

Number of Nambu-Goldstone bosons and its relation to charge densitiesHaruki Watanabe^{1,2,*} and Tomáš Brauner^{3,4,†}¹*Department of Physics, University of Tokyo, Hongo, Tokyo 113-0033, Japan*²*Department of Physics, University of California, Berkeley, California 94720, USA*³*Faculty of Physics, University of Bielefeld, 33615 Bielefeld, Germany*⁴*Department of Theoretical Physics, Nuclear Physics Institute ASCR, 25068 Řež, Czech Republic*

(Received 30 September 2011; published 9 December 2011)

The low-energy physics of systems with spontaneous symmetry breaking is governed by the associated Nambu-Goldstone (NG) bosons. While NG bosons in Lorentz-invariant systems are well understood, the precise characterization of their number and dispersion relations in a general quantum many-body system is still an open problem. An inequality relating the number of NG bosons and their dispersion relations to the number of broken symmetry generators was found by Nielsen and Chadha. In this paper, we give a presumably first example of a system in which the Nielsen-Chadha inequality is actually not saturated. We suggest that the number of NG bosons is exactly equal to the number of broken generators minus the number of pairs of broken generators whose commutator has a nonzero vacuum expectation value. This naturally leads us to a proposal for a different classification of NG bosons.

DOI: [10.1103/PhysRevD.84.125013](https://doi.org/10.1103/PhysRevD.84.125013)

PACS numbers: 11.30.Qc, 14.80.Va

I. INTRODUCTION AND SUMMARY

Spontaneous symmetry breaking (SSB) is a ubiquitous phenomenon in nature. In quantum many-body theory, its significance is further boosted by the fact that it provides us with a rare example of a general exact result: the Goldstone theorem [1,2]. The low-energy physics of systems exhibiting SSB is governed by the associated soft excitations, the Nambu-Goldstone (NG) bosons (see Ref. [3] for a comprehensive early review). While NG bosons in Lorentz-invariant systems are well understood, the precise characterization of their number and dispersion relations in a general quantum many-body system is still an open problem.

The systematic study of NG bosons in Lorentz-noninvariant systems was initiated by Nielsen and Chadha in their seminal work [4]. They showed that under certain technical assumptions, including rotational invariance and the requirement of absence of long-range interactions, the energy of the NG boson is proportional to an integer power of momentum in the long-wavelength limit, $\varepsilon(\mathbf{k}) \propto |\mathbf{k}|^n$. The NG boson is then classified as type I if n is odd, and as type II if n is even. Nielsen and Chadha proved that the number of type I NG bosons plus twice the number of type II NG bosons is greater than or equal to the number of broken symmetry generators.

In Lorentz-invariant systems, the Nielsen-Chadha (NC) inequality is trivial since it is well-known that the number of NG bosons, all being naturally type I, equals the number of broken generators. In Lorentz-noninvariant systems, the NC inequality allows the number of NG bosons to be smaller than the number of broken generators provided

some of them have nonlinear dispersion relation. A renowned example is the ferromagnet where two generators broken by the spontaneous magnetization give rise to only one NG boson—the magnon—having a quadratic dispersion relation at low momentum.

It is perhaps worth emphasizing that the NC counting rule is formulated as an inequality and that it does not constrain in any way the power of momentum appearing in the dispersion relation. In fact, a quick sociological survey reveals that this generality of the Nielsen-Chadha theorem is not very well appreciated in publications citing the original work [4]. The reason for this probably is that in the concrete systems studied in literature, type I and type II NG bosons always have linear and quadratic dispersion relations, respectively, and the NC inequality is always saturated. In Appendix A of this paper, we point out a rather trivial class of theories where NG bosons with energy proportional to an in principle arbitrarily high power of momentum can appear, and the NC inequality is not necessarily saturated [5]. This ultimately leads us to the proposal for a different classification of NG bosons, and the evidence that this provides an exact equality for their count.

Our work is based on the dependence of the number of NG bosons and their dispersion relations on expectation values of conserved charges, which was first investigated at the general level by Leutwyler [7] and Schäfer *et al.* [8] (see also Ref. [9] for a recent review). Leutwyler [7] used effective field theory to argue that, in the absence of quantum anomalies, nonzero density of a non-Abelian conserved charge leads to the appearance of NG bosons with a quadratic dispersion relation. This connection was further elaborated in Ref. [10] where an exact equality (saturating the NC inequality) was established for the class of linear sigma models with chemical potential at tree

*hwatanabe@berkeley.edu

†tbrauner@physik.uni-bielefeld.de

level. Every pair of broken generators whose commutator has a nonzero expectation value was found to give rise to one type II NG boson with a quadratic dispersion relation. On the other hand, Schäfer *et al.* [8] related charge densities to the number of NG bosons by showing that this number equals the number of broken generators provided the expectation values of commutators of all pairs of broken generators vanish.

Our present proposal can in a sense be understood as a synthesis of these previous works. We argue that given the three characteristics of SSB, that is, the number of NG bosons, their dispersion relations, and the expectation values of charge densities, an exact equality can be obtained by focusing on the relation between the number of NG bosons and the charge densities. Thus, our work generalizes the theorem of Schäfer *et al.* [8]. This is in contrast to Nielsen and Chadha [4] as well as Leutwyler [7] who emphasized the role of the NG boson dispersion relations.

In order that the generality of our proposal is not obscured by the technical details, we formulate it here. Let Q_a be the set of (spontaneously broken) conserved charges of the theory. *The number of NG bosons n_{NG} is related to the number of broken symmetry generators n_{BS} by the equality*

$$n_{\text{BS}} - n_{\text{NG}} = \frac{1}{2} \text{rank } \rho, \quad (1)$$

where the matrix of commutators ρ is defined by $i\rho_{ab} \equiv \lim_{\Omega \rightarrow \infty} \frac{1}{\Omega} \langle 0 | [Q_a, Q_b] | 0 \rangle$ and Ω is the space volume. Note that $\text{rank } \rho$ is always even since ρ is real and antisymmetric. Unfortunately, we were not able to prove Eq. (1) in the full generality so it remains a conjecture at the moment. However, we can prove a one-sided inequality and provide evidence that the opposite inequality holds as well. We should also emphasize that our argument applies to continuum field theories as well as to models defined on a (space) lattice.

The paper is organized as follows. In Sec. II we make some general remarks on SSB, basically setting up the stage for the discussion of our main result. We also introduce a class of “uniform” symmetries to which the standard derivation of the Goldstone theorem applies. In Sec. III we give a partial proof of the conjecture (1) and provide evidence for the missing part of the proof. Finally, in Sec. IV we conclude and afford some speculations. Some technical details as well as specific examples that do not pertain to the general argument are relegated to the appendices.

II. GENERAL REMARKS ON SSB AND THE GOLDSTONE THEOREM

In this section we will briefly review the notion of SSB and the Goldstone theorem. Although we will not repeat in detail its proof, we would like to make a number of comments explaining under what conditions and technical assumptions this proof applies. This may at times look

like purely academic pedantry, yet the example of a sharp inequality in the NC counting rule presented in Appendix A shows that one should be prepared for the unexpected.

Let us first, following Nielsen and Chadha [4], define what we mean by the number of broken generators n_{BS} . We demand that there is a set of conserved charges Q_a and a set of (quasi)local fields $\Phi_a(x)$, or, on a space lattice, operators $\Phi_a(t, \mathbf{x}_i)$, where $a = 1, \dots, n_{\text{BS}}$, such that the matrix

$$M_{ab} \equiv \langle 0 | [Q_a, \Phi_b(0)] | 0 \rangle \quad (2)$$

is nonsingular (i.e., has a nonzero determinant). Moreover, we assume that the conserved charges can be expressed as a spatial integral or sum of local charge densities,

$$Q_a = \int d^d \mathbf{x} j_a^0(t, \mathbf{x}), \quad \text{or} \quad Q_a = \sum_{\mathbf{x}_i \in \text{lattice}} \rho_a(t, \mathbf{x}_i). \quad (3)$$

Strictly speaking, the broken charge operators are only well defined in a finite volume. Nevertheless, in the formal derivations they only appear in commutators which have a well-defined infinite-volume limit [3].

A. Uniform symmetries

The proof of the Goldstone theorem essentially proceeds by finding the spectral representation of the commutator (2). This heavily relies on the integral representation (3) and the following translational property of the charge densities,

$$j_a^0(x) = e^{iP \cdot x} j_a^0(0) e^{-iP \cdot x}, \quad \text{or} \quad \rho_a(t, \mathbf{x}_i) = e^{iHt} T_{\mathbf{x}_i}^\dagger \rho_a(0) T_{\mathbf{x}_i} e^{-iHt}, \quad (4)$$

where P^μ is the four-momentum operator, H the Hamiltonian, and $T_{\mathbf{x}_i}$ is the operator of a finite (lattice) translation by $\mathbf{x}_i$. Even though this property is usually taken for granted, it only holds provided that the charge density operator, as constructed in terms of the local operators of the theory, does not depend explicitly on the coordinate. We will call symmetries whose charge densities satisfy this condition uniform.

As an example, consider the simplest field theory exhibiting SSB and a NG boson: the free massless relativistic scalar field theory, defined by the Lagrangian

$$\mathcal{L} = \frac{1}{2} \partial^\mu \phi \partial_\mu \phi. \quad (5)$$

The action of this theory is invariant under the set of global transformations $\phi(x) \rightarrow \phi(x) + \theta(x)$ with $\theta(x) = a + b_\alpha x^\alpha$, where a, b_α are the parameters of the transformation. The corresponding five conserved Noether currents are given by

$$j^\mu(x) = \partial^\mu \phi(x), \quad (6)$$

$$j_\alpha^\mu(x) = x_\alpha \partial^\mu \phi(x) - \delta_\alpha^\mu \phi(x).$$

Thanks to the commutation relations with the field operator, $[iQ, \phi(x)] = 1$ and $[iQ_\alpha, \phi(x)] = x_\alpha$, all five integral charges Q, Q_α are spontaneously broken, while there is obviously only one massless mode in the spectrum. The resolution of this apparent paradox is that the charge densities $j_\alpha^0(x)$ do not satisfy Eq. (4). In fact, a short explicit calculation gives $j_\alpha^\mu(x) - e^{iP \cdot x} j_\alpha^\mu(0) e^{-iP \cdot x} = x_\alpha \partial^\mu \phi(x)$. Thus the standard proof of the Goldstone theorem does not apply to the charges Q_α , and we should not *a priori* expect additional NG bosons stemming from their spontaneous breaking.

An intuitive understanding of why there is only one NG boson instead of the naively expected five can be gained using the argument of Low and Manohar [11]. Since the NG boson can be generated by a local broken symmetry transformation acting on the order parameter for SSB, two spontaneously broken transformations will give rise to only one NG boson if their local forms coincide. This is exactly the case of the transformations generated by Q, Q_α which all have the same local form. A similar example is a crystalline solid where both continuous translations and rotations are spontaneously broken, yet only NG bosons corresponding to the translations—the phonons—appear in the spectrum.

Despite the above example of an internal nonuniform symmetry, typical representatives of the class of nonuniform symmetries according to our definition are spacetime symmetries such as rotational or conformal invariance. In fact, the only example of a uniform spacetime symmetry is obviously translational invariance. In the following, we will assume that there is at least a discrete unbroken translational invariance, such as in crystalline solids. This is necessary in order to have quasiparticles with well-defined (real or crystal) momentum. With the above in mind, our result will apply to all spontaneously broken global continuous uniform symmetries. Equation (4) then holds whether continuous translational invariance is spontaneously broken or not.

One might think that the restriction to uniform symmetries could be avoided by resorting to the variant of the proof of the Goldstone theorem using the quantum effective potential or action [2], which does not rely on the operator identity (4). As it may be of general interest, we provide in Appendix B a modification of this proof that applies to nonuniform symmetries, leading essentially to the same conclusion as the classical argument of Ref. [11]. However, this method of proof only gives the number of flat directions of the effective potential (or equivalently the number of zero modes of the inverse propagator of the theory at zero momentum), which in Lorentz-noninvariant theories is in general not equal to the number of NG bosons. This means that while it can be used to make statements within Lorentz-invariant theories, it can tell us very little about the number of NG bosons in Lorentz-noninvariant theories.

B. Conserved charges and their densities

Here we make a number of other short comments related to the conserved charges required by the Goldstone theorem. First, despite the fact that one (including ourselves) usually speaks of “broken generators,” it should be noted that the operators Q_a in Eq. (2) need not be generators of symmetry transformations in the Noether sense [12]. In fact, the only two properties needed for the proof of the Goldstone theorem are that Q_a are integrals or sums of charge densities as in Eq. (3) and that Q_a are time-independent. Of course, the most convenient way to ensure that Q_a is time-independent is starting from a four-current $j_a^\mu(x)$ which satisfies the continuity equation, $\partial_\mu j_a^\mu = 0$.

Second, our conjecture (1) is formulated in terms of a commutator of charges. Since generators of symmetry transformations span a Lie algebra, this strongly suggests that ρ_{ab} is actually a linear combination of expectation values of the charges themselves. However, this would disregard the possibility that the representation of the Lie algebra of conserved charges on the Hilbert space possesses a set of central charges. One can then expect a generalized commutation relation of the type $[Q_a, Q_b] = i c_{ab} \mathbb{1} + i f_{abc} Q_c$. This phenomenon is certainly rather rare since the central charges c_{ab} can be removed by a suitable redefinition of the generators Q_a for all semisimple Lie algebras [13]. Yet, once central charges do appear, it can even happen that ρ_{ab} is nonzero although at the classical level the Lie algebra of conserved charges is commutative. The simplest example is perhaps the free nonrelativistic particle which can be interpreted as a type II NG boson of an extended ISO(2) symmetry [9].

Finally, it may be instructive to emphasize that any proof of the Goldstone theorem or its generalizations, including ours, can be applied not only to the set of all broken generators of a given theory, but also to its subsets closed under commutation. Any conclusion about the number of NG bosons then refers to those NG bosons that couple to the broken charges considered.

III. CHARGE COMMUTATOR AND THE NUMBER OF NG BOSONS

The aim of this section is to provide evidence for our conjecture (1). We will actually prove a one-sided inequality and give a physical argument why the opposite inequality also holds. We will essentially follow the strategy of Nielsen and Chadha [4] and complement it with the relation to the charge commutator matrix ρ_{ab} .

The NG mode with zero momentum corresponding to the broken generator Q_a will be represented as $|a\rangle \equiv Q_a|0\rangle$. This is a consistent definition of zero-momentum NG states. On the one hand, all states $|a\rangle$ are created by the broken charges and have zero energy. On the other hand, assume that there is another zero-energy state which is orthogonal to all $|a\rangle$'s. By construction, it then has a zero coupling to all broken charges, hence does not contribute to

the commutator (2), and thus should not be counted among the NG bosons. Consequently, the number of NG bosons is equal to the number of linearly independent vectors $|a\rangle$. Note that this argument uses the implicit assumption that the NG states form a vector space, which is rigorously justified at least at zero momentum.

A. Proof of $n_{\text{BS}} - n_{\text{NG}} \leq (1/2) \text{rank } \rho$

Since out of the n_{BS} vectors $|a\rangle$, there are only n_{NG} linearly independent ones, there must be a set of complex coefficients C_{pa} , where $p = 1, \dots, \Delta n$ and $\Delta n \equiv n_{\text{BS}} - n_{\text{NG}}$, such that $C_{pa}|a\rangle = 0$ with $\text{rank } C = \Delta n$ (summations over repeated indices are implied). As pointed out by Nielsen and Chadha [4], the Δn vectors $C_{pa}^*|a\rangle$ are linearly independent. Indeed, if this were not the case, there would be a set of coefficients D_p such that $D_p C_{pa}^*|a\rangle = 0$. Then,

$$\begin{aligned} (D_p C_{pa}^*) M_{ab} &= D_p C_{pa}^* \langle 0 | [Q_a, \Phi_b(0)] | 0 \rangle \\ &= \langle a | C_{pa}^* D_p \Phi_b(0) | 0 \rangle - \langle 0 | \Phi_b(0) D_p C_{pa}^* | a \rangle \\ &= 0, \end{aligned} \quad (7)$$

where the matrix M is defined in Eq. (2). Equation (7) would imply that the matrix M has a zero eigenvalue, which would contradict the assumption of SSB that M is nonsingular.

We have observed that $C_{pa} Q_a |0\rangle = 0$ for all $p = 1, \dots, \Delta n$, and that $\{C_{pa}^* Q_a |0\rangle\}_{p=1}^{\Delta n}$ are linearly independent. This implies that $A_p \equiv C_{pa} Q_a$ ($A_p^\dagger \equiv C_{pa}^* Q_a$) acts as the annihilation (creation) operator of a NG boson labeled by p . Since $\{A_p^\dagger |0\rangle\}_{p=1}^{\Delta n}$ are linearly independent, the $\Delta n \times \Delta n$ matrix G defined as

$$G_{pq} \equiv \langle 0 | A_p A_q^\dagger | 0 \rangle = \langle 0 | [A_p, A_q^\dagger] | 0 \rangle = (C G C^\dagger)_{pq} \quad (8)$$

is Hermitian and positive definite; the Hermitian matrix g here is defined as $g_{ab} \equiv \langle a | b \rangle$. The matrix G can be diagonalized by a unitary matrix U so that $U G U^\dagger = \text{diag}(\lambda_1, \dots, \lambda_{\Delta n})$. From Eq. (8) we can see that the diagonalization is achieved by the replacement $C \rightarrow U C$. Thus, we can choose C from the beginning in such a way that G is already diagonal. We then have

$$\langle 0 | [A_p, A_q^\dagger] | 0 \rangle = \delta_{pq} \lambda_q \quad (\text{no sum over } q). \quad (9)$$

Note that this relation is reminiscent of a commutation relation between the annihilation and creation operators, $[A_p, A_q^\dagger] \propto \delta_{pq}$.

Now we are ready to show the desired inequality. Let us define

$$\begin{aligned} Q_p^+ &= \frac{1}{2} (A_p + A_p^\dagger) = \text{Re}(C_{pa}) Q_a, \\ Q_p^- &= \frac{1}{2i} (A_p - A_p^\dagger) = \text{Im}(C_{pa}) Q_a. \end{aligned} \quad (10)$$

These make $2\Delta n$ independent broken generators. In order to obtain a basis of the space spanned by the broken generators $\{Q_a\}_{a=1}^{n_{\text{BS}}}$, we need to add $n_{\text{BS}} - 2\Delta n$ other broken generators, $Q'_s = B_{sa} Q_a$, with real B_{sa} . Introducing finally the notation for the new basis

$$\begin{aligned} &(\bar{Q}_1, \bar{Q}_2, \dots, \bar{Q}_{n_{\text{BS}}}) \\ &\equiv (Q_1^+, Q_1^-, \dots, Q_{\Delta n}^+, Q_{\Delta n}^-, Q'_1, \dots, Q'_{n_{\text{BS}}-2\Delta n}), \end{aligned} \quad (11)$$

we infer that the antisymmetric matrix $\bar{\rho}$, defined by $i\bar{\rho}_{ab} \equiv \lim_{\Omega \rightarrow \infty} \frac{1}{\Omega} \langle 0 | [\bar{Q}_a, \bar{Q}_b] | 0 \rangle$, equals

$$\bar{\rho} = \lim_{\Omega \rightarrow \infty} \frac{1}{2\Omega} \times \begin{pmatrix} 0 & \lambda_1 & & & & \\ -\lambda_1 & 0 & & & & \\ & & \ddots & & & \\ & & & 0 & \lambda_{\Delta n} & \\ & & & -\lambda_{\Delta n} & 0 & \\ & & * & & & * \end{pmatrix}, \quad (12)$$

where the asterisks indicate in turn $2\Delta n \times (n_{\text{BS}} - 2\Delta n)$, $(n_{\text{BS}} - 2\Delta n) \times 2\Delta n$, and $(n_{\text{BS}} - 2\Delta n) \times (n_{\text{BS}} - 2\Delta n)$ blocks of unknown, possibly nonzero entries. Since the transformation $\{\bar{Q}_a\}_{a=1}^{n_{\text{BS}}} \leftrightarrow \{Q_a\}_{a=1}^{n_{\text{BS}}}$ is nothing but a change of basis, $\bar{\rho}$ must have the same rank as ρ ; hence

$$\text{rank } \rho = \text{rank } \bar{\rho} \geq 2\Delta n, \quad (13)$$

which was to be proven.

B. Explanation of $n_{\text{BS}} - n_{\text{NG}} = (1/2) \text{rank } \rho$ and the relation to the NC classification

In Ref. [14], Nambu pointed out that $\langle 0 | [Q_a, Q_b] | 0 \rangle \neq 0$ implies that the corresponding zero modes behave like canonical conjugates of each other; the same observation was made in a special case in Ref. [8]. We have seen this above; Q_p^+ and Q_p^- excite the same mode $A_p^\dagger |0\rangle$, and they have the commutation relation $\langle 0 | [Q_p^+, Q_p^-] | 0 \rangle = (i/2) \langle 0 | [A_p, A_p^\dagger] | 0 \rangle \neq 0$ [see Eqs. (9) and (10)].

By construction, $\{Q_p^+\}_{p=1}^{\Delta n} \cup \{Q'_s\}_{s=1}^{n_{\text{BS}}-2\Delta n}$ (or equivalently $\{Q_p^-\}_{p=1}^{\Delta n} \cup \{Q'_s\}_{s=1}^{n_{\text{BS}}-2\Delta n}$) excite linearly independent modes. So, if we assume that Nambu's argument is valid, all of $\langle 0 | [Q_p^+, Q'_s] | 0 \rangle$, $\langle 0 | [Q_p^-, Q'_s] | 0 \rangle$, and $\langle 0 | [Q'_s, Q'_s] | 0 \rangle$ should vanish, which means all the asterisks in Eq. (12) should be replaced with zero block matrices. We then arrive at the equality $n_{\text{BS}} - n_{\text{NG}} = (1/2) \text{rank } \rho$.

As for the dispersion relation $\varepsilon_p(\mathbf{k})$ of the mode $A_p^\dagger |0\rangle$, we can apply the same analyticity argument as Nielsen and Chadha [4] and conclude that $\varepsilon_p(\mathbf{k})$ is an analytic function of $\mathbf{k}$. In the special case of an unbroken rotational invariance, we then have $\varepsilon_p(\mathbf{k}) \propto |\mathbf{k}|^{2m_p}$ where m_p is a positive integer. Thus, all $A_p^\dagger |0\rangle$ modes are type II NG bosons. How

about the modes excited by $\{Q'_s\}_{s=1}^{n_{\text{BS}}-2\Delta n}$? Based on the examples discussed in literature, we can say that they usually have a linear dispersion relation, that is, are type I. However, there is no general argument that would guarantee this. It is possible to modify the dispersion relation to a quadratic one by fine-tuning the parameters of the theory. Indeed, this is demonstrated by the example we propose in Appendix A. All in all, the number of NG bosons with an even power of momentum in the dispersion relation is at least Δn . This leads immediately to the NC inequality.

In practice, the classification of the various types of NG bosons can be significantly simplified by choosing an appropriate basis of the broken generators. To that end, note that one can always choose the ground state in such a way that only mutually commuting charges have nonzero vacuum expectation value. In other words, only generators from the Cartan subalgebra of the Lie algebra of the symmetry group can have a nonzero vacuum expectation value. A detailed proof of this statement is given in Appendix C at the level of generality sufficient for all practical purposes. By means of the general root decomposition of Lie algebras, it then follows immediately that the set of broken generators splits into pairs whose commutator has possibly nonzero vacuum expectation value (see Appendix D for details).

A cautious reader may have noticed that, strictly speaking, the above choice of basis may not be compatible with the choice made in Eq. (11). Therefore, the material of Appendix C cannot be directly used in our proof. However, as already stressed, it is of great practical utility, and in addition, it can be used for an alternative derivation of the inequality $n_{\text{BS}} - n_{\text{NG}} \leq (1/2)\text{rank } \rho$, which is given in Appendix D. This derivation provides us with a new insight into the nature of the NG bosons associated with charge density.

IV. SOME SPECULATIONS AND OUTLOOK

In this paper we pointed out the intimate connection between the number of NG bosons in a quantum many-body system exhibiting SSB and the densities of conserved charges in the ground state. Our main result is summarized in Eq. (1). The method used to (partially) prove it, following closely Ref. [4], suggests that a different classification of NG bosons than that based on the dispersion relation might be useful. Let us denote a NG boson that couples to some combination of broken charges $C_{pa}^* Q_a$ introduced in Sec. III as type C (“charged”), and the remaining NG bosons as type N (“neutral”). The argument in Sec. III then immediately proves that

$$n_{\text{N}} + 2n_{\text{C}} = n_{\text{BS}}, \quad (14)$$

with obvious notation. Moreover, our conjecture (1) can then be cast in the form $n_{\text{C}} = (1/2)\text{rank } \rho$.

As follows from the proof of the Nielsen-Chadha theorem [4], all type C NG bosons are necessarily type II. On

the other hand, as pointed out already in Sec. III B, type N NG bosons can be both type I and type II. In the systems usually discussed in literature, all type N NG bosons actually are type I; hence the two classifications coincide and the NC inequality is saturated as a consequence of Eq. (1). However, the explicit example in Appendix A shows that there may exist NG bosons which are simultaneously type N and type II. We may say that such NG bosons are “accidentally” type II since their nonlinear, typically quadratic, dispersion relation is achieved by tuning the parameters of the model. On the contrary, all type C NG bosons are “robustly” type II due to the connection with the conserved charge densities. The physical origin of these two different kinds of type II NG bosons is obviously very different.

Of course, in order that this classification of NG bosons is consistently defined, one first has to prove that it does not depend on the choice of the operators Q_a . For instance, the type II NG boson of Appendix A would be classified as type N. However, one could imagine that there is another conserved charge which couples to the same NG boson and has a nonzero commutator with the Noether charge of the shift symmetry of the theory (A1). The same NG boson would then be classified as type C. Of course, the conjecture (1) would be satisfied in both cases. That is why we only propose this classification here in the form of a speculation. It would be interesting to investigate this issue further.

Another interesting problem is what happens when symmetry whose spontaneous breaking gives rise to a reduced number of NG bosons is gauged. How many massive spin-one bosons appear in the spectrum as a consequence of the Anderson-Higgs mechanism? Is their number equal to n_{NG} or rather to n_{BS} ? This question has been addressed in Refs. [15,16], but so far only for one particular model, whose global version was introduced earlier in Refs. [8,17]. The problem would certainly deserve a more general investigation.

Finally, note that in this paper we concentrated solely on NG bosons of spontaneously broken uniform symmetries. While a generalization of the Goldstone theorem to spontaneously broken nonuniform symmetries, which ensures the existence of at least one NG boson, was given in Appendix B, the issue of the count of the NG bosons in such a case is still open. Any result with the same level of generality as well as rigor as the Goldstone theorem would significantly improve our understanding of spacetime as well as other nonuniform symmetries.

ACKNOWLEDGMENTS

We thank H. Abuki, H. Aoki, T. Hatsuda, T. Hayata, T. Kanazawa, and T. Kugo for fruitful discussions. Part of the work was carried out during the stay of H.W. at the University of Bielefeld. H.W. acknowledges the University of Tokyo exchange program for a partial

support of his visit to Bielefeld. The work of T.B. was supported by the Sofja Kovalevskaja program of the Alexander von Humboldt Foundation.

Note added.—After this work was completed, we learned that the result proven in Appendix C is a special case of the following general theorem known in mathematical literature: If G is a compact connected Lie group and T is a maximal torus, then each element of G is conjugate to a member of T (theorem 4.36 in Ref. [22]). We are indebted to S. Naito and R. Sakamoto for bringing this theorem to our attention.

APPENDIX A: INEQUALITY IN THE NC COUNTING RULE

Here we propose an example in which a strict inequality in the NC counting rule holds. Similarly to the example in Sec. II A, let us consider a class of free scalar field theories defined by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_0\phi\partial_0\phi - \frac{1}{2}\phi\mathcal{D}\phi, \quad (\text{A1})$$

where $\mathcal{D} = \sum_{n=1}^{\infty} c_n (-\nabla^2)^n$. This Lagrangian is invariant under a (real) constant shift of the field, $\phi(x) \rightarrow \phi(x) + \theta$.

By choosing the ground state $|0\rangle$ for simplicity as the Fock vacuum, we get

$$\langle 0 | [iQ, \phi(x)] | 0 \rangle = 1. \quad (\text{A2})$$

Thus, the shift symmetry is always spontaneously broken. The corresponding NG boson is nothing but the one-particle state in the Fock space of this noninteracting field theory, $|n_k\rangle = a_k^\dagger |0\rangle$ with the dispersion relation $\varepsilon(\mathbf{k}) = \sqrt{\sum_n c_n |\mathbf{k}|^{2n}}$. For the usual Lorentz-invariant case ($c_n = \delta_{n,1}$), we have $\varepsilon(\mathbf{k}) = |\mathbf{k}|$ which represents a type I NG boson so that the NC inequality is saturated. However, despite being somewhat pathological, it is also possible to choose $c_n \propto \delta_{n,2}$. In this case, $\varepsilon(\mathbf{k}) \propto |\mathbf{k}|^2$ which describes a type II NG boson; hence a strict inequality occurs in the NC counting rule. The punch line is that the power of momentum in the dispersion relation of the NG boson can be modified by tuning the parameters of the Lagrangian. We expect the same is also true in other models such as the Heisenberg ferromagnet with a sufficient number of coupling constants.

APPENDIX B: GOLDSTONE THEOREM FOR NONUNIFORM SYMMETRIES

The alternative proof of the Goldstone theorem [2] usually makes use of the quantum effective potential, which is a function of classical uniform fields, and thus cannot be applied directly to nonuniform symmetry transformations. We offer here a generalization of the argument that relies on the full quantum effective action.

Following essentially Weinberg [18], we consider a theory of a set of (not necessarily scalar) fields, $\phi_i(x)$,

and assume that the classical action of the theory is invariant under the infinitesimal shift $\delta\phi_i(x) = \theta F_i[x; \phi]$. Here θ is a parameter of the transformation, and F_i is in general a functional of the fields and a function of the coordinate x . This classical symmetry is shared by the quantum effective action, $\Gamma[\phi]$, that is,

$$\int d^4y \frac{\delta\Gamma[\phi]}{\delta\phi_j(y)} F_j[y; \phi] = 0, \quad (\text{B1})$$

provided F_i is linear in the fields. Taking now the variational derivative of Eq. (B1) with respect to $\phi_i(x)$ and setting the field equal to its expectation value, $\langle 0 | \phi_i(x) | 0 \rangle \equiv v_i(x)$, we obtain

$$\int d^4y \frac{\delta^2\Gamma[v]}{\delta\phi_i(x)\delta\phi_j(y)} F_j[y; v] = 0. \quad (\text{B2})$$

Finally, we assume for the sake of simplicity full translational invariance of the vacuum. The second derivative of the effective action can then be expressed in terms of the inverse propagator of the theory, $G_{ij}^{-1}(k)$.

For uniform symmetries we can immediately infer that $G_{ij}^{-1}(0)F_j[v] = 0$. In other words, SSB implies the existence of a gapless pole in the propagator. This is equivalent to the more common formulation of the proof using the effective potential. Our argument is nevertheless more general since it also applies to theories with a nonlocal action which cannot be expressed as a spacetime integral of a local Lagrangian density.

For nonuniform symmetries, we have to make one further step, that is, multiply Eq. (B2) by $e^{ik\cdot x}$ and integrate over the coordinate x , which yields

$$G_{ij}^{-1}(k) \int d^4y e^{ik\cdot y} F_j[y; v] = 0. \quad (\text{B3})$$

(Nonzero momentum k is used just to ensure the convergence of the integral.) Once again, in the limit $k \rightarrow 0$, the integral of $F_j[y; v]$ represents a zero mode of the inverse propagator, which establishes the existence of a massless particle in the spectrum.

Applying Eq. (B3) to the example discussed in Sec. II A, we can immediately see that all five spontaneously broken charges of Eq. (6) give rise to the same zero mode of the inverse propagator. Generally, it follows from Eq. (B3) that two broken symmetries that have the same local form will give the same zero mode of the inverse propagator, that is, the same NG boson. This is in accord with the conclusion of Low and Manohar [11]. Nevertheless, in contrast to their classical field theory argument, our derivation actually proves the existence of the massless mode in the spectrum of the quantum theory, and is only limited by the assumed translational invariance and the linearity of the symmetry transformation in the fields.

APPENDIX C: CHARGE DENSITIES AND THE CARTAN SUBALGEBRA

In this appendix we will discuss in detail the statement made in Sec. III B that one can always choose the ground state in such a way that only mutually commuting charges have nonzero vacuum expectation value. As explained there, this is of great practical help in the classification of the NG bosons. Even though we have not been able to prove this claim in the full generality, we give below a detailed proof that applies to all compact semisimple Lie groups which are given by direct products of classical simple Lie groups. This is sufficient for virtually all practical purposes. Our claim can be reformulated mathematically as the statement that every adjoint orbit of the symmetry group passes through the Cartan subalgebra of its Lie algebra. We will prove it in turn for all classes of classical simple Lie groups and address the generalization to semisimple groups in the end.

Let us first introduce some notation. By T_a and $\hat{Q}_a$ we will denote the generators of the symmetry group in its defining representation, and its representation on the Hilbert space of the quantum system, respectively. The corresponding finite symmetry transformations will then be $U(\theta) \equiv e^{i\theta_a T_a}$ and $\hat{U}(\theta) \equiv e^{i\theta_a \hat{Q}_a}$. Finally, $R(\theta)$ will stand for the adjoint representation of the same transformation. Note that for the sake of clarity, we use in this appendix hats to distinguish operators on the Hilbert space from finite-dimensional matrices.

We now introduce the charge expectation values, $q_a \equiv \langle 0 | \hat{Q}_a | 0 \rangle$, and construct the matrix $\mathcal{M} \equiv q_a T_a$ belonging to the Lie algebra of the symmetry group. Upon a symmetry transformation, the ground state $|0\rangle$ becomes $|\theta\rangle \equiv \hat{U}(\theta)|0\rangle$. The charges q_a then transform to [19]

$$\begin{aligned} q'_a &\equiv \langle \theta | \hat{Q}_a | \theta \rangle = \langle 0 | \hat{U}(\theta)^\dagger \hat{Q}_a \hat{U}(\theta) | 0 \rangle \\ &= \langle 0 | R(\theta)_{ab}^{-1} \hat{Q}_b | 0 \rangle = R(\theta)_{ba} q_b. \end{aligned} \quad (C1)$$

Hence $\mathcal{M}$ transforms to

$$\mathcal{M}' \equiv q'_b T_b = q_a R(\theta)_{ab} T_b = U(\theta) \mathcal{M} U(\theta)^\dagger. \quad (C2)$$

We can see that $\mathcal{M}$ transforms in the adjoint representation, which provides a link between the physical and the mathematical formulation of the problem; the fact that only q'_a 's corresponding to mutually commuting generators are nonzero simply means that the matrix $\mathcal{M}'$ lies in the Cartan subalgebra. In the following, we will use the more concise mathematical formulation of the problem.

1. Proof for the groups SU(N)

Since the Cartan subalgebra is in this case formed by all real diagonal traceless matrices, our claim follows trivially from the well-known fact that every Hermitian matrix can be diagonalized by a suitable (unimodular) unitary transformation. To the best of our knowledge, the proof in this

case was given for the first time in Ref. [20] using an argument essentially identical to ours.

2. Proof for the groups SO(N)

The Cartan subalgebra in this case can be chosen as the set of all (purely imaginary and antisymmetric) matrices whose only nonzero entries are localized in 2×2 blocks along the diagonal. Our task is to prove that every purely imaginary antisymmetric matrix $\mathcal{M}$ can be brought into such block-diagonal form by a special orthogonal transformation.

We know that the (real) eigenvalues of a purely imaginary antisymmetric matrix come in pairs with opposite signs and that the corresponding eigenvectors are related by complex conjugation. Let us denote these eigenvectors as $\mathbf{u}_k, \mathbf{u}_k^*$. Every such pair of vectors spans an invariant subspace of $\mathcal{M}$. We can trade them for real vectors $\mathbf{v}_k \equiv (\mathbf{u}_k + \mathbf{u}_k^*)/\sqrt{2}$ and $\mathbf{w}_k \equiv -i(\mathbf{u}_k - \mathbf{u}_k^*)/\sqrt{2}$. (In case of odd N , there will be an additional real eigenvector associated with zero eigenvalue.) Since the basis of eigenvectors $\{\mathbf{u}_k, \mathbf{u}_k^*\}_{k=1}^{N/2}$ is orthonormal, so is the basis $\{\mathbf{v}_k, \mathbf{w}_k\}_{k=1}^{N/2}$. It is real and hence connected to the basis we started from by a real orthogonal transformation. In this basis, $\mathcal{M}$ takes the desired block-diagonal form. Finally, by flipping the sign of one of the basis vectors if necessary, we can ensure that the similarity transformation leading to $\{\mathbf{v}_k, \mathbf{w}_k\}_{k=1}^{N/2}$ is unimodular.

3. Proof for the groups Sp(2N)

Recall that the group $\text{Sp}(2N)$ consists of all $2N \times 2N$ unitary matrices U such that $UIU^T = I$ where $I \equiv i\sigma_2 \otimes \mathbb{1}_{N \times N}$ with σ_2 the second Pauli matrix. We can say that $\text{Sp}(2N)$ contains all unitary matrices which are simultaneously I -orthogonal. Likewise, the generators of $\text{Sp}(2N)$ are Hermitian and I -antisymmetric, $T_a^T = -IT_a I^{-1}$. As a consequence, the generators of $\text{Sp}(2N)$ take the form

$$T = \begin{pmatrix} A & B \\ B^\dagger & -A^T \end{pmatrix}, \quad (C3)$$

where A is a Hermitian and B a symmetric complex $N \times N$ matrix. The Cartan subalgebra then consists of all matrices of this form with A real and diagonal and $B = 0$.

We are now to prove that every I -antisymmetric (and Hermitian) matrix $\mathcal{M}$ can be diagonalized by a unitary symplectic transformation. First, observe that if $\mathbf{u}_k$ is its eigenvector with the (real) eigenvalue λ_k , then its I conjugate, $I\mathbf{u}_k^*$, is also an eigenvector with the eigenvalue $-\lambda_k$. Hence the eigenvalues of $\mathcal{M}$ once again come in pairs with opposite sign and we can thus construct an orthonormal basis of eigenvectors and organize them in the columns of a unitary matrix [21],

$$P \equiv (\mathbf{u}_1, \dots, \mathbf{u}_N, I\mathbf{u}_1^*, \dots, I\mathbf{u}_N^*). \quad (C4)$$

In this basis, $\mathcal{M}$ acquires the desired diagonal form and a simple calculation shows that P itself is symplectic as required. This completes the proof.

4. Proof for semisimple groups

Let us prove our claim for a direct product of two groups of the type discussed in the preceding three subsections. The general case will then follow by induction. The two (commuting) sets of generators will be denoted $T_a \otimes \mathbb{1}$ and $\mathbb{1} \otimes T_\alpha$, where we use Latin and Greek indices to distinguish the two groups. The matrix $\mathcal{M}$ then becomes

$$\mathcal{M} = (q_a T_a) \otimes \mathbb{1} + \mathbb{1} \otimes (q_\alpha T_\alpha). \quad (\text{C5})$$

The transformations from the two groups are labeled by independent parameters θ_a and σ_α , that is, $U(\theta) \equiv e^{i\theta_a T_a}$ and $V(\sigma) \equiv e^{i\sigma_\alpha T_\alpha}$. Under a general transformation from the product group, $U(\theta) \otimes V(\sigma)$, the matrix $\mathcal{M}$ transforms to

$$\begin{aligned} \mathcal{M}' &= [U(\theta) \otimes V(\sigma)] \mathcal{M} [U(\theta) \otimes V(\sigma)]^\dagger \\ &= [U(\theta) q_a T_a U(\theta)^\dagger] \otimes \mathbb{1} + \mathbb{1} \otimes [V(\sigma) q_\alpha T_\alpha V(\sigma)^\dagger]. \end{aligned} \quad (\text{C6})$$

Obviously, one can choose the transformations in the two groups independently to diagonalize the respective parts of $\mathcal{M}$ so that, in the end, $\mathcal{M}'$ will lie in the Cartan subalgebra of the product group, which is just a direct sum of the two Cartan subalgebras.

APPENDIX D: CLASSIFICATION OF NG BOSONS VIA CARTAN DECOMPOSITION

In order to view the NG bosons associated with charge densities from a different perspective, let us recall the root decomposition from the theory of Lie algebras. (Our argument will apply to all Lie algebras to which the result of Appendix C applies, that is, to all compact semisimple Lie algebras which are given by direct products of classical simple Lie algebras.) There, one constructs linear combinations of the Lie algebra generators in the form of root generators E_α , such that $[E_\alpha, E_\beta^\dagger] = \delta_{\alpha\beta} \alpha_i H_i$, where α is the associated root vector and H_i is the basis of the Cartan subalgebra.

Using the result of Appendix C, we now choose the basis of the Lie algebra so that only the generators H_i can have a nonzero vacuum expectation value. Concretely, let us consider all the roots α_c , $c = 1, \dots, \Delta n'$ such that

$\lambda'_c \equiv (\alpha_c)_i \langle 0 | H_i | 0 \rangle \neq 0$. Defining analogously to Eq. (10) the set of $2\Delta n'$ generators $Q_c^\pm$ as

$$Q_c^+ = \frac{1}{2}(E_{\alpha_c} + E_{\alpha_c}^\dagger), \quad Q_c^- = \frac{1}{2i}(E_{\alpha_c} - E_{\alpha_c}^\dagger), \quad (\text{D1})$$

and adding $n_{\text{BS}} - 2\Delta n'$ generators Q'_s in order to obtain a complete basis of generators, the commutator matrix ρ becomes

$$\rho = \lim_{\Omega \rightarrow \infty} \frac{1}{2\Omega} \times \begin{pmatrix} 0 & \lambda'_1 & & & & \\ -\lambda'_1 & 0 & & & & \\ & & \ddots & & & \\ & & & 0 & \lambda'_{\Delta n'} & \\ & 0 & & -\lambda'_{\Delta n'} & 0 & \\ & & & & & 0 \end{pmatrix}. \quad (\text{D2})$$

All nonzero entries of the matrix are located in $\Delta n' \times 2 \times 2$ blocks along the diagonal. Ordering the generators as in Eq. (11), we next choose the first $2\Delta n'$ interpolating fields Φ_a in the commutator (2) as $(j_1^{0+}, j_1^{0-}, \dots, j_{\Delta n'}^{0+}, j_{\Delta n'}^{0-})$. For $s = 2\Delta n' + 1, \dots, n_{\text{BS}}$ and $a = 1, \dots, 2\Delta n'$, we now have $M_{sa} = 0$, and the SSB condition $\det M \neq 0$ demands that $\det M' \neq 0$ where M'_{st} is the $(n_{\text{BS}} - 2\Delta n') \times (n_{\text{BS}} - 2\Delta n')$ lower-right corner of M_{ab} . Finally, we observe that since the expectation values of commutators of all pairs of generators Q'_s vanish, according to the theorem of Schäfer *et al.* [8] there must be $n_{\text{BS}} - 2\Delta n'$ NG bosons as intermediate states in the commutator M'_{st} . At the same time, at least one NG boson must appear as an intermediate state in the commutator $\langle 0 | [Q_c^+, j_c^{0-}(0)] | 0 \rangle$ for each $c = 1, \dots, \Delta n'$. Therefore, the number of NG bosons is bounded from below as $n_{\text{NG}} \geq (n_{\text{BS}} - 2\Delta n') + \Delta n' = n_{\text{BS}} - \Delta n' = n_{\text{BS}} - (1/2) \text{rank } \rho$, as proved by a different method in Sec. III A. While the above argument uses the notation for charge densities introduced in Sec. II for systems with continuous translational invariance, it goes through without change if we replace j_a^0 with ρ_a —a charge density on a spatial lattice—as in Eq. (3).

-
- [1] J. Goldstone, *Nuovo Cimento* **19**, 154 (1961).
 - [2] J. Goldstone, A. Salam, and S. Weinberg, *Phys. Rev.* **127**, 965 (1962).
 - [3] G.S. Guralnik, C.R. Hagen, and T.W.B. Kibble, in *Advances in Particle Physics*, edited by R.L. Cool and R.E. Marshak (Wiley, New York, 1968), Vol. 2, p. 567.

- [4] H.B. Nielsen and S. Chadha, *Nucl. Phys.* **B105**, 445 (1976).
- [5] The power of momentum in the dispersion relation is bounded from above by a value depending on the space dimension d though. Generalizing naively the argument of the classic paper by Coleman [6] to Lorentz-noninvariant

- systems, the correlation function of the NG field would diverge at large distance if the power of momentum exceeded this limit.
- [6] S.R. Coleman, *Commun. Math. Phys.* **31**, 259 (1973).
 - [7] H. Leutwyler, *Phys. Rev. D* **49**, 3033 (1994).
 - [8] T. Schäfer, D.T. Son, M.A. Stephanov, D. Toublan, and J.J.M. Verbaarschot, *Phys. Lett. B* **522**, 67 (2001).
 - [9] T. Brauner, *Symmetry* **2**, 609 (2010).
 - [10] T. Brauner, *Phys. Rev. D* **72**, 076002 (2005).
 - [11] I. Low and A. V. Manohar, *Phys. Rev. Lett.* **88**, 101602 (2002).
 - [12] To avoid misunderstanding, let us emphasize that every conserved charge of course generates a symmetry transformation on the phase space of the classical theory. Nevertheless, it may not be possible to obtain this charge as a consequence of an off-shell invariance of the action in the Lagrangian formalism. This is what we mean by saying that the charge is not of the Noether type.
 - [13] S. Weinberg, *The Quantum Theory of Fields* (Cambridge University Press, Cambridge, UK, 1995), Vol. 1.
 - [14] Y. Nambu, *J. Stat. Phys.* **115**, 7 (2004).
 - [15] V.P. Gusynin, V.A. Miransky, and I.A. Shovkovy, *Phys. Lett. B* **581**, 82 (2004).
 - [16] Y. Hama, T. Hatsuda, and S. Uchino, *Phys. Rev. D* **83**, 125009 (2011).
 - [17] V. Miransky and I. Shovkovy, *Phys. Rev. Lett.* **88**, 111601 (2002).
 - [18] S. Weinberg, *The Quantum Theory of Fields* (Cambridge University Press, Cambridge, UK, 1996), Vol. 2.
 - [19] The last step here makes use of the assumption that the adjoint representation matrices $R(\theta)$ are orthogonal, which in turn follows from the full antisymmetry of the structure constants of the Lie algebra. This can be achieved for any compact semisimple Lie algebra by a proper choice of basis.
 - [20] K. Rajagopal and A. Schmitt, *Phys. Rev. D* **73**, 045003 (2006).
 - [21] While the orthogonality is trivial in the case of a nonzero eigenvalue with multiplicity one, in general one has to be a bit careful. First, note that when $\lambda_k = 0$, $\mathcal{I}\mathbf{u}_k^*$ is orthogonal to $\mathbf{u}_k$ by construction, thanks to the antisymmetry of $\mathcal{I}$. Second, suppose that some eigenvalues are equal, say, $\lambda_{k+1} = \lambda_k$. We can then choose the eigenvector $\mathbf{u}_{k+1}$ to be orthogonal to both $\mathbf{u}_k$ and $\mathcal{I}\mathbf{u}_k^*$. This already guarantees that $\mathcal{I}\mathbf{u}_{k+1}^*$ will be orthogonal to all three of these vectors. One can thus construct an orthonormal basis by induction.
 - [22] A.W. Knap, *Lie Groups beyond an Introduction* (Birkhauser, Boston, 2005), 2nd ed.