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Propagators and matrix basis on noncommutative Minkowski space

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We describe an analytic continuation of the Euclidean Grosse-Wulkenhaar and Langmann-Szabo-Zarembo models which defines a one-parameter family of duality covariant noncommutative field theories interpolating between Euclidean and Minkowski space versions of these models, and provides an alternative regularization to the usual Feynman prescription. This regularization allows for a matrix model representation of the field theories in terms of a complex generalization of the usual basis of Landau wave functions. The corresponding propagators are calculated and identified with the Feynman propagators of the field theories. The regulated quantum field theories are shown to be UV/IR-duality covariant. We study the asymptotics of the regularized propagators in position and matrix space representations, and confirm that they generically possess a comparably good decay behavior as in the Euclidean case.

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I. INTRODUCTION AND SUMMARY

This paper is devoted to an in-depth study of the perturbative properties and renormalizability of noncommutative $\phi^{\star 4}$-type scalar field theories on real vector spaces subjected to a Moyal deformation. The vast majority of the literature on this subject has been devoted to Euclidean quantum field theory. The most prominent feature of these models is the notorious mixing of ultraviolet and infrared modes, which renders the noncommutative $\phi^{\star n}$ field theories nonrenormalizable [1] (see e.g. [2] for a review). Grosse and Wulkenhaar demonstrated how to obtain field theories which are renormalizable to all orders in perturbation theory by extending the kinetic term of the $\phi^{\star 4}$ Lagrangian by an additional harmonic oscillator potential [3,4]. The Grosse-Wulkenhaar model also has vanishing beta-functions and its perturbation series is likely to be Borel summable [5]; in two dimensions this has been recently established in [6]. In four dimensions, the Euclidean Grosse-Wulkenhaar model is the first rigorous four-dimensional quantum field theory without unnatural cutoff which is expected to exist nonperturbatively and is not asymptotically free.

The continuation of the Euclidean models to noncommutative Minkowski space is presently an open problem which is plagued by both conceptual and technical difficulties. The original problems were unveiled in [7], where it was found that the standard perturbative expansion in terms of Feynman diagrams leads to a violation of unitarity if space and time do not commute. As subsequently pointed out in [8], this is due to the failure of Wick's theorem, which does not apply to nonlocal interactions in general. By using canonical quantization in the Hamiltonian framework involving the Dyson series and time-ordered products of the interaction Hamiltonian, the resulting field theory is still unitary but no longer equivalent to the Lagrangian formulation of the quantum field theory in the path integral framework. For models built on the Hamiltonian framework see e.g. [9–11]. Yet another inequivalent perturbative approach is based on the Yang-Feldman formalism [8,9], which also gives a unitary noncommutative quantum field theory on Minkowski space with timelike noncommutativity.[1]

The UV/IR mixing problem of the ordinary $\phi^{\star n}$ field theory is absent in the Hamiltonian framework to lowest orders, and it has long been an open question as to whether it exists at all. Only recently has UV/IR mixing been shown to still occur, albeit through a mechanism which is different from that of the Euclidean setting with modified Feynman rules [16]. It has also been shown that UV/IR mixing arises in the Yang-Feldman formalism [17].

Since the perturbative setups in the Hamiltonian and Yang-Feldman formalisms are quite complicated, it would be desirable to have an equivalent Euclidean path integral formalism which simplifies the combinatorial aspects of perturbation theory. However, the relationship between the Euclidean and Minkowski space theories when time and

[1]Still another approach is provided by the twist-deformation formalism for noncommutative quantum field theory on Minkowski space which is considered in [12–15]; here one first quantizes the classical field theory before deforming spacetime, and the free part of the quantum field theory also differs from its commutative counterpart.

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space do not commute is unclear. In [18] it has been shown that the Euclidean counterparts of the n-point functions for the Klein-Gordon theory on noncommutative Minkowski space in the Hamiltonian formalism are not those which follow from the standard Euclidean framework, but appear with on-shell twisting factors involving only on-shell momenta.

In this paper we will pursue the other direction of this correspondence, starting with specific field theories in Euclidean space. In order to find noncommutative field theories in hyperbolic signature which are free from UV/IR mixing, we construct Minkowski space counterparts of the Grosse-Wulkenhaar model and its generalizations known as the Langmann-Szabo-Zarembo (LSZ) models [19,20]. These models differ from the standard noncommutative $\phi^{\star 4}$ field theory by the introduction of an external background field into the kinetic term of the Lagrangian, making them covariant under the duality comprising Fourier transformation plus a rescaling of the fields [21]. This duality is believed to be related to the improved asymptotic behaviors of the propagators, which suppresses the UV/IR mixing. This UV/IR duality may thus be responsible for the renormalizability of these field theories. The Euclidean Grosse-Wulkenhaar and LSZ models are defined via path integral quantization, which leads to a violation of unitarity for the usual noncommutative field theories in Minkowski space. Here we will be interested in the renormalization properties of their hyperbolic counterparts; unitarity of these quantum field theories will be addressed elsewhere.

In Euclidean space the introduction of an external field has the useful additional effect that the corresponding wave operators have discrete spectra and the models can be analyzed with the help of a matrix basis for expansion of fields; the countably infinite set of eigenfunctions are the Landau wave functions which diagonalize the free parts of the action. This basis defines a mapping of the duality covariant field theories onto matrix models, which permits a simple and natural regularization of the field theories while maintaining duality manifestly at quantum level. In this way Grosse and Wulkenhaar were able to prove the renormalizability of their model to all orders of perturbation theory. In addition, it has been used to solve the LSZ model exactly and demonstrate the vanishing of the beta-function.

However, in passing to hyperbolic signature, the background field, which is a magnetic field in the Euclidean metric, now plays the role of an electric field. This yields a qualitative change due to the work done on the particles by the field. The electric field accelerates and splits virtual dipole pairs leading to pair production. This is reflected in the spectra of the wave operators, which now have a continuous part and are unbounded from below.

In [22] the perturbative expansion in terms of modified Feynman diagrams in the continuous eigenvalue representation has been investigated. At one-loop order unusual divergences arise which are very likely to be nonrenormalizable. Moreover, the retarded propagator in position space is no longer a tempered distribution in general.

In [23] a different approach has been investigated, where a set of resonance states has been used to expand the field theory in a discrete set of functions. In the following we will take yet another path, which is related to the resonance expansion found in [23], but which potentially avoids the associated technical problems. We will show that the Grosse-Wulkenhaar and LSZ models allow for well-defined analytic continuations to Minkowski space with the help of a special regularization, that we call the "ϑ regularization," which is a suitable replacement for Feynman's $\mathrm{i}\epsilon$ prescription.

This approach may also avoid the strange divergences found in [22]. These divergences come from squares of Dirac delta functions which arise from undetermined loop integrations. They are not ultraviolet divergences in the usual sense, as they occur before performing loop integrals, and they show up in every $\phi^{\star n}$ theory with $n \geq 3$ for graphs with an unbroken internal line. Using the ϑ regularization instead of the usual $\mathrm{i}\epsilon$ regularization, one gets a discrete spectrum instead of a continuous spectrum, leading to Kronecker delta-functions and sums rather than Dirac delta-distributions and integrals. This procedure renders these diagrams finite, and at the same time keeps the model duality covariant.

The outline of this paper is as follows. We will define duality covariant quantum field theories on Minkowski space based on the work [23,24], and describe some of their renormalization properties. In order to employ an expansion of the action functionals in terms of the resonance states found in [23], we will regularize the models such that the resonances turn into genuine eigenfunctions of the regularized wave operators; this new matrix basis and the corresponding matrix model representations of the LSZ and Grosse-Wulkenhaar models on Minkowski space are described in detail in Sec. II and III. These wave operators are related by Weyl-Wigner correspondence to the complex harmonic oscillator Hamiltonian, which interpolates between the ordinary and inverted harmonic oscillator Hamiltonians, and thus between the Euclidean and Minkowski space theories; this unifies both theories into a one-parameter family of duality covariant noncommutative quantum field theories. The Feynman graphs are analytic continuations of the Euclidean diagrams. We show that this regularized matrix basis is a bi-orthogonal system whose linear span is the space of square-integrable functions. At the quantum level and in the limit of vanishing electric background, this regularization turns into the usual $\mathrm{i}\epsilon$ prescription. For the special case of a Klein-Gordon theory in a constant external electric field, where the various propagators are known, we recalculate in Sec. IV the propagator using the complex matrix basis and verify that the regularization leads to Feynman propagators. This confirms the equivalence to the iϵ prescription, and demonstrates that the ϑ regularization is also connected to causality of the quantum field theory. Using the ϑ regularization, we show in Sec. V that a cutoff can be introduced that renders the duality covariant field theories finite at every order of perturbation theory and at the same time imposes duality covariance manifestly. In Sec. VI we derive the propagators for the regularized models which include the Euclidean space propagators and the Minkowski space causal propagators as special cases; away from the hyperbolic point our propagators have a good decay behavior in all directions and are singular at coincident positions. The ϑ regularization turns out to improve their asymptotic behavior and may thus be crucial for the renormalization programme. However, due to the oscillatory behaviors of the occurring integrands in Minkowski space, the corresponding asymptotics are much more difficult to derive than in the Euclidean case. For a special case of the LSZ model we find that the exponential decay in the short Euclidean space variables ceases if one goes over to Minkowski space, but persists in a neighborhood of the hyperbolic point in the one-parameter family of field theories. The regularization thus gives a means to control the decay behavior of the propagators. The applicability of the matrix basis in this context, however, is still an open question; the detailed analysis of the removal of the matrix regularization will not be addressed in this paper. As we discuss in the following, the Minkowski limit of our one-parameter family is very singular; see [25] for a detailed analysis of some of the uncontrollable divergences which arise. In the following we will simply regard the ϑ regularized field theories as the appropriate well-defined analytic continuations of the Euclidean space models.

The derivations of propagators with the help of the matrix basis may be compared to calculations using other methods, such as Schwinger's proper time formalism [26], the "sum over solutions method" [27], or the eigenvalue method using the continuous eigenbasis [28]. Compared to the latter technique the matrix basis involves only polynomials and sums instead of complicated integral expressions, and thus brings along a huge simplification. In Sec. IV B the causal propagator for a massive complex scalar field in four dimensions in the background of a constant electric field in the ϑ regularization is computed. As a further demonstration of how the matrix basis can be applied, we calculate the one-loop effective action of the Klein-Gordon theory in a background electric field (see Appendix B). Finally, in Lemma D.13 we demonstrate that the ϑ regularization can likewise be used regulate the standard mass-shell singularities in the Feynman propagator for the free Klein-Gordon theory. We propose that going beyond the case of a constant background field might be possible using our alternative regularization and the matrix basis, by perturbing varying field configurations around a uniform background. This might help in probing quantum electrodynamics in the nonperturbative regime (see e.g. [29–32]). We conclude that the matrix basis may serve as a powerful computational tool in simplifying some otherwise cumbersome calculations.

II. COVARIANT RELATIVISTIC NONCOMMUTATIVE FIELD THEORY

In this section we will introduce the duality covariant models in Minkowski space. We show that it is possible to construct a well-defined matrix model representation of the corresponding quantum field theories through a suitable regularization, that we call ϑ *regularization*, which is an alternative to the usual iϵ prescription. For this, we will use the Weyl-Wigner transformation to map the eigenvalue problem of the ϑ-regularized wave operators to that of the *complex* harmonic oscillator.

A. Formulation of the duality covariant models

We work in $D = 2n$ spacetime dimensions with metric of signature $(1, -1, \ldots, -1)$. Tensors will be labeled by Greek indices $\mu, \nu, \ldots$ ranging from 0 to $d = D - 1$. Throughout we use the Einstein summation convention. For simplicity we will denote the hyperbolic norm square of vectors $\mathsf{a} = (a^\mu)$ as

$$\| \mathsf{a}\|^2_{\mathrm{M}} = a_0^2 - a_1^2 - \cdots - a_d^2 = a_\mu a^\mu =: a_\mu^2. \quad (2.1)$$

Euclidean space dimensions are labeled by Latin indices $i, j, \ldots$ ranging from 1 to D, and norm squares of vectors $\mathsf{a} = (a^i)$ with respect to the D-dimensional Euclidean metric are denoted

$$\| \mathsf{a}\|^2_{\mathrm{E}} = a_1^2 + \cdots + a_D^2 = a_i a^i =: a_i^2. \quad (2.2)$$

Position vectors are denoted $\boldsymbol{x} = (x^\mu)$, with derivatives $\partial_\mu := \frac{\partial}{\partial x^\mu}$; in two dimensions we often write $\boldsymbol{x} = (t, x)$. The dual pairing between a covariant vector $\boldsymbol{x} = (x^\mu) \in \mathbb{R}^D$ and a contravariant vector $\boldsymbol{k} = (k_\mu) \in (\mathbb{R}^D)^*$ is written $\boldsymbol{k} \cdot \boldsymbol{x} = k_\mu x^\mu$.

The LSZ model is a complex $\phi^{\star 4}_D$ theory defined by the action

$$\mathcal{S}_{\mathsf{LSZ}} = \mathcal{S}_0 + \mathcal{S}_{\mathsf{int}} \quad (2.3)$$

with

$$\mathcal{S}_0 = \int d^D\boldsymbol{x}\, \phi^*(\boldsymbol{x})(\sigma K_\mu^2 + (1-\sigma)\tilde{K}_\mu^2 - \mu^2)\phi(\boldsymbol{x}), \quad (2.4)$$

$$\mathcal{S}_{\mathsf{int}} = -g \int d^D\boldsymbol{x} (\alpha(\phi^* \star_\Theta \phi \star_\Theta \phi^* \star_\Theta \phi)(\boldsymbol{x}) + \beta(\phi^* \star_\Theta \phi^* \star_\Theta \phi \star_\Theta \phi)(\boldsymbol{x})), \quad (2.5)$$

where $\sigma \in [0, 1]$, $\alpha, \beta \in \mathbb{R}_+ := [0, \infty)$, and $\mu^2, g > 0$ are the mass and coupling parameters. The generalized momentum operators K_μ and generalized dual momentum operators $\tilde{\mathsf{K}}_\mu$ are given by

$$\mathsf{K}_\mu = \mathrm{i}\partial_\mu - F_{\mu\nu}x^\nu \quad \text{and} \quad \tilde{\mathsf{K}}_\mu = \mathrm{i}\partial_\mu + F_{\mu\nu}x^\nu, \tag{2.6}$$

and they obey the commutation relations

$$\begin{aligned} &[\mathsf{K}_\mu, \mathsf{K}_\nu] = 2\mathrm{i}F_{\mu\nu}, \\ &[\tilde{\mathsf{K}}_\mu, \tilde{\mathsf{K}}_\nu] = -2\mathrm{i}F_{\mu\nu} \quad \text{and} \quad [\mathsf{K}_\mu, \tilde{\mathsf{K}}_\nu] = 0. \end{aligned} \tag{2.7}$$

The star product of arbitrary Schwartz functions $f(\boldsymbol{x})$, $g(\boldsymbol{x}) \in \mathcal{S}(\mathbb{R}^D)$ is given by

$$\begin{aligned} (f \star_\Theta g)(\boldsymbol{x}) &:= \frac{1}{\pi^D|\det\Theta|}\int d^D\boldsymbol{y} \\ &\quad \times \int d^D\boldsymbol{z} f(\boldsymbol{x}+\boldsymbol{y})g(\boldsymbol{x}+\boldsymbol{z})\mathrm{e}^{-2\mathrm{i}\boldsymbol{y}\cdot\Theta^{-1}\boldsymbol{z}} \\ &= (g^* \star_\Theta f^*)(\boldsymbol{x}) \end{aligned} \tag{2.8}$$

with respect to a constant, real-valued, antisymmetric and nondegenerate $D \times D$ deformation matrix Θ.

The coordinate system is chosen such that Θ takes the canonical skew-diagonal form

$$\Theta = (\Theta^{\mu\nu}) = \begin{pmatrix} 0 & \theta_0 & & & & & & 0 \\ -\theta_0 & 0 & & & & & & \\ & & 0 & \theta_1 & & & & \\ & & -\theta_1 & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & & 0 & \theta_{n-1} \\ 0 & & & & & & -\theta_{n-1} & 0 \end{pmatrix} \tag{2.9}$$

with $\theta_k > 0$ for $k = 0, 1, \ldots, \frac{D}{2} - 1$. The constant electromagnetic field strength tensor $F_{\mu\nu}$ is likewise given by

$$F = (F_{\mu\nu}) = \begin{pmatrix} 0 & E & & & & & & 0 \\ -E & 0 & & & & & & \\ & & 0 & B_1 & & & & \\ & & -B_1 & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & & 0 & B_{n-1} \\ 0 & & & & & & -B_{n-1} & 0 \end{pmatrix}, \tag{2.10}$$

with $E, B_k > 0$ and

$$E\theta_0 = B_k\theta_k = 2\Omega \tag{2.11}$$

for $k = 1, \ldots, \frac{D}{2} - 1$ and $0 < \Omega \leq 1$. We will sometimes regard Θ and F as invertible linear maps Θ: $(\mathbb{R}^D)^* \to \mathbb{R}^D$ and F: $\mathbb{R}^D \to (\mathbb{R}^D)^*$. The LSZ models with $\sigma = 1$, where only the generalized momentum operator K_μ appears, are called "critical" models, while those with $\Omega = 1$, where the field theory is invariant under the UV/IR duality between position and momentum space representations, are called "self-dual" models.

We will solve the eigenvalue equation for the wave operator and relate it to that of the Euclidean case. For this, we note that the $2n$-dimensional wave operators break up into n blocks with

$$\mathsf{K}_\mu^2 = \sum_{k=0}^{n-1}(\mathsf{P}_\mu^2)_k \quad \text{and} \quad \tilde{\mathsf{K}}_\mu^2 = \sum_{k=0}^{n-1}(\tilde{\mathsf{P}}_\mu^2)_k. \tag{2.12}$$

The operators

$$\begin{aligned} (\mathsf{P}_\mu^2)_k &= (\partial_{2k}^2 + \partial_{2k+1}^2) + 2\mathrm{i}B_k(x^{2k+1}\partial_{2k} - x^{2k}\partial_{2k+1}) \\ &\quad - B_k^2(x_{2k}^2 + x_{2k+1}^2), \\ (\tilde{\mathsf{P}}_\mu^2)_k &= (\partial_{2k}^2 + \partial_{2k+1}^2) - 2\mathrm{i}B_k(x^{2k+1}\partial_{2k} - x^{2k}\partial_{2k+1}) \\ &\quad - B_k^2(x_{2k}^2 + x_{2k+1}^2) \end{aligned} \tag{2.13}$$

for $k = 1, \ldots, n-1$ act on two-dimensional Euclidean Klein-Gordon fields in a constant external magnetic background of field strengths $\pm 2B_k$, respectively, while the operators

$$\begin{aligned} (\mathsf{P}_\mu^2)_0 &= -(\partial_0^2 - \partial_1^2) - 2\mathrm{i}E(x^1\partial_0 + x^0\partial_1) - E^2(x_0^2 - x_1^2), \\ (\tilde{\mathsf{P}}_\mu^2)_0 &= -(\partial_0^2 - \partial_1^2) + 2\mathrm{i}E(x^1\partial_0 + x^0\partial_1) - E^2(x_0^2 - x_1^2) \end{aligned} \tag{2.14}$$

act on $1 + 1$-dimensional Klein-Gordon fields in a constant electric background with field strengths $\pm 2E$, respectively. Since all component operators $(\mathsf{P}_\mu^2)_k$ and $(\tilde{\mathsf{P}}_\mu^2)_k$ for $k = 0, 1, \ldots, n-1$ mutually commute, the diagonalization of the full wave operators amounts to diagonalizing each of their two-dimensional blocks independently.

As is well known the spectra of the operators (2.13) are discrete with corresponding eigenfunctions the Landau wave functions $f_{mn}^{(B)}$ satisfying

$$\begin{aligned} (\mathsf{P}_\mu^2)_k f_{mn}^{(B_k)}(\boldsymbol{x}_k) &= -4B_k(m + \tfrac{1}{2})f_{mn}^{(B_k)}(\boldsymbol{x}_k), \\ (\tilde{P}_\mu^2)_k f_{mn}^{(B_k)}(\boldsymbol{x}_k) &= -4B_k(n + \tfrac{1}{2})f_{mn}^{(B_k)}(\boldsymbol{x}_k) \end{aligned} \tag{2.15}$$

for $m, n \in \mathbb{N}_0$, where we write $\boldsymbol{x}_k = (x^{2k}, x^{2k+1}) \in \mathbb{R}^2$ for $k = 0, 1, \ldots, n-1$. These functions are Wigner transformations of tensor products of harmonic oscillator number basis states

$$f_{mn}^{(B)}(x, y) = \mathsf{W}[|m\rangle\langle n|](x, y) \tag{2.16}$$

of frequency Ω, where, in two dimensions with deformation parameter $\Theta^{01} = \theta$, the Wigner distribution function of a compact operator $\hat{\boldsymbol{\rho}}$ on Fock space is the Schwartz function on $\mathbb{R}^2$ given by [2]

$$\mathsf{W}[\hat{\boldsymbol{\rho}}](x, y) = \int dk \mathrm{e}^{\mathrm{i}ky/\theta}\langle x + k/2|\hat{\boldsymbol{\rho}}|x - k/2\rangle. \tag{2.17}$$

On the other hand, the spectra of the operators (2.14) are continuous, with corresponding eigenfunctions given by Wigner transformations of tensor products of parabolic cylinder functions [22,23], denoted χ_{pq} with $p, q \in \mathbb{R}$, which solve the eigenvalue equations

$$(\mathsf{P}^2_\mu)_0 \chi_{pq}(\boldsymbol{x}_0) = 4Ep\chi_{pq}(\boldsymbol{x}_0) \quad \text{and} \quad (\tilde{\mathsf{P}}^2_\mu)_0 \chi_{pq}(\boldsymbol{x}_0) = 4Eq\chi_{pq}(\boldsymbol{x}_0). \tag{2.18}$$

For generic σ the free part of the LSZ action (2.4) can be rewritten as

$$\mathcal{S}_0 = \int d^D\boldsymbol{x}\,\phi^*(\boldsymbol{x})(\mathsf{K}^2_\mu|_{F\to\tilde{F}} + \Omega^2\tilde{x}^2_\mu - \mu^2)\phi(\boldsymbol{x}) \tag{2.19}$$

with $\tilde{F} = (2\sigma - 1)F = (2\sigma - 1)(F_{\mu\nu})$ and $\tilde{x}_\mu = 2\Theta^{-1}_{\mu\nu}x^\mu$. The free action thus describes a massive complex scalar field coupled to a constant electromagnetic background and in an oscillator potential proportional to $\Omega^2\tilde{x}^2_\mu$. The Grosse-Wulkenhaar model in $D = 2n$ spacetime dimensions is the LSZ model for $\sigma = \frac{1}{2}$ and $\alpha = \beta = \frac{1}{2}$ with real scalar fields. The action is thus

$$\mathcal{S}_{\mathsf{GW}} = \int d^D\boldsymbol{x}\frac{1}{2}\phi(\boldsymbol{x})(-\partial^2_\mu + \Omega^2\tilde{x}^2_\mu - \mu^2)\phi(\boldsymbol{x}) - g\int d^D\boldsymbol{x}(\phi \star_\Theta \phi \star_\Theta \phi \star_\Theta \phi)(\boldsymbol{x}). \tag{2.20}$$

The D-dimensional wave operator again reduces to a sum of $n - 1$ Euclidean wave operators plus a two-dimensional wave operator in Minkowski signature

$$\tfrac{1}{2}(\mathsf{P}^2_\mu)_0 + \tfrac{1}{2}(\tilde{\mathsf{P}}^2_\mu)_0 - \mu^2 = -(\partial^2_0 - \partial^2_1) - \Omega^2(x^2_0 - x^2_1) - \mu^2 \tag{2.21}$$

with frequency $\Omega = E\theta_0/2$. The main difference, besides the hyperbolic signature, is an extra minus sign in front of the Ω term. The corresponding wave operator is given by the Hamiltonian of a harmonic oscillator with imaginary frequency, known as the inverted harmonic oscillator.

The corresponding models on Euclidean space are defined in terms of the wave operator $\sigma\mathsf{K}^2_i + (1 - \sigma)\tilde{\mathsf{K}}^2_i + \mu^2$, which also split up into n blocks made up of the operators

$$\mathsf{K}^2_i = \sum_{k=1}^n(\mathsf{P}^2_i)_k \quad \text{and} \quad \tilde{\mathsf{K}}^2_i = \sum_{k=1}^n(\tilde{\mathsf{P}}^2_i)_k. \tag{2.22}$$

After relabelling of coordinates, one can relate $(\mathsf{P}^2_i)_k = -(\mathsf{P}^2_\mu)_{k-1}$ and $(\tilde{\mathsf{P}}^2_i)_k = -(\tilde{\mathsf{P}}^2_\mu)_{k-1}$ for $k = 2, \ldots, n$, whereas $(\mathsf{P}^2_i)_1$ and $(\tilde{\mathsf{P}}^2_i)_1$ are of the same form as (2.13). Thus in contrast to the mixed discrete and continuous spectrum of the hyperbolic space wave operator, the Euclidean case deals with purely discrete spectrum. This situation is responsible for the powerful application of the matrix model representation of Grosse and Wulkenhaar [3,4,33].

The duality covariant field theories involve two parameters Θ and F. In the commutative limit $\Theta = 0$, one recovers the field theory for an interacting scalar field in a constant electromagnetic background; in Sec. IV B and Appendix B we demonstrate how to reproduce the known standard results in the literature using the novel regularization we propose below. In the vanishing background limit $F = 0$, we recover the usual ϕ^4 theories on noncommutative Minkowski space together with their UV/IR mixing problems as discussed in Sec. I; in Lemma D.13 we illustrate how our regularization is applicable in this case as well. Neither of these two limits possess duality covariance. In the self-dual limit $F = (2\Theta)^{-1}$ the field theory is duality invariant; the matrix representation we obtain below at the self-dual point makes no sense in the limit $F = 0$.

B. Spectral decomposition and ϑ regularization

The external electromagnetic background will be treated by considering all terms quadratic in the fields as being part of the free action. Then the path integral quantization gives the usual (modified) Feynman diagrams but with the dressed propagator for the scalar field moving in this background. It is a feature of most field theories defined on hyperbolic space that there is more than one propagator, i.e. a distribution whose kernel $\Delta(\boldsymbol{x}, \boldsymbol{y})$ solves the partial differential equation $\mathsf{D}_{\boldsymbol{x}}\Delta(\boldsymbol{x}, \boldsymbol{y}) = \delta(\boldsymbol{x} - \boldsymbol{y})$ with $\mathsf{D}_{\boldsymbol{x}}$ the wave operator of the field theory. This is due to the occurrence of zero eigenvalues of $\mathsf{D}_{\boldsymbol{x}}$, which prevents the naive inversion of the operator to give a propagator. It is therefore necessary to impose further conditions so as to make the solution of this problem unique. This may be done either by imposing boundary conditions, by postulating a spectral representation, or by extending the wave operator so as to make the solution of the partial differential equation unique.

For the ordinary scalar field theory, the $\mathrm{i}\epsilon$ prescription is a method to single out a specific propagator, namely, the Feynman propagator. In the commutative field theories, this prescription enhances the action by an additional term $\mathrm{i}\epsilon\int\phi^2$ which for $\epsilon > 0$ ensures the required asymptotic damping of the integrand in the partition function at $|\phi| \to \infty$ (rather than an oscillatory behavior), and at the same time regularizes the singularity of the free propagator and furthermore imposes causality. In this particular case it is also the infinitesimal version of the Wick rotation to Euclidean space $t \mapsto \mathrm{e}^{\mathrm{i}\epsilon}t$.

However, in our case the field theories defined on the two different spacetimes are not related by this *ordinary* Wick rotation—it has to be accompanied by an additional transformation $E \mapsto \pm\mathrm{i}B$. This is not surprising, since the model can be viewed as a field theory on a curved nonstationary spacetime, for which this is a generic feature [34]. Another characteristic of those field theories is that the multitude of different equivalent definitions of the Feynman propagator is resolved [35]; we return to this point in Sec. IV A. Since we are interested in an analytic continuation of the Euclidean space models in a path integral framework, it is the propagator we obtain by this transformation that we are concerned with. The extra transformation of the magnetic field strength is also in harmony with the fact that in order to ensure the commutation relation $[x^0, x^i] = \mathrm{i}\Theta^{0i}$ for both Euclidean and Minkowski space, the deformation parameter Θ^{0i} has to transform accordingly to compensate the phase coming from the Wick rotation. For the duality invariant field theories, i.e. at the self-dual point $\Omega = 1$, the deformation matrix is proportional to the field strength tensor, which in turn implies a rotation of the field strength.

In [23] it was shown that the actions of the Minkowski and Euclidean space wave operators, $(\mathsf{P}^2_\mu)_0$ and $(\mathsf{P}^2_i)_1$, can be represented as a star-product with a classical Hamiltonian at the self-dual point $\Omega = 1$. To compare to the Euclidean version, we have to identify $B = E$ and the ordered coordinate pairs $\boldsymbol{x} = (x^1, x^2) = (t, x)$ to find[2]

$$(\mathsf{P}^2_i)_1 f(\boldsymbol{x}) = E^2(x^2 + t^2) \star_{2/E} f(\boldsymbol{x}) \quad \text{and} \\ (\mathsf{P}^2_\mu)_0 f(\boldsymbol{x}) = E^2(x^2 - t^2) \star_{2/E} f(\boldsymbol{x}) \tag{2.23}$$

and likewise

$$(\tilde{\mathsf{P}}^2_i)_1 f(\boldsymbol{x}) = f(\boldsymbol{x}) \star_{2/E} E^2(x^2 + t^2) \quad \text{and} \\ (\tilde{\mathsf{P}}^2_\mu)_0 f(\boldsymbol{x}) = f(\boldsymbol{x}) \star_{2/E} E^2(x^2 - t^2), \tag{2.24}$$

which can be verified by explicitly writing out the individual terms

$$x^2 \star_\theta f(\boldsymbol{x}) = (x^2 - \mathrm{i}\theta x \partial_t - \tfrac{1}{4}\theta^2 \partial_t^2) f(\boldsymbol{x}), \\ t^2 \star_\theta f(\boldsymbol{x}) = (t^2 + \mathrm{i}\theta t \partial_x - \tfrac{1}{4}\theta^2 \partial_x^2) f(\boldsymbol{x}). \tag{2.25}$$

Consequently, there is a one-parameter family of operators which continuously interpolates between the Euclidean and the Minkowski space wave operators. They are denoted by $\mathsf{P}^2(\vartheta)$ and $\tilde{\mathsf{P}}^2(\vartheta)$, with $\vartheta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, and are defined by

$$\mathsf{P}^2(\vartheta) = \mathrm{e}^{\mathrm{i}\vartheta}(\cos(\vartheta)(\mathsf{P}^2_i)_1 - \mathrm{i}\sin(\vartheta)(\mathsf{P}^2_\mu)_0), \\ \tilde{\mathsf{P}}^2(\vartheta) = \mathrm{e}^{\mathrm{i}\vartheta}(\cos(\vartheta)(\tilde{\mathsf{P}}^2_i)_1 - \mathrm{i}\sin(\vartheta)(\tilde{\mathsf{P}}^2_\mu)_0). \tag{2.26}$$

Using (2.25) one easily checks

$$\mathsf{P}^2(\vartheta) f(\boldsymbol{x}) = H(\vartheta) \star_{2/E} f(\boldsymbol{x}) \quad \text{and} \\ \tilde{\mathsf{P}}^2(\vartheta) f(\boldsymbol{x}) = f(\boldsymbol{x}) \star_{2/E} H(\vartheta), \tag{2.27}$$

where

$$H(\vartheta) := E^2(x^2 + \mathrm{e}^{2\mathrm{i}\vartheta} t^2). \tag{2.28}$$

The wave operators (2.26) relate both signatures, with $\vartheta = 0$ corresponding to Euclidean signature and $\vartheta = \pm\frac{\pi}{2}$ to hyperbolic signature. In the limit $E \to 0$ one easily verifies that this regularization reduces to the $\mathrm{i}\epsilon$ prescription for the usual Klein-Gordon operator. Hence it can be regarded as a generalization of the $\mathrm{i}\epsilon$ prescription to the case with an external electromagnetic field. To distinguish both schemes we will call this alternative prescription the *ϑ regularization*.

Using the Weyl-Wigner correspondence, the eigenvalue equations of our original operators can be represented on the space of Weyl symbols by

$$\mathsf{P}^2(\vartheta) f^{(E_\vartheta)}_{mn}(\boldsymbol{x}) = \mathsf{W}[\hat{\mathsf{H}}(\vartheta)\hat{\mathbf{f}}^{(E_\vartheta)}_{mn}](\boldsymbol{x}) = \lambda^{(E_\vartheta)}_m f^{(E_\vartheta)}_{mn}(\boldsymbol{x}), \\ \tilde{\mathsf{P}}^2(\vartheta) f^{(E_\vartheta)}_{mn}(\boldsymbol{x}) = \mathsf{W}[\hat{\mathbf{f}}^{(E_\vartheta)}_{mn}\hat{\mathsf{H}}(\vartheta)](\boldsymbol{x}) = \lambda^{(E_\vartheta)}_n f^{(E_\vartheta)}_{mn}(\boldsymbol{x}), \tag{2.29}$$

with $\hat{\mathbf{f}}^{(E_\vartheta)}_{mn} = \mathsf{W}^{-1}[f^{(E_\vartheta)}_{mn}]$ and the Weyl symbol

$$\hat{\mathsf{H}}(\vartheta) = \tfrac{1}{2}(\mathsf{W}^{-1}[\sqrt{2}Ex]^2 + \mathrm{e}^{2\mathrm{i}\vartheta}\mathsf{W}^{-1}[\sqrt{2}Et]^2) \\ = \tfrac{1}{2}(\hat{\mathbf{p}}^2 + \mathrm{e}^{2\mathrm{i}\vartheta}\hat{\mathbf{q}}^2). \tag{2.30}$$

The eigenvalues will turn out to depend on E and ϑ only through the combination

$$E_\vartheta := E\mathrm{e}^{\mathrm{i}\vartheta}, \tag{2.31}$$

which explains the notation. The Hermitian symbols $\mathsf{W}^{-1}[\sqrt{2}Ex] = \hat{\mathbf{p}}$ and $\mathsf{W}^{-1}[\sqrt{2}Et] = \hat{\mathbf{q}}$ obey the commutation relation of the Heisenberg algebra

$$[\hat{\mathbf{q}}, \hat{\mathbf{p}}] = 2E^2\mathsf{W}^{-1}[t \star_{2/E} x - x \star_{2/E} t] = 4\mathrm{i}E, \tag{2.32}$$

where we used the fundamental property

$$\mathsf{W}[\hat{\mathbf{f}}] \star_{2/E} \mathsf{W}[\hat{\mathbf{g}}] = \mathsf{W}[\hat{\mathbf{f}}\,\hat{\mathbf{g}}] \tag{2.33}$$

of the Weyl-Wigner correspondence.

The operators $\hat{\mathsf{H}}(\vartheta)$ for $\vartheta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ are known as *complex harmonic oscillator Hamiltonians*. When defined on $\mathcal{S}(\mathbb{R})$ they have a *discrete* spectrum given by

$$\sigma(\hat{\mathsf{H}}(\vartheta)) = \{\lambda^{(E_\vartheta)}_m = 4E_\vartheta(m + \tfrac{1}{2}) | m \in \mathbb{N}_0\}. \tag{2.34}$$

The spectrum of $\hat{\mathsf{H}}(\vartheta)$ and its eigenoperators $\hat{\mathbf{f}}^{(E_\vartheta)}_{mn}$ will be investigated in Sec. III. The simultaneous eigenfunctions of $\mathsf{P}^2(\vartheta)$ and $\tilde{\mathsf{P}}^2(\vartheta)$ are given by the Wigner transformation

$$f^{(E_\vartheta)}_{mn}(\boldsymbol{x}) = \mathsf{W}[\hat{\mathbf{f}}^{(E\vartheta)}_{mn}](\boldsymbol{x}). \tag{2.35}$$

These functions are calculated explicitly in Appendix A. They have an exponential decay for $x, t \to \infty$ and are Schwartz functions. This is in contrast to the functions one obtains in the limit $\vartheta \to \pm\frac{\pi}{2}$, which are tempered distributions and have been found in [23]. As shown in [25], Appendix D, these functions span a dense subspace of $L^2(\mathbb{R}^2)$, thus every square-integrable function on $\mathbb{R}^2$ can be expanded pointwise into functions lying in the span. In addition, they fulfill the important projector property

$$f^{(E_\vartheta)}_{mn}(\boldsymbol{x}) \star_{2/E} f^{(E_\vartheta)}_{kl}(\boldsymbol{x}) = \sqrt{\frac{E}{4\pi}}\delta_{nk} f^{(E_\vartheta)}_{ml}(\boldsymbol{x}). \tag{2.36}$$

The generalization of the ϑ-regularized wave operators to $D = 2n$ dimensions are given by

[2]These identities are taken in the multiplier algebra corresponding to the Schwartz space $\mathcal{S}(\mathbb{R}^2)$.

$$\mathsf{K}^2(\vartheta) = \mathrm{e}^{\mathrm{i}\vartheta}(\cos(\vartheta)\mathsf{K}_i^2 - \mathrm{i}\sin(\vartheta)\mathsf{K}_\mu^2) \quad \text{and} \\ \tilde{\mathsf{K}}^2(\vartheta) = \mathrm{e}^{\mathrm{i}\vartheta}(\cos(\vartheta)\tilde{\mathsf{K}}_i^2 - \mathrm{i}\sin(\vartheta)\tilde{\mathsf{K}}_\mu^2), \tag{2.37}$$

which again split up into two-dimensional wave operators defined by (2.12), (2.13), (2.14), and (2.22). In $D = 2n$ dimensions the components $(\mathsf{P}_i^2)_k$ and $(\mathsf{P}_\mu^2)_{k-1}$, and likewise $(\tilde{\mathsf{P}}_i^2)_k$ and $(\tilde{\mathsf{P}}_\mu^2)_{k-1}$, differ only by a sign for $k = 2, \ldots, n$ up to a relabelling of the coordinates. We thus have

$$\mathsf{K}^2(\vartheta) = \mathsf{P}^2(\vartheta) + \mathrm{e}^{2\mathrm{i}\vartheta}\sum_{k=2}^n (\mathsf{P}_i^2)_k \quad \text{and} \\ \tilde{\mathsf{K}}^2(\vartheta) = \tilde{\mathsf{P}}^2(\vartheta) + \mathrm{e}^{2\mathrm{i}\vartheta}\sum_{k=2}^n (\tilde{\mathsf{P}}_i^2)_k \tag{2.38}$$

according to (2.26). The eigenfunctions of the operators $(\mathsf{P}_i^2)_k$ and $(\tilde{\mathsf{P}}_i^2)_k$ are just Landau wave functions. What remains is to find the eigenfunctions of the remaining parts of the wave operators. Since all of these two-dimensional differential operators commute, the total eigenfunctions will be a product of the individual two-dimensional eigenfunctions.

C. Perturbative quantum field theory

Without loss of generality we will choose in the following always $\vartheta > 0$ and define $\vartheta = \frac{\pi}{2} - \kappa > 0$ for small $\kappa > 0$. Denoting

$$(\mathsf{K}_\mu^2 - \mu^2)_\kappa := \mathrm{e}^{\mathrm{i}\kappa}\mathsf{K}^2\Big(\frac{\pi}{2} - \kappa\Big) - \mathrm{e}^{-\mathrm{i}\kappa}\mu^2 \quad \text{and} \\ (\tilde{\mathsf{K}}_\mu^2 - \mu^2)_\kappa := \mathrm{e}^{\mathrm{i}\kappa}\tilde{\mathsf{K}}^2\Big(\frac{\pi}{2} - \kappa\Big) - \mathrm{e}^{-\mathrm{i}\kappa}\mu^2 \tag{2.39}$$

the regularized LSZ model is defined by the classical action

$$\mathcal{S}_{\mathsf{LSZ}}^{(\kappa)} = \int d^D\boldsymbol{x}\,\phi^*(\boldsymbol{x})(\sigma(\mathsf{K}_\mu^2 - \mu^2)_\kappa + (1-\sigma)(\tilde{\mathsf{K}}_\mu^2 - \mu^2)_\kappa) \\ \times \phi(\boldsymbol{x}) - g\Big(\alpha\int d^D\boldsymbol{x}(\phi^* \star_\Theta \phi \star_\Theta \phi^* \star_\Theta \phi)(\boldsymbol{x}) \\ + \beta\int d^D\boldsymbol{x}(\phi^* \star_\Theta \phi^* \star_\Theta \phi \star_\Theta \phi)(\boldsymbol{x})\Big), \tag{2.40}$$

and the regularized Grosse-Wulkenhaar model by

$$\mathcal{S}_{\mathsf{GW}}^{(\kappa)} = \int d^D\boldsymbol{x}\frac{1}{2}\phi(\boldsymbol{x})\Big(\frac{1}{2}(\mathsf{K}_\mu^2 - \mu^2)_\kappa + \frac{1}{2}(\tilde{\mathsf{K}}_\mu^2 - \mu^2)_\kappa\Big)\phi(\boldsymbol{x}) \\ - g\int d^D\boldsymbol{x}(\phi \star_\Theta \phi \star_\Theta \phi \star_\Theta \phi)(\boldsymbol{x}). \tag{2.41}$$

The Minkowski space duality covariant noncommutative quantum field theory of the regularized LSZ model is defined by the partition function, which is the generating functional obtained by adding external sources $J(\boldsymbol{x})$ and $J^*(\boldsymbol{x})$ to the action (2.40) with

$$Z[J, J^*] = \lim_{\kappa\to 0^+}\mathcal{N}\int\mathcal{D}\phi\mathcal{D}\phi^*\exp\Big(\mathrm{i}\mathcal{S}_{\mathsf{LSZ}}^{(\kappa)} \\ + \int d^D\boldsymbol{x}J^*(\boldsymbol{x})\phi(\boldsymbol{x}) + \int d^D\boldsymbol{x}\phi^*(\boldsymbol{x})J(\boldsymbol{x})\Big) \tag{2.42}$$

and analogously for the real Grosse-Wulkenhaar model, where $\mathcal{N}$ is a normalization constant. The precise definition of the path integral measure is not required to determine $Z[J, J^*]$ perturbatively, since only the vanishing of the integrand for $|\phi| \to \infty$ is needed to find a functional differential equation for the partition function via formal integration by parts in field space; this is ensured by the ϑ regularization. The "free" partition function $Z_0[J, J^*] := Z[J, J^*]|_{g=0}$ is then the solution of

$$\lim_{\kappa\to 0^+}(\sigma(\mathsf{K}_\mu^2 - \mu^2)_\kappa + (1-\sigma)(\tilde{\mathsf{K}}_\mu^2 - \mu^2)_\kappa)\frac{\delta Z_0[J, J^*]}{\delta J^*(\boldsymbol{x})} \\ = \mathrm{i}J(\boldsymbol{x})Z_0[J, J^*], \\ \lim_{\kappa\to 0^+}(\sigma(\mathsf{K}_\mu^2 - \mu^2)_\kappa + (1-\sigma)(\tilde{\mathsf{K}}_\mu^2 - \mu^2)_\kappa)\frac{\delta Z_0[J, J^*]}{\delta J(\boldsymbol{x})} \\ = \mathrm{i}J^*(\boldsymbol{x})Z_0[J, J^*] \tag{2.43}$$

given by

$$Z_0[J, J^*] = \lim_{\kappa\to 0^+}\exp\Big(\mathrm{i}\int d^D\boldsymbol{x}\int d^D\boldsymbol{y}J^*(\boldsymbol{x})\Delta^{(\kappa,\sigma)}(\boldsymbol{x}, \boldsymbol{y})J(\boldsymbol{y})\Big), \tag{2.44}$$

with $\Delta^{(\kappa,\sigma)}(\boldsymbol{x}, \boldsymbol{y})$ the regularized dressed propagator defined through the equation

$$(\sigma(\mathsf{K}_\mu^2 - \mu^2)_\kappa + (1-\sigma)(\tilde{\mathsf{K}}_\mu^2 - \mu^2)_\kappa)\Delta^{(\kappa,\sigma)}(\boldsymbol{x}, \boldsymbol{y}) \\ = \delta(\boldsymbol{x} - \boldsymbol{y}). \tag{2.45}$$

An explicit expression for $\Delta^{(\kappa,\sigma)}(\boldsymbol{x}, \boldsymbol{y})$ will be derived in Sec. VI A. The full interacting quantum field theory is given by the partition function

$$Z[J, J^*] = \lim_{\kappa\to 0^+}\mathcal{N}\exp\Big[\mathrm{i}\mathcal{S}_{\mathsf{int}}\Big(\frac{\delta}{\delta J^*}, \frac{\delta}{\delta J}\Big)\Big] \\ \times\exp\Big(\mathrm{i}\int d^D\boldsymbol{x}\int d^D\boldsymbol{y}J^*(\boldsymbol{x})\Delta^{(\kappa,\sigma)}(\boldsymbol{x}, \boldsymbol{y})J(\boldsymbol{y})\Big) \tag{2.46}$$

leading to a perturbative expansion in Feynman diagrams corresponding to the interaction part $\mathcal{S}_{\mathsf{int}}[\phi, \phi^*]$ of the action (2.5) and the dressed propagator $\Delta^{(\kappa,\sigma)}(\boldsymbol{x}, \boldsymbol{y})$. The corresponding Green's functions contain products of distributions and have to be regularized; this is described in Sec. V. For real scalar fields we get

$$Z[J] = \lim_{\kappa\to 0^+}\mathcal{N}\exp\Big[\mathrm{i}\mathcal{S}_{\mathsf{int}}\Big(\frac{\delta}{\delta J}\Big)\Big]\exp\Big(\frac{\mathrm{i}}{2} \\ \times\int d^D\boldsymbol{x}\int d^D\boldsymbol{y}J(\boldsymbol{x})\Delta^{(\kappa)}(\boldsymbol{x}, \boldsymbol{y})J(\boldsymbol{y})\Big), \tag{2.47}$$

with regularized dressed propagator $\Delta^{(\kappa)}(\boldsymbol{x}, \boldsymbol{y})$ given by (2.45) for $\sigma = \frac{1}{2}$.

In the following we will construct dynamical matrix models representing the regularized duality covariant quantum field theories. In fact, our construction has killed two birds with one stone. First of all, we have regularized the wave operator such that no zero eigenvalues occur, so we can invert it to get a unique propagator. On the other hand, we have also found a discrete spectrum for the regulated wave operator which, together with the projector relation (2.36), is needed to define a proper matrix model formulation of the quantum field theory. This is in marked contrast to the usual $\mathrm{i}\epsilon$ prescription that gives a regulated wave operator $\mathsf{D}_{\boldsymbol{x}}^{(\epsilon)} = \sigma \mathsf{K}_\mu^2 + (1-\sigma)\tilde{\mathsf{K}}_\mu^2 - \mu^2 + \mathrm{i}\epsilon$, which simply amounts to adding the constant $\mathrm{i}\epsilon$ to the continuous spectrum of the electric part of the wave operators, but otherwise leaves its continuous character unaltered. A perturbative quantum field theory amenable for the continuous basis approach with functions χ_{pq} is analyzed in this way in [22].

In the following we shall address the following questions:

(i) What is the interpretation of the sign of ϑ?
(ii) Which propagator do we obtain in the limit $\vartheta \to \pm\frac{\pi}{2}$?
(iii) Is it possible to prove duality covariance for $\vartheta \neq \pm\frac{\pi}{2}$ at the quantum level?
(iv) Are the Feynman diagrams finite in the limit $\vartheta \to \pm\frac{\pi}{2}$?

In Sec. IV we shall argue that, like the $\mathrm{i}\epsilon$ prescription at $E = 0$, the ϑ regularization is related to causality, and flipping the sign of the regulator corresponds to interchanging the Feynman (causal) propagator with the Dyson (anticausal) propagator, i.e. interchanging the particle and antiparticle descriptions in a background electric field. The proof of duality covariance at the classical level follows easily from the Euclidean and Minkowski space proofs given in [21,23], respectively. The spacetime metric plays no role in the proof, which relies solely on Fourier expansion techniques and Gaussian integrations; in Sec. V we will calculate the partition function of duality covariant noncommutative quantum field theories, and hand in the proof of the duality invariance at quantum level.

III. DYNAMICAL MATRIX MODELS

In this section we will work out the matrix model representations of the perturbative quantum field theories defined above. In Sec. II we used the Weyl-Wigner transformation to map the eigenvalue problem for the ϑ-regularized wave operators to that of the complex harmonic oscillator. Below we investigate its spectrum and eigenfunctions, and construct the appropriate generalizations of the Landau wave functions. Using their Fock space representation, we will finally arrive at the matrix model representation for the two-dimensional classical models and their corresponding quantum field theories. The generalization to higher dimensions is also presented.

A. Complex harmonic oscillator wave functions

We will begin by investigating the spectrum of the complex harmonic oscillator Hamiltonian $\hat{\mathsf{H}}(\vartheta)$ defined in (2.30), with commutation relation (2.32) and positive real frequency $E \in \mathbb{R}_+$, which turns out to have a discrete spectrum (2.34) resembling the usual harmonic oscillator spectrum rotated into the complex plane by a phase factor $\mathrm{e}^{\mathrm{i}\vartheta}$. Since $\hat{\mathsf{q}} = \mathsf{W}^{-1}[\sqrt{2}Et]$, it is natural to work in the representation defined by the eigenbasis of $\hat{\mathsf{q}}$ such that

$$\langle q'|\hat{\mathsf{q}}|q\rangle = \sqrt{2}Eq\langle q'|q\rangle \quad \text{and} \quad \langle q'|\hat{\mathsf{p}}|q\rangle = -\mathrm{i}\sqrt{8}\partial_q\langle q'|q\rangle \tag{3.1}$$

and thus

$$\langle q'|\hat{\mathsf{H}}(\vartheta)|q\rangle = (-4\partial_q^2 + E_\vartheta^2 q^2)\langle q'|q\rangle \tag{3.2}$$

with the condensed notation (2.31). Firstly, note that the eigenvalue differential equation

$$(-4\partial_q^2 + E_\vartheta^2 q^2) f_m^{(E_\vartheta)}(q) = 4E_\vartheta\Big(m + \frac{1}{2}\Big) f_m^{(E_\vartheta)}(q) \tag{3.3}$$

is fulfilled for complex values E_ϑ if $f_m^{(E_\vartheta)}(q)$ represent the usual Hermite oscillator wave functions $f_m^{(E)}(q)$ with the complex frequency E_ϑ substituted for E,

$$f_m^{(E_\vartheta)}(q) = \bigg(\frac{\sqrt{E_\vartheta}}{2^m m! \sqrt{2\pi}}\bigg)^{1/2} \mathrm{e}^{-E_\vartheta q^2/4} H_m(\sqrt{E_\vartheta/2}q), \tag{3.4}$$

where $H_m(z) = (-1)^m \mathrm{e}^{z^2}\partial_z^m \mathrm{e}^{-z^2}$ are the Hermite polynomials. These functions will be called complex harmonic oscillator wave functions, as a generalization of the harmonic oscillator wave functions to complex frequencies E_ϑ. They possess an exponential decay due to the Gaussian factor, and are thus Schwartz functions on $\mathbb{R}$ for $|\vartheta| < \frac{\pi}{2}$. We expect that by continuity, for $|\vartheta|$ small enough the eigenvalues of the complex harmonic oscillator Hamiltonian are given by the set (2.34); the values (2.34) are indeed the eigenvalues of $\hat{\mathsf{H}}(\vartheta)$ for $|\vartheta| < \frac{\pi}{2}$ [36].

The complex harmonic oscillator wave functions (3.4) are not orthogonal, and thus do not serve as a usual Hilbert space basis for $\mathcal{S}(\mathbb{R})$. But together with their complex conjugated functions and for $\Re\mathrm{e}(E_\vartheta) > 0$, they constitute a bi-orthogonal system with respect to the L^2-inner product $\langle - | - \rangle$. This means that the two sets of functions $(f_m^{(E_\vartheta)})_{m\in\mathbb{N}_0}$ and $(f_m^{(E_{-\vartheta})})_{m\in\mathbb{N}_0}$ with nonzero E_ϑ and $\Re\mathrm{e}(E_\vartheta) > 0$ fulfill

$$\langle f_n^{(E_{-\vartheta})}|f_m^{(E_\vartheta)}\rangle = \int dq f_n^{(E_\vartheta)}(q) f_m^{(E_\vartheta)}(q) = \delta_{nm}, \tag{3.5}$$

which follows immediately from the orthogonality of the Hermite functions on $\mathbb{R}$ by a deformation of the integration contour to a straight line from $-\infty e^{i\vartheta}$ to $+\infty e^{i\vartheta}$. This rotation is possible due to the Gaussian factor $e^{-E_\vartheta q^2/2}$ in the integrand, ensuring an exponential decay for $\Re e(E_\vartheta) > 0$. In addition their linear span is dense in $L^2(\mathbb{R})$, which means that every square-integrable function on $\mathbb{R}$ can be approximated pointwise by a linear combination of these functions; the proof can be found in [25], Appendix D. But the series which occur are not convergent in the L^2 norm and thus do not build a Riesz basis [36,37].

To ensure the applicability of this basis to arbitrary quantum field theories, however, one has also to be able to deal with scalar products and (tempered) distributions. The problem of uniform convergence for $|\vartheta| \leq \frac{\pi}{2}$ might be circumvented by considering a smaller space than the Schwartz space, like the space of smooth functions with compact support, or by considering the Sturm-Liouville problem for the complex harmonic oscillator Hamiltonian on a finite interval $[-L, L]$ in $\mathbb{R}$; the expansion on $\mathcal{S}(\mathbb{R})$ might then be defined in some limiting procedure. The applicability of the Gel'fand-Shilov space of type $\mathcal{S}^\alpha_\alpha(\mathbb{R}) \subset \mathcal{S}(\mathbb{R})$ with $\alpha = \frac{1}{2}$ as an appropriate dense subspace of fields on $\mathbb{R}$ is discussed in [25], Appendix C, see also [38,39]; but the question of which precise spaces of functions this complex oscillator basis is applicable in a distributional sense is still an open problem.

Although it would be desirable to have a general rule which tells us for which functions the matrix basis is applicable, for a given field theory it suffices to derive the asymptotics of the matrix space representation of the corresponding propagator in order to ensure the convergence of the sums in Feynman diagrams; this is investigated in Sec. VI, but at present it is an open issue. In the following we will use the matrix basis to derive the propagators of the various field theories and find that they coincide with the position space propagators in all cases for which results are already known in the literature. In Appendix B the one-loop effective action of the Klein-Gordon theory in a constant background electric field is calculated with the help of the matrix basis, and shown to also coincide with the known results. By picking up the regularization scheme imposed on the position space propagator in the Euclidean case, which effectively cuts off the matrix element summations at some finite rank N, the occurring Feynman diagrams of the ϑ-regularized field theories are well defined and duality covariant. Whether or not new divergences arise in the limit $N \to \infty$ remains to be investigated.

B. Complex Landau wave functions

We will now construct the complex Landau wave functions $f^{(E_\vartheta)}_{mn}$, defined by (2.35), through Wigner distribution of the tensor product of two complex oscillator wave functions $f^{(E_\vartheta)}_m$. We will also derive a "ladder operator" type construction, which allows us to obtain the matrix model representation of the duality covariant field theories. For the moment we set $\theta = 2/E$.

We will first relate the ordinary and complex harmonic oscillator wave functions using complex scaling methods. Introducing the Hermitian scaling operator

$$\hat{\mathsf{V}}(\vartheta) = \exp\left(-\frac{\vartheta}{2E}(\hat{\mathsf{p}}\,\hat{\mathsf{q}} + \hat{\mathsf{q}}\,\hat{\mathsf{p}})\right) \tag{3.6}$$

we see that

$$\begin{aligned} \hat{\mathsf{V}}(\vartheta)\hat{\mathsf{q}}\,\hat{\mathsf{V}}(\vartheta)^{-1} &= e^{i\vartheta/2}\hat{\mathsf{q}} \quad \text{and} \\ \hat{\mathsf{V}}(\vartheta)\hat{\mathsf{p}}\,\hat{\mathsf{V}}(\vartheta)^{-1} &= e^{-i\vartheta/2}\hat{\mathsf{p}}. \end{aligned} \tag{3.7}$$

The complex harmonic oscillator Hamiltonian (2.30) is thus related to the ordinary oscillator Hamiltonian by

$$\hat{\mathsf{H}}(\vartheta) = e^{i\vartheta}\hat{\mathsf{V}}(\vartheta)\tfrac{1}{2}(\hat{\mathsf{p}}^2 + \hat{\mathsf{q}}^2)\hat{\mathsf{V}}(\vartheta)^{-1}, \tag{3.8}$$

while the complex eigenfunctions can now easily be obtained from the orthonormal oscillator number basis states $|m\rangle$, $m \in \mathbb{N}_0$, where $\langle m|n\rangle = \delta_{mn}$ and $\langle q|m\rangle = f^{(E)}_m(q)$ are the ordinary harmonic oscillator wave functions

$$f^{(E)}_m(q) = \left(\frac{\sqrt{E}}{2^m m!\sqrt{2\pi}}\right)^{1/2} e^{-Eq^2/4} H_m(\sqrt{E/2}\,q). \tag{3.9}$$

By noting that

$$\begin{aligned} \hat{\mathsf{H}}(\vartheta)\hat{\mathsf{V}}(\vartheta)|m\rangle &= e^{i\vartheta}\hat{\mathsf{V}}(\vartheta)\hat{\mathsf{H}}(0)|m\rangle \\ &= e^{i\vartheta}4E(m + \tfrac{1}{2})\hat{\mathsf{V}}(\vartheta)|m\rangle, \end{aligned} \tag{3.10}$$

the corresponding eigenfunctions are related to the oscillator wave functions by

$$f^{(E_\vartheta)}_m(q) = \langle q|f^{(E_\vartheta)}_m\rangle := \langle q|\hat{\mathsf{V}}(\vartheta)|m\rangle = e^{i\vartheta/4} f^{(E)}_m(e^{i\vartheta/2}q). \tag{3.11}$$

Clearly the left/right eigenoperators of $\hat{\mathsf{H}}(\vartheta)$ are tensor products of the form

$$\hat{\mathsf{f}}^{(E_\vartheta)}_{mn} = \sqrt{\frac{E}{4\pi}}\hat{\mathsf{V}}(\vartheta)|m\rangle\langle n|\hat{\mathsf{V}}(-\vartheta), \tag{3.12}$$

and the complex Landau wave functions are thus given by

$$f^{(E_\vartheta)}_{mn}(\boldsymbol{x}) = \sqrt{\frac{E}{4\pi}}\mathsf{W}[\hat{\mathsf{V}}(\vartheta)|m\rangle\langle n|\hat{\mathsf{V}}(-\vartheta)](\boldsymbol{x}). \tag{3.13}$$

The normalization has been chosen such that

$$\int d^2\boldsymbol{x} f_{mn}^{(E_\vartheta)}(\boldsymbol{x}) = \sqrt{\frac{E}{4\pi}}\int dt\int dx\int dk\,\mathrm{e}^{\mathrm{i}Ekx/2}\langle t + k/2|\hat{\mathbf{V}}(\vartheta)|m\rangle\langle n|\hat{\mathbf{V}}(-\vartheta)|t-k/2\rangle = \sqrt{\frac{E}{4\pi}}\frac{4\pi}{E}\langle f_n^{(E_{-\vartheta})}|f_m^{(E_\vartheta)}\rangle = \sqrt{\frac{4\pi}{E}}\delta_{mn}, \tag{3.14}$$

where we have used the explicit representation for the Wigner transformation (2.17). From (2.17) we also see that complex conjugation yields

$$f_{mn}^{(E_\vartheta)}(\boldsymbol{x})^* = \sqrt{\frac{E}{4\pi}}\int dk\,\mathrm{e}^{-\mathrm{i}Ekx/2}\langle t+k/2|\hat{\mathbf{V}}(-\vartheta)|n\rangle \times\langle m|\hat{\mathbf{V}}(\vartheta)|t-k/2\rangle = f_{nm}^{(E_{-\vartheta})}(\boldsymbol{x}) \tag{3.15}$$

and the projector property takes the form

$$(f_{mn}^{(E_\vartheta)} \star_{2/E} f_{kl}^{(E_\vartheta)})(\boldsymbol{x}) = \frac{E}{4\pi}\mathsf{W}[\hat{\mathbf{V}}(\vartheta)|m\rangle\langle n|k\rangle\langle l|\hat{\mathbf{V}}(-\vartheta)](\boldsymbol{x}) = \sqrt{\frac{E}{4\pi}}\delta_{nk}f_{ml}^{(E_\vartheta)}(\boldsymbol{x}). \tag{3.16}$$

Together with the normalization condition this implies the bi-orthogonality of the system of complex Landau wave functions with respect to the L^2-inner product

$$\langle f_{mn}^{(E_\vartheta)}|f_{kl}^{(E_{-\vartheta})}\rangle = \int d^2\boldsymbol{x} f_{nm}^{(E_{-\vartheta})}(\boldsymbol{x})f_{kl}^{(E_{-\vartheta})}(\boldsymbol{x}) = \int d^2\boldsymbol{x}(f_{nm}^{(E_{-\vartheta})}\star_{2/E} f_{kl}^{(E_{-\vartheta})})(\boldsymbol{x}) = \sqrt{\frac{E}{4\pi}}\int d^2\boldsymbol{x}\,\delta_{mk}f_{nl}^{(E_{-\vartheta})}(\boldsymbol{x}) = \delta_{mk}\delta_{nl}. \tag{3.17}$$

The explicit expressions for the matrix basis functions are given by

Proposition 3.18. The complex Landau wave functions $f_{mn}^{(E_\vartheta)}(\boldsymbol{x})$ for $m, n\in\mathbb{N}_0$ are given by

$$f_{mn}^{(E_\vartheta)}(t,x) = (-1)^{\min(m,n)}\sqrt{\frac{E}{\pi}}\sqrt{\frac{\min(m!,n!)}{\max(m!,n!)}} \times E_\vartheta^{|m-n|/2}\mathrm{e}^{-E_\vartheta x_+^{(\vartheta)}x_-^{(\vartheta)}/2}(x_{-\mathrm{sgn}(m-n)}^{(\vartheta)})^{|m-n|} \times L_{\min(m,n)}^{|m-n|}(E_\vartheta x_+^{(\vartheta)}x_-^{(\vartheta)}), \tag{3.19}$$

where

$$x_\pm^{(\vartheta)} = t \pm \mathrm{i}\,\mathrm{e}^{-\mathrm{i}\vartheta}x \tag{3.20}$$

are complex light-cone coordinates and $L_n^k(z)$ are the associated Laguerre polynomials.

The proof of proposition 3.18 is found in Appendix A. Setting $\vartheta = 0$ this result coincides with the well-known expression for the Landau wave functions in the Euclidean case. The coordinates (3.20) continuously interpolate between the complex coordinates $t\pm\mathrm{i}x$ in the Euclidean case $\vartheta = 0$ and the light-cone coordinates $t\pm x$ in the hyperbolic case $\vartheta = \pm\frac{\pi}{2}$. Since

$$E_\vartheta x_+^{(\vartheta)}x_-^{(\vartheta)} = E(\mathrm{e}^{\mathrm{i}\vartheta}t^2 + \mathrm{e}^{-\mathrm{i}\vartheta}x^2) = E(\cos(\vartheta)(t^2+x^2) + \mathrm{i}\sin(\vartheta)(t^2-x^2)), \tag{3.21}$$

we see that similarly to $f_m^{(E_\vartheta)}$ these functions are Schwartz functions only for $|\vartheta| < \frac{\pi}{2}$; in particular, they belong to the Gel'fand-Shilov spaces $\mathcal{S}_\alpha^\alpha(\mathbb{R}^2)$ for all $\alpha\geq\frac{1}{2}$. At $\vartheta = \pm\frac{\pi}{2}$ they have a polynomial increase and are thus tempered distributions.

The Fock space representation of the harmonic oscillator wave functions has a counterpart in the complex scaled version, which will prove very useful in the explicit determination of the dynamical matrix models. For this, we note that

$$|f_n^{(E_\vartheta)}\rangle\langle f_m^{(E_{-\vartheta})}| = \hat{\mathbf{V}}(\vartheta)\frac{(\hat{\mathbf{a}}^\dagger)^m}{\sqrt{m!}}|0\rangle\langle 0|\frac{(\hat{\mathbf{a}})^n}{\sqrt{n!}}\hat{\mathbf{V}}(\vartheta)^{-1} = \frac{1}{\sqrt{m!n!}}(\hat{\mathbf{V}}(\vartheta)\hat{\mathbf{a}}^\dagger\hat{\mathbf{V}}(\vartheta)^{-1})^m|f_0^{(E_\vartheta)}\rangle \times\langle f_0^{(E_{-\vartheta})}|(\hat{\mathbf{V}}(\vartheta)\hat{\mathbf{a}}\,\hat{\mathbf{V}}(\vartheta)^{-1})^n \tag{3.22}$$

with $\hat{\mathbf{a}} = \frac{1}{\sqrt{8E}}(\hat{\mathbf{q}} + \mathrm{i}\hat{\boldsymbol{p}})$. We can use the relations (3.7) to get

$$\hat{\mathbf{V}}(\vartheta)\hat{\mathbf{a}}^\dagger\hat{\mathbf{V}}(\vartheta)^{-1} = \frac{1}{\sqrt{8E}}(\mathrm{e}^{\mathrm{i}\vartheta/2}\hat{\mathbf{q}} - \mathrm{i}\,\mathrm{e}^{-\mathrm{i}\vartheta/2}\hat{\mathbf{p}}),\quad \hat{\mathbf{V}}(\vartheta)\hat{\mathbf{a}}\,\hat{\mathbf{V}}(\vartheta)^{-1} = \frac{1}{\sqrt{8E}}(\mathrm{e}^{\mathrm{i}\vartheta/2}\hat{\mathbf{q}} + \mathrm{i}\,\mathrm{e}^{-\mathrm{i}\vartheta/2}\hat{\mathbf{p}}). \tag{3.23}$$

Since $\mathsf{W}^{-1}[\sqrt{2}Et] = \hat{\mathbf{q}}$ and $\mathsf{W}^{-1}[\sqrt{2}Ex] = \hat{\mathbf{p}}$ we find

$$\mathsf{W}[\hat{\mathbf{V}}(\vartheta)\hat{\mathbf{a}}^\dagger\hat{\mathbf{V}}(\vartheta)^{-1}] = \sqrt{\frac{E_\vartheta}{4}}x_-^{(\vartheta)}\quad\text{and}\quad \mathsf{W}[\hat{\mathbf{V}}(\vartheta)\hat{\mathbf{a}}\,\hat{\mathbf{V}}(\vartheta)^{-1}] = \sqrt{\frac{E_\vartheta}{4}}x_+^{(\vartheta)}, \tag{3.24}$$

where we used the complex light-cone coordinates (3.20). The corresponding derivatives are given by

$$\partial_\pm^{(\vartheta)} = \partial_t \mp \mathrm{i}\,\mathrm{e}^{\mathrm{i}\vartheta}\partial_x, \tag{3.25}$$

with $\partial_\pm^{(\vartheta)}x_\pm^{(\vartheta)} = 2$ and $\partial_\pm^{(\vartheta)}x_\mp^{(\vartheta)} = 0$. The matrix basis functions on $\mathbb{R}^2$ can now be obtained via the Weyl-Wigner correspondence

$$
\begin{aligned}
f_{mn}^{(E_\vartheta)}(\boldsymbol{x}) &= \sqrt{\frac{E}{4\pi}}\mathsf{W}[|f_m^{(E_\vartheta)}\rangle\langle f_n^{(E_{-\vartheta})}|](\boldsymbol{x}) \\
&= \frac{1}{\sqrt{m!n!}}\left(\sqrt{\frac{E_\vartheta}{4}}x_-^{(\vartheta)}\right)^{\star_{2/E}m} \star_{2/E} \sqrt{\frac{E}{4\pi}}\mathsf{W}[|f_0^{(E_\vartheta)}\rangle \\
&\quad \times \langle f_0^{(E_{-\vartheta})}|] \star_{2/E} \left(\sqrt{\frac{E_\vartheta}{4}}x_+^{(\vartheta)}\right)^{\star_{2/E}n}. \qquad (3.26)
\end{aligned}
$$

We define "ladder operators" through the equations[3]

$$
\begin{aligned}
&\left(\sqrt{\frac{E_\vartheta}{4}}x_-^{(\vartheta)}\right)\star_{2/E} g(\boldsymbol{x}) = a_{(E_\vartheta)}^+ g(\boldsymbol{x}) \quad \text{and} \\
&\left(\sqrt{\frac{E_\vartheta}{4}}x_+^{(\vartheta)}\right)\star_{2/E} g(\boldsymbol{x}) = a_{(E_\vartheta)}^- g(\boldsymbol{x}), \\
&g(\boldsymbol{x}) \star_{2/E}\left(\sqrt{\frac{E_\vartheta}{4}}x_+^{(\vartheta)}\right) = b_{(E_\vartheta)}^+ g(\boldsymbol{x}) \quad \text{and} \\
&g(\boldsymbol{x}) \star_{2/E}\left(\sqrt{\frac{E_\vartheta}{4}}x_-^{(\vartheta)}\right) = b_{(E_\vartheta)}^- g(\boldsymbol{x}).
\end{aligned} \qquad (3.27)
$$

The differential operators on the right-hand sides of these equations can most easily be obtained by expressing the star-product in terms of the complex light-cone coordinates. Inverting the relations (3.20) and (3.25), after a bit of algebra we find that the ladder operators are then given by

$$
\begin{aligned}
a_{(E_\vartheta)}^{\pm} &= \frac{1}{2}\left(\sqrt{E_\vartheta}x_{\mp}^{(\vartheta)} \mp \sqrt{\frac{1}{E_\vartheta}}\partial_{\pm}^{(\vartheta)}\right) \quad \text{and} \\
b_{(E_\vartheta)}^{\pm} &= \frac{1}{2}\left(\sqrt{E_\vartheta}x_{\pm}^{(\vartheta)} \mp \sqrt{\frac{1}{E_\vartheta}}\partial_{\mp}^{(\vartheta)}\right),
\end{aligned} \qquad (3.28)
$$

and that they fulfill the commutation relations

$$
[a_{(E_\vartheta)}^-, a_{(E_\vartheta)}^+] = 1 \quad \text{and} \quad [b_{(E_\vartheta)}^-, b_{(E_\vartheta)}^+] = 1, \qquad (3.29)
$$

with all other commutators equal to zero. As expected, we arrive at the usual Euclidean case from [20] when substituting E_ϑ by E.

The ground state wave function is determined by the differential equations

$$
a_{(E_\vartheta)}^- f_{00}^{(E_\vartheta)}(\boldsymbol{x}) = b_{(E_\vartheta)}^- f_{00}^{(E_\vartheta)}(\boldsymbol{x}) = 0 \qquad (3.30)
$$

plus the normalization condition (3.14) with $m = n = 0$, which has the solution

$$
f_{00}^{(E_\vartheta)}(\boldsymbol{x}) = \sqrt{\frac{E}{\pi}}\mathrm{e}^{-E_\vartheta x_+^{(\vartheta)} x_-^{(\vartheta)}/2}. \qquad (3.31)
$$

The wave functions $f_{mn}^{(E_\vartheta)}$ have the ladder operator representation

$$
f_{mn}^{(E_\vartheta)}(\boldsymbol{x}) = \frac{(a_{(E_\vartheta)}^+)^m}{\sqrt{m!}}\frac{(b_{(E_\vartheta)}^+)^n}{\sqrt{n!}} f_{00}^{(E_\vartheta)}(\boldsymbol{x}). \qquad (3.32)
$$

It immediately follows that

$$
\begin{aligned}
a_{(E_\vartheta)}^- f_{mn}^{(E_\vartheta)}(\boldsymbol{x}) &= \sqrt{m} f_{m-1,n}^{(E_\vartheta)}(\boldsymbol{x}) \quad \text{and} \\
a_{(E_\vartheta)}^+ f_{mn}^{(E_\vartheta)}(\boldsymbol{x}) &= \sqrt{m+1} f_{m+1,n}^{(E_\vartheta)}(\boldsymbol{x}), \\
b_{(E_\vartheta)}^- f_{mn}^{(E_\vartheta)}(\boldsymbol{x}) &= \sqrt{n} f_{m,n-1}^{(E_\vartheta)}(\boldsymbol{x}) \quad \text{and} \\
b_{(E_\vartheta)}^+ f_{mn}^{(E_\vartheta)}(\boldsymbol{x}) &= \sqrt{n+1} f_{m,n+1}^{(E_\vartheta)}(\boldsymbol{x}).
\end{aligned} \qquad (3.33)
$$

We will use these relations to obtain the desired matrix model representations. Note that, by [23], Lem. 5, the problem of the right test function space is the same as in the complex oscillator case of Sec. III A; we can relate the subspaces of Gel'fand-Shilov spaces $\mathcal{S}_\alpha^\alpha(\mathbb{R})$ to subspaces of $\mathcal{S}_\alpha^\alpha(\mathbb{R}^2)$ via Wigner transformation.

C. Matrix space representation

Using the Fock space representation of Sec. III B, we will now derive the matrix model representation of the classical regularized actions of the LSZ model. In the following we denote

$$
f_{mn}^\kappa := f_{mn}^{(2/\theta_{-\vartheta})} \qquad (3.34)
$$

with $\vartheta = \frac{\pi}{2} - \kappa$ and $\theta \neq 2/E$ in general; in this case the complex Landau wave functions diagonalize the interaction part of the action, but not necessarily the free part of the action.

We expand the scalar fields in terms of the complex Landau basis[4]

$$
\begin{aligned}
\phi(\boldsymbol{x}) &= \sum_{m,n=0}^{\infty} f_{mn}^\kappa(\boldsymbol{x})\phi_{mn}^\kappa \quad \text{and} \\
\phi(\boldsymbol{x})^* &= \sum_{m,n=0}^{\infty} f_{mn}^\kappa(\boldsymbol{x})\bar{\phi}_{mn}^\kappa,
\end{aligned} \qquad (3.35)
$$

where the complex expansion coefficients are given by

$$
\begin{aligned}
\phi_{mn}^\kappa &= \langle f_{mn}^{-\kappa}|\phi\rangle = \int d^2\boldsymbol{x} f_{nm}^\kappa(\boldsymbol{x})\phi(\boldsymbol{x}) \quad \text{and} \\
\bar{\phi}_{mn}^\kappa &= \langle f_{mn}^{-\kappa}|\phi^*\rangle = \int d^2\boldsymbol{x} f_{nm}^\kappa(\boldsymbol{x})\phi(\boldsymbol{x})^*
\end{aligned} \qquad (3.36)
$$

with $\bar{\phi}_{mn}^\kappa = (\phi_{nm}^{-\kappa})^*$. The free parts of the actions can be deduced from

Lemma 3.37. The ϑ-regularized wave operator of the 1 + 1-dimensional LSZ model in matrix space is given by

[3] Since $(a_{(E_\vartheta)}^+)^\dagger \neq a_{(E_\vartheta)}^-$ and $(b_{(E_\vartheta)}^+)^\dagger \neq b_{(E_\vartheta)}^-$ they are strictly speaking not ladder operators in the conventional sense, but we will nevertheless refer to them as such.

[4] This expansion is defined for L^2-functions in a limiting procedure which can be found in [25], Appendix D.

$$D^{(\kappa,\sigma)}_{mn;kl}=\Big(-\mathrm{e}^{-\mathrm{i}\kappa}\mu^2+2\mathrm{i}\frac{(1+\Omega^2)}{\theta}(m+n+1)+\frac{4\mathrm{i}\tilde{\Omega}}{\theta}(n-m)\Big)\delta_{ml}\delta_{nk}+2\mathrm{i}\frac{\Omega^2-1}{\theta}(\sqrt{nm}\delta_{m,l+1}\times\delta_{n,k+1}+\sqrt{(n+1)(m+1)}\delta_{m,l-1}\delta_{n,k-1}) \quad (3.38)$$

with frequencies $\Omega = E\theta/2$ and $\tilde{\Omega} = (2\sigma - 1)\Omega$.

Proof. The wave operator is defined in the matrix basis by

$$D^{(\kappa,\sigma)}_{mn;kl}=\int d^2\boldsymbol{x} f^{\kappa}_{mn}(\boldsymbol{x})\Big(\sigma\mathrm{e}^{\mathrm{i}\kappa}\mathsf{P}^2\Big(\frac{\pi}{2}-\kappa\Big)+(1-\sigma)\mathrm{e}^{\mathrm{i}\kappa}\tilde{\mathsf{P}}^2\Big(\frac{\pi}{2}-\kappa\Big)-\mathrm{e}^{-\mathrm{i}\kappa}\mu^2\Big)f^{\kappa}_{kl}(\boldsymbol{x}). \quad (3.39)$$

One has

$$\mathsf{P}^2(\vartheta)=\frac{\mathrm{e}^{\mathrm{i}\vartheta}}{2\theta}\Big((2+E\theta)^2\Big(a^{+}_{(2/\theta_\vartheta)}a^{-}_{(2/\theta_\vartheta)}+\frac{1}{2}\Big)+(2-E\theta)^2\Big(b^{+}_{(2/\theta_\vartheta)}b^{-}_{(2/\theta_\vartheta)}+\frac{1}{2}\Big)+(\theta^2E^2-4)\times(a^{+}_{(2/\theta_\vartheta)}b^{+}_{(2/\theta_\vartheta)}+a^{-}_{(2/\theta_\vartheta)}b^{-}_{(2/\theta_\vartheta)})\Big), \quad (3.40)$$

together with a similar expression for $\tilde{\mathsf{P}}^2(\vartheta)$ with $a^{\pm}_{(2/\theta_\vartheta)}$ and $b^{\pm}_{(2/\theta_\vartheta)}$ interchanged. These formulas are verified directly by inserting (3.28). The matrix space representation of the partial differential operators $\mathsf{P}^2(\vartheta)$ and $\tilde{\mathsf{P}}^2(\vartheta)$ away from the self-dual point can be obtained from (3.40) with the help of (3.33) to get

$$\mathsf{P}^2_{mn;kl}(\vartheta)=\frac{\mathrm{e}^{\mathrm{i}\vartheta}}{2\theta}\Big((2+E\theta)^2\Big(m+\frac{1}{2}\Big)\delta_{ml}\delta_{nk}+(2-E\theta)^2\times\Big(n+\frac{1}{2}\Big)\delta_{ml}\delta_{nk}+(\theta^2E^2-4)\times(\sqrt{nm}\delta_{m,l+1}\delta_{n,k+1}+\sqrt{(n+1)(m+1)}\delta_{m,l-1}\delta_{n,k-1})\Big) \quad (3.41)$$

and

$$\tilde{\mathsf{P}}^2_{mn;kl}(\vartheta)=\frac{\mathrm{e}^{\mathrm{i}\vartheta}}{2\theta}\Big((2+E\theta)^2\Big(n+\frac{1}{2}\Big)\delta_{ml}\delta_{nk}+(2-E\theta)^2\times\Big(m+\frac{1}{2}\Big)\delta_{ml}\delta_{nk}+(\theta^2E^2-4)\times(\sqrt{nm}\delta_{m,l+1}\delta_{n,k+1}+\sqrt{(n+1)(m+1)}\delta_{m,l-1}\delta_{n,k-1})\Big), \quad (3.42)$$

which can be combined to give (3.38).

For the LSZ interaction terms, we use the projector property (3.16) to get

$$f^{\kappa}_{m_1n_1}\star_\theta f^{\kappa}_{m_2n_2}\star_\theta f^{\kappa}_{m_3n_3}\star_\theta f^{\kappa}_{m_4n_4}=\frac{1}{(2\pi\theta)^{3/2}}\delta_{n_1m_2}\delta_{n_2m_3}\delta_{n_3m_4}f^{\kappa}_{m_1n_4}. \quad (3.43)$$

The regularized LSZ model in two-dimensional Minkowski space can then be represented in the matrix basis as

$$\mathcal{S}^{(\kappa)}_{\mathsf{LSZ}}=\sum_{m,n,k,l}\bar{\phi}^{\kappa}_{mn}D^{(\kappa,\sigma)}_{mn;kl}\phi^{\kappa}_{lk}-\frac{g}{2\pi\theta}\sum_{m,n,k,l}(\alpha\bar{\phi}^{\kappa}_{mn}\phi^{\kappa}_{nk}\bar{\phi}^{\kappa}_{kl}\phi^{\kappa}_{lm}+\beta\bar{\phi}^{\kappa}_{mn}\bar{\phi}^{\kappa}_{nk}\phi^{\kappa}_{kl}\phi^{\kappa}_{lm}). \quad (3.44)$$

As a one-matrix model with infinite complex matrices $\phi_\kappa = (\phi^{\kappa}_{mn})_{m,n\in\mathbb{N}_0}$ this representation reads

$$\mathcal{S}^{(\kappa)}_{\mathsf{LSZ}}=\frac{1}{2\theta}\mathrm{Tr}\Big(((2-\theta E)^2+8\sigma\theta E)\phi^{\dagger}_{-\kappa}\mathcal{E}\phi_\kappa+((2+\theta E)-8\sigma\theta E)\phi_\kappa\mathcal{E}\phi^{\dagger}_{-\kappa}+\mathrm{i}(\theta^2E^2-4)(\phi^{\dagger}_{-\kappa}\Gamma^{\dagger}\phi_\kappa\Gamma+\phi_\kappa\Gamma^{\dagger}\phi^{\dagger}_{-\kappa}\Gamma)-2\theta\mathrm{e}^{-\mathrm{i}\kappa}\mu^2\phi^{\dagger}_{-\kappa}\phi_\kappa-\frac{g}{2\pi\theta}(\alpha\phi^{\dagger}_{-\kappa}\phi_\kappa\phi^{\dagger}_{-\kappa}\phi_\kappa+\beta\phi^{\dagger}_{-\kappa}\phi^{\dagger}_{-\kappa}\phi_\kappa\phi_\kappa)\Big), \quad (3.45)$$

with the diagonal matrix

$$\mathcal{E}_{mn}=4\mathrm{i}(m+\tfrac{1}{2})\delta_{mn} \quad (3.46)$$

and the infinite shift matrix

$$\Gamma_{mn}=\sqrt{n-1}\delta_{m,n-1}. \quad (3.47)$$

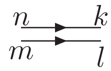
Using the perturbative solution (2.46), the duality covariant field theory can be defined perturbatively in the matrix basis by the partition function

$$Z[J,\bar{J}]=\lim_{\kappa\to0^+}\mathcal{N}\exp\Big(-\frac{\mathrm{i}\alpha g}{2\pi\theta}\sum_{m,n,k,l}\frac{\partial^4}{\partial J^{\kappa}_{ml}\partial\bar{J}^{\kappa}_{lk}\partial J^{\kappa}_{kn}\partial\bar{J}^{\kappa}_{nm}}\Big)\times\exp\Big(-\frac{\mathrm{i}\beta g}{2\pi\theta}\sum_{m,n,k,l}\frac{\partial^4}{\partial J^{\kappa}_{ml}\partial J^{\kappa}_{lk}\partial\bar{J}^{\kappa}_{kn}\partial\bar{J}^{\kappa}_{nm}}\Big)\times\exp\Big(\mathrm{i}\sum_{m,n,k,l}\bar{J}^{\kappa}_{mn}\Delta^{(\kappa,\sigma)}_{mn;kl}J^{\kappa}_{kl}\Big), \quad (3.48)$$

with J^{κ}_{mn}, $\bar{J}^{\kappa}_{mn}$ external sources in the matrix basis and the propagator $\Delta^{(\kappa,\sigma)}_{mn;kl}$ defined as the inverse of $D^{(\kappa,\sigma)}_{mn;kl}$,

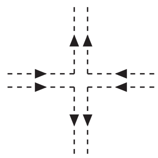
$$\sum_{k,l}D^{(\kappa,\sigma)}_{mn;kl}\Delta^{(\kappa,\sigma)}_{lk;sr}=\sum_{k,l}\Delta^{(\kappa,\sigma)}_{nm;lk}D^{(\kappa,\sigma)}_{kl;rs}=\delta_{mr}\delta_{ns}. \quad (3.49)$$

An explicit expression for the propagator $\Delta^{(\kappa,\sigma)}_{mn;kl}$ will be derived in Sec. VI B. The modified Feynman rules are presented in the double line formalism and are exactly as in the Euclidean case [2,4]. The generically nonlocal propagators are represented by double lines with orientation pointing from ϕ^* to ϕ as



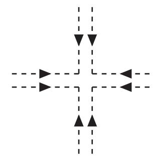
$$= \Delta^{(\kappa,\sigma)}_{nm;lk} \; .$$

The two interaction terms $\phi^* \star_\theta \phi \star_\theta \phi^* \star_\theta \phi$ and $\phi^* \star_\theta \phi^* \star_\theta \phi \star_\theta \phi$ are represented by different diagonal vertices given, respectively, by



$$= -\frac{\mathrm{i}\,g\,\alpha}{2\pi\,\theta}\,\delta_{mp}\,\delta_{nq}\,\delta_{kr}\,\delta_{ls}$$

and



$$= -\frac{\mathrm{i}\,g\,\beta}{2\pi\,\theta}\,\delta_{mp}\,\delta_{nq}\,\delta_{kr}\,\delta_{ls} \; .$$

Restricting to one of these interactions reduces the number of possible diagrams for the complex matrix model.

For real fields, one can apply Lemma 3.37 by setting $\sigma = \frac{1}{2}$ to immediately get

Lemma 3.50. The ϑ-regularized Grosse-Wulkenhaar wave operator in $1 + 1$ dimensions has the matrix space representation given by

$$\begin{aligned} D^{(\kappa)}_{mn;kl} &= \Big(-\mathrm{e}^{-\mathrm{i}\kappa}\mu^2 + 2\mathrm{i}\frac{\Omega^2+1}{\theta}(m+n+1)\Big)\delta_{ml}\delta_{nk} \\ &\quad + 2\mathrm{i}\frac{\Omega^2-1}{\theta}(\sqrt{nm}\,\delta_{m,l+1}\delta_{n,k+1} \\ &\quad + \sqrt{(n+1)(m+1)}\,\delta_{m,l-1}\delta_{n,k-1}) \end{aligned} \tag{3.51}$$

with frequency $\Omega = E\theta/2$.

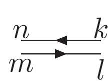
The Minkowski space Grosse-Wulkenhaar action in the matrix basis then reads

$$\mathcal{S}^{(\kappa)}_{\mathsf{GW}} = \sum_{m,n,k,l}\Big(\frac{1}{2}\phi^\kappa_{mn}D^{(\kappa)}_{mn;kl}\phi^\kappa_{kl} - \frac{g}{2\pi\theta}\phi^\kappa_{mn}\phi^\kappa_{nk}\phi^\kappa_{kl}\phi^\kappa_{lm}\Big) \tag{3.52}$$

with $\bar{\phi}^\kappa_{mn} = \phi^\kappa_{nm}$; it thus corresponds to a Hermitian one-matrix model. From the perturbative solution (2.47) the partition function in matrix space is given by

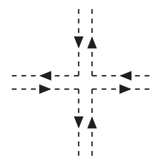
$$\begin{aligned} Z[J] &= \lim_{\kappa\to 0^+}\mathcal{N}\exp\Big(-\frac{\mathrm{i}g}{2\pi\theta}\sum_{m,n,k,l}\frac{\partial^4}{\partial J^\kappa_{ml}\partial J^\kappa_{lk}\partial J^\kappa_{kn}\partial J^\kappa_{nm}}\Big) \\ &\quad\times\exp\Big(\frac{\mathrm{i}}{2}\sum_{m,n,k,l}J^\kappa_{mn}\Delta^{(\kappa)}_{mn;kl}J^\kappa_{kl}\Big) \end{aligned} \tag{3.53}$$

where the propagator $\Delta^{(\kappa)}_{mn;kl}$ is the inverse of $D^{(\kappa)}_{mn;kl}$ and is represented by the unoriented double line



$$= \Delta^{(\kappa)}_{nm;lk} \; .$$

The vertex of the $\phi^{\star 4}$ interaction is given by the graph



$$= -\frac{\mathrm{i}\,g}{2\pi\,\theta}\,\delta_{mp}\,\delta_{nq}\,\delta_{kr}\,\delta_{ls} \; .$$

Since the vertex is unoriented there are as many diagrams as in the LSZ model with both parameters α and β turned on.

D. Generalization to higher dimensions

The spectra of both partial differential operators in (2.38) in generic $D = 2n$ dimensions are given by the set

$$\Big\{4E\mathrm{e}^{\mathrm{i}\vartheta}\Big(l_0+\frac{1}{2}\Big)+\sum_{k=1}^{n-1}4B_k\mathrm{e}^{2\mathrm{i}\vartheta}\times\Big(l_k+\frac{1}{2}\Big)\Big|l_0,l_1,\ldots,l_{n-1}\in\mathbb{N}_0\Big\}, \tag{3.54}$$

where the eigenfunctions are products of the complex Landau wave functions $f^{(E_\vartheta)}_{m_0n_0}$ from Sec. III B and the ordinary Landau wave functions $f^{(B_k)}_{m_kn_k}$, so that

$$f^{(\boldsymbol{F}_\vartheta)}_{\boldsymbol{mn}}(\boldsymbol{x}) := f^{(E_\vartheta)}_{m_0n_0}(\boldsymbol{x}_0)\prod_{k=1}^{n-1}f^{(B_k)}_{m_kn_k}(\boldsymbol{x}_k) \tag{3.55}$$

with $\boldsymbol{x}_k = (x^{2k}, x^{2k+1}) \in \mathbb{R}^{2k}$ for $k = 0, 1, \ldots, n-1$, $\boldsymbol{x} = (x^\mu) = (x^0, x^1, \ldots, x^d) \in \mathbb{R}^D$, $\boldsymbol{m} = (m_k)$, $\boldsymbol{n} = (n_k) \in \mathbb{N}_0^n$, and $\boldsymbol{F}_\vartheta = (E_\vartheta, B_1, \ldots, B_{n-1}) \in \mathbb{C}_+ \times \mathbb{R}_+^{n-1}$ where $\mathbb{C}_+ := \{z \in \mathbb{C} | \Re\mathrm{e}(z) \geq 0\}$. The star-product of two multi-dimensional complex Landau wave functions (3.55) with respect to the deformation matrix (2.9) decomposes into star-products of Landau wave functions depending on $\boldsymbol{x}_k$ for $k = 0, 1, \ldots, n-1$. If in addition $E = 2/\theta_0$ and $B_k = 2/\theta_k$ for $k = 1, \ldots, n-1$, then

$$(f^{(\boldsymbol{F}_\vartheta)}_{\boldsymbol{mn}} \star_\Theta f^{(\boldsymbol{F}_\vartheta)}_{\boldsymbol{kl}})(\boldsymbol{x}) = \frac{1}{\det(2\pi\Theta)^{1/4}}\delta_{\boldsymbol{nk}}f^{(\boldsymbol{F}_\vartheta)}_{\boldsymbol{ml}}(\boldsymbol{x}) \tag{3.56}$$

with $\delta_{\boldsymbol{mn}} = \prod_k \delta_{m_kn_k}$.

The generalization of the matrix model representation to higher spacetime dimensions is now straightforward.

To conform with our previous conventions we again set $\vartheta = \frac{\pi}{2} - \kappa > 0$ and use the notation

$$f^{\kappa}_{\boldsymbol{mn}}(\boldsymbol{x}) = f^{(2/(\theta_0)-\vartheta)}_{m_0n_0}(\boldsymbol{x}_0)\prod_{k=1}^{n-1} f^{(2/\theta_k)}_{m_kn_k}(\boldsymbol{x}_k). \tag{3.57}$$

The functions $f^{\kappa}_{\boldsymbol{mn}}$ are arranged so as to diagonalize the interaction part but not necessarily the free part of the action. The scalar fields on $\mathbb{R}^D$ are expanded in the complex Landau basis

$$\phi(\boldsymbol{x}) = \sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{N}_0^n} f^{\kappa}_{\boldsymbol{mn}}(\boldsymbol{x})\phi^{\kappa}_{\boldsymbol{mn}} \quad \text{and} \quad \phi(\boldsymbol{x})^* = \sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{N}_0^n} f^{\kappa}_{\boldsymbol{mn}}(\boldsymbol{x})\bar{\phi}^{\kappa}_{\boldsymbol{mn}}, \tag{3.58}$$

where the complex expansion coefficients are given by

$$\phi^{\kappa}_{\boldsymbol{mn}} = \langle f^{-\kappa}_{\boldsymbol{mn}}|\phi\rangle = \int d^D\boldsymbol{x} f^{\kappa}_{\boldsymbol{nm}}(\boldsymbol{x})\phi(\boldsymbol{x}) \quad \text{and} \quad \bar{\phi}^{\kappa}_{\boldsymbol{mn}} = \langle f^{-\kappa}_{\boldsymbol{mn}}|\phi^*\rangle = \int d^D\boldsymbol{x} f^{\kappa}_{\boldsymbol{nm}}(\boldsymbol{x})\phi(\boldsymbol{x})^*. \tag{3.59}$$

The matrix space representation of the LSZ model in $D = 2n$ dimensions away from the self-dual point can be obtained by comparing the operators (2.38) with their two-dimensional constituents, and their matrix representations given by (3.41) and (3.42) together with their Euclidean counterparts for $\vartheta = 0$ [20]. The matrix space LSZ wave operator is thus the sum of the two-dimensional Minkowski space wave operator given by (3.38) plus $n-1$ copies of the massless Euclidean wave operator for $\vartheta = 0$ [20] times the factor $-e^{-i\kappa}$. Noting that the massless LSZ wave operators in Euclidean and Minkowski space differ only by a factor of the imaginary unit i, we can write

$$D^{(\kappa,\sigma)}_{\boldsymbol{mn};\boldsymbol{kl}} = i\mathcal{D}^{0(\sigma)}_{m_0n_0;k_0l_0} - e^{-i\kappa}\sum_{i=1}^{n-1}\mathcal{D}^{i(\sigma)}_{m_in_i;k_il_i} - e^{-i\kappa}\mu^2\delta_{\boldsymbol{ml}}\delta_{\boldsymbol{nk}}, \tag{3.60}$$

where $\boldsymbol{m} = (m_0, m_1, \ldots, m_n)$, $\boldsymbol{n} = (n_0, n_1, \ldots, n_n)$, $\boldsymbol{k} = (k_0, k_1, \ldots, k_n)$, $\boldsymbol{l} = (l_0, l_1, \ldots, l_n) \in \mathbb{N}_0^n$ and $\mathcal{D}^{j(\sigma)}_{mn;kl}$ are the two-dimensional massless Euclidean LSZ matrix space wave operators

$$\mathcal{D}^{j(\sigma)}_{mn;kl} = \Big(2\frac{\Omega^2+1}{\theta_j}(m+n+1) + \frac{4\tilde{\Omega}}{\theta_j}(n-m)\Big)\delta_{ml}\delta_{nk} + 2\frac{\Omega^2-1}{\theta_j}(\sqrt{nm}\,\delta_{m,l+1}\delta_{n,k+1} + \sqrt{(n+1)(m+1)}\,\delta_{m,l-1}\delta_{n,k-1}), \tag{3.61}$$

with frequencies $\Omega = E\theta_0/2 = B_i\theta_i/2$ and $\tilde{\Omega} = (2\sigma - 1)\Omega$. The $2n$-dimensional regularized LSZ action is then given in the usual matrix space form

$$\mathcal{S}^{(\kappa)}_{\mathsf{LSZ}} = \sum_{\boldsymbol{m},\boldsymbol{n},\boldsymbol{k},\boldsymbol{l}} \bar{\phi}^{\kappa}_{\boldsymbol{mn}} D^{(\kappa,\sigma)}_{\boldsymbol{mn};\boldsymbol{kl}}\phi^{\kappa}_{\boldsymbol{lk}} - \frac{g}{\sqrt{\det(2\pi\Theta)}}\sum_{\boldsymbol{m},\boldsymbol{n},\boldsymbol{k},\boldsymbol{l}}(\alpha\bar{\phi}^{\kappa}_{\boldsymbol{mn}}\phi^{\kappa}_{\boldsymbol{nk}}\bar{\phi}^{\kappa}_{\boldsymbol{kl}}\phi^{\kappa}_{\boldsymbol{lm}} + \beta\bar{\phi}^{\kappa}_{\boldsymbol{mn}}\bar{\phi}^{\kappa}_{\boldsymbol{nk}}\phi^{\kappa}_{\boldsymbol{kl}}\phi^{\kappa}_{\boldsymbol{lm}}). \tag{3.62}$$

Every other result of this section (and ensuing ones) can now formally be generalized to higher dimensions by substituting multi-indices $\boldsymbol{m}, \boldsymbol{n}, \ldots \in \mathbb{N}_0^n$ for the usual matrix indices $m, n, \ldots \in \mathbb{N}_0$.

IV. CAUSALITY

In this section we will treat the problem of determining the causal propagator of the duality covariant field theories in Minkowski space. Problematic for this issue is the lack of time translation invariance, which allows for transitions that violate energy conservation; this manifests itself in an instability of the vacuum with respect to production of particle-antiparticle pairs. We review how the standard techniques must be altered to take care of these features. We will then examine the corresponding propagators which one obtains by removing the ϑ regularization.

A. Causal propagators

The way we chose the propagator of the Minkowski space theory was to find the analytically continued propagator of the Euclidean case. This also brought about the possibility of finding a matrix space representation. In the following we will show that the propagator which is prescribed by the ϑ regularization is the causal propagator of the duality covariant quantum field theory. For this, we will first review the connection between regularization and propagators by describing the eigenvalue representation, and the related operator extension method.

The free partition function $Z_0[J, J^*]$ of a complex scalar field theory is defined as the vacuum-to-vacuum amplitude

$$Z_0[J, J^*] = \langle\Omega, \text{out}|\Omega, \text{in}\rangle^{J,J^*}, \tag{4.1}$$

where $|\Omega, \text{in}\rangle$ and $\langle\Omega, \text{out}|$ are the vacuum states at time instances t_{in} and t_{out} of the quantum field theory defined by the free action $\mathcal{S}_0[\phi, \phi^*]$ in the presence of the sources J and J^*. Using Schwinger's action principle, one can show that causality implies the relation

$$\frac{\delta^2 \log Z_0[J, J^*]}{\delta J^*(\boldsymbol{x})\delta J(\boldsymbol{y})}\bigg|_{J=J^*=0} = \frac{\langle 0, \text{out}|T(\hat{\phi}(\boldsymbol{x})\hat{\phi}(\boldsymbol{y})^\dagger)|0, \text{in}\rangle}{\langle 0, \text{out}|0, \text{in}\rangle}, \tag{4.2}$$

where $\hat{\phi}(\boldsymbol{x})$ is the second quantized field operator, T denotes time-ordering with respect to the time variables x^0 and y^0, and $|0, \text{in}\rangle$ and $\langle 0, \text{out}|$ are the in- and out- vacuum states for $J = J^* = 0$ which in the presence of further interactions are taken in the interaction picture where the field operators satisfy the equations of motion obtained from varying $\mathcal{S}_0[\phi, \phi^*]$. For field theories which allow for spontaneous particle-antiparticle pair production, like the covariant models we are considering, the in- and out- vacuum vectors are in general not dual to each other. Thus a nontrivial vacuum-to-vacuum probability $|\langle 0, \text{out}|0, \text{in}\rangle|^2 < 1$ may occur, since $|\langle 0, \text{out}|0, \text{in}\rangle|^2$ measures the vacuum persistence which is equal to 1 only if no spontaneous pair creation occurs.

The tempered distribution defined by the right-hand side of (4.2) is known as the *causal* propagator and will be denoted as $i\Delta_c(\boldsymbol{x}, \boldsymbol{y})$, where the imaginary unit has been factored out to conform with our previous conventions. Quite generally, for a Klein-Gordon field which may be free or moving in an external background which preserves vacuum stability, the expression (4.2) may be evaluated through the expansion

$$i\Delta_c(\boldsymbol{x}, \boldsymbol{y}) = \tau(x^0 - y^0)\int_{\mathcal{C}} dm(\nu)\phi_\nu^{(+)}(\boldsymbol{x})\phi_\nu^{(+)}(\boldsymbol{y})^* + \tau(y^0 - x^0)\int_{\mathcal{C}} dm(\nu)\phi_\nu^{(-)}(\boldsymbol{x})\phi_\nu^{(-)}(\boldsymbol{y})^* \quad (4.3)$$

with τ the Heaviside distribution function, $(\phi_\nu^{(\pm)})$ a complete set of distributional solutions of the classical field equation with positive and negative frequency, respectively, and $dm(\nu)$ a suitable measure on the set $\mathcal{C}$ of all quantum numbers ν parametrizing the space of solutions. One can check that the distribution (4.2) and (4.3) propagates particles (positive frequency solutions) forward in time and antiparticles (negative frequency solutions) backward in time. This is the imprint of causality and lends the causal propagator its name.

The situation is more complicated if the background field spoils vacuum persistence. A typical example is ordinary quantum electrodynamics in an external field which allows for pair creation. Crucial for the canonical quantization scheme and for the expression (4.3) is the existence of a complete set of classical solutions which have definite positive or negative frequency for all times. However, such a set of solutions only exists if we are working on a stationary spacetime, i.e. a spacetime which admits a global timelike Killing vector field [34]. In our case, there is no such vector field due to the absence of time translation symmetry; production of particle-antiparticle pairs manifests itself in an inevitable mixing of positive and negative frequencies at the level of solutions to the field equations. The requisite methods in this case have been developed in [40,41].

Since the asymptotic Hilbert spaces in the remote past and future (if they exist) are different, there are two complete sets of solutions denoted $(\phi_\nu^{(\pm)})$ and $(\phi_{\nu(\pm)})$, having definite positive/negative frequency at times t_{in} and t_{out}, respectively, which are the equivalent of the positive/negative frequency solutions above in the infinite future and past, i.e. in the limits $t_{\text{in}} \to -\infty$ and $t_{\text{out}} \to +\infty$. The generalization of the expansion into classical solutions (4.3) then reads

$$\begin{aligned} i\Delta_c(\boldsymbol{x}, \boldsymbol{y}) = \tau(x^0 - y^0)\int_{\mathcal{C}} dm(\mu) &\times \int_{\mathcal{C}} dm(\nu)\phi_\mu^{(+)}(\boldsymbol{x})\omega^+(\mu|\nu)\phi_{\nu(+)}(\boldsymbol{y})^* \\ &+ \tau(y^0 - x^0)\int_{\mathcal{C}} dm(\mu) \\ &\times \int_{\mathcal{C}} dm(\nu)\phi_{\nu(-)}(\boldsymbol{x})\omega^-(\mu|\nu)\phi_\mu^{(-)}(\boldsymbol{y})^*. \end{aligned} \quad (4.4)$$

Here $\omega^\pm(\mu|\nu)$ is the relative probability for a particle/antiparticle to be scattered by the vacuum in the external field, given by a generalized Wick contraction of creation-annihilation operators on Fock space which appear in the mode expansions of the in- and out- field operators, and which create the in- and out- particle/antiparticle states. For a field theory with a stable vacuum state one has $\omega^\pm(\mu|\nu) = \delta(\mu, \nu)$ and $\phi_\nu^{(\pm)} = \phi_{\nu(\pm)}$, where $\int_{\mathcal{C}} dm(\mu)\delta(\mu, \nu)f(\mu) = f(\nu)$. This construction determines the propagator uniquely and is equivalent to the definition (4.2), but can be quite technically cumbersome to carry out explicitly; hence it is desirable to have another method at hand.

Such an equivalent method, which will prove profitable for us, is the eigenvalue representation. Let $\varphi_\lambda(\boldsymbol{x})$ be a complete orthonormal set of eigenfunctions of the wave operator $\mathsf{D}_{\boldsymbol{x}}$ of the field theory with eigenvalues $\lambda \in \sigma(\mathsf{D}_{\boldsymbol{x}})$, i.e.

$$\mathsf{D}_{\boldsymbol{x}}\varphi_\lambda(\boldsymbol{x}) = \lambda\varphi_\lambda(\boldsymbol{x}) \quad (4.5)$$

with

$$\int_{\sigma(\mathsf{D}_x)} d\ell(\lambda)\varphi_\lambda(\boldsymbol{x})^*\varphi_\lambda(\boldsymbol{y}) = \delta(\boldsymbol{x} - \boldsymbol{y}) \quad \text{and} \quad \int d^D\boldsymbol{x}\varphi_\lambda(\boldsymbol{x})^*\varphi_{\lambda'}(\boldsymbol{x}) = \delta(\lambda, \lambda'), \quad (4.6)$$

where the measure $d\ell(\lambda)$ is discrete measure on the point spectrum and Lebesgue measure on the absolutely continuous spectrum in $\sigma(\mathsf{D}_{\boldsymbol{x}}) \subset \mathbb{C}$ with $\int_{\sigma(\mathsf{D}_x)} d\ell(\lambda)\delta(\lambda, \lambda')f(\lambda) = f(\lambda')$. Contrary to the functions $\phi_\nu^{(\pm)}$ above, these eigenfunctions need not solve the field equations. Decomposing the propagator into these eigenfunctions gives the formal expansion

$$\Delta(\boldsymbol{x}, \boldsymbol{y}) = \int_{\sigma(\mathsf{D}_x)} d\ell(\lambda)\varphi_\lambda(\boldsymbol{x})^*\lambda^{-1}\varphi_\lambda(\boldsymbol{y}). \quad (4.7)$$

However, the potential poles at $\lambda = 0$ make this definition problematic, which reflects the existence of more than one propagator for a given field theory. Usually one modifies the denominator by adding an adiabatic cutoff $\lambda \to \lambda + i\epsilon F(\lambda)$ with small $\epsilon > 0$ and a function $F: \sigma(\mathsf{D}_{\boldsymbol{x}}) \to \mathbb{R}$ on the spectrum of $\mathsf{D}_{\boldsymbol{x}}$, so that

$$\lambda + i\epsilon F(\lambda) \neq 0 \quad (4.8)$$

for all $\lambda \in \sigma(\mathsf{D}_{\boldsymbol{x}})$. A Green's function for the partial differential operator $\mathsf{D}_{\boldsymbol{x}}$ is finally obtained by taking the adiabatic limit $\epsilon \to 0^+$. Equivalently, one can regularize the operator $\mathsf{D}_{\boldsymbol{x}} \to \mathsf{D}_{\boldsymbol{x}}^{(\epsilon)}$ with $\lim_{\epsilon\to 0^+} \mathsf{D}_{\boldsymbol{x}}^{(\epsilon)} = \mathsf{D}_{\boldsymbol{x}}$ and solve the equation $\mathsf{D}_{\boldsymbol{x}}^{(\epsilon)} \Delta^{(\epsilon)}(\boldsymbol{x}, \boldsymbol{y}) = \delta(\boldsymbol{x} - \boldsymbol{y})$, where $\lim_{\epsilon\to 0^+} \Delta^{(\epsilon)}(\boldsymbol{x}, \boldsymbol{y})$ is a Green's function of the original wave operator $\mathsf{D}_{\boldsymbol{x}}$.

Hence any well-defined operator which is continuously connected to the original wave operator and has no zero eigenvalues gives rise to a propagator for $\mathsf{D}_{\boldsymbol{x}}$. However, apart from the requirement of absence of zero eigenvalues of $\mathsf{D}_{\boldsymbol{x}}^{(\epsilon)}$ (or equivalently the condition (4.8)), the regularization is arbitrary and different regularizations may lead to different propagators. For example, in the free Klein-Gordon theory the choice of $F(\boldsymbol{k})$ as a positive constant leads to the Feynman propagator, while $F(\boldsymbol{k}) = 2k^0$ yields the retarded propagator. In general, one cannot be sure whether one obtains the causal propagator unless one compares it by hand to the result obtained from (4.2). This is the obvious drawback of the eigenvalue method, and while the problem is easily solved in free field theory, it is still unsolved for the general case of an arbitrary propagator and arbitrary external field; in particular, the equivalence of the propagators in the different representations for generic electromagnetic backgrounds is still an open question.

For the LSZ model we already encountered the two regularized wave operators $\mathsf{D}_{\boldsymbol{x}}^{(\epsilon)}$ given by the ϑ regularization and the iϵ prescription. The question of which propagator they lead to in the limit $\epsilon \to 0^+$ has been answered for the iϵ prescription for several related models. For the Klein-Gordon field moving in crossed or parallel uniform electric and magnetic fields, or in an electric field with an additional plane wave background, this method gives the causal propagator [28,42,43]. Since an additional uniform magnetic background should not change the pole structure of the propagator, the iϵ prescription should also give the causal propagator in the background of a pure electric field. In Sec. IV B we will confirm that the ϑ regularization also gives the causal propagator in the background of a uniform electric field along one direction.

B. Causal propagator in matrix space

Using the "sum over solutions method" (4.4), the causal propagator for a massive complex scalar field of charge e in four dimensions in the background of a constant electric field along one space direction has been calculated in [27], Eq. (6.2.40) with the result

$$\Delta_c(\boldsymbol{x}, \boldsymbol{y}) = \frac{eE}{16\pi^2}\, \mathrm{e}^{-\mathrm{i}e\boldsymbol{x}_\parallel\cdot \boldsymbol{E}\boldsymbol{y}_\parallel/2} \int_0^\infty \frac{ds}{s}\, \frac{1}{\sinh(seE)} \times \exp\Big(-\mathrm{i}s\mu^2 - \frac{\mathrm{i}}{4} eE \parallel \boldsymbol{x}_\parallel - \boldsymbol{y}_\parallel\parallel_{\mathrm{M}}^2 \coth(seE) + \mathrm{i}\frac{\parallel \boldsymbol{x}_\perp - \boldsymbol{y}_\perp\parallel_{\mathrm{E}}^2}{4s}\Big). \tag{4.9}$$

Here we defined $\boldsymbol{x} = (\boldsymbol{x}_\parallel, \boldsymbol{x}_\perp) \in \mathbb{R}^4$ with $\boldsymbol{x}_\perp \in \mathbb{R}^2$ denoting the two space components perpendicular to the electric field and

$$\boldsymbol{x}_\parallel \cdot \boldsymbol{E}\boldsymbol{y}_\parallel := E x_\parallel^\mu \epsilon_{\mu\nu} y_\parallel^\nu, \tag{4.10}$$

where $\epsilon_{\mu\nu}$ is the rank two antisymmetric tensor with $\epsilon_{01} = 1$ and $E > 0$ the electric field strength. Below we will start with this four-dimensional wave operator, with the electric part regularized as in (2.45), and calculate its (unique) propagator. For $\kappa \to 0^+$ we find exact agreement with (4.9) confirming that this is the causal propagator. The calculations performed here using the matrix basis are comparably simple, so that the matrix basis can be alternatively regarded as a powerful computational tool in ordinary field theory.

We begin with some notation and a preliminary result. We define the symmetric bilinear form $(-,-)_\vartheta$: $\mathbb{R}^2 \otimes \mathbb{R}^2 \to \mathbb{C}$ for $\vartheta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ by

$$\begin{aligned}(\boldsymbol{x}, \boldsymbol{y})_\vartheta &= \cos(\vartheta)(\boldsymbol{x}, \boldsymbol{y})_{\mathrm{E}} + \mathrm{i}\sin(\vartheta)(\boldsymbol{x}, \boldsymbol{y})_{\mathrm{M}} \\ &= \tfrac{1}{2}\mathrm{e}^{\mathrm{i}\vartheta}(x_+^{(\vartheta)} y_-^{(\vartheta)} + x_-^{(\vartheta)} y_+^{(\vartheta)}),\end{aligned} \tag{4.11}$$

where $(-,-)_{\mathrm{E}}$ is the two-dimensional Euclidean scalar product and $(-,-)_{\mathrm{M}}$ the two-dimensional hyperbolic inner product. We also define the map $\|-\|_\vartheta$: $\mathbb{R}^2 \to \mathbb{C}$ by

$$\begin{aligned}\|\boldsymbol{x}\|_\vartheta^2 &= (\boldsymbol{x}, \boldsymbol{x})_\vartheta = \cos(\vartheta)\|\boldsymbol{x}\|_{\mathrm{E}}^2 + \mathrm{i}\sin(\vartheta)\|\boldsymbol{x}\|_{\mathrm{M}}^2 \\ &= \mathrm{e}^{\mathrm{i}\vartheta} x_+^{(\vartheta)} x_-^{(\vartheta)}\end{aligned} \tag{4.12}$$

with $\|-\|_{\mathrm{E}}$ the two-dimensional Euclidean norm (2.2) and $\|-\|_{\mathrm{M}}$ the two-dimensional hyperbolic norm (2.1). For arbitrary two-dimensional vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^2$ we denote as above $\boldsymbol{x}\cdot\boldsymbol{E}\boldsymbol{y} = E x^\mu \epsilon_{\mu\nu} y^\nu$. In Appendix C we prove

Lemma 4.13. For any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^2$ and $a \in \mathbb{C}^*$, one has

$$\begin{aligned}\sum_{n=0}^\infty f_{mn}^{(E_\vartheta)}(\boldsymbol{x}) f_{nm}^{(E_\vartheta)}(\boldsymbol{y}) a^n &= \frac{Ea^m}{\pi} \exp\Big(-\frac{E}{2}\|\boldsymbol{x}-\boldsymbol{y}\|_\vartheta^2 + (a-1)E(\boldsymbol{x},\boldsymbol{y})_\vartheta - a\mathrm{i}\boldsymbol{x}\cdot\boldsymbol{E}\boldsymbol{y}\Big) \\ &\quad \times L_m^0(E\|\boldsymbol{x}-\boldsymbol{y}\|_\vartheta^2 - a(1-a^{-1})^2 E(\boldsymbol{x},\boldsymbol{y})_\vartheta \\ &\quad + (a-a^{-1})\mathrm{i}\boldsymbol{x}\cdot\boldsymbol{E}\boldsymbol{y}).\end{aligned} \tag{4.14}$$

Now we determine the propagator of the Klein-Gordon field in four dimensions coupled to a constant electric field, where the wave operator parallel to the electric field is given by the two-dimensional ϑ-regularized operator $(\mathsf{P}_\mu^2 - \mu^2)_\kappa$. The coordinate vector is again written as $\boldsymbol{x} = (\boldsymbol{x}_\parallel, \boldsymbol{x}_\perp) \in \mathbb{R}^4$, with $\boldsymbol{x}_\perp \in \ker(\boldsymbol{E})$ the components perpendicular to the electric field, and analogously for the momenta $\boldsymbol{p} = (\boldsymbol{p}_\parallel, \boldsymbol{p}_\perp) \in (\mathbb{R}^4)^*$ and derivatives $\partial_\mu = (\partial_\parallel, \partial_\perp)$.

Proposition 4.15. The propagator of the ϑ-regularized wave operator $(\mathsf{K}_\mu^2 - \mu^2)_\kappa = (\mathsf{P}_\mu^2 - \mu^2)_\kappa + (\mathrm{i}\partial_\perp)^2$ coincides in the limit $\kappa \to 0^+$ with the causal propagator (4.9).

Proof. The inverse of $(\mathsf{K}_\mu^2 - \mu^2)_\kappa$ is given by

$$\Delta^{(\kappa,1)}(\boldsymbol{x},\boldsymbol{y}) = \langle \boldsymbol{x}| \frac{1}{(\mathsf{P}_\mu^2 - \mu^2)_\kappa + (\mathrm{i}\partial_\perp)^2} |\boldsymbol{y}\rangle, \tag{4.16}$$

where $(\mathsf{K}_\mu^2 - \mu^2)_\kappa = \mathrm{e}^{\mathrm{i}\kappa}\mathsf{P}^2(\frac{\pi}{2} - \kappa) - \mathrm{e}^{-\mathrm{i}\kappa}\mu^2 + (\mathrm{i}\partial_\perp)^2$ with $\kappa > 0$ has the eigenvalue equation

$$\begin{aligned}(\mathsf{K}_\mu^2 - \mu^2)_\kappa &(f_{mn}^{(E_\vartheta)}(\boldsymbol{x}_\|)\mathrm{e}^{-\mathrm{i}\boldsymbol{p}_\perp\cdot\boldsymbol{x}_\perp}) \\ &= \Big(4\mathrm{i}E(m+\tfrac{1}{2}) + \|\boldsymbol{p}\|_{\mathrm{E}}^2 - \mathrm{e}^{-\mathrm{i}\kappa}\mu^2\Big) f_{mn}^{(E_\vartheta)}(\boldsymbol{x}_\|)\mathrm{e}^{-\mathrm{i}\boldsymbol{p}_\perp\cdot\boldsymbol{x}_\perp}\end{aligned} \tag{4.17}$$

with $\vartheta = \frac{\pi}{2} - \kappa$. We write $\mu_\kappa^2 := \mathrm{e}^{-\mathrm{i}\kappa}\mu^2$ for brevity, remembering that it has a small negative imaginary part. Using the identity

$$\frac{1}{a} = -\mathrm{i}\int_0^\infty ds\,\mathrm{e}^{\mathrm{i}sa} \quad \text{for } \Im\mathfrak{m}(a) > 0, \tag{4.18}$$

we obtain

$$\begin{aligned}\Delta^{(\kappa,1)}(\boldsymbol{x},\boldsymbol{y}) = -\mathrm{i}\int_0^\infty ds \int \frac{d^2\boldsymbol{p}_\perp}{(2\pi)^2} &\times \sum_{m,n=0}^{\infty} f_{mn}^{(E_\vartheta)}(\boldsymbol{x}_\|) f_{nm}^{(E_\vartheta)}(\boldsymbol{y}_\|)\mathrm{e}^{-\mathrm{i}s\mu_\kappa^2}\mathrm{e}^{-4sE(m+\frac{1}{2})} \\ &\times \mathrm{e}^{\mathrm{i}s\|\boldsymbol{p}_\perp\|_{\mathrm{E}}^2 - \mathrm{i}(\boldsymbol{x}_\perp - \boldsymbol{y}_\perp)\cdot\boldsymbol{p}_\perp}.\end{aligned} \tag{4.19}$$

The sum over n is given by Lemma 4.13 with $a = 1$, and the resulting sum over m follows from the identity [[44], Eq. (48.4.1)]

$$\mathrm{e}^{-y/2}\sum_{m=0}^{\infty} L_m^0(y)t^m = \frac{1}{1-t}\exp\Big(\frac{y}{2}\frac{t^{1/2}+t^{-1/2}}{t^{1/2}-t^{-1/2}}\Big) \text{ for } |t|<1 \tag{4.20}$$

which yields

$$\begin{aligned}\Delta^{(\kappa,1)}(\boldsymbol{x},\boldsymbol{y}) = -\mathrm{i}\frac{E}{2\pi}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{x}_\|\cdot\boldsymbol{E}\boldsymbol{y}_\|}\int_0^\infty ds &\times \frac{\exp(-\mathrm{i}s\mu_\kappa^2 - \frac{1}{2}E\,\|\,\boldsymbol{x}_\| - \boldsymbol{y}_\|\|_\vartheta^2\coth(2sE))}{\sinh(2sE)} \\ &\times \int \frac{d^2\boldsymbol{p}\perp}{(2\pi)^2}\,\mathrm{e}^{\mathrm{i}s\|\boldsymbol{p}\|_{\mathrm{E}}^2 - \mathrm{i}(\boldsymbol{x}-\boldsymbol{y})\cdot\boldsymbol{p}\perp\perp\perp\perp}.\end{aligned} \tag{4.21}$$

The integration over the perpendicular momenta can now be performed by using

$$\int dp\,\mathrm{e}^{\mathrm{i}sp^2 - \mathrm{i}(x-y)p} = \sqrt{\frac{\mathrm{i}\pi}{s}}\mathrm{e}^{\mathrm{i}(x-y)^2/4s}, \tag{4.22}$$

to get

$$\begin{aligned}\Delta^{(\kappa,1)}(\boldsymbol{x},\boldsymbol{y}) = \frac{E}{8\pi^2}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{x}_\|\cdot\boldsymbol{E}\boldsymbol{y}_\|}\int_0^\infty \frac{ds}{s}\frac{1}{\sinh(2sE)} &\times \exp\Big(-\mathrm{i}s\mu_\kappa^2 - \frac{1}{2}E\,\|\,\boldsymbol{x}_\| - \boldsymbol{y}_\|\|_\vartheta^2\coth(2sE) \\ &+ \mathrm{i}\frac{\|\,\boldsymbol{x}_\perp - \boldsymbol{y}_\perp\|_{\mathrm{E}}^2}{4s}\Big).\end{aligned} \tag{4.23}$$

Taking the limit $\kappa \to 0^+$, thus $\vartheta \to \frac{\pi}{2}$, and substituting $E \to eE/2$ to conform to the conventions of [27], this result is identical to (4.9).

The eigenfunctions for the full regularized wave operator $(\mathsf{K}_\mu^2 - \mu^2)_\kappa$ factorize into components perpendicular to the electric field times the eigenfunctions of $(\mathsf{P}_\mu^2 - \mu^2)_\kappa$. Since the eigenvalues of the perpendicular momentum operators do not produce new pole singularities, we can neglect them in this calculation and also in the calculation leading to (4.9). This result thus easily carries over to the two-dimensional case confirming that the ϑ regularization imposes causality of the critical LSZ model. We expect that the ϑ regularization also leads to the causal propagators for $\sigma \neq 1$.

The Schwinger parameter $s > 0$ introduced in (4.18) only allows for the regularizations $\vartheta > 0$ and $\mu^2 - \mathrm{i}\epsilon$ because of the requirement $\Im\mathfrak{m}(a) > 0$, where the latter regularization is normally associated to the Feynman boundary condition on the propagator. The other choices $\vartheta < 0$ and $\mu^2 + \mathrm{i}\epsilon$ can be applied by using

$$\frac{1}{a} = \mathrm{i}\int_{-\infty}^0 ds\,\mathrm{e}^{\mathrm{i}sa} \quad \text{for } \Im\mathfrak{m}(a) < 0. \tag{4.24}$$

The regularization $\mu^2 + \mathrm{i}\epsilon$ is known as the Dyson boundary condition, which leads to an anticausal propagator where antiparticles travel forward and particles backward in time. This strongly suggests that the regularization $\vartheta < 0$ leads to the Dyson propagator.

The regularization of the mass $\mu_\kappa^2 = \mathrm{e}^{-\mathrm{i}\kappa}\mu^2$ is actually irrelevant for the analysis above. Its only function is to provide a continuous interpolation between the hyperbolic and Euclidean wave operators with the help of the parameter ϑ alone, without the need to keep track of additional minus signs in front of the mass term. This means that the interpretation in terms of Feynman/Dyson propagators for the cases $\vartheta \to \pm\frac{\pi}{2}$ still holds by regularizing only the operator P_μ^2.

V. QUANTUM DUALITY

In this section we will treat the problem of implementing duality covariance at quantum level for our field theories on Minkowski space. The ϑ regularization allows us to regularize the covariant field theories such that the duality is preserved at quantum level. This is done in the same spirit as in [21,23], with the ϑ regularization now being the only new ingredient. In the following this will be demonstrated for the two-dimensional Grosse-Wulkenhaar model. The more general case of the LSZ model is treated in exactly the same way.

We only need to address how the ϑ regularization affects the behavior of quantities under the duality transformation. The regularized propagator with $\vartheta = \frac{\pi}{2} - \kappa > 0$ reads

$$\Delta^{(\kappa)}(\boldsymbol{x}, \boldsymbol{y}) = \langle \boldsymbol{x} | \Big(\frac{1}{2}\mathsf{P}_\mu^2 + \frac{1}{2}\tilde{\mathsf{P}}_\mu^2 - \mu^2\Big)_\kappa^{-1} | \boldsymbol{y}\rangle = \sum_{m,n} \frac{f_{mn}^{(E_\vartheta)}(\boldsymbol{x}) f_{nm}^{(E_\vartheta)}(\boldsymbol{y})}{2\mathrm{i}E(m+n+1) - \mathrm{e}^{-\mathrm{i}\kappa}\mu^2}. \tag{5.1}$$

In Appendix D we show that the Fourier transformation of the matrix basis functions is given by

$$\mathcal{F}[f_{mn}^{(E_\vartheta)}](\boldsymbol{k}) = f_{nm}^{(1/E_\vartheta)}(\boldsymbol{k}) = \frac{(-\mathrm{i})^{m-n}}{E} f_{mn}^{(E_\vartheta)}(\boldsymbol{E}^{-1}\boldsymbol{k}) \tag{5.2}$$

with $\boldsymbol{E}^{-1}\boldsymbol{k} = -E^{-1}(k^1, k^0)$.[5] Since

$$\mathcal{F}[(\mathsf{P}^2(\vartheta) + \tilde{\mathsf{P}}^2(\vartheta)) f_{mn}^{(E_\vartheta)}](\boldsymbol{k}) = 4E_\vartheta(m+n+1)\mathcal{F}[f_{mn}^{(E_\vartheta)}](\boldsymbol{k}), \tag{5.3}$$

we find that Fourier transformation relates the propagator in position space to the momentum space propagator even in the regularized case as

$$(\mathcal{F}\otimes\mathcal{F})[\Delta^{(\kappa)}](\boldsymbol{k}, \boldsymbol{p}) = \frac{1}{E^2}\Delta^{(\kappa)}(\boldsymbol{E}^{-1}\boldsymbol{k}, \boldsymbol{E}^{-1}\boldsymbol{p}). \tag{5.4}$$

This relation just reflects the classical duality covariance for $g = 0$.

Analogously to the Euclidean case [21], the UV/IR-symmetric regularization now amounts to cutting off the matrix element sums at some finite rank N by modifying the regularized position space propagator to

$$\Delta_\Lambda^{(\kappa)}(\boldsymbol{x}, \boldsymbol{y}) = \langle \boldsymbol{x} | (\tfrac{1}{2}\mathsf{P}_\mu^2 + \tfrac{1}{2}\tilde{P}_\mu^2 - \mu^2)_\kappa^{-1} L(\Lambda^{-2}|\mathsf{P}^2(\vartheta) + \tilde{\mathsf{P}}^2(\vartheta)|) | \boldsymbol{y}\rangle, \tag{5.5}$$

where $\Lambda \in \mathbb{R}_+$ is a cutoff parameter, and $L\colon \mathbb{R}_+ \to [0,1]$ a smooth cutoff function which is monotonically decreasing with $L(z) = 1$ for $z < 1$ and $L(z) = 0$ for $z > 2$. We adjust the matrix basis functions so as to diagonalize the regulated Grosse-Wulkenhaar propagator

$$\Delta_{\Lambda|mn;kl}^{(\kappa)} = \int d^2\boldsymbol{x} f_{mn}^\kappa(\boldsymbol{x}) \Big(\frac{1}{2}\mathsf{P}_\mu^2 + \frac{1}{2}\tilde{\mathsf{P}}_\mu^2 - \mu^2\Big)_\kappa^{-1} \times L(\Lambda^{-2}|\mathsf{P}^2(\vartheta) + \tilde{\mathsf{P}}^2(\vartheta)|) f_{kl}^\kappa(\boldsymbol{x}) = \frac{\delta_{ml}\delta_{nk}}{2\mathrm{i}E(m+n+1) - \mathrm{e}^{-\mathrm{i}\kappa}\mu^2} \times L(4\Lambda^{-2}E(m+n+1)). \tag{5.6}$$

The interaction vertices in the matrix space representation are now quite complicated; they are proportional to

$$v^\kappa(m_1, n_1; \ldots; m_4, n_4) = \int d^2\boldsymbol{x} (f_{m_1n_1}^\kappa \star_\theta f_{m_2n_2}^\kappa \star_\theta f_{m_3n_3}^\kappa \star_\theta f_{m_4n_4}^\kappa)(\boldsymbol{x}) \tag{5.7}$$

with $\theta \neq 2/E$ in general. Since for $\kappa > 0$ the complex Landau wave functions f_{mn}^κ are elements of the Gel'fand-Shilov spaces $\mathcal{S}_\alpha^\alpha(\mathbb{R}^2)$ with $\alpha \geq \frac{1}{2}$, which are closed under multiplication of functions with the star-product, the interaction vertex (5.7) is well defined.

Feynman diagrams can now be obtained by taking suitable combinations of derivatives of the partition function (3.53) with respect to the external sources involving the regularized propagator. Denoting

$$\Delta_{\Lambda|mn;kl}^{(\kappa)} =: \delta_{mk}\delta_{nl}\Delta_\Lambda^{(\kappa)}(m, n), \tag{5.8}$$

they have the schematic form

$$\sum_{n_1,m_1,\ldots,n_K,m_K} \prod_{k=1}^K \Delta_\Lambda^{(\kappa)}(m_k, n_k)(\cdots), \tag{5.9}$$

where $(\ldots)$ denotes the contributions from products of the noncommutative interaction vertices (5.7) and combinatorial factors. Since the propagator $\Delta_\Lambda^{(\kappa)}(m, n)$ is nonzero only for $4E(m+n+1) < 2\Lambda^2$, which at finite Λ is only true for a finite number of distinct values of $(m, n) \in \mathbb{N}_0^2$, every Feynman amplitude is represented by a finite sum and thus constitutes well-defined duality covariant Green's functions in the matrix basis; this circumvents the issue of the appropriate test function space for the time being. By multiplying these expressions with $f_{m_in_i}^\kappa(\boldsymbol{x}_i)$ for $i = 1, \ldots, M$, we get back the position space Green's functions with M external legs by summing over all m_i, n_i. They are also well-defined and duality covariant, since they are built from finite sums of well-defined covariant objects. This establishes the quantum duality in Minkowski space for the case $\kappa > 0$.

To prove the duality covariance at $\kappa = 0$ in the same manner as above, one has to ensure that the interaction vertex (5.7) away from the self-dual point is well defined. In the absence of further analysis, the ϑ regularization should be kept while the matrix cutoff is removed, and all summations and integrations have been performed. Of course the limit $\Lambda \to \infty$ can still be ill-defined and may require renormalization; removing this regularization

[5]There is a subtle difference here between the Euclidean and hyperbolic cases. Contrary to the ordinary Landau wave functions in Euclidean space, the (unscaled) Fourier transforms of the complex Landau wave functions have interchanged indices $m \leftrightarrow n$ and a reflected regularization parameter $\vartheta \to -\vartheta$. The interchange is equivalent to time reversal (or parity), see Appendix D. The reflection corresponds to charge conjugation, i.e. exchange of particles with antiparticles; this follows from the results of Sec. IV B, where the regularization $\vartheta > 0$ is identified with the Feynman boundary condition and $\vartheta < 0$ with the Dyson boundary condition. The specific rescalings of momenta from $(\mathbb{R}^2)^*$ to $\mathbb{R}^2$ in both cases, which are formally identical but differ by the signature of the metric applied, compensates for this difference.

requires a good decay behavior of the matrix space propagator for large values of its indices, see Sec. VI. In addition, the results from Sec. III A are not able to exclude the possibility that even at finite $\kappa > 0$ there might be extra divergences at $\Lambda \to \infty$ if we work in matrix space, stemming from the complex matrix basis itself. This, however, does not affect the duality covariance of the quantum field theory, which has been achieved for the Green's functions in position space through the regularization of the propagators in (5.5). This result is independent of the matrix basis.

VI. ASYMPTOTIC ANALYSIS OF PROPAGATORS

One of the most intriguing features of Euclidean duality covariant field theories is their renormalizability. We will not attempt to prove here the renormalizability of their Minkowski space counterparts, but start this program by deriving their propagators in position and matrix space representations, and studying their asymptotics. We begin by extending the formulas given in [45] to the hyperbolic setting, giving the propagators for the general LSZ models in generic $D = 2n$ spacetime dimensions in the position and matrix bases.

A. Position space representation

In the notations of Sec. III D and IV B, the main result from which all causal propagators in Minkowski space and their Euclidean counterparts can be derived is

Proposition 6.1. The propagator of the regularized LSZ model in $D = 2n$ spacetime dimensions is given by

$$\begin{aligned}\Delta^{(\kappa,\sigma)}(\boldsymbol{x},\boldsymbol{y}) = {}& -\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\vartheta}\frac{E}{2\pi}\int_0^\infty ds\,\frac{\mathrm{e}^{-s\mu_\kappa^2}}{\sinh(2sE_{-\vartheta})}\exp\Big(-\frac{\sinh(2s\tilde{E}_{-\vartheta})}{\sinh(2sE_{-\vartheta})}\,\mathrm{i}\boldsymbol{x}_0\cdot\boldsymbol{E}\boldsymbol{y}_0\Big)\exp\Big(-\frac{1}{2}\coth(2sE_{-\vartheta})E(\|\boldsymbol{x}_0\|^2_\vartheta+\|\boldsymbol{y}_0\|^2_\vartheta)\\ &+\frac{\cosh(2s\tilde{E}_{-\vartheta})}{\sinh(2sE_{-\vartheta})}E(\boldsymbol{x}_0,\boldsymbol{y}_0)_\vartheta\Big)\prod_{k=1}^{n-1}\frac{B_k}{2\pi}\frac{1}{\sinh(2sB_k)}\exp\Big(-\frac{\sinh(2s\tilde{B}_k)}{\sinh(2sB_k)}\,\mathrm{i}\boldsymbol{x}_k\cdot\boldsymbol{B}_k\boldsymbol{y}_k\Big)\\ &\times\exp\Big(-\frac{1}{2}\coth(2sB_k)B_k(\|\boldsymbol{x}_k\|^2_\mathrm{E}+\|\boldsymbol{y}_k\|^2_\mathrm{E})+\frac{\cosh(2s\tilde{B}_k)}{\sinh(2sB_k)}B_k(\boldsymbol{x}_k,\boldsymbol{y}_k)_\mathrm{E}\Big)\end{aligned}\tag{6.2}$$

with $\vartheta = \frac{\pi}{2} - \kappa > 0$, $\mu_\kappa^2 = \mathrm{e}^{-\mathrm{i}\kappa}\mu^2$, $\tilde{E} = (2\sigma - 1)E$ and $\tilde{B}_k = (2\sigma - 1)B_k$.

The proof of proposition 6.1 is found in Appendix E. We can now read off the causal propagators for the four-dimensional LSZ and Grosse-Wulkenhaar models. Since $(-,-)_{\pi/2} = \mathrm{i}(-,-)_\mathrm{M}$ and thus $\|-\|^2_{\pi/2} = \mathrm{i}\|-\|^2_\mathrm{M}$, one finds

Corollary 6.3. The causal propagator of the LSZ model for generic $\sigma \in [0, 1]$ in four-dimensional Minkowski space is given by

$$\Delta^{(0,\sigma)}(\boldsymbol{x},\boldsymbol{y}) = -\frac{\mathrm{i}EB}{(2\pi)^2}\int_0^\infty ds\,\frac{\mathrm{e}^{-s\mu^2-A_\mathrm{M}-A_\mathrm{E}}}{\sin(2sE)\sinh(2sB)}\exp\Big(-\frac{\sin(2s\tilde{E})}{\sin(2sE)}\,\mathrm{i}\boldsymbol{x}_0\cdot\boldsymbol{E}\boldsymbol{y}_0-\frac{\sinh(2s\tilde{B})}{\sinh(2sB)}\,\mathrm{i}\boldsymbol{x}_1\cdot\boldsymbol{B}\boldsymbol{y}_1\Big)\tag{6.4}$$

with

$$\begin{aligned}A_\mathrm{M} &= -\frac{E}{2}\cot(2sE)(\|\boldsymbol{x}_0\|^2_\mathrm{M}+\|\boldsymbol{y}_0\|^2_\mathrm{M})+\frac{\cos(2s\tilde{E})}{\sin(2sE)}E(\boldsymbol{x}_0,\boldsymbol{y}_0)^2_\mathrm{M},\\ A_\mathrm{E} &= \frac{B}{2}\coth(2sB)(\|\boldsymbol{x}_1\|^2_\mathrm{E}+\|\boldsymbol{y}_1\|^2_\mathrm{E})-\frac{\cosh(2s\tilde{B})}{\sinh(2sB)}B(\boldsymbol{x}_1,\boldsymbol{y}_1)^2_\mathrm{E}.\end{aligned}\tag{6.5}$$

Corollary 6.6. The causal propagator of the four-dimensional critical LSZ model in Minkowski space is given by

$$\begin{aligned}\Delta^{(0,1)}(\boldsymbol{x},\boldsymbol{y}) = {}& -\frac{\mathrm{i}EB}{(2\pi)^2}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{x}_0\cdot\boldsymbol{E}\boldsymbol{y}_0-\mathrm{i}\boldsymbol{x}_1\cdot\boldsymbol{B}\boldsymbol{y}_1}\int_0^\infty ds\,\frac{\mathrm{e}^{-s\mu^2}}{\sin(2sE)\sinh(2sB)}\\ &\times\exp\Big(\frac{1}{2}E\|\boldsymbol{x}_0-\boldsymbol{y}_0\|^2_\mathrm{M}\cot(2sE)-\frac{1}{2}B\|\boldsymbol{x}_1-\boldsymbol{y}_1\|^2_\mathrm{E}\coth(2sB)\Big).\end{aligned}\tag{6.7}$$

Corollary 6.8. The causal propagator of the four-dimensional Grosse-Wulkenhaar model in hyperbolic signature is given by

$$\Delta^{(0)}(\boldsymbol{x},\boldsymbol{y}) = -\frac{\mathrm{i}EB}{(2\pi)^2}\int_0^\infty ds\,\frac{\mathrm{e}^{-s\mu^2}}{\sin(2sE)\sinh(2sB)}\exp\left(\frac{1}{2}E\cot(2sE)(||\boldsymbol{x}_0||_{\mathrm{M}}^2+||\boldsymbol{y}_0||_{\mathrm{M}}^2)-\frac{E}{\sin(2sE)}(\boldsymbol{x}_0,\boldsymbol{y}_0)_{\mathrm{M}}\right)\times\exp\left(-\frac{1}{2}B\coth(2sB)(||\boldsymbol{x}_1||_{\mathrm{E}}^2+||\boldsymbol{y}_1||_{\mathrm{E}}^2)+\frac{B}{\sinh(2sB)}(\boldsymbol{x}_1,\boldsymbol{y}_1)_{\mathrm{E}}\right). \tag{6.9}$$

The Euclidean parts of the propagators here coincide with those found in [45] after suitable redefinitions of parameters.

B. Matrix space representation

Below we set $\theta_0=\theta_1=\ldots=\theta_{n-1}=:\theta$ for simplicity.

Proposition 6.10. The matrix space propagator for the $2n$-dimensional regularized LSZ model in Minkowski space is given by

$$\Delta^{(\kappa,\sigma)}_{\boldsymbol{m},\boldsymbol{m}+\boldsymbol{\alpha};\boldsymbol{l}+\boldsymbol{\alpha},\boldsymbol{l}} = -\mathrm{e}^{\mathrm{i}\kappa}\frac{\theta}{8\Omega}\int_0^1 ds\,s^{-\mathrm{i}\,\mathrm{e}^{\mathrm{i}\kappa}(\sigma\alpha_0+(1/2))+\sum_{i=1}^{n-1}(\sigma\alpha_i+(1/2))-1+\frac{\theta\mu^2}{8\Omega}}\Delta^{(\kappa)}_{m_0,m_0+\alpha_0;l_0+\alpha_0,l_0}(s)\prod_{i=1}^{n-1}\Delta^{\mathrm{E}}_{m_i,m_i+\alpha_i;l_i+\alpha_i,l_i}(s) \tag{6.11}$$

with hyperbolic part

$$\Delta^{(\kappa)}_{m,m+\alpha;l+\alpha,l}(s) = \sum_{u=\max(0,-\alpha)}^{\min(m,l)}\frac{s^{-\mathrm{i}\,\mathrm{e}^{\mathrm{i}\kappa}u}(1-s^{-\mathrm{i}\,\mathrm{e}^{\mathrm{i}\kappa}})^{m+l-2u}}{(1-\frac{(1-\Omega)^2}{(1+\Omega)^2}s^{-\mathrm{i}\,\mathrm{e}^{\mathrm{i}\kappa}})^{\alpha+m+l+1}}\left(\frac{4\Omega}{(1+\Omega)^2}\right)^{\alpha+2u+1}\left(\frac{1-\Omega}{1+\Omega}\right)^{m+l-2u}\mathcal{A}(m,l,\alpha,u) \tag{6.12}$$

and Euclidean part

$$\Delta^{\mathrm{E}}_{m,m+\alpha;l+\alpha,l}(s) = \sum_{u=\max(0,-\alpha)}^{\min(m,l)}\frac{s^{u}(1-s)^{m+l-2u}}{(1-\frac{(1-\Omega)^2}{(1+\Omega)^2}s)^{\alpha+m+l+1}}\left(\frac{4\Omega}{(1+\Omega)^2}\right)^{\alpha+2u+1}\left(\frac{1-\Omega}{1+\Omega}\right)^{m+l-2u}\mathcal{A}(m,l,\alpha,u), \tag{6.13}$$

where

$$\mathcal{A}(m,l,\alpha,u) = \sqrt{\binom{\alpha+m}{\alpha+u}\binom{\alpha+l}{\alpha+u}\binom{m}{u}\binom{l}{u}} \tag{6.14}$$

and $\boldsymbol{\alpha}=(\alpha_0,\alpha_1,\ldots,\alpha_{n-1})\in\mathbb{Z}^n$ with $\alpha_j=n_j-m_j=k_j-l_j$.

The proof of proposition 6.10 is found in Appendix F. The respective special cases, like the four-dimensional Grosse-Wulkenhaar model, can easily be read off from this general expression.

C. Power counting

The power counting theorem for general nonlocal matrix models was proven by Grosse and Wulkenhaar in [33]. In the matrix basis, every Feynman diagram of the duality covariant field theory is represented by a ribbon graph, whose topology is decisive for the question of whether or not it is divergent. Power counting in a dynamical matrix model depends crucially on this topological data. For a *regular* matrix model, the power counting degree of divergence for an N-leg ribbon graph G of genus g with V vertices and B loops carrying external legs is given by [33]

$$\omega(G) = D + V(D-4) - \tfrac{1}{2}N(D-2) - D(2g+B-1). \tag{6.15}$$

Here we briefly recall the role played by the asymptotic behavior of the propagator in the derivation of this power counting theorem. For this, one uses multiscale analysis of the Schwinger parametric representation of the propagator, which works in both position and matrix space.

The slicing of the propagator is defined as

$$\Delta = \sum_{i=0}^{\infty}\Delta^i \quad \text{through} \quad \int_0^1 ds = \sum_{i=0}^{\infty}\int_{M^{-2i}}^{M^{-2(i-1)}} ds \tag{6.16}$$

with an arbitrary constant $M>1$. This leads to a scale decomposition of the amplitude $\mathcal{A}_G$ of any given Feynman graph G as

$$\mathcal{A}_G = \sum_I \mathcal{A}_G^I, \tag{6.17}$$

where $I=\{i_\ell\}$ runs through all assignments of positive integers i_ℓ to each line ℓ of G. One then seeks appropriate bounds on the sliced propagators.

For the i-th slice, the main bounds in matrix space are given by [46,47]

$$|\Delta^i_{\boldsymbol{mn};\boldsymbol{kl}}| \le KM^{-2i}\mathrm{e}^{-cM^{-2i}||\boldsymbol{m}+\boldsymbol{n}+\boldsymbol{k}+\boldsymbol{l}||_1}, \tag{6.18}$$

$$\sum_{\boldsymbol{l}}\max_{\boldsymbol{n},\boldsymbol{k}}|\Delta^i_{\boldsymbol{mn};\boldsymbol{kl}}| \le K'M^{-2i}\mathrm{e}^{-c'M^{-2i}||\boldsymbol{m}||_1} \tag{6.19}$$

for some positive constants K, K' and c, c', where we have introduced the ℓ^1-norm $||\boldsymbol{m}||_1 := m_0 + m_1 + \ldots + m_{n-1}$. Perturbative power counting amounts to finding which summations cost a factor M^{2ni} through (6.18),

$$\sum_{\boldsymbol{m}\in\mathbb{N}_0^n} \mathrm{e}^{-cM^{-2i}||\boldsymbol{m}||_1} = \frac{1}{(1-\mathrm{e}^{-cM^{-2i}})^n} = \frac{M^{2ni}}{c^n}(1+O(M^{-2i})), \tag{6.20}$$

and which cost $O(1)$ due to the bound (6.19). Integrating out loops at higher scales of a graph then gives effective coupling constants in powers of M. The important point is that the faster the propagator decays, the smaller is the contribution of the integration over internal lines to effective coupling constants. This in turn reduces the number of divergent graphs which require renormalization.

As an immediate application of this result in the present context, we can straightforwardly establish the power counting theorem for the 1 + 1-dimensional self-dual Grosse-Wulkenhaar model. In this case the matrix model is local with propagator $\Delta^{(\kappa)}_{mn;kl} = \delta_{mk}\delta_{nl}\Delta^{(\kappa)}(m,n)$ given by

$$\Delta^{(\kappa)}(m,n) = -\frac{\mathrm{i}}{2E(m+n+1)+\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\kappa}\mu^2} = -\mathrm{i}\int_0^\infty ds\mathrm{e}^{-2E(m+n+1)s-\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\kappa}\mu^2 s} = K\int_0^1 ds\mathrm{e}^{-2E(m+n+1)s-\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\kappa}\mu^2 s}, \tag{6.21}$$

where $K = \mathrm{i}(\mathrm{e}^{-2E(m+n+1)-\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\kappa}\mu^2} - 1)^{-1}$. Slicing this propagator as in (6.16), we easily find that the i-th slice can be bounded as

$$|\Delta^{(\kappa)i}(m,n)| \leq |K|M^{-2i}(M^2 - 1)\mathrm{e}^{-2E(m+n+1)M^{-2i}}\mathrm{e}^{-\sin(\kappa)\mu^2 M^{-2i}}. \tag{6.22}$$

The Minkowski space propagator thus has the requisite exponential decays (6.18) and (6.19), and hence the perturbative multiscale renormalization in this case can be treated exactly as in the Euclidean setting [47]; we expect renormalizability to hold in this case. The case $\Omega < 1$ is much more difficult; in the Euclidean case the coupling Ω flows very rapidly to the self-dual point $\Omega = 1$, and it would be interesting to see if this is also the case for the hyperbolic self-dual point.

D. Asymptotics

As discussed in Sec. VI C, the asymptotics of the propagators play an important role in perturbative renormalization. In this paper we are also interested in determining to what extent the complex matrix basis is applicable to the perturbative analysis of the duality covariant field theories; here the asymptotics also give crucial information. However, the asymptotic behaviours of the hyperbolic parts of the propagators are difficult to investigate due to the oscillatory behaviors of the integrands.

For example, consider the Grosse-Wulkenhaar model in four-dimensional Minkowski space with propagator given by corollary 6.8. Introducing *short variables* $\boldsymbol{u}_k = \boldsymbol{x}_k - \boldsymbol{y}_k$ and *long variables* $\boldsymbol{v}_k = \boldsymbol{x}_k + \boldsymbol{y}_k$ for $k = 0$, 1, and using elementary hyperbolic and trigonometric identities, we can write this propagator in the form

$$\Delta^{(0)}(\boldsymbol{u},\boldsymbol{v}) = -\frac{\mathrm{i}B}{(2\pi)^2}\int_0^\infty ds\mathrm{e}^{-s\mu^2/B}\frac{1}{\sin(2s)}\frac{1}{\sinh(2s)}\exp\Big(\frac{B}{4}\cot(s)||\boldsymbol{u}_0||^2_{\mathrm{M}} - \frac{B}{4}\tan(s)||\boldsymbol{v}_0||^2_{\mathrm{M}}\Big) \times\exp\Big(-\frac{B}{4}\coth(s)||\boldsymbol{u}_1||^2_{\mathrm{E}} - \frac{B}{4}\tanh(s)||\boldsymbol{v}_1||^2_{\mathrm{E}}\Big), \tag{6.23}$$

where we set $E = B$ for simplicity. The integral is sliced in the usual way to get

$$\Delta^{(0)j}(\boldsymbol{u},\boldsymbol{v}) = -\frac{\mathrm{i}B}{(2\pi)^2}\int_{M^{-2j}}^{M^{-2(j-1)}} ds\mathrm{e}^{-s\mu^2/B}\frac{1}{\sin(2s)}\frac{1}{\sinh(2s)}\exp\Big(\frac{B}{4}\cot(s)||\boldsymbol{u}_0||^2_{\mathrm{M}} - \frac{B}{4}\tan(s)||\boldsymbol{v}_0||^2_{\mathrm{M}}\Big) \times\exp\Big(-\frac{B}{4}\coth(s)||\boldsymbol{u}_1||^2_{\mathrm{E}} - \frac{B}{4}\tanh(s)||\boldsymbol{v}_1||^2_{\mathrm{E}}\Big) \tag{6.24}$$

with $M > 1$.

The Euclidean part of the modulus of the integral (6.24) can be easily bounded from above by maximizing each of the hyperbolic functions in the integrand on the interval $[M^{-2j}, M^{-2(j-1)}]$. The factor $\mathrm{e}^{-\frac{B}{4}\tanh(s)||\boldsymbol{v}_1||^2_{\mathrm{E}}1}$ takes its maximum at $s = M^{-2j}$ where $\tanh(s) = M^{-2j} - \frac{1}{3}M^{-6j} + O((M^{-2j})^5) < c'M^{-2j}$ for some constant $c' > 0$, while $\mathrm{e}^{-\frac{B}{4}\coth(s)||\boldsymbol{u}_1||^2_{\mathrm{E}}}$ takes its maximum value at $s = M^{-2(j-1)}$ with $\coth(s) < M^{2(j-1)} + M^{-2(j-1)} < c''M^{2j}$ and some constant $c'' > 0$. The function $\sinh(2s)^{-1}$ can be bounded from above by M^{2j}, and in this way one arrives at the very rough bound

$$|\Delta^{(0)j}(\boldsymbol{u},\boldsymbol{v})| \le K M^{2j} e^{-c(M^{2j}||\boldsymbol{u}_1||_{\rm E}^2 + M^{-2j}||\boldsymbol{v}_1||_{\rm E}^2)} \int_{M^{-2j}}^{M^{-2(j-1)}} ds \frac{e^{-s\mu^2/B}}{|\sin(2s)|} \exp\Big(\frac{B}{4}\cot(s)||\boldsymbol{u}_0||_{\rm M}^2 - \frac{B}{4}\tan(s)||\boldsymbol{v}_0||_{\rm M}^2\Big) \quad (6.25)$$

for some positive constants K and c. This reproduces the first required bound which gives exponential decay in both short and long variables in the Euclidean plane [46]; in particular, integrating over long Euclidean coordinates costs a factor M^{2j} while short Euclidean coordinates cost M^{-2j}. However, the asymptotic behavior of the full propagator remains unclear; the hyperbolic part of the integrand is oscillatory, so that more sophisticated methods are needed to bound this integral.

There is a special case in which one can deduce the qualitative behavior. The propagator of the critical, regularized massless LSZ model in 1 + 1 dimensions can be written using proposition 6.1 as

$$\Delta^{(\kappa,1)}_{\mu^2=0}(\boldsymbol{x},\boldsymbol{y}) = -\frac{iE}{2\pi}\int_0^\infty ds \frac{e^{-i\boldsymbol{x}\cdot\boldsymbol{E}\boldsymbol{y}}}{\sinh(2sE)} \times \exp\Big(-\frac{E}{2}\coth(2sE)||\boldsymbol{x}-\boldsymbol{y}||^2\vartheta\Big), \quad (6.26)$$

where the integration contour has been rotated as $s \to s e^{i\vartheta}$. Substituting

$$u = \tfrac{1}{2}E||\boldsymbol{x}-\boldsymbol{y}||_\vartheta^2(\coth(2sE)-1) \quad (6.27)$$

we get

$$\Delta^{(\kappa,1)}_{\mu^2=0}(\boldsymbol{x},\boldsymbol{y}) = -\frac{i}{4\pi}e^{-i\boldsymbol{x}\cdot\boldsymbol{E}\boldsymbol{y}}\int_0^\infty du \frac{e^{-u-\frac{E}{2}||\boldsymbol{x}-\boldsymbol{y}||_\vartheta^2}}{\sqrt{u^2+Eu||\boldsymbol{x}-\boldsymbol{y}||_\vartheta^2}} = -\frac{i}{4\pi}e^{-i\boldsymbol{x}\cdot\boldsymbol{E}\boldsymbol{y}}K_0\Big(\frac{E}{2}||\boldsymbol{x}-\boldsymbol{y}||_\vartheta^2\Big), \quad (6.28)$$

with $K_0(z)$ the modified Bessel function of the second kind of order 0.

This implies that there is still a logarithmic ultraviolet divergence at $\boldsymbol{x}=\boldsymbol{y}$ due to the singular behavior of $K_0(z)$ at $z=0$. Since [[48], 9.7.2]

$$K_0(z) = \sqrt{\frac{\pi}{2z}}e^{-z}(1+O(z^{-1})) \quad \text{for } z\to\infty, \quad (6.29)$$

this also implies that the propagator $\Delta^{(\kappa,1)}_{\mu^2=0}(\boldsymbol{x},\boldsymbol{y})$ has an asymptotic exponential decay in the short variable $\boldsymbol{u}=\boldsymbol{x}-\boldsymbol{y}$ only for

$$\Re e(||\boldsymbol{u}||_\vartheta^2) > 0, \quad (6.30)$$

and thus only for $|\vartheta| < \frac{\pi}{2}$. We believe that for $\sigma < 1$ the asymptotic exponential decay in the long variable $\boldsymbol{v}=\boldsymbol{x}+\boldsymbol{y}$ also persists as long as $|\vartheta|<\frac{\pi}{2}$. We conclude that the propagator has a worse behavior in Minkowski space than in Euclidean space, but we can control its asymptotic behavior with the help of the parameter ϑ. Regarding the restriction $|\vartheta|<\frac{\pi}{2}$ as being part of the regularization of the field theory, one could then try to carry out the perturbative multiscale renormalization of the Minkowski space duality covariant field theory.

In the matrix space representation there is a similar problem, since the integrand in (6.11) is oscillatory. Thus bounding the magnitude of the integral by an integral over the magnitude of the integrand possibly produces a big error and might lead to poor estimates of the asymptotic behavior. One can use this approximation to show that the Minkowski space Grosse-Wulkenhaar propagator at $|\vartheta| = \frac{\pi}{2}$ has an exponential decay in each index separately, as in (6.18). To find the other bounds, however, one has to take care of the oscillating behavior of the integrand. The asymptotics of the special case (6.28) for $|\vartheta|<\frac{\pi}{2}$ raises the hope that the propagators at hand may have such an asymptotic behavior in position space so that the matrix basis is applicable.[6]

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APPENDIX A: PROOF OF PROPOSITION 3.18

The complex Landau wave functions are built on tensor products of the complex harmonic oscillator wave functions $f_m^{(E_\vartheta)}$ as

$$f_{mn}^{(E_\vartheta)}(\boldsymbol{x}) = \sqrt{\frac{E}{4\pi}}\mathsf{W}[|f_m^{(E_\vartheta)}\rangle\langle f_n^{(E_{-\vartheta})}|](\boldsymbol{x}). \quad (A1)$$

Using (2.17) and (3.4) we get

$$\begin{aligned} f_{mn}^{(E_\vartheta)}(t,x) &= \sqrt{\frac{E}{4\pi}}\int dk e^{iEkx/2} f_m^{(E_\vartheta)}(t+k/2) f_n^{(E_\vartheta)}(t-k/2) \\ &= \sqrt{\frac{E}{4\pi}}\sqrt{\frac{E_\vartheta}{2\pi}}\frac{1}{\sqrt{2^{m+n}m!n!}}\int dk e^{iEkx/2} \\ &\quad\times e^{-(1/4)E_\vartheta((t+k/2)^2+(t-k/2)^2)} \\ &\quad\times H_m(\sqrt{E_\vartheta/2}(t+k/2))H_n(\sqrt{E_\vartheta/2}(t-k/2)). \end{aligned} \quad (A2)$$

The generating function for the Hermite polynomials

[6]As these propagators are duality covariant, they also have a similar decay in momentum space.

$$\mathrm{e}^{-a^2(\xi^2-2\xi q)} = \sum_{m=0}^{\infty} \frac{1}{m!}(a\xi)^m H_m(aq) \tag{A3}$$

is used to obtain the generating function for the complex matrix basis functions as

$$\begin{aligned} K^{(E_\vartheta)}(\xi,\eta;t,x) &:= \sqrt{\frac{4\pi}{E}} \sum_{m,n=0}^{\infty} \sqrt{\frac{2^{m+n}}{m!n!}} (\sqrt{E_\vartheta/2}\xi)^m \\ &\quad \times (\sqrt{E_\vartheta/2}\eta)^n f_{mn}^{(E_\vartheta)}(t,x) \\ &= \sqrt{\frac{E_\vartheta}{2\pi}} \int dk \mathrm{e}^{\mathrm{i}Ekx/2} \mathrm{e}^{-(1/4)E_\vartheta((t+k/2)^2+(t-k/2)^2)} \\ &\quad \times \mathrm{e}^{-(1/2)E_\vartheta(\xi^2-2\xi(t+k/2)+\eta^2-2\eta(t-k/2))} \\ &= 2\mathrm{e}^{(1/2)E_\vartheta(-x_+^{(\vartheta)}x_-^{(\vartheta)}+2\xi x_-^{(\vartheta)}+2\eta x_+^{(\vartheta)}-2\eta\xi)}, \end{aligned} \tag{A4}$$

where we used the complex light-cone coordinates (3.20). The complex matrix basis functions can now be obtained by taking suitable derivatives with respect to the variables ξ and η to get

$$\begin{aligned} f_{mn}^{(E_\vartheta)}(t,x) &= \sqrt{\frac{E_\vartheta}{4\pi}} \frac{1}{\sqrt{m!n!}} \left(\frac{1}{E_\vartheta}\right)^{(m+n)/2} \frac{\partial^m}{\partial\xi^m} \\ &\quad \times \frac{\partial^n}{\partial\eta^n} K^{(E_\vartheta)}(\xi,\eta;t,x)\big|_{\xi=\eta=0} \\ &= \sqrt{\frac{E}{\pi}} \sqrt{m!n!} (E_\vartheta)^{(m-n)/2} \mathrm{e}^{-E_\vartheta x_+^{(\vartheta)}x_-^{(\vartheta)}/2} (x_-^{(\vartheta)})^{m-n} \\ &\quad \times \sum_{p=0}^{n} (E_\vartheta x_+^{(\vartheta)}x_-^{(\vartheta)})^{n-p} \frac{(-1)^p}{(m-p)!(n-p)!p!}, \end{aligned} \tag{A5}$$

where we assumed $m \geq n$. This last sum can be identified with an associated Laguerre polynomial

$$L_n^k(z) = \sum_{q=0}^{n} \frac{(n+k)!}{(n-q)!(k+q)!q!}(-z)^q \tag{A6}$$

by shifting $p \to q = n - p$. We finally arrive at

$$\begin{aligned} f_{mn}^{(E_\vartheta)}(t,x) &= (-1)^n \sqrt{\frac{E}{\pi}} \sqrt{\frac{n!}{m!}} (E_\vartheta)^{(m-n)/2} \mathrm{e}^{-E_\vartheta x_+^{(\vartheta)}x_-^{(\vartheta)}/2} \\ &\quad \times (x_-^{(\vartheta)})^{m-n} L_n^{m-n}(E_\vartheta x_+^{(\vartheta)}x_-^{(\vartheta)}). \end{aligned} \tag{A7}$$

An identical calculation for $n \geq m$ leads to the same result with $+ \leftrightarrow -$ and $m \leftrightarrow n$, yielding (3.19).

APPENDIX B: ONE-LOOP EFFECTIVE ACTION IN MATRIX SPACE

We will now reconstruct a classic result in quantum electrodynamics using the ϑ regularization and the complex matrix basis. In his seminal paper [26] Schwinger calculated the effective action for both a Dirac field and a Klein-Gordon field of charge e in a uniform external electromagnetic background in four spacetime dimensions. In a pure electric field E the one-loop correction for the Klein-Gordon theory (before charge renormalization) is given by the Lagrangian

$$\mathcal{L}^{(1)} = \frac{1}{16\pi^2} \int_0^\infty ds s^{-3} \mathrm{e}^{-\mu^2 s} \left(\frac{eEs}{\sin(eEs)} - 1\right). \tag{B1}$$

By deforming the contour of integration above the real axis one picks up the poles at $s = s_n = n\pi/eE$ for $n \in \mathbb{N}$ by the residue theorem; this leads to the famous formula for the probability per unit time and unit volume $2\Im\mathrm{m}(\mathcal{L}^{(1)})$ to create a particle-antiparticle pair in the scalar field theory. We will now show that the regularized matrix basis approach leads to the same result quite effortlessly. We work throughout in the notations of Sec. IV.

The generating functional for connected graphs $W[J, J^*]$ is defined via the vacuum-to-vacuum amplitude (4.1) in the presence of the external sources J and J^* as

$$W[J, J^*] = -\mathrm{i} \log Z_0[J, J^*]. \tag{B2}$$

It can be expressed in terms of the causal propagator (4.2) as [26]

$$\begin{aligned} W[J, J^*] &= \int d^4\boldsymbol{x} \int d^4\boldsymbol{y} J^*(\boldsymbol{x}) \Delta_c(\boldsymbol{x},\boldsymbol{y}) J(\boldsymbol{y}) \\ &\quad - \mathrm{i} \log \det(\Delta_F^{-1}\Delta_c), \end{aligned} \tag{B3}$$

with $\Delta_F = \Delta_c|_{E=0}$ the usual Feynman propagator. By using the ϑ regularization we can write

$$\begin{aligned} W[J, J^*] &= \int d^4\boldsymbol{x} \int d^4\boldsymbol{y} J^*(\boldsymbol{x}) \Delta^{(\kappa,1)}(\boldsymbol{x},\boldsymbol{y}) J(\boldsymbol{y}) \\ &\quad - \mathrm{i} \log \det\left(\frac{-\partial_\mu^2 - \mu^2}{\mathsf{K}_\mu^2 - \mu^2}\right)_\kappa, \end{aligned} \tag{B4}$$

which is understood in the limit $\kappa \to 0^+$ with

$$\left(\frac{-\partial_\mu^2 - \mu^2}{\mathsf{K}_\mu^2 - \mu^2}\right)_\kappa = \frac{-\partial_\mu^2 - \mathrm{e}^{-\mathrm{i}\kappa}\mu^2}{\mathrm{e}^{\mathrm{i}\kappa}\mathsf{P}^2(\frac{\pi}{2}-\kappa) - \mathrm{e}^{-\mathrm{i}\kappa}\mu^2 + (\mathrm{i}\partial_\perp)^2}. \tag{B5}$$

The effective action is now defined as the Legendre transformation

$$\begin{aligned} \Gamma[\phi_{\mathrm{cl}}, \phi_{\mathrm{cl}}^*] &= W[J, J^*] - \int d^4\boldsymbol{x} J(\boldsymbol{x}) \phi_{\mathrm{cl}}^*(\boldsymbol{x}) \\ &\quad - \int d^4\boldsymbol{x} J^*(\boldsymbol{x}) \phi_{\mathrm{cl}}(\boldsymbol{x}) \end{aligned} \tag{B6}$$

of $W[J, J^*]$ with respect to the "classical" fields $\phi_{\mathrm{cl}}(\boldsymbol{x})$ and $\phi_{\mathrm{cl}}^*(\boldsymbol{x})$ defined by

$$\begin{aligned} \phi_{\mathrm{cl}}(\boldsymbol{x}) &= \frac{\delta W[J, J^*]}{\delta J^*(\boldsymbol{x})} = \int d^4\boldsymbol{y} \Delta^{(\kappa,1)}(\boldsymbol{x},\boldsymbol{y}) J(\boldsymbol{y}), \\ \phi_{\mathrm{cl}}^*(\boldsymbol{x}) &= \frac{\delta W[J, J^*]}{\delta J(\boldsymbol{x})} = \int d^4\boldsymbol{y} J^*(\boldsymbol{y}) \Delta^{(\kappa,1)}(\boldsymbol{y},\boldsymbol{x}). \end{aligned} \tag{B7}$$

These equations may be inverted to give

$$J(\boldsymbol{x}) = (\mathsf{K}_\mu^2 - \mu^2)_\kappa \phi_{\rm cl}(\boldsymbol{x}) \quad \text{and} \\ J^*(\boldsymbol{x}) = -(\mathsf{K}_\mu^2 - \mu^2)_\kappa \phi^*_{\rm cl}(\boldsymbol{x}), \tag{B8}$$

and inserting this into (B6) yields

$$\begin{aligned}\Gamma[\phi_{\rm cl}, \phi^*_{\rm cl}] = &-\int d^4\boldsymbol{x}\int d^4\boldsymbol{y} (\mathsf{K}_\mu^2 - \mu^2)_\kappa \phi^*_{\rm cl}(\boldsymbol{x})\Delta^{(\kappa,1)}(\boldsymbol{x},\boldsymbol{y}) \\ &\times (\mathsf{K}_\mu^2 - \mu^2)_\kappa \phi_{\rm cl}(\boldsymbol{y}) - {\rm i}\log\det\Big(\frac{-\partial_\mu^2 - \mu^2}{\mathsf{K}_\mu^2 - \mu^2}\Big)_\kappa \\ &- \int d^4\boldsymbol{x}((\mathsf{K}_\mu^2 - \mu^2)_\kappa \phi_{\rm cl}(\boldsymbol{x}))\phi^*_{\rm cl}(\boldsymbol{x}) \\ &+ \int d^4\boldsymbol{x}((\mathsf{K}_\mu^2 - \mu^2)_\kappa \phi^*_{\rm cl}(\boldsymbol{x}))\phi_{\rm cl}(\boldsymbol{x}) \\ = &\,\mathcal{S}_0[\phi_{\rm cl}, \phi^*_{\rm cl}] + {\rm i}\log\det\Big(\frac{\mathsf{K}_\mu^2 - \mu^2}{-\partial_\mu^2 - \mu^2}\Big)_\kappa. \end{aligned} \tag{B9}$$

This is the *full* effective action of the quantum field theory; the quantum mechanical content is completely captured by the one-loop correction

$${\rm i}\,\log\,\det\Big(\frac{\mathsf{K}_\mu^2 - \mu^2}{-\partial_\mu^2 - \mu^2}\Big)_\kappa = W[0,0]. \tag{B10}$$

We define $W[0,0] =: \int d^4\boldsymbol{x}\,\mathcal{L}^{(1)}(\boldsymbol{x})$, with $\mathcal{L}^{(1)}$ the one-loop effective Lagrangian. The probability that no pair gets produced out of the vacuum is given by $|\langle 0, {\rm out}|0, {\rm in}\rangle|^2 = {\rm e}^{-2\Im\mathfrak{m}(W[0,0])}$.

The effective action is given by

$$W[0,0] = {\rm i}\,{\rm Tr}\log\Big(\frac{\mathsf{K}_\mu^2 - \mu^2}{-\partial_\mu^2 - \mu^2}\Big)_\kappa, \tag{B11}$$

and the eigenvalue equation for the operator $(\mathsf{K}_\mu^2 - \mu^2)_\kappa$ is given by (4.17). We adhere to Schwinger's convention by substituting $E \to eE/2$. Using the identity

$$\log\Big(\frac{a}{b}\Big) = \int_0^\infty \frac{ds}{s}({\rm e}^{{\rm i}sa} - {\rm e}^{{\rm i}sb}) \tag{B12}$$

which is valid for $\Im\mathfrak{m}(a) > 0$ and $\Im\mathfrak{m}(b) > 0$, the effective Lagrangian can be obtained through

$$\begin{aligned}\mathcal{L}^{(1)}(\boldsymbol{x}) &= {\rm i}\langle\boldsymbol{x}|\log\Big(\frac{\mathsf{K}_\mu^2 - \mu^2}{-\partial_\mu^2 - \mu^2}\Big)_\kappa|\boldsymbol{x}\rangle \\ &= {\rm i}\int_0^\infty\frac{ds}{s}\int\frac{d^2\boldsymbol{p}_\perp}{(2\pi)^2}{\rm e}^{-{\rm i}s\mu_\kappa^2}{\rm e}^{{\rm i}s||\boldsymbol{p}_\perp||^2_{\rm E}} \\ &\quad\times\Big(\sum_{m,n=0}^\infty f_{nm}^{(E_\vartheta)}(\boldsymbol{x}_\|)f_{mn}^{(E_\vartheta)}(\boldsymbol{x}_\|){\rm e}^{-2seE(m+(1/2))} \\ &\quad - \int\frac{d^2\boldsymbol{p}_\|}{(2\pi)^2}\,{\rm e}^{{\rm i}s||\boldsymbol{p}_\||^2_{\rm M}}\Big).\end{aligned} \tag{B13}$$

The integration over parallel momenta gives $\frac{1}{4\pi s}$.

We can now use Lemma 4.13 with $\boldsymbol{x} = \boldsymbol{y}$ and $a = 1$ to obtain

$$\begin{aligned}\mathcal{L}^{(1)}(\boldsymbol{x}) &= {\rm i}\int_0^\infty\frac{ds}{s}\int\frac{d^2\boldsymbol{p}_\perp}{(2\pi)^2}\,{\rm e}^{-{\rm i}s\mu_\kappa^2} \\ &\quad\times\Big(\frac{eE}{2\pi}{\rm e}^{-seE}\sum_{m=0}^\infty{\rm e}^{-2seEm} - \frac{1}{4\pi s}\Big){\rm e}^{{\rm i}s||\boldsymbol{p}_\perp||^2_{\rm E}} \\ &= \frac{1}{16\pi^2}\int_0^\infty\frac{ds}{s^2}\,{\rm e}^{-{\rm i}s\mu_\kappa^2}\Big(\frac{eE}{\sinh(eEs)} - \frac{1}{s}\Big),\end{aligned} \tag{B.14}$$

which is independent of $\boldsymbol{x}$. The integral converges near infinity since μ_κ^2 has a small imaginary part, and near 0 due to the $\frac{1}{s}$ subtraction of the free scalar propagator. By rotating the integration contour as $s \to -{\rm i}s$, and taking the limit $\kappa \to 0^+$, this Lagrangian coincides with Schwinger's result (B1). The case of four-dimensional Dirac fields can be treated in the same way, by transforming the spinor propagator to the scalar propagator; see [25], Appendix F for details. This analysis again exemplifies the fact that the matrix basis provides an easy way of doing otherwise cumbersome calculations in quantum electrodynamics.

APPENDIX C: PROOF OF LEMMA 4.13

For $m \geq n$ the explicit expression for the first eigenfunctions on the left-hand side of (4.14) is

$$\begin{aligned}f_{mn}^{(E_\vartheta)}(\boldsymbol{x}) &= (-1)^n\sqrt{\frac{E}{\pi}}\sqrt{\frac{n!}{m!}}{\rm e}^{-E_\vartheta x_+^{(\vartheta)}x_-^{(\vartheta)}/2} \\ &\quad\times(\sqrt{E_\vartheta}x_-^{(\vartheta)})^{m-n}L_n^{m-n}(E_\vartheta x_+^{(\vartheta)}x_-^{(\vartheta)}),\end{aligned} \tag{C1}$$

while the second eigenfunctions have a similar representation

$$\begin{aligned}f_{nm}^{(E_\vartheta)}(\boldsymbol{y}) &= (-1)^n\sqrt{\frac{E}{\pi}}\sqrt{\frac{n!}{m!}}{\rm e}^{-E_\vartheta y_+^{(\vartheta)}y_-^{(\vartheta)}/2} \\ &\quad\times(\sqrt{E_\vartheta}y_+^{(\vartheta)})^{m-n}L_n^{m-n}(E_\vartheta y_+^{(\vartheta)}y_-^{(\vartheta)}),\end{aligned} \tag{C2}$$

with the notations (2.31) and (3.20). These representations can also be used for $n > m$ due to the identity

$$(-1)^n r^{m-n}L_n^{m-n}(r^2) = (-1)^m r^{n-m}\frac{m!}{n!}L_m^{n-m}(r^2). \tag{C3}$$

The sum over n thus has the form

$$\begin{aligned}&\sum_{n=0}^\infty f_{mn}^{(E_\vartheta)}(\boldsymbol{x})f_{nm}^{(E_\vartheta)}(\boldsymbol{y})a^n \\ &= \frac{E}{\pi}\frac{(E_\vartheta x_-^{(\vartheta)}x_+^{(\vartheta)})^m}{m!}{\rm e}^{-E_\vartheta(x_+^{(\vartheta)}x_-^{(\vartheta)}+y_+^{(\vartheta)}y_-^{(\vartheta)})/2} \\ &\quad\times\sum_{n=0}^\infty n!\Big(\frac{a}{E_\vartheta x_-^{(\vartheta)}y_+^{(\vartheta)}}\Big)^n L_n^{m-n}(E_\vartheta x_+^{(\vartheta)}x_-^{(\vartheta)}) \\ &\quad\times L_n^{m-n}(E_\vartheta y_+^{(\vartheta)}y_-^{(\vartheta)}).\end{aligned} \tag{C4}$$

It can be done explicitly by using the identity [[44], 48.23.11]

$$\sum_{n=0}^{\infty} n!c^n L_n^{m-n}(\xi)L_n^{k-n}(\eta) = k!\mathrm{e}^{c\xi\eta}(1-\eta c)^{m-k}c^m L_k^{m-k}((1-\xi c)(\eta c-1)/c) \tag{C5}$$

with $k=m$, $\xi = E_\vartheta x_+^{(\vartheta)} x_-^{(\vartheta)}$, $\eta = E_\vartheta y_+^{(\vartheta)} y_-^{(\vartheta)}$, and $c = a/E_\vartheta x_-^{(\vartheta)} y_+^{(\vartheta)}$. This yields

$$\sum_{n=0}^{\infty} f_{mn}^{(E_\vartheta)}(\boldsymbol{x}) f_{nm}^{(E_\vartheta)}(\boldsymbol{y}) a^n = \frac{E}{\pi}\mathrm{e}^{-(\xi+\eta)/2}\mathrm{e}^{c\xi\eta}a^m L_m^0(\eta+\xi-c\xi\eta-c^{-1}), \tag{C6}$$

which after some elementary algebra gives (4.14).

APPENDIX D: MATRIX BASIS IN MOMENTUM SPACE

The complex Landau wave functions have special symmetries which will be useful in analyzing their Fourier transforms.

Proposition D.1. The complex Landau wave functions satisfy the relations

$$f_{mn}^{(E_\vartheta)}(E^{-1}t, E^{-1}x) = E f_{mn}^{(1/E_{-\vartheta})}(t,x), \tag{D2}$$

$$f_{mn}^{(E_\vartheta)}(-t, x) = (-1)^{m-n} f_{nm}^{(E_\vartheta)}(t,x), \tag{D3}$$

$$f_{mn}^{(E_\vartheta)}(t, -x) = f_{nm}^{(E_\vartheta)}(t,x), \tag{D4}$$

$$f_{mn}^{(E_\vartheta)}(x, t) = (-\mathrm{i})^{m-n} f_{nm}^{(E_{-\vartheta})}(t,x). \tag{D5}$$

Proof. The relation (D2) follows directly from the explicit expression (3.19) by noting that E and $x_\pm^{(\vartheta)}$ occur only in the combinations $\sqrt{E}x_\pm^{(\vartheta)}$ and $Ex_+^{(\vartheta)}x_-^{(\vartheta)}$. Time reversal $t \to -t$ only affects the term involving $x_{-\mathrm{sgn}(m-n)}^{(\vartheta)} \to -x_{\mathrm{sgn}(m-n)}^{(\vartheta)} = -x_{-\mathrm{sgn}(n-m)}^{(\vartheta)}$, which gives (D3). Parity $x \to -x$ sends $x_{-\mathrm{sgn}(m-n)}^{(\vartheta)} \to x_{\mathrm{sgn}(m-n)}^{(\vartheta)} = x_{-\mathrm{sgn}(n-m)}^{(\vartheta)}$, which shows (D4). Under interchange of t and x, we find $x_\pm^{(\vartheta)} \to \pm\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\vartheta}x_\mp^{(-\vartheta)}$, and thus $\sqrt{E_\vartheta}x_\pm^{(\vartheta)} \to \sqrt{E_{-\vartheta}}(\pm\mathrm{i}x_\mp^{(-\vartheta)})$ and $E_\vartheta x_+^{(\vartheta)}x_-^{(\vartheta)} \to E_{-\vartheta}x_+^{(-\vartheta)}x_-^{(-\vartheta)}$. Putting these transformations into (3.19) proves (D5).

Proposition D.6. The Fourier transformation of the complex Landau wave function $f_{mn}^{(E_\vartheta)}(\boldsymbol{x})$ is given by

$$\mathcal{F}[f_{mn}^{(E_\vartheta)}](\boldsymbol{k}) = f_{nm}^{(1/E_\vartheta)}(\boldsymbol{k}) = \frac{(-\mathrm{i})^{m-n}}{E} f_{mn}^{(E_\vartheta)}(\boldsymbol{E}^{-1}\boldsymbol{k}) \tag{D7}$$

with $\boldsymbol{E}^{-1}\boldsymbol{k} = -E^{-1}(k^1, k^0)$.

Proof. Denote momentum space derivatives as $\hat\partial_\mu := \frac{\partial}{\partial k^\mu}$. Using the explicit forms of the hyperbolic and Euclidean space wave operators given in Sec. II A, in Fourier space we find that these operators have the form

$$\begin{aligned}\frac{1}{2\pi}\int d^2\boldsymbol{x}(\mathsf{P}_\mu^2\phi)(\boldsymbol{x})\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} &= \frac{1}{2\pi}\int d^2\boldsymbol{x}\phi(\boldsymbol{x})\tilde{\mathsf{P}}_\mu^2\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \\ &= ((k_0^2-k_1^2)+2\mathrm{i}E(k^0\hat\partial^1 - k^1\hat\partial^0) \\ &\quad + E^2(\hat\partial_0^2-\hat\partial_1^2))\mathcal{F}[\phi](\boldsymbol{k})\end{aligned} \tag{D8}$$

and

$$\begin{aligned}\frac{1}{2\pi}\int d^2\boldsymbol{x}(\mathsf{P}_i^2\phi)(\boldsymbol{x})\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} &= \frac{1}{2\pi}\int d^2\boldsymbol{x}\phi(\boldsymbol{x})\tilde{\mathsf{P}}_i^2\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \\ &= ((k_0^2+k_1^2)+2\mathrm{i}E(k^0\hat\partial^1 + k^1\hat\partial^0) \\ &\quad - E^2(\hat\partial_0^2+\hat\partial_1^2))\mathcal{F}[\phi](\boldsymbol{k}).\end{aligned} \tag{D9}$$

From the explicit forms of the regularized wave operators (2.26), this gives

$$\begin{aligned}\mathcal{F}[\mathsf{P}^2(\vartheta)\phi](\boldsymbol{k}) &= \mathrm{e}^{\mathrm{i}\vartheta}E^2(-(\mathrm{e}^{\mathrm{i}\vartheta}\hat\partial_0^2 + \mathrm{e}^{-\mathrm{i}\vartheta}\hat\partial_1^2) \\ &\quad + 2\mathrm{i}E^{-1}(\mathrm{e}^{\mathrm{i}\vartheta}k^1\hat\partial^0 + \mathrm{e}^{-\mathrm{i}\vartheta}k^0\hat\partial^1) \\ &\quad + (\mathrm{e}^{-\mathrm{i}\vartheta}k_0^2 + \mathrm{e}^{\mathrm{i}\vartheta}k_1^2))\mathcal{F}[\phi](\boldsymbol{k}) \\ &= \mathrm{e}^{2\mathrm{i}\vartheta}E^2\tilde{\mathcal{P}}^2(-\vartheta)\mathcal{F}[\phi](\boldsymbol{k}),\end{aligned} \tag{D10}$$

where the differential operator $\tilde{\mathcal{P}}^2(-\vartheta)$ has the same form as $\tilde{\mathsf{P}}^2(-\vartheta)$ with the substitutions $\partial_\mu \to \hat\partial_\mu$, $x^\mu \to k^\mu$ and $E \to E^{-1}$. On the other hand, by substituting $\phi = f_{mn}^{(E_\vartheta)}$ we find

$$\mathcal{F}[\mathsf{P}^2(\vartheta)f_{mn}^{(E_\vartheta)}](\boldsymbol{k}) = 4E_\vartheta(m+\tfrac{1}{2})\mathcal{F}[f_{mn}^{(E_\vartheta)}](\boldsymbol{k}) \tag{D11}$$

and thus

$$\tilde{\mathcal{P}}^2(-\vartheta)\mathcal{F}[f_{mn}^{(E_\vartheta)}](\boldsymbol{k}) = 4E_\vartheta^{-1}(m+\tfrac{1}{2})\mathcal{F}[f_{mn}^{(E_\vartheta)}](\boldsymbol{k}). \tag{D12}$$

By Parseval's theorem the Fourier transforms of the matrix basis functions have the same normalization as the position space wave functions, from which we conclude the first equality of (D7). The second equality of (D7) follows from the symmetry relations (D2)–(D5).

As a simple application of this result, we can establish that the Feynman propagator for the free Klein-Gordon theory in the complex matrix basis possesses the same mass-shell singularities as in momentum space.

Lemma D.13. The Feynman propagator in the complex matrix basis is given by

$$(\Delta_F^{(\kappa)})_{mn;kl} := (G^{(\kappa)-1})_{mn;kl} = \int d^2\boldsymbol{k} \frac{f_{mn}^{(1/E_\vartheta)}(\boldsymbol{k}) f_{kl}^{(1/E_\vartheta)}(\boldsymbol{k})}{-||\boldsymbol{k}||_{\rm M}^2 + \mu_\kappa^2}, \tag{D14}$$

with

$$G_{mn;kl}^{(\kappa)} = \langle f_{nm}^{(E_{-\vartheta})} | (\partial_\mu^2 + \mu^2)_\kappa | f_{kl}^{(E_\vartheta)} \rangle \tag{D15}$$

and $\mu_\kappa^2 := {\rm e}^{-{\rm i}\kappa}\mu^2$ for $\vartheta = \frac{\pi}{2} - \kappa > 0$.

Proof. We simply relate the Klein-Gordon operator in the different basis sets. One has

$$(\partial_\mu^2 + \mu^2)_\kappa \delta(\boldsymbol{x} - \boldsymbol{y}) = \langle \boldsymbol{x} | (\partial_\mu^2 + \mu^2)_\kappa | \boldsymbol{y} \rangle = \sum_{n,m,k,l} f_{mn}^{(E_\vartheta)}(\boldsymbol{x}) G_{mn;kl}^{(\kappa)} f_{lk}^{(E_\vartheta)}(\boldsymbol{y}), \tag{D16}$$

and thus

$$\begin{aligned} G_{mn;kl}^{(\kappa)} &= \int d^2\boldsymbol{x} \int d^2\boldsymbol{y} f_{mn}^{(E_\vartheta)}(\boldsymbol{x}) \langle \boldsymbol{x} | (\partial_\mu^2 + \mu^2)_\kappa | \boldsymbol{y} \rangle f_{kl}^{(E_\vartheta)}(\boldsymbol{y}) \\ &= \int \frac{d^2\boldsymbol{k}}{(2\pi)^2} \int d^2\boldsymbol{x} \int d^2\boldsymbol{y} f_{mn}^{(E_\vartheta)}(\boldsymbol{x}) {\rm e}^{{\rm i}\boldsymbol{k}\cdot\boldsymbol{x}} (-||\boldsymbol{k}||_{\rm M}^2 + \mu_\kappa^2) \\ &\quad \times {\rm e}^{-{\rm i}\boldsymbol{k}\cdot\boldsymbol{y}} f_{kl}^{(E_\vartheta)}(\boldsymbol{y}) \\ &= \int d^2\boldsymbol{k} \mathcal{F}[f_{nm}^{(E_{-\vartheta})}](\boldsymbol{k})^* (-||\boldsymbol{k}||_{\rm M}^2 + \mu_\kappa^2) \mathcal{F}[f_{kl}^{(E_\vartheta)}](\boldsymbol{k}). \end{aligned} \tag{D17}$$

It follows that the Fourier transforms of the functions $f_{mn}^{(E_\vartheta)}$ diagonalize $G_{mn;kl}^{(\kappa)}$, and the result now follows from (D7).

APPENDIX E: PROOF OF PROPOSITION 6.1

The propagator is given by

$$\begin{aligned} \Delta^{(\kappa,\sigma)}(\boldsymbol{x},\boldsymbol{y}) &= \langle \boldsymbol{x} | (\sigma {\rm e}^{{\rm i}\kappa} {\sf K}^2(\vartheta) + \tilde\sigma {\rm e}^{{\rm i}\kappa} \tilde{\sf K}^2(\vartheta) - {\rm e}^{-{\rm i}\kappa}\mu^2)^{-1} | \boldsymbol{y} \rangle \\ &= {\rm e}^{-{\rm i}\kappa} \langle \boldsymbol{x} | \Big(\sigma {\sf P}^2(\vartheta) + \tilde\sigma \tilde{\sf P}^2(\vartheta) + {\rm e}^{2{\rm i}\vartheta} \sum_{k=2}^n (\sigma({\sf P}_i^2)_k \\ &\quad + \tilde\sigma(\tilde{\sf P}_i^2)_k) + {\rm e}^{2{\rm i}\vartheta}\mu^2 \Big)^{-1} | \boldsymbol{y} \rangle \end{aligned} \tag{E1}$$

where $\vartheta = \frac{\pi}{2} - \kappa > 0$ and we have set $\tilde\sigma = 1 - \sigma$. The (regularized) wave operators have the eigenvalue equations

$$\begin{aligned} &(\sigma {\sf P}^2(\vartheta) + \tilde\sigma \tilde{\sf P}^2(\vartheta)) f_{m_0 n_0}^{(E_\vartheta)}(\boldsymbol{x}_0) \\ &\quad = 4E_\vartheta(\sigma m_0 + \tilde\sigma n_0 + \tfrac{1}{2}) f_{m_0 n_0}^{(E_\vartheta)}(\boldsymbol{x}_0), \\ &(\sigma({\sf P}_i^2)_{k+1} + \tilde\sigma(\tilde{\sf P}_i^2)_{k+1}) f_{m_k n_k}^{(B_k)}(\boldsymbol{x}_k) \\ &\quad = 4B_k(\sigma m_k + \tilde\sigma n_k + \tfrac{1}{2}) f_{m_k n_k}^{(B_k)}(\boldsymbol{x}_k), \end{aligned} \tag{E2}$$

with $f_{m_k n_k}^{(B_k)}(\boldsymbol{x}_k)$ the usual Landau wave functions and $B_k \in \mathbb{R}_+$ for $k = 1, \ldots, n-1$. Using the identity

$$a^{-1} = \int_0^\infty ds {\rm e}^{-sa} \tag{E3}$$

which is valid for $\Re{\rm e}(a) > 0$, we find

$$\begin{aligned} \Delta^{(\kappa,\sigma)}(\boldsymbol{x},\boldsymbol{y}) &= -{\rm i}\,{\rm e}^{-{\rm i}\vartheta} \int_0^\infty ds {\rm e}^{-s\mu_\kappa^2} \sum_{m_0,n_0=0}^\infty f_{m_0 n_0}^{(E_\vartheta)}(\boldsymbol{x}_0) f_{n_0 m_0}^{(E_\vartheta)}(\boldsymbol{y}_0) {\rm e}^{-4sE_{-\vartheta}(\sigma m_0 + \tilde\sigma n_0 + (1/2))} \\ &\quad \times \prod_{k=1}^{n-1} \Big(\sum_{m_k,n_k=0}^\infty f_{m_k n_k}^{(B_k)}(\boldsymbol{x}_k) f_{n_k m_k}^{(B_k)}(\boldsymbol{y}_k) {\rm e}^{-4sB_k(\sigma m_k + \tilde\sigma n_k + (1/2))} \Big). \end{aligned} \tag{E4}$$

By Lemma 4.13 the sum over n_0 gives

$$\begin{aligned} &\frac{E}{\pi} \sum_{m_0=0}^\infty {\rm e}^{-4sE_{-\vartheta}(m_0 + (1/2))} \exp\Big(-\frac{E}{2} ||\boldsymbol{x}_0 - \boldsymbol{y}_0||_\vartheta^2 + ({\rm e}^{-4sE_{-\vartheta}\tilde\sigma} - 1) E(\boldsymbol{x}_0,\boldsymbol{y}_0)_\vartheta - {\rm e}^{-4sE_{-\vartheta}\tilde\sigma} {\rm i}\boldsymbol{x}_0 \cdot \boldsymbol{E}\boldsymbol{y}_0 \Big) \\ &\quad \times L_{m_0}^0(E||\boldsymbol{x}_0 - \boldsymbol{y}_0||_\vartheta^2 - 4\sinh^2(2sE_{-\vartheta}\tilde\sigma) E(\boldsymbol{x}_0,\boldsymbol{y}_0)_\vartheta - 2\sinh(4sE_{-\vartheta}\tilde\sigma) {\rm i}\boldsymbol{x}_0 \cdot \boldsymbol{E}\boldsymbol{y}_0). \end{aligned} \tag{E5}$$

The sum over m_0 can be performed by using the identity (4.20) with $t = {\rm e}^{-4sE_{-\vartheta}}$ to get

$$\begin{aligned} &\frac{E}{2\pi\sinh(2sE_{-\vartheta})} \exp\Big(-\frac{\cosh(2sE_{-\vartheta})}{2\sinh(2sE_{-\vartheta})} E||\boldsymbol{x}_0 - \boldsymbol{y}_0||_\vartheta^2 + \Big({\rm e}^{-4sE_{-\vartheta}\tilde\sigma} - 1 + 2{\rm e}^{-2sE_{-\vartheta}} \frac{\sinh^2(2sE_{-\vartheta}\tilde\sigma)}{\sinh(2sE_{-\vartheta})} \Big) E(\boldsymbol{x}_0,\boldsymbol{y}_0)_\vartheta \\ &\quad + \Big(-{\rm e}^{-4sE_{-\vartheta}\tilde\sigma} + {\rm e}^{-2sE_{-\vartheta}} \frac{\sinh(4sE_{-\vartheta}\tilde\sigma)}{\sinh(2sE_{-\vartheta})} \Big) {\rm i}\boldsymbol{x}_0 \cdot \boldsymbol{E}\boldsymbol{y}_0 \Big). \end{aligned} \tag{E6}$$

By using elementary hyperbolic identities, the terms proportional to $(\boldsymbol{x}_0,\boldsymbol{y}_0)_\vartheta$ can be simplified to

$${\rm e}^{-4sE_{-\vartheta}\tilde\sigma} - 1 + 2{\rm e}^{-2sE_{-\vartheta}} \frac{\sinh^2(2sE_{-\vartheta}\tilde\sigma)}{\sinh(2sE_{-\vartheta})} = \frac{\cosh(2s\tilde{E}_{-\vartheta})}{\sinh(2sE_{-\vartheta})} - \frac{\cosh(2sE_{-\vartheta})}{\sinh(2sE_{-\vartheta})} \tag{E7}$$

where we defined $\tilde{E}_{-\vartheta} := (1-2\tilde{\sigma})E_{-\vartheta} = (2\sigma - 1)E_{-\vartheta}$. Likewise, the terms proportional to $\mathrm{i}\boldsymbol{x}_0 \cdot \boldsymbol{E}\boldsymbol{y}_0$ can be rearranged to

$$-\,\mathrm{e}^{-4sE_{-\vartheta}\tilde{\sigma}} + \mathrm{e}^{-2sE_{-\vartheta}} \frac{\sinh(4sE_{-\vartheta}\tilde{\sigma})}{\sinh(2sE_{-\vartheta})} = -\frac{\sinh(2s\tilde{E}_{-\vartheta})}{\sinh(2sE_{-\vartheta})}. \tag{E8}$$

The triangle relation $\|\boldsymbol{x}_0 - \boldsymbol{y}_0\|^2_\vartheta = \|\boldsymbol{x}_0\|^2_\vartheta + \|\boldsymbol{y}_0\|^2_\vartheta - 2(\boldsymbol{x}_0, \boldsymbol{y}_0)_\vartheta$ allows us to combine further terms. The sums over n_k and m_k for $k = 1, \ldots, n-1$ are treated in exactly the same way, and putting everything together we finally get (6.2).

APPENDIX F: PROOF OF PROPOSITION 6.10

The $2n$-dimensional regularized LSZ wave operator in the matrix basis is given by (3.60) and (3.61) with $\theta_j = \theta$ and $\mathcal{D}^{(\sigma)}_{mn;kl} := \mathcal{D}^{j(\sigma)}_{mn;kl}$ for $j = 0, 1, \ldots, n-1$. Each of these operators have nonvanishing matrix elements only for

$$n_j - m_j = k_j - l_j =: \alpha_j \quad \text{for } j = 0, 1, \ldots, n-1. \tag{F1}$$

This is due to the $SO(1,1) \times SO(2)^{n-1}$ symmetry of the action. We can thus eliminate n components and write instead

$$D^{(\kappa,\sigma)}_{\boldsymbol{m},\boldsymbol{m}+\boldsymbol{\alpha};\boldsymbol{l}+\boldsymbol{\alpha},\boldsymbol{l}} = \mathrm{i}\mathcal{D}^{(\sigma)}_{m_0,m_0+\alpha_0;l_0+\alpha_0,l_0} - \mathrm{e}^{-\mathrm{i}\kappa}\sum_{i=1}^{n-1}\mathcal{D}^{(\sigma)}_{m_i,m_i+\alpha_i;l_i+\alpha_i,l_i} - \mathrm{e}^{-\mathrm{i}\kappa}\mu^2\delta_{\boldsymbol{ml}} \tag{F2}$$

with $\boldsymbol{\alpha} \in \mathbb{Z}^n$.

The n components of the wave operator (F2) are independent and its eigenvectors are therefore products of the eigenvectors of the individual matrices. The mass term is already diagonal and so are the terms proportional to $\tilde{\Omega}$. Thus for every $\alpha \in \mathbb{Z}$ we seek solutions of the eigenvalue equations

$$\sum_{l=0}^{\infty} \mathcal{D}^{(1/2)}_{m,m+\alpha;l+\alpha,l} U^{(\alpha)}_{lv} = v U^{(\alpha)}_{mv}. \tag{F3}$$

This equation has been solved in [4]. The eigenvectors are given by

$$U^{(\alpha)}_{mv} = \sqrt{\binom{\alpha+m}{m}\binom{\alpha+y}{y}}\left(\frac{2\sqrt{\Omega}}{1+\Omega}\right)^{\alpha+1} \times \left(\frac{1-\Omega}{1+\Omega}\right)^{m+y} {}_2F_1\left(\begin{matrix}-m, -y\\ 1+\alpha\end{matrix}\,\middle|\, -\frac{4\Omega}{(1-\Omega)^2}\right) \tag{F4}$$

and the eigenvalues are

$$v = \frac{4\Omega}{\theta}(2y + \alpha + 1) \tag{F5}$$

for $y \in \mathbb{N}_0$. As expected, this is the usual harmonic oscillator spectrum. The hypergeometric function ${}_2F_1$ appearing in (F4) with negative integer values in its first two arguments is an orthogonal Meixner polynomial. In particular, $U^{(\alpha)}_{mv}$ is symmetric in its lower indices.

For the full wave matrix the addition of the $\tilde{\Omega}$-term modifies the eigenvalues $v \to v'$ with

$$v' = \frac{4\Omega}{\theta}(2y + 2\sigma\alpha + 1). \tag{F6}$$

The complete matrix space wave operator in $D = 2n$ dimensions has the representation

$$D^{(\kappa,\sigma)}_{\boldsymbol{m},\boldsymbol{m}+\boldsymbol{\alpha};\boldsymbol{l}+\boldsymbol{\alpha},\boldsymbol{l}} = \sum_{\boldsymbol{v}} U^{(\boldsymbol{\alpha})}_{\boldsymbol{mv}}\left(\mathrm{i}v_0' - \mathrm{e}^{-\mathrm{i}\kappa}\sum_{i=1}^{n-1} v_i' - \mathrm{e}^{-\mathrm{i}\kappa}\mu^2\right) \times (U^{(\boldsymbol{\alpha})-1})_{\boldsymbol{lv}}, \tag{F.7}$$

where

$$U^{(\boldsymbol{\alpha})}_{\boldsymbol{mv}} = \prod_{j=0}^{n-1} U^{(\alpha_j)}_{m_j v_j} \tag{F8}$$

and

$$\mathrm{i}v_0' - \mathrm{e}^{-\mathrm{i}\kappa}\sum_{i=1}^{n-1} v_i' - \mathrm{e}^{-\mathrm{i}\kappa}\mu^2 = \frac{8\Omega}{\theta}\Bigg(\mathrm{i}y_0 + \mathrm{i}\left(\sigma\alpha_0 + \frac{1}{2}\right) - \mathrm{e}^{-\mathrm{i}\kappa}\mu^2\frac{\theta}{8\Omega} - \mathrm{e}^{-\mathrm{i}\kappa}\sum_{i=1}^{n-1} y_i - \mathrm{e}^{-\mathrm{i}\kappa}\sum_{i=1}^{n-1}\left(\sigma\alpha_i + \frac{1}{2}\right)\Bigg) \tag{F9}$$

with $y_j \in \mathbb{N}_0$. From the orthogonality relations for the Meixner polynomials it follows that

$$(U^{(\boldsymbol{\alpha})-1})_{\boldsymbol{mv}} = U^{(\boldsymbol{\alpha})}_{\boldsymbol{mv}}. \tag{F10}$$

In the following we will use the notation $U^{(\alpha)}_{mv} = U^{(\alpha)}_m(y)$ where v and y are related by (F5). Using the Schwinger parametrization this yields the propagator

$$\Delta^{(\kappa,\sigma)}_{\boldsymbol{m},\boldsymbol{m}+\boldsymbol{\alpha};\boldsymbol{l}+\boldsymbol{\alpha},\boldsymbol{l}} = \sum_{y_0,y_1,\ldots,y_{n-1}=0}^{\infty}\Big(\mathrm{i}v_0' - \mathrm{e}^{-\mathrm{i}\kappa}\sum_{i=1}^{n-1}v_i' - \mathrm{e}^{-\mathrm{i}\kappa}\mu^2\Big)^{-1}\prod_{j=0}^{n-1}(U^{(\alpha_j)}_{m_j}(y_j)U^{(\alpha_j)}_{l_j}(y_j))$$
$$= -\mathrm{e}^{\mathrm{i}\kappa}\frac{\theta}{8\Omega}\int_0^\infty dt\,\mathrm{e}^{\mathrm{i}t\mathrm{e}^{\mathrm{i}\kappa}(\sigma\alpha_0+(1/2))-t\sum_{i=1}^{n-1}(\sigma\alpha_i+(1/2))-t\frac{\theta\mu^2}{8\Omega}}\Big(\sum_{y_0=0}^{\infty}\mathrm{e}^{\mathrm{i}t\mathrm{e}^{\mathrm{i}\kappa}y_0}U^{(\alpha_0)}_{n_0}(y_0)U^{(\alpha_0)}_{l_0}(y_0)\Big)$$
$$\times\prod_{i=1}^{n-1}\Big(\sum_{y_i=0}^{\infty}\mathrm{e}^{-ty_i}U^{(\alpha_i)}_{m_i}(y_i)U^{(\alpha_i)}_{l_i}(y_i)\Big). \tag{F11}$$

The sum over y_0 can be performed by using the explicit formula for the eigenvectors (F4), and the hypergeometric identity [4]

$$\sum_{y=0}^{\infty}\binom{\alpha+y}{y}{}_2F_1\Big(\begin{matrix}-m,-y\\1+\alpha\end{matrix}\Big|w\Big){}_2F_1\Big(\begin{matrix}-l,-y\\1+\alpha\end{matrix}\Big|w\Big)z^y = \frac{(1-(1-w)z)^{m+l}}{(1-z)^{\alpha+m+l+1}}{}_2F_1\Big(\begin{matrix}-m,-l\\1+\alpha\end{matrix}\Big|\frac{zw^2}{(1-(1-w)z)^2}\Big)\quad\text{for } |z|<1 \tag{F12}$$

with $z=\mathrm{e}^{\mathrm{i}t\mathrm{e}^{\mathrm{i}\kappa}}(1-\Omega)^2(1+\Omega)^{-2}$ and $w=-4\Omega(1-\Omega)^{-2}$.

After some algebra this leads to

$$\sum_{y_0=0}^{\infty}\mathrm{e}^{\mathrm{i}t\mathrm{e}^{\mathrm{i}\kappa}y_0}U^{(\alpha_0)}_{m_0}(y_0)U^{(\alpha_0)}_{l_0}(y_0) = \frac{(1-\mathrm{e}^{\mathrm{i}t\mathrm{e}^{\mathrm{i}\kappa}})^{m_0+l_0}}{(1-\frac{\mathrm{e}^{\mathrm{i}t\mathrm{e}^{\mathrm{i}\kappa}}(1-\Omega)^2}{(1+\Omega)^2})^{\alpha_0+m_0+l_0+1}}\sqrt{\binom{\alpha_0+m_0}{m_0}\binom{\alpha_0+l_0}{l_0}}{}_2F_1$$
$$\times\Big(\begin{matrix}-m_0,-l_0\\1+\alpha_0\end{matrix}\Big|\Big(\frac{4\Omega}{(1+\Omega)^2}\Big)^2\Big(\frac{1+\Omega}{1-\Omega}\Big)^2\frac{\mathrm{e}^{\mathrm{i}t\mathrm{e}^{\mathrm{i}\kappa}}}{(1-\mathrm{e}^{\mathrm{i}t\mathrm{e}^{\mathrm{i}\kappa}})^2}\Big). \tag{F13}$$

Now we substitute $s=\mathrm{e}^{-t}$ (with Jacobian s^{-1}) and use the expansion of the hypergeometric functions

$${}_2F_1\Big(\begin{matrix}-m,-l\\1+\alpha\end{matrix}\Big|z\Big) = \sum_{u=\max(0,-\alpha)}^{\min(m,l)}\frac{m!l!\alpha!}{(m-u)!(l-u)!(\alpha+u)!u!}z^u. \tag{F14}$$

After a bit of algebra the various factorial terms can be recombined into the quantity (6.14), and we find

$$\sum_{y_0=0}^{\infty}\mathrm{e}^{\mathrm{i}t\mathrm{e}^{\mathrm{i}\kappa}y_0}U^{(\alpha_0)}_{m_0}(y_0)U^{(\alpha_0)}_{l_0}(y_0) = \sum_{u_0=\max(0,-\alpha_0)}^{\min(m_0,l_0)}\frac{s^{-\mathrm{i}\,\mathrm{e}^{\mathrm{i}\kappa}u_0}(1-s^{-\mathrm{i}\,\mathrm{e}^{\mathrm{i}\kappa}})^{m_0+l_0-2u_0}}{(1-\frac{(1-\Omega)^2}{(1+\Omega)^2}z^{-\mathrm{i}\,\mathrm{e}^{\mathrm{i}\kappa}})^{\alpha_0+m_0+l_0+1}}\Big(\frac{4\Omega}{(1+\Omega)^2}\Big)^{\alpha_0+2u_0+1}$$
$$\times\Big(\frac{1-\Omega}{1+\Omega}\Big)^{m_0+l_0-2u_0}\mathcal{A}(m_0,l_0,\alpha_0,u_0). \tag{F15}$$

The sums over y_i for $i=1,\ldots,n-1$ are performed in a completely analogous way. The only difference between the Euclidean and hyperbolic parts of the propagator is the additional factor $-\mathrm{i}\,\mathrm{e}^{\mathrm{i}\kappa}$ in the exponential of y_0. This simply changes $\mathrm{i}t\mathrm{e}^{\mathrm{i}\kappa}\to -t$ and $s^{-\mathrm{i}\,\mathrm{e}^{\mathrm{i}\kappa}}\to s$ everywhere in the above derivation, and we arrive finally at the expression (6.11).

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