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Critical behavior of supersymmetric $O(N)$ models in the large-N limit

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We derive a supersymmetric renormalization group (RG) equation for the scale-dependent superpotential of the supersymmetric $O(N)$ model in three dimensions. For a supersymmetric optimized regulator function, we solve the RG equation for the superpotential exactly in the large-N limit. The fixed-point solutions are classified by an exactly marginal coupling. In the weakly coupled regime there exists a unique fixed-point solution, for intermediate couplings we find two separate fixed-point solutions and in the strong coupling regime no globally defined fixed-point potentials exist. We determine the exact critical exponents both for the superpotential and the associated scalar potential. Finally, we relate the high-temperature limit of the four-dimensional theory to the Wilson-Fisher fixed point of the purely scalar theory.

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I. INTRODUCTION

Fixed points of the renormalization group (RG) play a fundamental role in statistical physics and quantum field theory [1,2]. Infrared (IR) fixed points dominate the long-distance behavior of correlation functions and are relevant for the understanding of continuous phase transitions and universal scaling laws [3]. Ultraviolet (UV) fixed points control the short-distance behavior of quantum field theories. It is widely believed that the existence of a UV fixed point is mandatory for a definition of quantum field theory on a microscopic level, e.g. asymptotic freedom of QCD or asymptotic safety of gravity [4,5]. In general, the fixed-point structure of a given theory depends on its field content, the spacetime dimensionality, the long-range or short-range nature of its interactions and the symmetries of the action.

Scalar field theories with a global $O(N)$ symmetry provide an important testing ground for fixed-point studies. In three dimensions, the $(\phi^2)^2$ theory displays a nontrivial IR fixed point which determines the second-order phase transition between an $O(N)$ symmetric and the symmetry broken phase as realized in many physical systems ranging from entangled polymers and water to ferromagnets or QCD with two massless flavors of quarks [3,6]. The $(\phi^2)^3$ theory also displays a line of first-order phase transitions whose end point, in the limit of many scalar fields, qualifies as a UV fixed point [7,8].

Supersymmetry represents the global symmetry which relates bosonic to fermionic degrees of freedom. Supersymmetric theories are important candidates for extensions of the standard model. It is important to understand how the fixed-point structure of a nonsupersymmetric theory differs from that of its supersymmetric extension, both in view of the IR and the UV behavior of the theory.

In this paper, we study fixed points of supersymmetric $O(N)$ models which consist of an N-component scalar field coupled to N Majorana fermions. We employ nonperturbative renormalization group methods a la Wilson, based on the integrating-out of momentum modes from a path-integral representation of the theory [9–11]. A particular strength of this continuum method is its flexibility, allowing for the study of theories with strong correlations and large couplings. Furthermore, optimization techniques are available to control the physics content within systematic approximations [12–14]. In the past, this method has been successfully employed for the study of critical phenomena in a variety of settings including scalar theories, fermions, gauge theories and gravity [5,15–24]. It has recently been extended to include supersymmetric theories [25–36]. Our prime interest here concerns the limit of many scalar fields $1/N \to 0$, where effects induced by the fields' anomalous dimensions are suppressed and a local potential approximation (LPA) becomes exact. Then full analytical fixed-point results are obtained for the fixed points in the supersymmetric theory, allowing for a complete analytical understanding of the theory, analogous to the purely scalar theory [37–39].

Supersymmetric $O(N)$ models have previously been investigated with Dyson-Schwinger equations [40] and with the large-N expansion [41,42]. The three-dimensional theory has also been studied at finite temperatures, where supersymmetry is softly broken [43,44]. The model has a peculiar phase structure concerning the breaking of the $O(N)$ symmetry: Additionally to the normal phases with a broken and an unbroken symmetry, a phase with two $O(N)$ symmetric ground states and a phase with one symmetric and one nonsymmetric ground state have been found. In addition, there exists a supersymmetric analogue of the Bardeen-Moshe-Bander phenomenon [7]. The fate of this phenomenon at finite N remains yet to be resolved [45–47].

The paper is organized as follows: First, we introduce the supersymmetric $O(N)$ model (Sec. II) and derive the nonperturbative flow equation for the superpotential in

LPA (Sec. III). We then solve this equation analytically in the large-N limit (Sec. III C) and analyze the resulting fixed-point solutions (Sec. IV). We compute the universal scaling exponents and compare our results with those in the nonsupersymmetric theory without fermions (Sec. V). We close with a discussion of our results (Sec. VI). Our conventions and a derivation of supersymmetric flow equations in superspace is found in the Appendix.

II. SUPERSYMMETRY

In this section, we recall the definition of three-dimensional supersymmetric $O(N)$ models, which are built from N real superfields

$$\Phi^i(x,\theta) = \phi^i + \bar\theta\psi^i(x) + \frac{1}{2}\bar\theta\theta F^i(x), \qquad i = 1,\ldots,N. \tag{1}$$

Each component of the superfield contains a real scalar field, a two-component Majorana spinor field and a real auxiliary field, $\Phi^i \sim (\phi^i, \psi^i, F^i)$. We shall use a Majorana representation with imaginary γ-matrices $\{\gamma^\mu\} = \{\sigma_2, i\sigma_3, i\sigma_1\}$. Then the metric in $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ takes the form $\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1)$. A Majorana spinor is real in this representation and $\bar\psi = (i\psi_2, -i\psi_1)$. The supersymmetry variation of the superfield is generated by the supercharge $\mathcal{Q}$ via $\delta_\epsilon\Phi^i = i\bar\epsilon\mathcal{Q}\Phi^i$, where the explicit form of the supercharge and further conventions are collected in appendixAppendix A. To construct a supersymmetric invariant action, we note that the F-term in the expansion (1) transforms under supersymmetry transformations into a spacetime divergence such that its spacetime integral is invariant.

In order to define an $O(N)$ symmetric, supersymmetric action we introduce the supercovariant derivatives

$$\mathcal{D} = \frac{\partial}{\partial\bar\theta} + i\not{\partial}\theta \quad \text{and} \quad \bar{\mathcal{D}} = -\frac{\partial}{\partial\theta} - i\bar\theta\not{\partial}, \tag{2}$$

which anticommute with the supercharges and thus map superfields into superfields. Since the theory should be $O(N)$ invariant, the superpotential only depends on the invariant composite superfield $R \equiv \frac{1}{2}\Phi^i\Phi_i$. In component form, it reads

$$R = \bar\varrho + (\bar\theta\psi_i)\phi^i + \frac{1}{2}\bar\theta\theta\Big(\phi^i F_i - \frac{1}{2}\bar\psi^i\psi_i\Big), \tag{3}$$

where the quantity $\bar\varrho \equiv \frac{1}{2}\phi^i\phi_i$ has been introduced. The starting point for further investigations will be the supersymmetric action

$$\mathcal{S} = \int d^3x\Big[-\frac{1}{2}\Phi^i\bar{\mathcal{D}}\mathcal{D}\Phi_i + 2NW\Big(\frac{R}{N}\Big)\Big]\Big|_{\bar\theta\theta} \tag{4}$$

which contains a kinetic term with supercovariant Laplacian $\bar{\mathcal{D}}\mathcal{D}$ as well as an interaction term, given by the superpotential W. We have already rescaled the fields and the superpotential with N. An expansion in component fields yields the Lagrangian density

$$\mathcal{L}_{\rm off} = \frac{1}{2}(-\phi^i\Box\phi_i - i\bar\psi^i\not{\partial}\psi_i + F^2) + W'\Big(\frac{\bar\varrho}{N}\Big)\phi_i F^i - \frac{1}{2}W'\Big(\frac{\bar\varrho}{N}\Big)\bar\psi^i\psi_i - W''\Big(\frac{\bar\varrho}{N}\Big)\frac{(\bar\psi^i\phi_i)(\psi^j\phi_j)}{2N}, \tag{5}$$

where primes denote derivatives with respect to $\bar\varrho/N$. Eliminating the auxiliary field F^i by its algebraic equation of motion, $F^i = -W'(\bar\varrho/N)\phi^i$, yields the on-shell Lagrangian density

$$\mathcal{L}_{\rm on} = -\frac{1}{2}\phi^i\Box\phi_i - \frac{i}{2}\bar\psi^i\not{\partial}\psi_i - \frac{1}{2}W'\Big(\frac{\bar\varrho}{N}\Big)\bar\psi^i\psi_i - \bar\varrho W'^2\Big(\frac{\bar\varrho}{N}\Big) - W''\Big(\frac{\bar\varrho}{N}\Big)\frac{(\bar\psi^i\phi_i)(\psi^j\phi_j)}{2N}. \tag{6}$$

From (6), we conclude that the potential for the bosonic field follows from the superpotential W via

$$V(\bar\varrho) = \bar\varrho W'^2\Big(\frac{\bar\varrho}{N}\Big). \tag{7}$$

Note that for a polynomial superpotential $W(\bar\varrho/N)$ which for large $\bar\varrho$ tends to $W \sim \bar\varrho^n$, we do *not* expect supersymmetry breaking in our nonperturbative renormalization group studies.

III. RENORMALIZATION GROUP

A. Supersymmetric flows

In order to analyze the phase transition and the low-energy behavior of supersymmetric sigma models, we resort to Wilsonian renormalization group techniques. Specifically, we adopt the framework of the effective average action based on the infinitesimal integrating-out of degrees of freedom with momenta q^2 larger than some infrared momentum scale k^2. In consequence, the effective action becomes a scale-dependent effective action Γ_k which interpolates between the microscopic action S in the UV and the full quantum effective action in the IR, where $k \to 0$. The scale dependence of Γ_k is given by an exact functional differential equation [48]

$$\partial_t\Gamma_k = \frac{1}{2}\,\mathrm{STr}\{\partial_t R_k(\Gamma_k^{(2)} + R_k)^{-1}\}, \tag{8}$$

where $t = \ln(k/\Lambda)$. The function $R_k(q^2)$ denotes the momentum cutoff. It obeys $R_k(q^2) \to 0$ for $k^2/q^2 \to 0$, $R_k(q^2) > 0$ for $q^2/k^2 \to 0$, and $R_k(q^2) \to \infty$ for $k \to \Lambda \to \infty$, where $k = \Lambda$ stands for the initial scale in the UV. The stability and convergence of the RG flow (8) is controlled through adapted, optimized choices of the momentum cutoff [12,39,49]. Furthermore, $\Gamma_k^{(2)}$ denotes the second functional derivative of Γ_k with respect to the fields according to

$$(\Gamma_k^{(2)})_{ab} = \frac{\vec{\delta}}{\delta \Psi^a} \Gamma_k \frac{\overleftarrow{\delta}}{\delta \Psi^b}, \tag{9}$$

where the indices a, b summarize field components, internal and Lorentz indices as well as coordinates. Note that Ψ is merely a collection of fields and not a superfield.

Following the construction in [27–31], it is essential that the regulator term ΔS_k preserves both the $O(N)$-symmetry and supersymmetry of the classical theory. Being quadratic in the fields it should be the superspace integral of $\Phi^i R_k(\bar{\mathcal{D}}\mathcal{D})\delta_{ij}\Phi^j$. Using the anticommutation relation $\{\mathcal{D}_k, \bar{\mathcal{D}}_l\} = -2i(\gamma^\mu)_{kl}\partial_\mu$ for the supercovariant derivatives, we have

$$\left(\frac{1}{2}\bar{\mathcal{D}}\mathcal{D}\right)^{2n} = (-\Box)^n, \tag{10}$$

such that a supersymmetric and $O(N)$-invariant regulator term is the superspace integral of

$$\Phi_i R_k(\bar{\mathcal{D}}\mathcal{D})\Phi^i = \Phi_i\left(r_1(-\Box) - r_2(-\Box)\frac{\bar{\mathcal{D}}\mathcal{D}}{2}\right)\Phi^i. \tag{11}$$

Expressed in component fields, we find

$$\Delta S_k = \frac{1}{2}\int (\phi, F) R_k^B \begin{pmatrix} \phi \\ F \end{pmatrix} + \frac{1}{2}\int \bar{\psi} R_k^F \psi. \tag{12}$$

In momentum space, $i\partial_\mu$ is replaced by p_μ and the bosonic and fermionic momentum cutoffs R_k^B and R_k^F, respectively, are of the form

$$R_k^B = \begin{pmatrix} p^2 r_2 & r_1 \\ r_1 & r_2 \end{pmatrix} \otimes \mathbb{1}_N$$
$$R_k^F = -(r_1 + r_2 \not{p}) \otimes \mathbb{1}_N. \tag{13}$$

Note that the requirements of manifest supersymmetry imposes a link between the bosonic and fermionic momentum cutoffs, leaving two free functions $r_1 \equiv r_1(p^2/k^2)$ and $r_2 \equiv r_2(p^2/k^2)$ at our disposal. Such supersymmetric cutoffs have been introduced for the $N = 1$ model in two and three dimensions in [28,31].

There exist no Majorana fermions in three Euclidean spacetime dimensions. With respect to the supersymmetric $O(N)$ model, we could thus analytically continue the flow equation in Minkowski spacetime to imaginary time or alternatively just ignore the fact that the Majorana condition is not compatible with Lorentz invariance in Euclidean spacetime [43]. Both approaches lead to identical flow equations in Euclidean spacetime, cf. [31].

B. Local potential approximation

Next, we turn to the supersymmetric RG flow in the local potential approximation. Here, one keeps the leading order term in a superderivative expansion such that the effective action (with Lorentzian signature) reads

$$\begin{aligned}\Gamma_k[\Phi] &= \int d^3x\left[-\frac{1}{2}\Phi^i\bar{\mathcal{D}}\mathcal{D}\Phi_i + 2NW_k\left(\frac{R}{N}\right)\right]\Bigg|_{\bar{\theta}\theta} \\ &= \frac{1}{2}\int d^3x(\partial_\mu\phi^i\partial^\mu\phi_i - i\bar{\psi}^i\not{\partial}\psi_i + F^2) \\ &\quad + \int d^3x\left(W_k'\frac{2\phi^iF_i - \bar{\psi}^i\psi_i}{2} - \frac{W_k''(\bar{\psi}^i\phi_i)(\psi^j\phi_j)}{2N}\right),\end{aligned} \tag{14}$$

where the prime denotes the derivative with respect to $\bar{\varrho}/N$. The flow of the renormalized superpotential $W_k(\frac{\bar{\varrho}}{N})$ in Euclidean space is obtained by projecting the flow (8) onto the term linear in the auxiliary field F and performing a Wick rotation (see Appendix B for its derivation in superspace). The function r_1 acts as IR regulator but not as UV regulator, in contrast to r_2 which serves both as IR and UV regulator. Thus, we use r_2 as regulator in what follows.[1] Then we find

$$\begin{aligned}\partial_t W_k = -\frac{1}{2}\int\frac{d^3p}{(2\pi)^3}\partial_t r_2\bigg(&\frac{N-1}{N}\frac{W_k'}{(1+r_2)^2p^2 + W_k'^2} \\ &+ \frac{1}{N}\frac{W_k' + 2(\bar{\varrho}/N)W_k''}{(1+r_2)^2p^2 + (W_k' + 2(\bar{\varrho}/N)W_k'')^2}\bigg).\end{aligned} \tag{15}$$

Similar to the bosonic $O(N)$ model, the flow receives contributions from the $N - 1$ Goldstone modes (the first term) and from the radial mode (second term).

Next, we specify the function $r_2(p^2/k^2)$. Following [12–14,31], we choose the optimized regulator function

$$r_2(p^2) = \left(\frac{k}{|p|} - 1\right)\theta(k^2 - p^2). \tag{16}$$

This choice implies $\partial_t r_2$ to vanish identically for $p^2 > k^2$, and the inverse propagators

$$(1 + r_2)^2p^2 + X = \begin{cases} p^2 + X & \text{for } p^2 > k^2 \\ k^2 + X & \text{for } p^2 < k^2 \end{cases}$$

become flat (momentum independent) in the regime where the right-hand side of (15) is nonvanishing. In the LPA, this is a solution to the general optimization condition of [12–14] and is therefore expected to lead to improved convergence and stability of the RG flow. Equally important, the momentum integrals in (15) can be performed analytically, leading to

$$\begin{aligned}\partial_k W_k = &-\frac{k^2}{8\pi^2}\left(1 - \frac{1}{N}\right)\frac{W_k'}{k^2 + W_k'^2} \\ &- \frac{k^2}{8\pi^2}\frac{1}{N}\frac{W_k' + 2(\bar{\varrho}/N)W_k''}{k^2 + (W_k' + 2(\bar{\varrho}/N)W_k'')^2}.\end{aligned} \tag{17}$$

With given initial condition $W_{k=\Lambda}(\bar{\varrho}/N) \equiv W(\bar{\varrho}/N)$ at the UV scale Λ, this flow equation uniquely determines the

[1] In preliminary studies, we did include the regulator r_1 and got almost identical results.

superpotential in the infrared limit $k \to 0$. For $N = 1$, it reduces to the three-dimensional Wess-Zumino model studied in [31].

In order to write the flow Eq. (17) in a scale-invariant form, it is convenient to define a dimensionless field variable ρ as well as a dimensionless superpotential w and a dimensionless scalar potential v. The canonical mass dimension of the fields and potentials are $[\bar{\varrho}] = d - 2$, $[V] = d$ and $[W] = d - 1$ in d spacetime dimensions. We therefore introduce the dimensionless quantities

$$\rho = \frac{8\pi^2}{N}\frac{\bar{\varrho}}{k} \quad \text{and} \quad w(\rho) = 8\pi^2 \frac{W(\frac{\bar{\varrho}}{N})}{k^2}. \tag{18}$$

Note that we have also rescaled an irrelevant numerical factor into the potential and the fields. It is understood that w is also a function of the RG scale parameter, though this is not spelled out explicitly. Similarly, we define the dimensionless bosonic potential v as

$$v(\rho) = \frac{8\pi^2}{N}\frac{\bar{\varrho}}{k}\bigg(\frac{W'(\frac{\bar{\varrho}}{N})}{k}\bigg)^2 \equiv \rho w'^2(\rho), \tag{19}$$

where (7) and (18) have been used. Thus, by substituting (18) into (17) we end up with the following flow equation for the dimensionless superpotential,

$$\partial_t w - \rho w' + 2w = -\frac{(1-\frac{1}{N})w'}{1+w'^2} - \frac{\frac{1}{N}(w' + 2\rho w'')}{1 + (w' + 2\rho w'')^2}. \tag{20}$$

C. Large-N limit

In the large-N limit, the Goldstone modes fully dominate the dynamics and the contribution of the radial mode becomes a subleading effect. It follows that the anomalous dimension of the Goldstone modes vanish, as no momentum-dependent two-point function exists that contributes to the running of the kinetic term of these modes to leading order in N. This is a particular feature of the bosonic $O(N)$ models [3] and their supersymmetric extensions.[2] Consequently, the LPA approximation becomes exact for $N \to \infty$.

In this limit, the RG equation for the first derivative of the superpotential $u(\rho) \equiv w'(\rho)$ becomes

$$\partial_t u + \partial_\rho u[1 - \rho - u^2 f(u^2)] = -u \tag{21}$$

with $f(x) = (3 + x)/(1 + x)^2$. We note that the second-order partial differential Eq. (20) has turned into a first-order one in this limit, which is solved analytically with the method of characteristics. The first characteristic reads $ue^t = \text{const.}$ and the second one is

$$\frac{\rho - 1}{u} - F(u) = \text{const.} \tag{22}$$

with

$$F(u) = \frac{u}{1 + u^2} + 2\arctan u \tag{23}$$

and $F'(u) = f(u^2)$. Altogether, we find

$$\frac{\rho - 1}{u} - F(u) = G(ue^t) \tag{24}$$

for all $\rho \geq 0$, where the function $G(ue^t)$ is determined by the boundary conditions for $u(\rho)$, imposed at the initial UV scale $k = \Lambda$. The validity of the solution (24) is confirmed by direct insertion into (21). For completeness, we also give the RG equation for the bosonic potential. Using (19) and (22), we obtain

$$\partial_t v + 3v - \rho v' = (v - \rho v')\frac{\rho - v}{(\rho + v)^2}. \tag{25}$$

In passing, we note that up to minor modifications Eq. (21) holds for general spacetime dimensions away from $d = 3$. The canonical mass dimension of u is one for all dimensions and the dependence on spacetime dimensionality, therefore only enters via the field variable leading to the replacement of $(-\rho)$ by $(2 - d)\rho$ in (21). This modifies the second characteristic equation whose solution is expressed in terms of the hypergeometric function for arbitrary dimension $d \neq 1$. Below, we restrict ourselves to the case $d = 3$.

IV. FIXED POINTS

A. Supersymmetric fixed points

Fixed points are the scale-independent solutions of (21), i.e. solutions satisfying $\partial_t u = 0$. Besides the Gaussian fixed-point solution $u_* \equiv 0$, nontrivial fixed points follow from (24) in the limit where $G(ue^t)$ becomes a t-independent constant. The classification of solutions of

$$\rho = 1 + H(u_*) + cu_*, \qquad H(u_*) = u_* F(u_*), \tag{26}$$

where $F(u_*)$ is given by (23), then depends only on the real parameter c. With $|u_*| \in [0, \infty)$ and for a fixed c, (26) identifies the range of achievable field values. Candidates for physical fixed points $u_*(\rho)$ are those solutions which extend over all fields $\rho \in [0, \infty)$. Figures 1 and 2 display the entire set of solutions to (26) for all c. Note that Fig. 1 shows the function $\rho(u_*)$, whereas the relation $u_*(\rho)$ is displayed in Fig. 2.

The space of solutions enjoys some internal symmetry. Since $H(u_*)$ is an even function, solutions only depend on the absolute value of c, i.e. any solution $u_*(\rho)$ with parameter c is equivalent to the reflected solution $-u_*(\rho)$ with parameter $-c$. Both solutions lead to identical scalar potentials v_* and therefore we may restrict our discussion to $c \geq 0$.

[2] For example, Yukawa-type systems may have large anomalous dimensions in the large-N limit [50].

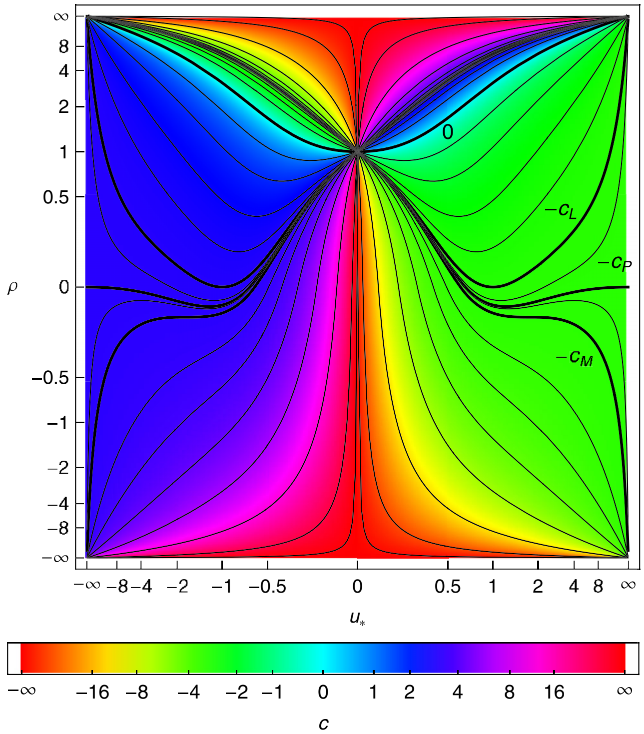

FIG. 1 (color online). Supersymmetric fixed-point solutions $\rho(u_*)$ for all fields ρ and all superfield potentials u_*, color-coded by the free parameter c (both axes are rescaled as $x \to \frac{x}{1+|x|}$ for display purposes). Thin lines are included to guide the eye, thick lines correspond to distinguished values for c ($|c| = 0, c_L, c_P, c_M$) as defined in main text.

We now discuss (26) in more detail. All curves pass through

$$(\rho, u_*) = (1, 0) \tag{27}$$

which follows immediately from (26) due to $H(0) = 0$. As can be seen from Fig. 2, the fixed-point solutions fall into two distinct classes, and solutions in the same class show the same global behavior. Depending on the value of c, the solution u_* is either defined for all real ρ or it has a turning point at $|\rho_s| < \infty$ and is only defined for $\rho \in [\rho_s, \infty)$. In the latter case, the solution has two branches bifurcating at $\rho = \rho_s$. The value of ρ_s will be determined below.

Next, we discuss some limiting cases of interest. For small u_*, we conclude from (26) that

$$\rho - 1 = cu_* + 3u_*^2 + \mathcal{O}(u_*^4). \tag{28}$$

Hence, the potential is analytical in $\rho - 1$ in the vicinity of $\rho = 1$ for all c, except for $c = 0$ where it becomes non-analytical with $u_* \propto \sqrt{\rho - 1}$. Equation (28) implies that all fixed-point solutions have one simple zero at $\rho = 1$ with finite $u_*'(1)$ except for $c = 0$ where $u_*'(1)$ diverges. Consequently, the scalar fixed-point potentials $v_* = \rho u_*^2$ possess two minima at

$$\rho = 0 \quad \text{and} \quad \rho = 1, \tag{29}$$

the first one being a simple zero. The second minimum is a double zero for $c \neq 0$ and a simple zero for $c = 0$.

In the large-u_* limit of (27), we find

$$\rho = \pi|u_*| + cu_* + \mathcal{O}(1/u_*^2). \tag{30}$$

Thus, the asymptotic behavior of u_* is given by

$$\begin{aligned} u_* &= \frac{\rho}{c + \pi} + \text{subleading } (u_* > 0), \\ u_* &= \frac{\rho}{c - \pi} + \text{subleading } (u_* < 0). \end{aligned} \tag{31}$$

If $|c| > c_P$, with

$$c_P = \pi, \tag{32}$$

the expansions extend towards $\rho \to \pm\infty$, respectively. Together with the boundedness of $H(u_*)$, we conclude that $u_*(\rho)$ is defined for all real ρ. The expansions correspond to asymptotically large fields $\rho \gg 1$ in the physical regime. At $|c| = c_P$ the leading term in (30) vanishes and, depending on the sign of c, one of the asymptotic solutions is replaced by $u_* \sim \rho^{-1/2}$, thus corresponding to a small field regime $\rho \ll 1$. For $|c| < c_P$, both expansions extend toward $\rho \to +\infty$. We conclude that u_* has, simultaneously, two asymptotic expansions for large positive ρ. This implies that v_* displays a loop consisting of two branches $v_<$ and $v_>$, which coincide at $\rho = \infty$ and possibly at some $\rho = \rho_s < \infty$ where u_* has infinite slope. The latter condition determines the turning point ρ_s as the simultaneous solution of

$$\rho_s = \frac{1 - u_s^2}{(1 + u_s^2)^2} \tag{33}$$

together with (26), leading to

$$|c| = \frac{1}{|u_s|}\left(\frac{u_s^2(3 + u_s^2)}{(1 + u_s^2)^2} + H(u_s)\right), \tag{34}$$

where $u_s \equiv u_*(\rho_s)$. The degenerate solutions extend over the whole physical regime $\rho \geq 0$, provided that $\rho_s \leq 0$. From (33) it follows that the equal sign holds for $u_s^2 = 1$ leading with (34) to $|c| = c_L$, where

$$c_L = \frac{1}{2}(\pi + 3) \approx 3.071. \tag{35}$$

For $|c| = c_L$, both $u_<$ and $u_>$ have infinite slope at vanishing field with the nonanalytical behavior

$$\frac{du_*}{d\rho} = \pm\frac{1}{\sqrt{\rho}} + \text{subleading} \tag{36}$$

and $u_s = \mp 1$ for $c = \pm c_L$ (see Fig. 3, left panel). In contrast, for $c_L < |c| < c_P$, the behavior at vanishing field is analytic. The turning point (33) exists for small $0 \leq |c| \leq c_M$ as long as $d^2\rho/du_*^2|_{\rho_s}$ does not vanish, which happens at $u_s^2 = 3$ leading with (34) to $|c| = c_M$, where

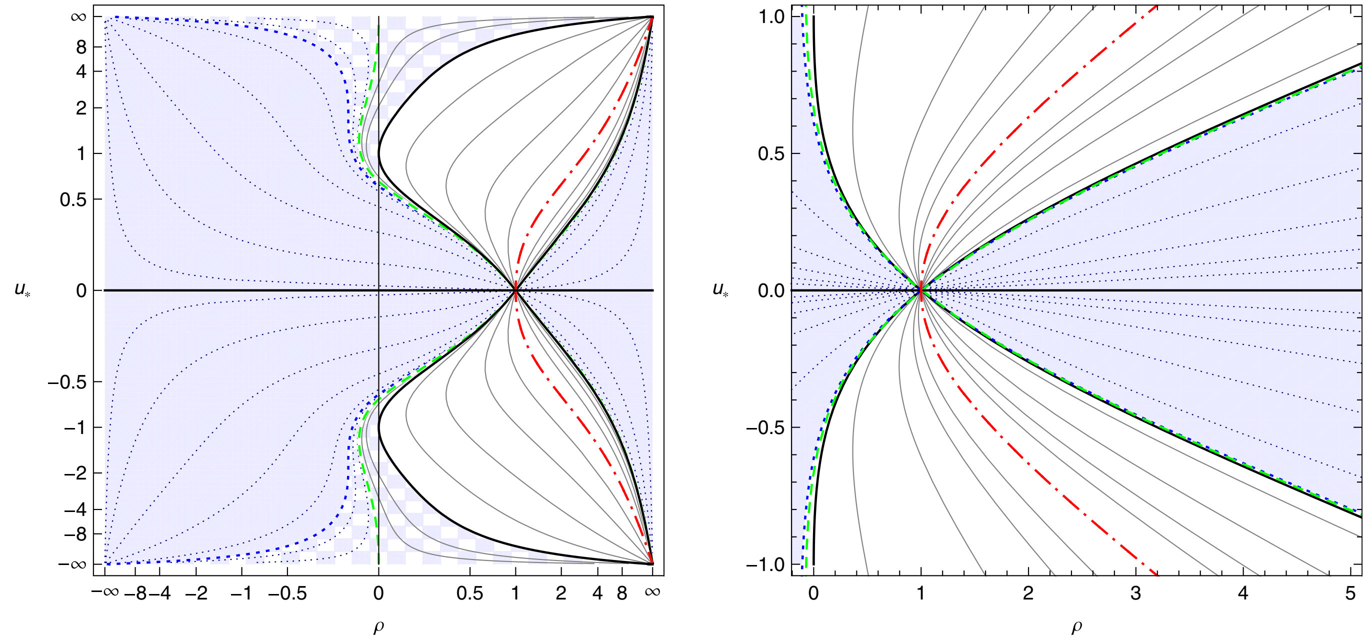

FIG. 2 (color online). Supersymmetric fixed-point solutions $u_*(\rho)$ according to (26), covering the entire parameter range for c. With decreasing c, fixed-point curves rotate counterclockwise around $(\rho, u_*) = (1, 0)$ starting with $c = \infty$ where $u_* = 0$ (horizontal line), passing through $c = 0$ (red, dash-dotted line), completing a rotation of 180° at $c = -\infty$ (horizontal line). Further special lines refer to $|c| = c_M$ (blue dashed line), $|c| = c_P$ (green, long dashed line), $|c| = c_L$ (black, thick solid lines), see main text. Left panel: fixed-point solutions for all fields (both axes are rescaled as $x \longrightarrow \frac{x}{1+|x|}$ for display purposes). Right panel: fixed-point solutions for physical fields in the vicinity of $\rho = 1$.

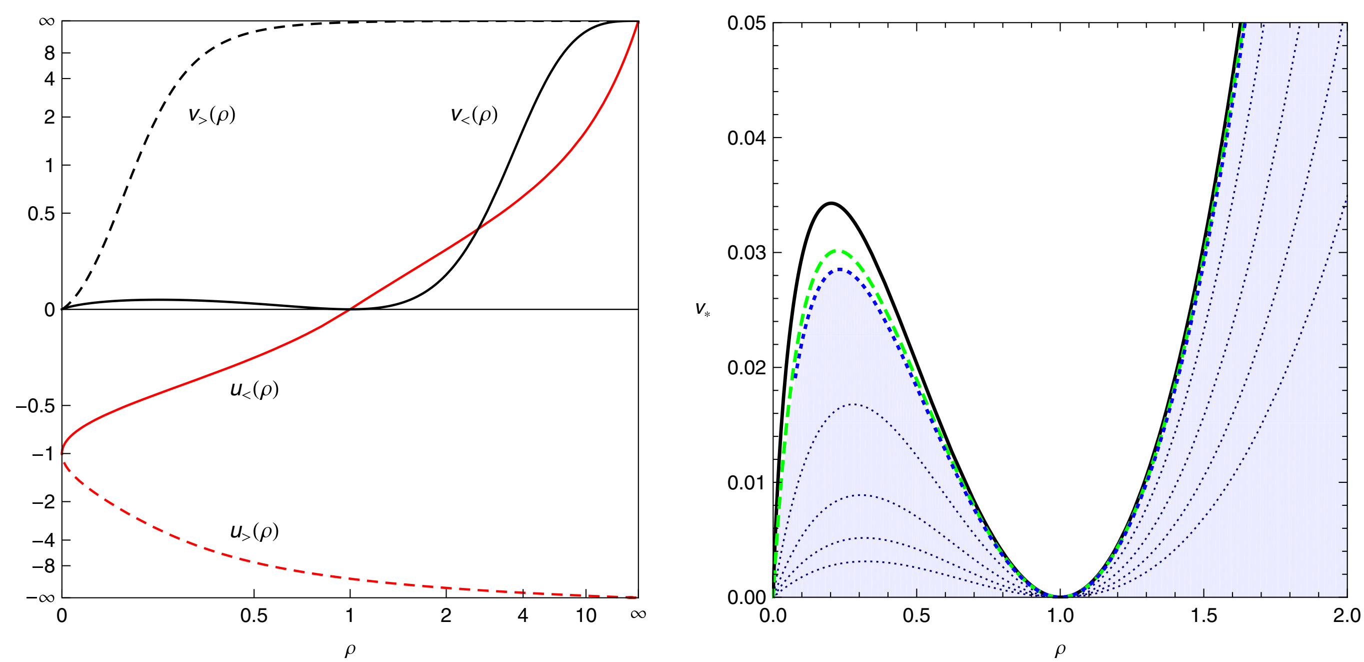

FIG. 3 (color online). Left panel: Fixed-point solutions u_* and fixed-point potentials $v_* = \rho u_*^2$ at $|c| = c_L$ showing the two branches $v_<, u_<$ (solid lines) and $v_>, u_>$ (dashed lines). Right panel: The scalar fixed-point potential $v_*(\rho)$ as a function of the parameter c with c_L (black, thick solid line), c_P (green, long dashed line), c_M (blue, short dashed line) and $c = a^n c_L$, $a = 2^{1/4}$ with $n = 1.0, 2.3, 3.6, 4.9$ (blue, dotted line). For c_L and c_P just one branch is plotted.

$$c_M = 2\left(\frac{\pi}{3} + \frac{5\sqrt{3}}{16}\right) \approx 3.177. \tag{37}$$

We note that $c_P < c_M$ and conclude that the fixed-point solutions in the parameter regime $c_P \leq |c| < c_M$ are single-valued in the physical regime but multivalued in the nonphysical regime $\rho < 0$. For all $c_M \leq |c|$, fixed-point solutions are single-valued on the entire real axis. In Fig. 3, right panel, the scalar fixed-point potential v_* for different values of c is displayed.

B. Exactly marginal coupling

Next, we discuss the physical meaning of the parameter c. To this end, we employ the polynomial expansion of the RG-time dependent superpotential $u(t, \rho)$ which satisfies the flow Eq. (21). For a typical initial condition $u_\Lambda = \tau_1(\rho - \rho_0)$, there always exists a node $\rho_0(t)$ around which we can perform a Taylor expansion:

$$u(t, \rho) = \sum_{n=1} \frac{1}{n!} \tau_n(t)(\rho - \rho_0(t))^n. \tag{38}$$

Inserting this ansatz into the flow Eq. (21), we read off the flow equations for ρ_0 and the couplings τ_n entering the Taylor expansion

$$\partial_t \rho_0 = 1 - \rho_0 \tag{39}$$

$$\partial_t \tau_1 \equiv 0 \tag{40}$$

$$\partial_t \tau_2 = 6\tau_1^3 + \tau_2 \tag{41}$$

and similarly to higher order. Several comments are in order at this point. First, the running of the vev $\rho_0(t)$ is independent of all the other local couplings. This property is typical for a supersymmetric flow and has previously been observed in [29,31]. The fixed point is obtained for $\rho_0 = 1$. Second, the system of algebraic equations describing the t-independent fixed-point couplings can be solved recursively. This leads to fixed-point couplings $\tau_n(\tau_1)$ for all $n \geq 2$ as functions of τ_1. Inserting (38) into the expansion of the scalar field potential $v = \rho u^2 = \sum_{n=2} \lambda_n / n!(\rho - \rho_0)^n$ and evaluating it on the fixed point leads to the fixed-point values

$$\lambda_2 = 2\tau_1^2 \tag{42}$$

$$\lambda_3 = 6\tau_1^2(1 - 6\tau_1^2) \tag{43}$$

$$\lambda_3 = -24\tau_1^4(1 - 45\tau_1^2) \tag{44}$$

and similarly to higher order. Clearly, the weak (strong) coupling regimes correspond to small (large) λ_2 and hence small (large) τ_1, respectively. Also, on the level of the scalar field potential the critical behavior is independent of the sign of τ_1. Finally, and most importantly, the coupling τ_1 remains unrenormalized under the supersymmetric RG flow (40). Therefore, τ_1 corresponds to an *exactly marginal coupling*, and fixed points can be classified according to the value of the linear (dimensionless) superfield interaction τ_1 which relates to the free parameter c in the analytical solution (26) as

$$c = \frac{1}{\tau_1}. \tag{45}$$

This relation can be shown by inserting expansion (38) into the fixed-point Eq. (26). The presence of the exactly marginal coupling τ_1 explains the existence of a line of fixed points.

C. Line of fixed points

In summary, the following picture has emerged. Fixed-point solutions are characterized by the dimensionless linear superfield coupling $\tau_1 = 1/c$ in the vicinity of the node $\rho_0 \neq 0$. In the *weakly coupled regime*

$$c_P \leq |c|, \tag{46}$$

a unique fixed-point solution exists covering the whole physical domain $\rho \geq 0$. This includes the Gaussian fixed-point $\tau_1 = 0$. In the *intermediate coupling regime*

$$c_L \leq |c| < c_P, \tag{47}$$

two separate fixed-point solutions $u_<$ and $u_>$ exist. The former solution has a node at $\rho_0 = 1$ whereas the other solution has no node, see Fig. 3, left panel. Therefore, the corresponding scalar field potentials $v_<$ ($v_>$) have two minima at (29) (one minimum at $\rho = 0$). Both are analytical functions of ρ in the vicinity of their global minima. For $|c| = c_L$, the potential becomes nonanalytical for either of them at $\rho = 0$ in a manner reminiscent of the Bardeen-Moshe-Bander phenomenon in the purely scalar theory [7]. In the *strong coupling regime*

$$|c| < c_L, \tag{48}$$

the theory becomes so strongly coupled that $du/d\rho|_{\rho_s}$ diverges in the physical regime, and hence no fixed-point solution exists which extends over all fields. Therefore, the supersymmetric $O(N)$ model displays a line of fixed points which bifurcates at $|c| = c_P$ into two fixed points, and then terminates at $|c| = c_L$.

Finally, we note that the solution with $c = 0$ is closely linked to the Wilson-Fisher fixed point in the purely bosonic model [37–39]. The precise relation is discussed in Sec. V C below.

V. UNIVERSALITY

A. Critical exponents

Fixed-point solutions are characterized by universal critical scaling exponents. The exponents can be deduced from the RG equations in several ways. Within a polynomial approximation up to order n, we expand

$u(\rho) = \sum_{i=1}^{n} b_i(\rho - b_0)^i/i!$ in terms of the $n+1$ couplings $b_i(t)$. From their beta-functions $\beta_i \equiv \partial_t b_i$, the universal exponents θ^I follow as the negative of the eigenvalues λ^I of their stability matrix $B_i^j = \partial\beta_i/\partial b_j|_{b=b^*}$ as $Bv^I = \lambda^I v^I = -\theta^I v^I$ with eigenvectors v^I. Using the flow equation, we find

$$\theta = 1, 0, -1, -2, -3, \cdots \tag{49}$$

both numerically and analytically. In fact, the LPA approximation has become exact in the large-N limit, and hence the correct scaling exponents are achieved to every order in the polynomial approximation. We note that this analysis relies on local information of the RG flow in the vicinity of $u = 0$, showing that the scaling (49) is achieved mathematically for all $0 < |c| < \infty$. Physically, however, the analysis is not sensitive to the global behavior of the solution, and consequently cannot detect that $|c| = c_L$ denotes a physical endpoint. Also, the case $c = 0$ requires special care as an analytical expansion about $u = 0$ is no longer applicable.

B. Eigenperturbations

Interestingly, the critical exponents and eigenperturbations can also be calculated analytically without resorting to a polynomial expansion. To that end, we consider small fluctuations δu about the fixed-point superpotential such that $u(t,\rho) = u_*(\rho) + \delta u(t,\rho)$. Linearizing the flow equation in δu leads to the fluctuation equation

$$\partial_t \delta u = \frac{u_*}{u_*'}\left(\partial_\rho - \frac{(u_* u_*')'}{u_* u_*'}\right)\delta u, \tag{50}$$

where primes denote a derivative with respect to the function's argument. Since the right-hand side is independent of t, the differential Eq. (50) can be factorized via separation of variables $\delta u(t,\rho) = f(t)g(\rho)$ with

$$(\ln f)' = \lambda \qquad (\ln g)' = \lambda(\ln u_*)' + (\ln u_* u_*')', \tag{51}$$

where λ denotes the eigenvalue. Integration leads to the exact solution for the linear perturbation of the fixed-point superpotential

$$\delta u = Ce^{\lambda t} u_*^{\lambda+1} u_*'. \tag{52}$$

The allowed range of values for the eigenvalues λ is determined using regularity conditions for the eigenperturbations. To that end, we recall that the fixed-point potential u_* grows linearly with the field for large ρ, see (30), and hence $\delta u \propto e^{\lambda t}\rho^{\lambda+1}$. Furthermore, in the vicinity of the node we have (28), which for $c \neq 0$ leads to a finite u' (meaning $0 < u' < \infty$). We thus find

$$\delta u \propto e^{\lambda t}(\rho - 1)^{\lambda+1}. \tag{53}$$

Regularity of the perturbations requires non-negative integer values for the exponent $\lambda + 1$. Since the critical exponents are defined as the negative eigenvalues, we thus obtain (49).

Note that this line of reasoning assumes analyticity of the perturbation at the node which holds for all $c \neq 0$. For $c = 0$, u_* is nonanalytical at (27) but u_*^2 instead is analytical and has a simple zero with finite $(u_*^2)'|_{u_*=0}$. Therefore, we use (52) to relate the (regular) fluctuations of u^2 to u_*^2, leading to

$$\delta u^2 = Ce^{\lambda t}(u_*^2)^{1/2(\lambda+1)}(u_*^2)'. \tag{54}$$

Again, analyticity implies that the exponent $(\lambda+1)/2$ is a non-negative integer and hence

$$\theta = 1, -1, -3, -5, -7, \cdots \tag{55}$$

Here, we recognize the universal critical exponents of the $3d$ spherical model [49]. We stress, however, that this solution is not a proper fixed-point solution in the usual sense because it is limited to field values with $\rho \geq 1$.

Finally, we extend the analysis of linear perturbations to those of the function u^2 and the scalar potential $v = \rho u^2$. We begin with $u^2 = u_*^2 + \delta u^2$. An analytical solution is found by using the identity $\delta u^2 = 2u_*\delta u$ together with (52), leading to

$$\delta u^2 = 2Ce^{\lambda t} u_*^{\lambda+2} u_*'. \tag{56}$$

Note that the degree in u_* has increased by one unit. Employing the same reasoning as above for $c \neq 0$, we conclude that the set of available negative eigenvalues is

$$\theta = 2, 1, 0, -1, -2, -3, \cdots \tag{57}$$

Physically, the appearance of the eigenmode with eigenvalue -2 is due to the mass term squared, a term which on dimensional grounds is available in u^2 but not in u.

Finally, using (25), (56), and (21), the linear eigenperturbations about the scalar potential $v(t,\rho) = v_* + \delta v(t,\rho)$ are found as

$$\delta v = 2Ce^{\lambda t} u_*^{\lambda+2}\{u_* + u_*'[1 - u_*^2 f(u_*^2)]\}. \tag{58}$$

Close to $u_* = 0$, the term in square brackets reduces to 1, and the curly bracket becomes u_*' which is finite at $u_* = 0$. Therefore, regularity of eigenperturbations again implies (57). In a nonsupersymmetric scalar theory, the potential is not constrained to be of the product form (19) and an additional eigenvalue -3 becomes available related to redundant shifts of the potential.

We conclude that supersymmetry is responsible for the absence of the redundant eigenvalue -3 in the scalar potential, and for relating its two relevant eigendirections with eigenvalues -1 and -2 with the sole relevant eigendirection with eigenvalue -1 of the derivative of the superpotential.

C. Wilson-Fisher fixed point

It is interesting to clarify how the supersymmetric model and its fixed points fall back onto those of the $3d$ nonsupersymmetric scalar theory in the same approximation [37–39]. To that end, we consider the $4d$ supersymmetric

$O(N)$ at finite temperature. The temperature is implemented using the imaginary time formalism which on the level of the flow equation amounts to the replacements [10,12,15]

$$\int_{-\infty}^{\infty} \frac{dq_0}{2\pi} f(q_0) \to T \sum_{n=-\infty}^{\infty} f(q_0 = 2\pi c_n T). \quad (59)$$

Here, $2\pi c_n T$ denotes the nth Matsubara frequency with $c_n = n$ for bosons and $c_n = n + \frac{1}{2}$ for fermions. The temperature imposes periodic (antiperiodic) boundary conditions for bosons (fermions) and, consequently, softly breaks global supersymmetry. Within a derivative expansion, the relevant momentum integrals are performed analytically using the four-dimensional version of (15) together with (59) and the optimized momentum cutoff (16).

We are interested in the large-scale behavior $k/T \to 0$. Because of (59), all fermions and bosons with a nonvanishing Matsubara mass will decouple from the system, except for the bosonic zero mode. In this limit, the $4d$ supersymmetric model undergoes a dimensional reduction to a $3d$ nonsupersymmetric theory where all fermions have decoupled. In the large-N limit, the RG equation for the potential of the remaining bosonic zero mode in LPA is given by

$$\partial_t z = -2z + \rho z' - \frac{1 - z}{(1 + z)^2} z' \quad (60)$$

where z is related to the scalar field potential by $v(\rho) = \rho z(\rho)$. The key difference to the supersymmetric system studied previously is that the function z is no longer constrained to be the square of a superpotential derivative w'. Relaxing this constraint allows for an additional fixed-point solution, which follows from integrating (60) analytically. The general solution reads

$$\frac{\rho - 1}{\sqrt{z}} - \frac{\sqrt{z}}{1 + z} - 2 \arctan\sqrt{z} = B(ze^{2t}), \quad (61)$$

where $B(ze^{2t})$ is fixed through initial conditions. The solution for negative z is found by analytical continuation. In particular, (60) has a Wilson-Fisher fixed-point solution $z_* \neq 0$ with $z(\rho = 1) = 0$ corresponding to (61) with $B = 0$. The solution extends over all ρ with one unstable direction, see Fig. 4. The eigenperturbations $z = z_* + \delta z$ are found analytically leading to (54) with the replacements $\delta u \to \delta z$ and $u_*^2 \to z_*$. Hence, the universal eigenvalues are identical and given by (55).

The similarities and differences between the Wilson-Fisher fixed-point solution of the purely scalar theory and the $c = 0$ "would-be" Wilson-Fisher fixed point of the supersymmetric partner theory can also be appreciated from the behavior at small and large fields. In fact, for $\rho \geq 1$, $z_*(\rho)$ is positive and related to the real superpotential by

$$z_*(\rho) = w_*'(\rho)^2. \quad (62)$$

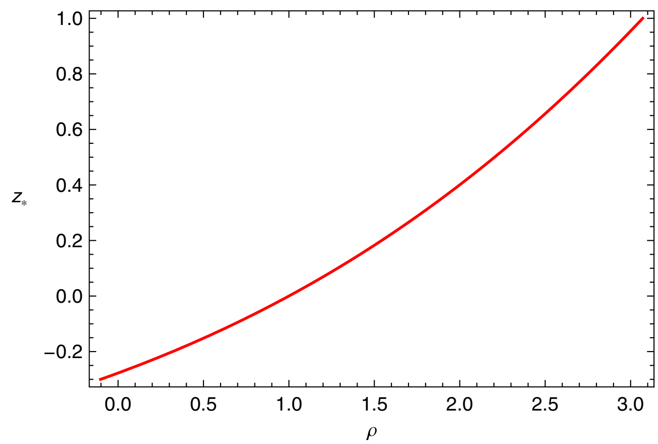

FIG. 4 (color online). The Wilson-Fisher fixed-point solution $z_*(\rho)$ of (60).

In turn, $z_*(\rho)$ is negative for all $\rho < 1$. Interestingly, this solution is still visible in the supersymmetric theory where it corresponds to a purely imaginary "superpotential" with

$$w_*'(\rho) = \pm i\sqrt{-z_*(\rho)}. \quad (63)$$

Hence, provided that a purely imaginary superpotential is meaningful in the supersymmetric theory, the $c = 0$ solution can be extended to a valid supersymmetric Wilson-Fisher fixed point for all ρ. However, the structure of the Lagrangian imposed by supersymmetry implies that the field-dependent fermion mass term is proportional to w_*' and the Yukawa-type fermion-boson interaction proportional to w_*'' all become purely imaginary. Most importantly, a purely imaginary w_*' for small fields implies that the scalar potential obeys $v_*(\rho) = \rho w_*'^2 < 0$ for all fields within $0 < \rho < 1$. Unbroken global supersymmetry requires that the dimensionful $V_k(\bar{\varrho})$ remains positive for all fields and scales. In the infrared limit $k \to 0$, reinserting powers of k, the dimensionful potential approaches $V(\bar{\varrho}) = 64\pi^2\bar{\varrho}^3/N^2 \geq 0$. Hence, our results state that this potential can be approached arbitrarily close from within a phase with $O(N)$ symmetry and global supersymmetry.

VI. CONCLUSIONS

We have studied fixed points of supersymmetric $O(N)$ symmetric Wess-Zumino models in the limit of many components $N \to \infty$ in three dimensions with the help of the renormalization group. We have solved the theory analytically, showing that it displays a line of nontrivial fixed points solely parametrized by the exactly marginal linear superfield coupling. The fixed points are non-Gaussian, yet they display Gaussian exponents similar to the line of fixed points observed in the bosonic $(\phi^2)^3$ theory. The line of fixed points contains the Gaussian fixed point and therefore all fixed points are continuously linked to the Gaussian one. With increasing superfield coupling, the line of fixed points bifurcates into two fixed-point solutions, both of which terminate at a critical coupling (35) below which no fixed-point solutions exist which extend over all physical fields. One of these solutions has its minimum at $\rho_0 = 0$, the other at $\rho_0 \neq 0$. Interestingly, remnants of the non-Gaussian scaling exponents of the $3d$ spherical model (55) become visible for asymptotically large superfield coupling. However, the fixed-point solution does not extend over all fields in the supersymmetric case, except if the superfield potential becomes purely imaginary for small fields.

From a structural point of view, the main impact of global supersymmetry on the critical behavior in comparison with the purely scalar theory is summarized as follows. First, for unbroken global supersymmetry the minimum value of the scalar potential V is zero. Hence, the irrelevant eigenmode with eigenvalue -3 corresponding to overall shifts in the potential is absent from the supersymmetric eigenvalue spectrum. Second, the quartic and sextic coupling of the scalar potential are no longer independent. Hence, in the supersymmetric theory criticality is achieved by tuning only one parameter as opposed to the tuning of two parameters in the corresponding purely bosonic theory. This is reflected in the sole negative eigenvalue for u as opposed to the two negative eigenvalues for both u^2 and v. Finally, at the coupling $|c| = c_L$ (35) the supersymmetric model shares similarities with the Bardeen-Moshe-Bander phenomenon in the bosonic theory [8]. The logarithmic singularity observed in [8] is superseded by a square-root behavior in the supersymmetric case, a difference which can be traced back to the underlying regularizations.

The fixed-point solutions discussed in this paper describe the phase transition for the breaking of the $O(N)$ symmetry. Analyzing the pattern of symmetry breaking and the phase transition between symmetric and broken phases in more detail, and relating our findings with earlier studies based on gap equations is deferred to an upcoming publication. Furthermore, stepping back to finite N, we expect modifications to the above picture, both within the local potential approximation studied here and to higher order in the derivative expansion. For example, it is known that the $N = 1$ model displays a superscaling relation linking the unstable direction with the anomalous dimension [28,31], a behavior which is quite different from the Ising universality class [24]. It will thus be interesting to see how these patterns generalize for supersymmetric $O(N)$ models with generic N.

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APPENDIX A: CONVENTIONS

Relevant symmetry relations and Fierz identities for Majorana spinors are $\bar\Psi\chi = \bar\chi\Psi$, $\bar\Psi\gamma^\mu\chi = -\bar\chi\gamma^\mu\Psi$ and $\theta_k\bar\theta_l = -\frac{1}{2}(\bar\theta\theta)\mathbb{1}_{kl}$. One of the main features of the action is its invariance under supersymmetry transformations. The latter are characterized by the supersymmetry variations $\delta_\epsilon\Phi^i$, generated by the $\mathcal{N} = 1$ fermionic generator $\mathcal{Q}$. We have

$$\delta_\epsilon\Phi^i(x) = i\bar\epsilon_k\mathcal{Q}_k\Phi^i(x) \quad \text{with} \quad \mathcal{Q}_k = -i\partial_{\bar\theta_k} - \gamma^\mu_{kl}\theta_l\partial_\mu, \quad \bar{\mathcal{Q}}_k = -i\partial_{\theta_k} - \bar\theta_l\gamma^\mu_{lk}\partial_\mu. \tag{A1}$$

Thus, (A1) leads to the supersymmetry variations

$$\delta\phi^i = \bar\epsilon\psi^i, \quad \delta\psi^i = (F^i + i\not\partial\phi^i)\epsilon \quad \text{and} \quad \delta F^i = i\bar\epsilon\not\partial\psi^i \tag{A2}$$

of the component fields. The anticommuting sector of the superalgebra is given by the anticommutator of two supercharges

$$\{\mathcal{Q}_k, \bar{\mathcal{Q}}_l\} = 2i\gamma^\mu_{kl}\partial_\mu. \tag{A3}$$

APPENDIX B: SUPERSPACE

Following [27], we consider the action of the three-dimensional supersymmetric $O(N)$ model in the local potential approximation

$$\Gamma_k[\Phi^i] = \int d^3x \frac{d\theta_1 d\theta_2}{2i}\left(-\frac{1}{2}\Phi^i K\Phi_i + 2W_k(R)\right), \tag{B1}$$

where $R = \frac{1}{2}\Phi^i\Phi_i$, $K = \frac{1}{2}(\bar{\mathcal{D}}\mathcal{D} - \mathcal{D}\bar{\mathcal{D}})$ and $i = 1, \ldots, N$. We derive the flow equation in the superspace $\mathbb{R}^{3|2}$ with coordinates $z = (x, \theta_1, \theta_2)$. Furthermore, we introduce the abbreviation $\int dz \equiv \int d^3x d\theta_1 d\theta_2/(2i)$. In Minkowski spacetime [31], the Wetterich equation in superspace may be written in the form

$$\partial_t\Gamma_k = \frac{i}{2}\int dz dz'(\partial_t R_k)_{mn}(z, z')(G_k)_{nm}(z', z), \quad t = \ln(k^2/\Lambda^2), \tag{B2}$$

where $(R_k)_{mn}$ represents a supersymmetric regulator term and $(G_k)_{nm}$ the connected Green's function. According to [27,31], we now choose a general regulator term quadratic in the superfields Φ^i and diagonal with respect to the field indices:

$$\Delta S_k = \frac{1}{2}\int dz \Phi^i R_{k,ij}(\mathcal{D}, \bar{\mathcal{D}})\Phi^j = \frac{1}{2}\int dz \Phi^i (2r_1(-\partial_x^2, k)\delta_{ij} - r_2(-\partial_x^2, k)K\delta_{ij})\Phi^j. \tag{B3}$$

Notice that this regulator conserves both the $O(N)$ symmetry and supersymmetry. The functional derivative with respect to a superfield is chosen according to the conventions $\frac{\vec{\delta}}{\delta\Phi^j(\tilde{z})}\int dz \Phi^i(z) = \delta^i_j$ with $\frac{\vec{\delta}\Phi^i(z)}{\delta\Phi^j(\tilde{z})} = 2i\delta^i_j\delta(x-\tilde{x})\times \delta(\theta_2 - \tilde{\theta}_2)\delta(\theta_1 - \tilde{\theta}_1) \equiv \delta^i_j\delta(z-\tilde{z})$. Thus, the second functional derivative of the effective average action with respect to the superfields reads

$$\Gamma^{(2)}_{k,nm}(z,z') \equiv \frac{\vec{\delta}}{\delta\Phi^n(z)}\Gamma_k\frac{\overleftarrow{\delta}}{\delta\Phi^m(z')} = [(-K + 2W_k'(R))\delta_{nm} + 2W_k''(R)\Phi_n\Phi_m](z)\delta(z-z'). \tag{B4}$$

Similarly, the second functional derivative $\Delta S_k^{(2)}(z,z')$ of the regulator term is given by

$$(R_k)_{nm}(z,z') = [2r_1 - r_2 K](z)\delta_{nm}\delta(z-z'). \tag{B5}$$

Now we assume the superfields to be constant, i.e. $\partial_x\Phi^i(x,\theta) = 0$, such that the regulator functions as well as the wave operator may be simply written in momentum space. However, note that the wave operator K still contains derivatives with respect to the Grassmann coordinates and thus acts on the adjacent delta functions. Hence, the flow of the effective average action may be written as

$$\begin{aligned}\partial_t\Gamma_k &= \frac{i}{2}\int dz dz'(\partial_t R_k)_{mn}(z,z')(\Gamma_k^{(2)} + R_k)^{-1}_{nm}(z',z)\\ &= \frac{i}{2}\int d^3x \frac{d\theta_1 d\theta_2}{2i}\frac{d\theta_1' d\theta_2'}{2i}\int\frac{d^3p}{(2\pi)^3}(2\partial_t r_1 - \partial_t r_2 K)\\ &\times (p,\theta_1,\theta_2)\delta_{mn}2i\delta(\theta_2-\theta_2')\delta(\theta_1-\theta_1')\\ &\times[(-hK_{(p,\theta_1',\theta_2')} + 2\mathcal{W}')\delta_{nm} + 2\mathcal{W}''\Phi_n\Phi_m]^{-1}\\ &\times 2i\delta(\theta_2'-\theta_2)\delta(\theta_1'-\theta_1).\end{aligned} \tag{B6}$$

We have thereby introduced the notation $\mathcal{W}'(R) \equiv W_k'(R) + r_1$, $h \equiv 1 + r_2$. The inverse of the $N\times N$-matrix

$$(M)^{-1}_{nm} \equiv (-hK + 2\mathcal{W}')\delta_{nm} + 2\mathcal{W}''\Phi_n\Phi_m \tag{B7}$$

is given by

$$(M)_{nm} = \frac{(-hK + 2\mathcal{W}')\delta_{nm} + 2\mathcal{W}''(\Phi^2\delta_{nm} - \Phi_n\Phi_m)}{4(h^2p^2 + \mathcal{W}'(\mathcal{W}' + 2\mathcal{W}''R) - hK(\mathcal{W}' + \mathcal{W}''R))}, \tag{B8}$$

where we have used the relation $K^2(p) = 4p^2$ resulting from the action of $K(p) = -\partial_\theta\partial_{\bar{\theta}} - (\partial_\theta p/\theta) - (\bar{\theta}p/\partial_{\bar{\theta}}) - p^2(\bar{\theta}\theta)$ on an arbitrary superfield. In order to eliminate the wave operator K in the denominator of (B8), we multiply both the numerator and the denominator with $[(h^2p^2 + \mathcal{W}'(\mathcal{W}' + 2\mathcal{W}''R)) + hK(\mathcal{W}' + \mathcal{W}''R)]$ and use again $K^2(p) = 4p^2$. Thus, we get

$$\begin{aligned}(M)_{nm} &= -2\frac{h^2p^2(\delta_{nm}\mathcal{W}' + \mathcal{W}''\Phi_n\Phi_m) - \mathcal{W}'(\mathcal{W}' + 2\mathcal{W}''R)(\mathcal{W}'\delta_{nm} + \mathcal{W}''(\Phi^2\delta_{nm} - \Phi_n\Phi_m))}{4(h^2p^2 - \mathcal{W}'^2)(h^2p^2 - (\mathcal{W}' + 2\mathcal{W}''R)^2)}\\ &\quad - hK\frac{\delta_{nm}(h^2p^2 - \mathcal{W}'^2) - 2\mathcal{W}''(\mathcal{W}' + \mathcal{W}''R)(\Phi^2\delta_{nm} - \Phi_n\Phi_m))}{4(h^2p^2 - \mathcal{W}'^2)(h^2p^2 - (\mathcal{W}' + 2\mathcal{W}''R)^2)}\\ &\equiv \frac{-2f_{nm} - hKg_{nm}}{\mathcal{R}}\end{aligned} \tag{B9}$$

with $\mathcal{R} = 4(h^2p^2 - \mathcal{W}'^2)(h^2p^2 - (\mathcal{W}' + 2\mathcal{W}''R)^2)$. For

$$(G_k)_{nm(p,\theta_1'-\theta_1,\theta_2'-\theta_2)} = (M)_{nm}(p,\theta_1',\theta_2')2i\delta(\theta_2'-\theta_2)\delta(\theta_1'-\theta_1) \tag{B10}$$

to be the Green's function, it has to fulfill the defining relation

$$\int dz(G_k)_{mn}(\tilde{z},z)(\Gamma_k^{(2)} + R_k)_{np}(z,z') = \delta(\tilde{z}-z')\delta_{mp}. \tag{B11}$$

This can be shown by directly inserting the explicit expressions on the left-hand side and working out the contributions to different orders in K.

The flow equation is calculated by inserting the regulator (B5) as well as the propagator (B10) into Eq. (B6). Note that the regulator $(R_k)_{mn} \propto \delta_{mn}$ is diagonal with respect to the field indices. Hence, we simply have evaluate the trace over the Green's function $(G_k)_{mm}$. This yields

$$\partial_t\Gamma_k = \frac{i}{2}\int d\theta_1 d\theta_2 d\theta_1' d\theta_2' \int d^3x \int \frac{d^3p}{(2\pi)^3}(2\partial_t r_1 - \partial_t r_2 K)(p,\theta_1,\theta_2)\delta(\theta_2-\theta_2')\delta(\theta_1-\theta_1')$$
$$\times\left(-2\frac{h^2p^2(N\mathcal{W}' + 2\mathcal{W}''R) - \mathcal{W}'(\mathcal{W}' + 2\mathcal{W}''R)(N\mathcal{W}' + 2(N-1)\mathcal{W}''R)}{4(h^2p^2 - \mathcal{W}'^2)(h^2p^2 - (\mathcal{W}' + 2\mathcal{W}''R)^2)}\right.$$
$$\left. - hK\frac{N(h^2p^2 - \mathcal{W}'^2) - 4(N-1)\mathcal{W}''(\mathcal{W}' + \mathcal{W}''R)R}{4(h^2p^2 - \mathcal{W}'^2)(h^2p^2 - (\mathcal{W}' + 2\mathcal{W}''R)^2)}\right)\delta(\theta_2' - \theta_2)\delta(\theta_1' - \theta_1). \tag{B12}$$

Now, only terms linear in K contribute to the flow of Γ_k after having integrated out the Grassmann variables. Those contributing terms lead to a multiplying factor of $2i$. Thus, the flow Eq. (B12) simplifies to

$$\partial_t\Gamma_k = -i\int d^3x d\theta_1 d\theta_2 \partial_t W_k(R)$$
$$= \frac{1}{2}\int d^3x d\theta_1 d\theta_2 \int \frac{d^3p}{(2\pi)^3}\left((N-1)\frac{(\partial_t r_1 h - \partial_t r_2 \mathcal{W}')}{h^2p^2 - \mathcal{W}'^2} + \frac{\partial_t r_1 h - \partial_t r_2(\mathcal{W}' + 2\mathcal{W}''R)}{h^2p^2 - (\mathcal{W}' + 2\mathcal{W}''R)^2}\right). \tag{B13}$$

Performing a Wick rotation of the zeroth component of the momentum, i.e. $p^0 \to ip_E^0$, $p^2 \to -p_E^2$, we obtain the Euclidean version of the flow Eq. (B13). Thus, the resulting flow equation in superspace reads

$$\int_{x,\theta_1,\theta_2}\partial_t W_k(R) = \frac{1}{2}\int_{x,\theta_1,\theta_2}\int\frac{d^3p_E}{(2\pi)^3}\left((N-1)\frac{(\partial_t r_1 h - \partial_t r_2\mathcal{W}')}{h^2p_E^2 + \mathcal{W}'^2} + \frac{\partial_t r_1 h - \partial_t r_2(\mathcal{W}' + 2\mathcal{W}''R)}{h^2p_E^2 + (\mathcal{W}' + 2\mathcal{W}''R)^2}\right). \tag{B14}$$

Notice that the truncation (14) involved a superpotential of the form $2NW_k(R/N)$ instead of $2W_k(R)$. The corresponding flow equation may be easily derived from the above result by performing the substitution $W_k(R) \to NW_k(R/N)$ in (B14). This yields the final result

$$\int_{x,\theta_1,\theta_2}\partial_t W_k(R/N) = \frac{1}{2}\int_{x,\theta_1,\theta_2}\int\frac{d^3p_E}{(2\pi)^3}\left(\frac{(N-1)}{N}\frac{(\partial_t r_1 h - \partial_t r_2\mathcal{W}')}{h^2p_E^2 + \mathcal{W}'^2} + \frac{1}{N}\frac{\partial_t r_1 h - \partial_t r_2(\mathcal{W}' + 2\mathcal{W}''R/N)}{h^2p_E^2 + (\mathcal{W}' + 2\mathcal{W}''R/N)^2}\right). \tag{B15}$$

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