}^\dagger_{-k}(t) + \nu^*_k(t_0, t)\hat{a}_k(t), \quad (13)$$

where

$$\mu_k(t_0, t) = i\mathrm{Wr}[\varphi^*_k(t_0), \varphi_k(t)], \\ \nu_k(t_0, t) = i\mathrm{Wr}[\varphi^*_k(t_0), \varphi^*_k(t)]. \quad (14)$$

The Bogoliubov coefficients satisfy the relation for bosons $|\mu_k(t_0, t)|^2 - |\nu_k(t_0, t)|^2 = 1$. Our main object of interest, the mean number of pairs present at time t assuming that this number was n_k at the initial time t_0, can now be read off from

$$\langle n_k, t|\hat{a}^\dagger_k(t_0)\hat{a}_k(t_0)|n_k, t\rangle = |\nu_k(t_0, t)|^2(2n_k + 1) + n_k. \quad (15)$$

Thus

$$\mathcal{N}_k(t) := |\nu_k(t_0, t)|^2(2n_k + 1) \quad (16)$$

is the number of pairs spontaneously produced from the initial vacuum by the electric field.

To obtain a time evolution equation for this quantity, we observe that the time-dependent Hamiltonian (1) has the spectrum generating algebra $SU(1, 1)$. Choosing the Hermitian basis

$$\hat{\mathcal{M}}^{(0)}_k(t_0) = \frac{1}{(2\pi)^3}[\hat{a}^\dagger_k(t_0)\hat{a}_k(t_0) + \hat{b}_{-k}(t_0)\hat{b}^\dagger_{-k}(t_0)], \\ \hat{\mathcal{M}}^{(+)}_k(t_0) = \frac{1}{(2\pi)^3}[\hat{a}_k(t_0)\hat{b}_{-k}(t_0) + \hat{a}^\dagger_k(t_0)\hat{b}^\dagger_{-k}(t_0)], \\ \hat{\mathcal{M}}^{(-)}_k(t_0) = \frac{i}{(2\pi)^3}[\hat{a}_k(t_0)\hat{b}_{-k}(t_0) - \hat{a}^\dagger_k(t_0)\hat{b}^\dagger_{-k}(t_0)], \quad (17)$$

this algebra becomes

$$[\hat{\mathcal{M}}^{(0)}_k(t_0), \hat{\mathcal{M}}^{(\pm)}_k(t_0)] = \pm 2i\hat{\mathcal{M}}^{(\mp)}_k(t_0), \\ [\hat{\mathcal{M}}^{(+)}_k(t_0), \hat{\mathcal{M}}^{(-)}_k(t_0)] = -2i\hat{\mathcal{M}}^{(0)}_k(t_0). \quad (18)$$

The correlators are the expectation values of Eq. (17) with respect to $|n_k, t\rangle$, that is, of the number of produced pairs and of pair creation and annihilation:

$$1 + 2\mathcal{N}_k(t) = (2|\nu_k(t_0, t)|^2 + 1)(2n_k + 1), \\ \mathcal{M}^{(+)}_k(t) = (\mu_k(t_0, t)\nu_k(t_0, t) + \mu^*_k(t_0, t)\nu^*_k(t_0, t)) \\ \times (2n_k + 1), \\ \mathcal{M}^{(-)}_k(t) = i(\mu_k(t_0, t)\nu_k(t_0, t) - \mu^*_k(t_0, t)\nu^*_k(t_0, t)) \\ \times (2n_k + 1). \quad (19)$$

Note that all three correlators are real and proportional to the quantum number $2n_k + 1$, and thus proportional to the ones defined by the vacuum state.

Using Eq. (14) and the mode equation (7), we find the first-order master equation

$$\frac{d}{dt}\begin{pmatrix} 1 + 2\mathcal{N}_k \\ \mathcal{M}^{(-)}_k \\ \mathcal{M}^{(+)}_k \end{pmatrix} = \begin{pmatrix} 0 & \Omega^{(-)}_k & 0 \\ \Omega^{(-)}_k & 0 & \Omega^{(+)}_k \\ 0 & -\Omega^{(+)}_k & 0 \end{pmatrix} \times \begin{pmatrix} 1 + 2\mathcal{N}_k \\ \mathcal{M}^{(-)}_k \\ \mathcal{M}^{(+)}_k \end{pmatrix}, \quad (20)$$

where

$$\Omega_k^{(\pm)}(t) := \frac{\omega_k^2(t) \pm \omega_k^2(t_0)}{\omega_k(t_0)}, \tag{21}$$

with the initial conditions $\mathcal{N}_k = n_k$, $\mathcal{M}_k^{(\pm)} = 0$ at $t = t_0$ (where t_0 may be $-\infty$). An immediate consequence of the master equation (20) is the conservation of the quantity

$$(1 + 2\mathcal{N}_k)^2 - (\mathcal{M}_k^{(+)})^2 - (\mathcal{M}_k^{(-)})^2 = (1 + 2n_k)^2. \tag{22}$$

This relates to the conservation of charge, as well as to the invariance of the Casimir operator for the $SU(1, 1)$ algebra [18]. The spinor QED case can be treated analogously [18]. As far as the master formula (20) is concerned, the generalization to the fermionic case requires only changing $1 + 2\mathcal{N}_k(t)$ to $1 - 2\mathcal{N}_k(t)$, and replacing $\omega_k^2(t)$ by

$$\omega_k^2(t) = (k_\parallel - qA_\parallel(t))^2 + iqE(t) + \mathbf{k}_\perp^2 + m^2. \tag{23}$$

III. ALTERNATIVE FORMULATIONS OF THE MASTER EQUATION

The first-order matrix equation (20) can be equivalently rewritten both as a single integral equation and as a third-order linear differential equation. Since we work with a fixed mode k, in this section we will generally suppress the index k and abbreviate $\omega_0 := \omega(t_0)$. We will now also set $n_k = 0$.

First, we combine the equations for $\mathcal{M}^{(\pm)}$ to a second-order inhomogeneous equation for $\mathcal{M}^{(-)}$,

$$\frac{d^2\mathcal{M}^{(-)}}{dt^2} - \frac{\dot{\Omega}^{(+)}}{\Omega^{(+)}}\frac{d\mathcal{M}^{(-)}}{dt} + (\Omega^{(+)})^2\mathcal{M}^{(-)} = \Omega^{(+)}\frac{d}{dt}\left[\frac{\Omega^{(-)}}{\Omega^{(+)}}(1 + 2\mathcal{N})\right]. \tag{24}$$

The homogeneous part of Eq. (24) has the exact solutions

$$\mathcal{M}^{(-)}(t) = C^{\pm}e^{\pm i\int_{t_0}^{t} dt'\Omega^{(+)}(t')} \tag{25}$$

with integration constants $C^{\pm}$. Using those in the usual way to construct the solution of the inhomogeneous equation with the appropriate initial conditions, we obtain the quantum Vlasov equation as the integral equation

$$\frac{d}{dt}(1 + 2\mathcal{N}(t)) = \Omega^{(-)}(t)\int_{t_0}^{t} dt'\left[\Omega^{(-)}(t')(1 + 2\mathcal{N}(t')) \times \cos\left(\int_{t'}^{t} dt''\Omega^{(+)}(t'')\right)\right]. \tag{26}$$

Second, inspection of the master equation (20) shows that its general solution can be parametrized by a function $f(t)$ fulfilling the integral equation

$$\dot{f}(t) = \frac{\Omega^{(-)}(t)}{\omega_0} - 2\int_{t_0}^{t} dt' f(t')(\omega^2(t) + \omega^2(t')) \tag{27}$$

with the initial condition $f(t_0) = \dot{f}(t_0) = 0$. Then, the correlators are given by

$$\begin{aligned} 1 + 2\mathcal{N} &= 1 + \omega_0\int_{t_0}^{t} dt' f(t')\Omega^{(-)}(t'),\\ \mathcal{M}^{(-)} &= \omega_0 f(t),\\ \mathcal{M}^{(+)} &= -\omega_0\int_{t_0}^{t} dt' f(t')\Omega^{(+)}(t'). \end{aligned} \tag{28}$$

Alternatively the integral equation (27) can, taking one derivative, be converted into a third-order linear differential equation,

$$\dddot{F} + 4\omega^2\dot{F} + 2(\omega^2)^{\cdot}F = \frac{(\omega^2)^{\cdot}}{\omega_0^2}, \tag{29}$$

where

$$F(t) := \int_{t_0}^{t} dt' f(t') \tag{30}$$

and the initial conditions are $F(t_0) = \dot{F}(t_0) = \ddot{F}(t_0) = 0$. Observe that $\ddot{F}$ is absent in Eq. (29), which by Abel's theorem implies that the Wronskian of the solutions of the corresponding homogeneous equation is constant.

The differential equation (29) bears an interesting relationship to the KdV equation. The form of the integral equation (27) suggests the ansatz

$$f(t) = \frac{(\omega^2)^{\cdot}(t)}{8\omega_0^4}, \qquad F(t) = \frac{\omega^2(t) - \omega_0^2}{8\omega_0^4}. \tag{31}$$

Defining $r(t) := \omega^2(t)/\omega_0^2$ and then $u(x, t) := -r(x - 10t)$, one can show that u satisfies the KdV equation,

$$u_{xxx} - 6uu_x + u_t = 0. \tag{32}$$

Thus we can use certain solutions of the KdV equation to calculate pair creation rates for the corresponding electric fields.

IV. EXACTLY SOLVABLE CASES

We will now study two exactly solvable cases. First, we consider the following soliton-type solution of the KdV equation (see, e.g., Refs. [23–25]):

$$u(x, t) = -1 - \frac{2}{\cosh^2(x - 10t)}, \tag{33}$$

which corresponds to

$$r(t) = 1 + \frac{2}{\cosh^2(\omega_0 t)}, \qquad F(t) = \frac{1}{4\omega_0^2\cosh^2(\omega_0 t)}. \tag{34}$$

This is a solution to Eq. (29) with the appropriate boundary conditions at $t_0 = -\infty$. The gauge potential is

$$qA(t) = k_\parallel - \sqrt{k_\parallel^2 + \frac{2\omega_0^2}{\cosh^2(\omega_0 t)}}. \tag{35}$$

From Eq. (28) we get the exact pair creation rate

$$\mathcal{N}(t) = \frac{1}{8\cosh^4(\omega_0 t)}. \tag{36}$$

Note that $\mathcal{N}(t)$ returns to zero for $t \to \infty$, which is due to the solitonic character that makes the scattering reflectionless. In fact, the mode solution to Eq. (7) is given by

$$\varphi(t) = \frac{e^{-i\omega_0 t}}{\sqrt{2\omega_0}} A(t), \tag{37}$$

where the amplitude is

$$A(t) = (e^{2\omega_0 t} + 1)^2 {}_2F_1(2, 2 - i; 1 - i; -e^{2\omega_0 t}) \tag{38}$$

with ${}_2F_1$ the hypergeometric function, and it does not have a negative frequency part in the future. The Bogoliubov coefficient (14) is

$$\nu(t) = \frac{e^{2i\omega_0 t}}{2\omega_0} \dot{A}^*(t), \tag{39}$$

which approximately leads to $\mathcal{N}(t) = 2e^{4\omega_0 t}$ for $\omega_0 t \ll -1$ and $\mathcal{N}(t) = 2e^{-4\omega_0 t}$ for $\omega_0 t \gg 1$, the leading approximation to the exact formula (36). Thus there is no pair creation in this case, contrary to the somewhat similarly looking Sauter field case [26]. This example also shows clearly that, as emphasized in Ref. [14], no direct physical meaning should be ascribed to $\mathcal{N}_k(t)$ at intermediate times.

Second, we consider an electric field consisting of two opposite delta function pulses,

$$E(t) = E_0\delta(t) - E_0\delta(t - t_1), \tag{40}$$

which has the gauge potential of a potential well [17]: $A_\parallel = 0$ for $t < 0$ and $t > t_1$, corresponding to $\omega_k(0)$, and $A_\parallel = -E_0$ for $0 < t < t_1$, corresponding to ω_k. The master equation (20), together with continuity at $t = 0$, leads to the pair production for the period $0 < t < t_1$,

$$1 + 2\mathcal{N}_k(t) = (2n_k + 1)\left(\frac{\Omega_k^{(+)}}{2\omega_k}\right)^2 \times \left[1 - \left(\frac{\Omega_k^{(-)}}{\Omega_k^{(+)}}\right)^2 \cos(2\omega_k t)\right], \tag{41}$$

and for the period $t > t_1$ it now remains constant, retaining its value for t_1. Note that for a single delta function pulse $\mathcal{N}_k(t)$ keeps oscillating, so that the limit $t \to \infty$ cannot be defined. This is presumably due to a combination of the unphysical character of such a field and the non-Markovian nature of the time evolution.

V. DISCUSSION AND CONCLUSIONS

The central results of this paper are the master equation (20) and associated quantum Vlasov equation (26), each describing the exact time evolution of the cumulative pair creation variable $\mathcal{N}_k(t)$ for an electric field that depends only on time, but is arbitrary otherwise. To the best of our knowledge, these equations are new. We have concentrated here on scalar QED, leaving the details of the spinor QED case to a more extensive publication [18].

In future work, we also plan to study the precise conditions under which a nonadiabatic treatment is really necessary. To define the adiabatic approximation, we write the mode solution in terms of the adiabatic basis [11]

$$\varphi_k(t) = \alpha_k(t)\frac{e^{-i\theta(t)}}{\sqrt{2\omega_k(t)}} + \beta_k(t)\frac{e^{i\theta(t)}}{\sqrt{2\omega_k(t)}}, \tag{42}$$

where $\theta(t) = \int_{t_0}^t dt' \omega_k(t')$ and the Bogoliubov relation $|\alpha_k|^2 - |\beta_k|^2 = 1$ holds. We then replace $|\nu_k(t_0, t)|^2$ by $|\beta_k(t)|^2$ in the definition (16) of $\mathcal{N}(t)$.

From Eq. (14) one can easily show that for this approximation to hold it is sufficient to assume that

$$\left|\sqrt{\frac{\omega_k(t)}{\omega_k(0)}} - \sqrt{\frac{\omega_k(0)}{\omega_k(t)}}\right|, \qquad \left|\frac{\dot{\omega}_k(t)}{\omega_k^2(t)}\right| \ll |\beta_k(t)| \tag{43}$$

throughout the time evolution. This criterion is similar, although not strictly equivalent, to the one given in [11],

$$\frac{\dot{\omega}}{\omega^2} \ll 1, \qquad \frac{\ddot{\omega}}{\omega^3} \ll 1. \tag{44}$$

In any case, all the inequalities in (43) and (44) are certainly fulfilled for even the strongest laser sources which are presently existing or in development. Those have a field strength still much lower than the critical strength $E_c = m^2/e$ and a characteristic time scale much longer than the Compton time [2].

Concerning the relation of the master equation to the KdV equation, although there is a well-known connection between the latter equation and one-dimensional quantum mechanical scattering (see, e.g., Refs. [23–25,27]), it appears not to have been previously applied to the Schwinger pair creation problem. It will be interesting to see whether the multisoliton solutions of the KdV equation may also be used in this context.

Finally, let us mention that it is straightforward to extend our master equation to the case of an initial state which is a thermal state at temperature T. As will be shown in Ref. [18], such a change leads again only to an overall factor $(\coth(\beta\omega_k(0)/2) + 1)$ multiplying all three correlators $1 + 2\mathcal{N}_k$, $\mathcal{M}_k^{(\pm)}$, so that the master equation itself remains unaffected.

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[1] J. Schwinger, Phys. Rev. **82**, 664 (1951).
[2] A. Ringwald, Phys. Lett. B **510**, 107 (2001); G. V. Dunne, Eur. Phys. J. D **55**, 327 (2009); T. Tajima, Eur. Phys. J. D **55**, 519 (2009).
[3] G. V. Dunne and C. Schubert, Phys. Rev. D **72**, 105004 (2005); G. V. Dunne, Q-h. Wang, H. Gies, and C. Schubert, Phys. Rev. D **73**, 065028 (2006).
[4] S. P. Kim and D. N. Page, Phys. Rev. D **65**, 105002 (2002); **73**, 065020 (2006); **75**, 045013 (2007).
[5] C. K. Dumlu and G. V. Dunne, Phys. Rev. Lett. **104**, 250402 (2010); Phys. Rev. D **83**, 065028 (2011).
[6] H. Gies and K. Klingmüller, Phys. Rev. D **72**, 065001 (2005).
[7] E. Brézin and C. Itzykson, Phys. Rev. D **2**, 1191 (1970).
[8] V. S. Popov, Sov. Phys. JETP **34**, 709 (1972); **35**, 659 (1972); M. S. Marinov and V. S. Popov, Fortschr. Phys. **25**, 373 (1977).
[9] V. G. Bagrov, D. M. Gitman, S. P. Gavrilov, and Sh. M. Shvartsman, Izv. Vuz. Fiz. **3**, 71 (1975); S. P. Gavrilov and D. M. Gitman, Phys. Rev. D **53**, 7162 (1996).
[10] Y. Kluger, J. M. Eisenberg, B. Svetitsky, F. Cooper, and E. Mottola, Phys. Rev. Lett. **67**, 2427 (1991); Phys. Rev. D **45**, 4659 (1992).
[11] Y. Kluger, E. Mottola, and J. M. Eisenberg, Phys. Rev. D **58**, 125015 (1998).
[12] J. Rau, Phys. Rev. D **50**, 6911 (1994).
[13] S. M. Schmidt, D. Blaschke, G. Ropke, S. A. Smolyansky, A. V. Prozorkevich, and V. D. Toneev, Int. J. Mod. Phys. E **7**, 709 (1998); S. M. Schmidt, D. Blaschke, G. Ropke, A. V. Prozorkevich, S. A. Smolyansky, and V. D. Toneev, Phys. Rev. D **59**, 094005 (1999).
[14] F. Hebenstreit, R. Alkofer, and H. Gies, Phys. Rev. D **78**, 061701 (2008).
[15] C. K. Dumlu, Phys. Rev. D **79**, 065027 (2009).
[16] H. R. Lewis and W. B. Riesenfeld, J. Math. Phys. (N.Y.) **10**, 1458 (1969).
[17] E. Akkermans and G. V. Dunne, arXiv:1109.3489 [Phys. Rev. Lett. (to be published)].
[18] A. Huet, S. P. Kim, and C. Schubert (unpublished).
[19] I. A. Malkin, V. I. Man'ko, and D. A. Trifonov, Phys. Rev. D **2**, 1371 (1970); J. Math. Phys. (N.Y.) **14**, 576 (1973).
[20] F. Cooper, J. F. Dawson, D. Meredith, and H. Shepard, Phys. Rev. Lett. **72**, 1337 (1994).
[21] L. Faccioli, F. Finelli, G. P. Vacca, and G. Venturi, Phys. Rev. Lett. **81**, 240 (1998).
[22] S. P. Kim and C. H. Lee, Phys. Rev. D **62**, 125020 (2000); S. P. Kim and D. N. Page, Phys. Rev. A **64**, 012104 (2001).
[23] A. Das, *Integrable Models*, World Scientific Lecture Notes in Physics, Vol. 30 (World Scientific, Singapore, 1989).
[24] P. G. Drazin and R. S. Johnson, *Solitons: An Introduction* (Cambridge University Press, Cambridge, England, 1989).
[25] F. Cooper, A. Khare, and U. Sukhatme, *Supersymmetry in Quantum Mechanics* (World Scientific, Singapore, 2001).
[26] F. Sauter, Z. Phys. **69**, 742 (1931).
[27] J. Feinberg, Nucl. Phys. **B433**, 625 (1995).