Interior structure of rotating black holes. III. Charged black holes

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This paper extends to the case of charged rotating black holes the conformally stationary, axisymmetric, conformally separable solutions presented for uncharged rotating black holes in a companion paper. In the present paper, the collisionless fluid accreted by the black hole may be charged. The charge of the black hole is determined self-consistently by the charge accretion rate. As in the uncharged case, hyper-relativistic counterstreaming between ingoing and outgoing streams drives inflation at (just above) the inner horizon, followed by collapse. If both ingoing and outgoing streams are charged, then conformal separability holds during early inflation, but fails as inflation develops. If conformal separability is imposed throughout inflation and collapse, then only one of the ingoing and outgoing streams can be charged: the other must be neutral. Conformal separability prescribes a hierarchy of boundary conditions on the ingoing and outgoing streams incident on the inner horizon. The dominant radial boundary conditions require that the incident ingoing and outgoing number densities be uniform with latitude, but the charge per particle must vary with latitude such that the incident charge densities vary in proportion to the radial electric field. The subdominant angular boundary conditions require specific forms of the incident number- and charge-weighted angular motions. If the streams fall freely from outside the horizon, then the prescribed angular conditions can be achieved by the charged stream, but not by the neutral stream. Thus, as in the case of an uncharged black hole, the neutral stream must be considered to be delivered *ad hoc* to just above the inner horizon.

I. INTRODUCTION

A companion paper [1], hereafter Paper 2, presents conformally stationary, axisymmetric, conformally separable solutions for the interior of an uncharged rotating black hole that undergoes inflation at its inner horizon and then collapses. The purpose of this paper is to extend these solutions to the case of a charged rotating black hole. A MATHEMATICA notebook containing many details of the calculations is at [2].

Because of the strength of electromagnetism and the overall charge neutrality of the Universe, real astronomical black holes are expected to have little electric charge. However, a black hole is likely to build up a residual positive charge because positively charged protons are more massive than negatively charged electrons, so protons are more able to overcome a Coulomb barrier against accretion. The charge-to-mass ratio of a proton is $e/m_p \approx 10^{18}$ in Planck units ($c = G = \hbar = 1$). A black hole might be able to build up a charge-to-mass of the order of the reciprocal of this ratio [3]. If so, then trajectories of charged particles falling into the black hole would be affected by the black hole's charge notwithstanding its small value.

As shown in Appendix A of Paper 2, given the assumptions of conformal time-translation invariance, axisymmetry, and conformal separability, the line element can be taken to be

$$ds^2 = \rho^2 \left[\frac{dx^2}{\Delta_x} - \frac{\Delta_x}{\sigma^4}(dt - \omega_y d\phi)^2 + \frac{dy^2}{\Delta_y} \right.$$
$$\left. + \frac{\Delta_y}{\sigma^4}(d\phi - \omega_x dt)^2 \right], \qquad (1)$$

where t is conformal time, ϕ is the azimuthal coordinate, x and y are radial and angular coordinates, Δ_x and Δ_y are radial and angular horizon functions, and $\sigma \equiv \sqrt{1 - \omega_x \omega_y}$. The conformal factor $\rho = \rho_s e^{vt - \xi}$ is a product of separable (electrovac) ρ_s, time-dependent e^{vt}, and inflationary $e^{-\xi}$ factors.

II. COLLISIONLESS STREAMS

As in Paper 2 [1], the present paper takes a general freely falling collisionless fluid as the source of energy that ignites and then drives inflation. In the present paper, collisionless streams are allowed to be electrically charged.

A. Conformal separability conditions

The tetrad-frame electromagnetic potential A_k is conveniently written in terms of a set of Hamilton-Jacobi potentials $\mathcal{A}_k$ [the following repeats Eq. (24) of Paper 2],

$$A_k \equiv \frac{1}{\rho} \left\{ \frac{\mathcal{A}_x}{\sqrt{-\Delta_x}}, \frac{\mathcal{A}_t}{\sqrt{-\Delta_x}}, \frac{\mathcal{A}_y}{\sqrt{\Delta_y}}, \frac{\mathcal{A}_\phi}{\sqrt{\Delta_y}} \right\}. \qquad (2)$$

As shown in Appendix A of Paper 2, conformal separability requires that

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$\omega_x,$ $\quad\Delta_x$ are functions of x only,

$\omega_y,$ $\quad\Delta_y$ are functions of y only, $\qquad$ (3)

and also that

$\mathcal{A}_x,$ $\quad\mathcal{A}_t$ are functions of x only,

$\mathcal{A}_y,$ $\quad A_\phi$ are functions of Y only. $\qquad$ (4)

However, dimensional analysis shows that the condition of conformal time-translation symmetry requires that the potentials $\mathcal{A}_k$ must be proportional to the time-dependent factor e^{vt} of the conformal factor,

$$\mathcal{A}_k \propto e^{vt}, \qquad (5)$$

contradicting conditions (4). The dimensional argument is robust; the proportionality (5) is correct. Thus, the separability conditions adopted in this paper are, in place of conditions (4),

$e^{-vt}\mathcal{A}_x,$ $\quad e^{-vt}\mathcal{A}_t$ are functions of x only,

$e^{-vt}\mathcal{A}_y,$ $\quad e^{-vt}\mathcal{A}_\phi$ are functions of y only. $\qquad$ (6)

B. Hamilton-Jacobi separation

The fact that the conformal separability conditions (4) fail and must be replaced by conditions (6) implies that the equations of motion of charged particles in conformally separable spacetimes are not exactly Hamilton-Jacobi separable. A similar situation occurred in Paper 2, where it was found that the equations of motion of massive particles, though not exactly Hamilton-Jacobi separable, are adequately so under the hyper-relativistic conditions of inflation. This suggests that the Hamilton-Jacobi equations might still provide an adequate approximation to the equations of motion of charged particles under the conditions peculiar to inflation. This subsection shows that the Hamilton-Jacobi equations do in fact provide an adequate approximation, but only subject to the special condition (14). Physically, these conditions require that only one of the ingoing and outgoing streams can be charged, the other being neutral, Sec. IV C.

As shown in Sec. IV of Paper 2, the tetrad-frame momentum p_k of a particle of rest mass m and charge q predicted by the Hamilton-Jacobi equations is

$$p_k = \frac{1}{\rho}\left\{\frac{P_x}{\sqrt{-\Delta_x}}, \frac{P_t}{\sqrt{-\Delta_x}}, \frac{P_y}{\sqrt{\Delta_y}}, \frac{P_\phi}{\sqrt{\Delta_y}}\right\}, \qquad (7)$$

where the Hamilton-Jacobi parameters P_t and P_ϕ are related to the particle's conserved energy $\pi_t = -E$ and angular momentum $\pi_\phi = L$ and to the potentials $\mathcal{A}_t$ and $\mathcal{A}_\phi$ by

$$P_t = \pi_t + \pi_\phi\omega_x - q\mathcal{A}_t,$$
$$P_\phi = \pi_\phi + \pi_t\omega_y - q\mathcal{A}_\phi, \qquad (8)$$

and P_x and P_y are then obtained from [the following are Eq. (35) of Paper 2]

$$P_x = \sqrt{P_t^2 - [m^2(\rho^2 - \rho_y^2) + \mathcal{K}]\Delta_x},$$
$$P_y = \sqrt{-P_\phi^2 - (m^2\rho_y^2 - \mathcal{K})\Delta_y}. \qquad (9)$$

As it stands, the tetrad-frame momentum p_k given by Eqs. (7)–(9) does not satisfy the Lorentz force law $dp_k/d\lambda = qp^l F_{lk}$ to adequate accuracy.

A fix that proves to work under inflationary conditions is, firstly, to replace $\mathcal{A}_t$ in Eq. (8) for P_t by $(\mathcal{A}_t \pm \mathcal{A}_x)/2$, respectively, for ingoing ($+$) and outgoing ($-$) particles, so that [given also that $\mathcal{A}_\phi = 0$, Eq. (28)]

$$P_t \equiv \pi_t + \pi_\phi\omega_x - q\frac{\mathcal{A}_t \pm \mathcal{A}_x}{2},$$
$$P_\phi = \pi_\phi + \pi_t\omega_y, \qquad (10)$$

and secondly, to replace the time-dependent factor e^{vt} in $\mathcal{A}_t \pm \mathcal{A}_x$ by its value as a function of x along the path of the particle predicted by the Hamilton-Jacobi equations,

$$\frac{dt}{dx} = -\frac{1}{P_x}\left(\frac{P_t}{-\Delta_x} + \frac{\omega_y P_\phi}{\Delta_y}\right). \qquad (11)$$

The equation of motion predicted by Eqs. (9)–(11) is [the following equation omits the dependency on rest mass m, given previously by Eq. (39) of Paper 2; the quantities Z_k are defined later, Eq. (24)]

$$\frac{dp_k}{d\lambda} - qp^l F_{lk} = \frac{q}{2\rho^3\sqrt{-\Delta_x}}\left[v(\mathcal{A}_t \pm \mathcal{A}_x)\frac{P_t \mp P_x}{P_x\Delta_x} - (\mathcal{A}_t \mp \mathcal{A}_x)\left(\frac{\partial}{\partial x}\ln\left(\frac{1}{\sigma^2}\frac{d\omega_x}{dx}\right) - \frac{P_\phi}{P_x}\frac{v\omega_y}{\Delta_y}\right) - Z_t \pm Z_x\right]$$

$$\times \{P_t, P_x, 0, 0\} + \frac{q}{2\rho^3\sqrt{\Delta_y}}(\mathcal{A}_t \mp \mathcal{A}_x)\frac{1}{\sigma^2}\frac{d\omega_y}{dy}\{0, 0, P_\phi, -P_y\}$$

$$+ \frac{qv\omega_y}{\rho^3\sqrt{-\Delta_x\Delta_y}}\frac{\mathcal{A}_x P_x - \mathcal{A}_t P_t}{P_x}\left\{\frac{P_\phi}{\sqrt{\Delta_y}}, 0, 0, \frac{P_x}{\sqrt{-\Delta_x}}\right\}. \qquad (12)$$

The right-hand side of Eq. (12), which would vanish if the equations of motion of charged particles were exactly Hamilton-Jacobi separable, does not vanish because the Hamilton-Jacobi approximation (9)–(11) is not exact. Appendix D of Paper 2 gives criteria under which integrals along the path of a particle may be deemed small, in the conformally stationary limit. By these criteria, all but one of the terms in Eq. (12) yield a small result when integrated over the path of a particle through inflation and collapse. The discrepant term is the azimuthal ϕ component of the last term, proportional to $\mathcal{A}_x P_x - \mathcal{A}_t P_t$, which has $\alpha = -1$ and $\beta = -3$ in the terminology of Appendix D of Paper 2, violating (marginally) condition (D4) during inflation. Integrated over the path of the particle, the term would produce a finite difference between the true azimuthal momentum p_ϕ and that predicted by Eqs. (9)–(11). The finite difference appears during inflation when $|\Delta_x| \ll 1$. The exception to this conclusion is that the term would vanish provided that the factor $\mathcal{A}_x P_x - \mathcal{A}_t P_t$ multiplying it is zero,

$$\mathcal{A}_x P_x = \mathcal{A}_t P_t. \tag{13}$$

But, particles are hyper-relativistic, $P_t = \pm P_x$, under the conditions $|\Delta_x| \ll 1$ where the difference occurs. Then, condition (13) holds provided that

$$\mathcal{A}_x = \pm \mathcal{A}_t. \tag{14}$$

In other words, the Hamilton-Jacobi approximation (9)–(11) works for ingoing particles only if $\mathcal{A}_t = \mathcal{A}_x$, and for outgoing particles only if $\mathcal{A}_t = -\mathcal{A}_x$. Later, Sec. IV C, it will be concluded that condition (14) is equivalent to requiring that only one of the ingoing and outgoing streams can be charged; the other stream must be neutral.

One might try to go beyond the Hamilton-Jacobi approximation, but there is no point. In Sec. III C, it will be found that the nonisotropic diagonal angular component of the electromagnetic energy-momentum tensor, which conformal separability requires must vanish, diverges unless condition (14) is true. Condition (14) appears necessary for the conformally separable solutions considered in this paper to hold.

C. Electric current

Equations governing the density N and number current n_k of a collisionless stream were derived in Sec. VII of Paper 2. For a single stream of particles of charge q with fixed constants of motion, the tetrad-frame current j_k is the particle charge q times the number current n_k, which is itself the number density N times the momentum p_k,

$$j_k = q n_k, \qquad n_k = N p_k. \tag{15}$$

The Hamilton-Jacobi equations predict that the number density N along a single stream satisfies

$$N \propto \frac{\sigma^2}{\rho^2 P_x P_y}. \tag{16}$$

As discussed in Sec. II B, the equations of motion of charged particles are not exactly Hamilton-Jacobi separable. The accuracy of the Hamilton-Jacobi approximation (9)–(11) can be checked by seeing how closely the covariant divergence $D^k n_k$ that they predict vanishes. The result is [the following equation omits the dependency on rest mass m given previously by Eq. (63) of Paper 2]

$$D^k n_k = 0, \tag{17}$$

which happens to vanish identically, confirming that the Hamilton-Jacobi approximation (9)–(11) is satisfactorily accurate.

III. ELECTROMAGNETISM

Maxwell's equations prove to separate in a manner consistent with the separation of Einstein's equations carried out in Sec. VIII of Paper 2 [1]. In this section, the spacetime is taken to be conformally time-translation symmetric (not necessarily conformally stationary) and axisymmetric, and to satisfy the conformal separability conditions (3) and (6).

Homogeneous solution of the stationary, separable Einstein equations leads to the usual electrovac solutions for the vierbein coefficients ω_x and ω_y of the line element, and, as is well known, homogeneous solution of the stationary, separable Maxwell equations leads to the same result [the following repeats Eq. (73) of Paper 2]:

$$\begin{aligned} \frac{d\omega_x}{dx} &= 2\sqrt{(f_0 + f_1 \omega_x)(g_0 - g_1 \omega_x)}, \\ \frac{d\omega_y}{dy} &= 2\sqrt{(f_1 + f_0 \omega_y)(g_1 - g_0 \omega_y)}, \end{aligned} \tag{18}$$

where f_0, f_1, g_0, and g_1 are constants set by boundary conditions. Equation (18) continues to hold through inflation and collapse in charged as well as neutral black holes. As found in Paper 2, inflation occurs generically at an inner horizon $\Delta_x \to -0$ regardless of the specific choice of the constants f_0, f_1, g_0, and g_1.

A. Electromagnetic field

The electromagnetic field F_{mn} is a bivector, and as such has a natural complex structure [4], with the real part being the electric field, which changes sign under parity transformation (a change of sign of all spatial coordinates), and the imaginary part being the magnetic field, which is unchanged by a parity transformation. The complex structure is manifest in a complexified electromagnetic field $\tilde{F}_{mn}$ defined by

$$\tilde{F}_{kl} \equiv \tfrac{1}{2}(F_{kl} + {}^*F_{kl}), \tag{19}$$

where ${}^*F_{kl}$ denotes the Hodge dual of F_{kl},

$$^{*}F_{kl} \equiv \frac{i}{2}\varepsilon_{kl}{}^{mn}F_{mn}, \qquad (20)$$

with ε_{klmn} the totally antisymmetric tensor, normalized here to $\varepsilon^{klmn} = [klmn]$ in an orthonormal tetrad frame.

The complexified electromagnetic field tensor is self-dual, $^{*}\tilde{F}_{kl} = \tilde{F}_{kl}$. Given conformal time-translation symmetry (not necessarily conformally stationarity), axisymmetry, and conformal separability, the tetrad-frame complexified electromagnetic field tensor $\tilde{F}_{mn}$ is

$$\tilde{F}_{xt} \equiv \frac{1}{2}(F_{xt} + iF_{\phi y}) = \frac{1}{2\rho^2}\left[-\left(\frac{\partial \mathcal{A}_t}{\partial x} + \frac{\omega_y \mathcal{A}_t - \mathcal{A}_\phi}{\sigma^2}\frac{d\omega_x}{dx}\right) + i\left(\frac{\partial \mathcal{A}_\phi}{\partial y} + \frac{\omega_x \mathcal{A}_\phi - \mathcal{A}_t}{\sigma^2}\frac{d\omega_y}{dy}\right) - v\left(\frac{\mathcal{A}_x}{\Delta_x} + i\frac{\omega_y \mathcal{A}_y}{\Delta_y}\right)\right],$$
$$(21a)$$

$$\tilde{F}_{xy} \equiv \frac{1}{2}(F_{xy} + iF_{t\phi}) = i\frac{v(\omega_y \mathcal{A}_t - \mathcal{A}_\phi)}{2\rho^2\sqrt{-\Delta_x \Delta_y}}, \qquad (21b)$$

$$\tilde{F}_{x\phi} \equiv \frac{1}{2}(F_{x\phi} + iF_{yt}) = \frac{v(\omega_y \mathcal{A}_x + i\mathcal{A}_y)}{2\rho^2\sqrt{-\Delta_x \Delta_y}}. \qquad (21c)$$

If the spacetime were strictly stationary, $v \equiv 0$, then the only nonvanishing component of the complexified electromagnetic field would be the radial component $\tilde{F}_{xt}$. In inflationary spacetimes, however, the radial horizon function Δ_x goes to zero at the inner horizon, and the angular components $\tilde{F}_{xy}$ and $\tilde{F}_{x\phi}$ of the electromagnetic field can grow large at the inner horizon, however small the accretion rate v may be.

The only electromagnetic gauge freedom that respects conformal time-translation symmetry and the conformal separability conditions (6) is $A_k \rightarrow A_k + \lambda \partial_k e^{vt}$ for some constant λ, which transforms

$$\mathcal{A}_t \rightarrow \mathcal{A}_t + \lambda v e^{vt}, \qquad \mathcal{A}_\phi \rightarrow \mathcal{A}_\phi + \lambda v \omega_y e^{vt}. \qquad (22)$$

Define the enclosed electric charge Q within radius x, and the enclosed magnetic charge $\mathcal{Q}$ above latitude y, by

$$Q \equiv -2(f_0 g_1 + f_1 g_0)\frac{\mathcal{A}_t}{d\omega_x/dx}, \qquad (23a)$$

$$\mathcal{Q} \equiv -2(f_0 g_1 + f_1 g_0)\frac{\mathcal{A}_\phi}{d\omega_y/dy}. \qquad (23b)$$

Further, define the quantities Z_k by

$$Z_x \equiv \frac{d\omega_x}{dx}\frac{\partial}{\partial x}\left(\frac{\mathcal{A}_x}{d\omega_x/dx}\right) + \frac{v\mathcal{A}_t}{\Delta_x}, \qquad Z_t \equiv \frac{d\omega_x}{dx}\frac{\partial}{\partial x}\left(\frac{\mathcal{A}_t}{d\omega_x/dx}\right) + \frac{v\mathcal{A}_x}{\Delta_x}, \quad (24a)$$

$$Z_y \equiv \frac{d\omega_y}{dy}\frac{\partial}{\partial y}\left(\frac{\mathcal{A}_y}{d\omega_y/dy}\right) + \frac{v\omega_y \mathcal{A}_\phi}{\Delta_y}, \qquad Z_\phi \equiv \frac{d\omega_y}{dy}\frac{\partial}{\partial y}\left(\frac{\mathcal{A}_\phi}{d\omega_y/dy}\right) - \frac{v\omega_y \mathcal{A}_y}{\Delta_y}. \quad (24b)$$

The conventional radial electric and magnetic fields E and B constitute the real and imaginary parts of (twice) the radial electromagnetic field. The radial electromagnetic field $\tilde{F}_{xt}$ can be written in terms of the enclosed electric and magnetic charges Q and $\mathcal{Q}$ and the Z_k as

$$2\tilde{F}_{xt} = E + iB = \frac{1}{\rho^2}\left[(Q + i\mathcal{Q})\frac{\rho_x + i\rho_y}{\rho_x - i\rho_y} - Z_t + iZ_\phi\right], \qquad (25)$$

where ρ_x and ρ_y are the x and y components of the separable conformal factor $\rho_s = \sqrt{\rho_x^2 + \rho_y^2}$, Eq. (72) of Paper 2.

B. Maxwell's equations

Maxwell's equations are embodied in the complex equation

$$D^m \tilde{F}_{mn} = 2\pi j_n, \qquad (26)$$

whose real (electric) and imaginary (magnetic) parts constitute, respectively, the source and source-free Maxwell's equations. Given conformal time-translation invariance and the conformal separability conditions (3) and (6), the source-free Maxwell's equations are satisfied identically with vanishing magnetic current. Since the magnetic current, and, in particular, its time component the magnetic charge, necessarily vanishes, the solutions preclude the accretion of any magnetic charge. Although strictly stationary solutions admit a black hole with magnetic charge, conformally stationary solutions do not.

In terms of the Z_k defined by Eq. (24), and given Eq. (18) for $d\omega_x/dx$ and $d\omega_y/dy$, the sourced Maxwell's equations can be written

$$j_x \pm j_t = \frac{1}{4\pi\rho^3}\left\{ \frac{v}{\sqrt{-\Delta_x}}\left[(\mathcal{A}_x \pm \mathcal{A}_t)\left(\pm\frac{\partial}{\partial x}\ln\left(\frac{1}{\sigma^2}\frac{d\omega_x}{dx}\right) - \frac{v\omega_y^2}{\Delta_y} \right) \pm \mathcal{A}_y\frac{\partial}{\partial y}\ln\left(\frac{1}{\sigma^2}\frac{d\omega_y}{dy}\right) - \mathcal{A}_\phi\frac{1}{\sigma^2}\frac{d\omega_x}{dx} \right] \right.$$
$$\left. + \frac{v}{\sqrt{-\Delta_x}}(Z_t \pm Z_y) \pm \sqrt{-\Delta_x}\left[\frac{\partial Z_t}{\partial x} + Z_t\frac{\partial}{\partial x}\ln\left(\frac{1}{\sigma^2}\frac{d\omega_x}{dx}\right) - Z_\phi\frac{1}{\sigma^2}\frac{d\omega_y}{dy} \right] \right\}, \tag{27a}$$

$$j_y \pm ij_\phi = \frac{1}{4\pi\rho^3}\left\{ \frac{v\omega_y}{\sqrt{\Delta_y}}\left[(\mathcal{A}_y \pm i\mathcal{A}_\phi)\left(\mp i\frac{\partial}{\partial y}\ln\left(\frac{1}{\sigma^2}\frac{d\omega_y}{dy}\right) + \frac{v}{\omega_y\Delta_x} \right) \mp i\mathcal{A}_x\frac{\partial}{\partial x}\ln\left(\frac{1}{\sigma^2}\frac{d\omega_x}{dx}\right) - \mathcal{A}_t\frac{1}{\sigma^2}\frac{d\omega_y}{dy} \right] \right.$$
$$\left. + \frac{v\omega_y}{\sqrt{\Delta_y}}(Z_\phi \mp iZ_x) \mp i\sqrt{\Delta_y}\left[\frac{\partial Z_\phi}{\partial y} + Z_\phi\frac{\partial}{\partial y}\ln\left(\frac{1}{\sigma^2}\frac{d\omega_y}{dy}\right) - Z_t\frac{1}{\sigma^2}\frac{d\omega_x}{dx} \right] \right\}. \tag{27b}$$

These equations yield both electrovac and inflationary solutions.

Electrovac solutions correspond to the case of strict stationarity, $v = 0$. In strictly stationary spacetimes, $\mathcal{A}_x$ and $\mathcal{A}_y$ can be set to zero by a gauge transformation, as is evident from the fact that for $v = 0$ the electromagnetic field, Eq. (21), is independent of $\mathcal{A}_x$ and $\mathcal{A}_y$. This gauge freedom is available in strictly stationary but not conformally stationary spacetimes. The homogeneous solutions of Maxwell's equations (27) are those with $Z_t = Z_\phi = 0$, which given the definitions (24) of Z_k, correspond to spacetimes with constant electric and magnetic charges Q and $\mathcal{Q}$, Eq. (23).

Inflationary solutions of Maxwell's equations (27) have small but nonzero accretion rate v. During inflation, the angular currents j_y and j_ϕ available from a collisionless source are small. The term proportional to $\mathcal{A}_y + i\mathcal{A}_\phi$ in Eq. (27b) involves a factor of v/Δ_x which diverges at the inner horizon $\Delta_x \to -0$ however small the accretion rate v

might be. The only way that this term can remain small is that

$$\mathcal{A}_y = \mathcal{A}_\phi = 0, \tag{28}$$

which is equivalent to requiring that the magnetic charge be identically zero [an apparent exception to this argument is that if $\mathcal{A}_\phi$ is chosen to be a constant times ω_y, then the divergent term can cancel against a corresponding term proportional to $\mathcal{A}_t$ in Z_x; but that simply reflects the gauge freedom (22) in $\mathcal{A}_\phi$]. That magnetic charge must vanish accords with the conclusion at the beginning of this subsection, that conformal time-translation invariance and conformal separability force the magnetic current to vanish identically, so the black hole cannot accrete magnetic charge, so its cumulative magnetic charge must be zero.

Given the vanishing of $\mathcal{A}_y$ and $\mathcal{A}_\phi$, it follows that Z_y and Z_ϕ vanish identically, and Maxwell's equations (27) reduce to

$$4\pi\rho^2(j_x \pm j_t) = \frac{1}{\rho}\left\{ \frac{v}{\sqrt{-\Delta_x}}(\mathcal{A}_x \pm \mathcal{A}_t)\left[\pm\frac{\partial}{\partial x}\ln\left(\frac{1}{\sigma^2}\frac{d\omega_x}{dx}\right) - \frac{v\omega_y^2}{\Delta_y} \right] \pm \sqrt{-\Delta_x}\left[\frac{\partial}{\partial x} \mp \frac{v}{\Delta_x} + \frac{\partial}{\partial x}\ln\left(\frac{1}{\sigma^2}\frac{d\omega_x}{dx}\right) \right]Z_t \right\}, \tag{29a}$$

$$4\pi\rho^2(j_y \pm ij_\phi) = \frac{1}{\rho}\left\{ \frac{v\omega_y}{\sqrt{\Delta_y}}\left[\mp i\mathcal{A}_x\frac{\partial}{\partial x}\ln\left(\frac{1}{\sigma^2}\frac{d\omega_x}{dx}\right) - \mathcal{A}_t\frac{1}{\sigma^2}\frac{d\omega_y}{dy} \right] \mp i\left(Z_x\frac{v\omega_y}{\sqrt{\Delta_y}} - Z_t\sqrt{\Delta_y}\frac{1}{\sigma^2}\frac{d\omega_x}{dx} \right) \right\}. \tag{29b}$$

C. Electromagnetic energy-momentum

In terms of the complexified electromagnetic field $\tilde{F}_{mn}$, the tetrad-frame electromagnetic energy-momentum tensor $T^{\rm e}_{kl}$ satisfies

$$4\pi T^{\rm e}_{kl} = \eta^{mn}(\tilde{F}_{km}\tilde{F}^*_{ln} + \tilde{F}_{kn}\tilde{F}^*_{lm}), \tag{30}$$

in which $*$ denotes the complex conjugate (not the Hodge dual). Given that the angular electromagnetic potentials $\mathcal{A}_y$ and $\mathcal{A}_\phi$ vanish, the tetrad-frame electromagnetic energy-momentum tensor $T^{\rm e}_{kl}$ is

$$T^e_{xx} - T^e_{tt} = T^e_{yy} + T^e_{\phi\phi} = \frac{E^2 + B^2}{4\pi}, \tag{31a}$$

$$\frac{T^e_{xx} + T^e_{tt}}{2} \pm T^e_{xt} = -\frac{v^2\omega_y^2(\mathcal{A}_x \pm \mathcal{A}_t)^2}{8\pi\rho^4\Delta_x\Delta_y}, \tag{31b}$$

$$T^e_{xy} \pm T^e_{ty} = -\frac{v\omega_y}{4\pi\rho^2\sqrt{-\Delta_x\Delta_y}}B(\mathcal{A}_x \pm \mathcal{A}_t), \tag{31c}$$

$$T^e_{x\phi} \pm T^e_{t\phi} = \mp\frac{v\omega_y}{4\pi\rho^2\sqrt{-\Delta_x\Delta_y}}E(\mathcal{A}_x \pm \mathcal{A}_t), \tag{31d}$$

$$\frac{T^e_{yy} - T^e_{\phi\phi}}{2} \pm iT^e_{y\phi} = -\frac{v^2\omega_y^2}{8\pi\rho^4\Delta_x\Delta_y}(\mathcal{A}_x + \mathcal{A}_t)(\mathcal{A}_x - \mathcal{A}_t), \tag{31e}$$

where E and B are the radial electric and magnetic fields from Eq. (25).

As discussed in Sec. VIII I of Paper 2, the condition of conformal separability requires that the 2×2 angular submatrix of the energy-momentum tensor must be isotropic (proportional to the unit 2×2 unit matrix), since the nonisotropic angular Einstein components depend only on angle y (modulo an overall conformal factor), and being initially negligible in the conformally stationary limit, must remain so at any radius x. The isotropy of the angular energy-momentum requires that the components given by Eq. (31e) must be negligible. On the other hand, the right-hand side of expression (31e) is proportional to $1/\Delta_x$, which diverges at the inner horizon $\Delta_x \rightarrow -0$. The only way out is that one of the remaining factors in the expression must vanish, which requires that

$$\mathcal{A}_x = \pm\mathcal{A}_t. \tag{32}$$

Condition (32) is the same as that (14) found previously in order that the equations of motion of charged particles be Hamilton-Jacobi separable to adequate accuracy.

IV. INFLATIONARY SOLUTIONS

This section presents inflationary solutions to the combined Maxwell and Einstein equations. Section IV A derives the evolution of the electric potentials $\mathcal{A}_x$ and $\mathcal{A}_t$ from the vanishing of Z_x and Z_t, which Maxwell's equations sourced by a collisionless current require to be small. The case of small but finite Z_k is deferred to the Appendix. Section IV B shows that, with these potentials, Maxwell's equations are satisfied by a sum of currents from ingoing and outgoing collisionless streams. Section IV C concludes that only one or other of the ingoing and outgoing streams can be charged. Section IV D reviews 8 of the Einstein components, and Sec. IV E shows that the energy-momentum required by these 8 Einsteins, after subtraction of the electromagnetic energy-momentum, is satisfied by the energy-momentum of ingoing and outgoing collisionless streams. The remaining 2 Einstein components were

shown in Sec. VIII D, E of Paper 2 to govern the evolution of the inflationary exponent ξ and horizon function Δ_x. Section IV F, following along the lines of Sec. VIII J of Paper 2, shows how the subdominant electromagnetic source for these 2 Einstein components can be taken into account by solving the Einstein equations to next higher order.

A. Evolution of the electromagnetic potential and enclosed charge

The definitions (24) of Z_k provide evolutionary equations for the electric potentials $\mathcal{A}_x$ and $\mathcal{A}_t$:

$$\left(\frac{\partial}{\partial x} \pm \frac{v}{\Delta_x}\right)\frac{\mathcal{A}_x \pm \mathcal{A}_t}{d\omega_x/dx} = \frac{Z_x \pm Z_t}{d\omega_x/dx}. \tag{33}$$

Since the angular components of the collisionless current must be small, Maxwell's equations (29b) require that Z_x and Z_t be small. The dominant driving term in Eq. (33) is then the one proportional to v/Δ_x, which diverges at the inner horizon $\Delta_x \rightarrow -0$ however small the accretion rate v may be. By comparison, the effect of a small but finite Z_x and Z_t is essentially negligible. In the Appendix, the effect of small but finite Z_k will be considered, but for the remainder of this paper Z_x and Z_t will be taken to vanish:

$$Z_x = Z_t = 0. \tag{34}$$

Thus, the solutions of interest are the homogeneous solutions of Eq. (33), those with vanishing right-hand side. Equation (33) depends on the horizon function Δ_x, whose behavior as a function of radius x was solved in Paper 2. Equation (93a) of Paper 2 gives

$$\frac{dx}{\Delta_x} = -\frac{dU_x}{2(U_x^2 - v^2)} = \frac{1}{4v}d\ln\left(\frac{U_x + v}{U_x - v}\right). \tag{35}$$

Consequently, the homogeneous solutions of Eq. (33) are

$$\frac{\mathcal{A}_x \pm \mathcal{A}_t}{d\omega_x/dx} = \mp \frac{Q^\pm}{4(f_0 g_1 + f_1 g_0)}, \qquad (36)$$

where $Q^\pm$ are ingoing (+) and outgoing (−) enclosed electric charges satisfying

$$Q^\pm = Q_\bullet^\pm e^{vt} \left[\frac{(U_x - v)(u + v)}{(U_x + v)(u - v)} \right]^{\pm 1/4}, \qquad (37)$$

with $Q_\bullet^\pm$ constants of integration. The total enclosed electric charge Q, which is related to the potential $\mathcal{A}_t$ by Eq. (23a), is a sum of the ingoing and outgoing enclosed charges,

$$Q = Q^+ + Q^-. \qquad (38)$$

The constants $Q_\bullet^\pm$ physically represent the cumulative ingoing and outgoing electric charge accreted by the black hole up to $t = 0$. The total electric charge $Q_\bullet$ of the black hole at $t = 0$, as seen by an observer well outside the horizon, is a sum of the ingoing and outgoing cumulative charges,

$$Q_\bullet = Q_\bullet^+ + Q_\bullet^-. \qquad (39)$$

The relation between time t and radius x along the path of a particle is given by Eq. (11). In the hyper-relativistic conditions $P_t = \pm P_x$ characteristic of inflation and collapse when $|\Delta_x| \ll 1$, Eq. (11) simplifies to $dt/dx = \pm 1/\Delta_x$, which given Eq. (35) integrates to

$$e^{vt} = \left[\frac{(U_x + v)(u - v)}{(U_x - v)(u + v)} \right]^{\pm 1/4}. \qquad (40)$$

The relation (40) continues to hold even at the end of collapse, when $|\Delta_x|$ ceases to be small, and U_x is growing exponentially huge, and the time coordinate t is frozen.

The solutions (37) for the enclosed electric charges $Q^\pm$ have the salient feature that, in view of Eq. (40), the ingoing charge Q^+ is constant along the path of an ingoing particle, while the outgoing charge Q^- is constant along the path of an outgoing particle. As found in Sec. VIII E of Paper 2, in the conformally stationary limit the coordinates x and y along the path of a freely falling stream remain frozen throughout inflation and collapse, so the factor $d\omega_x/dx$ in the relation (36) is constant. Consequently, the potentials $\mathcal{A}_x \pm \mathcal{A}_t$ are also constant along the paths of, respectively, ingoing and outgoing particles. Among other things, this implies that the Hamilton-Jacobi parameter P_t defined by Eq. (10) is constant along the path of an ingoing or outgoing particle. Likewise, the Hamilton-Jacobi parameter P_x defined by Eq. (9) is constant along

the path of an ingoing or outgoing particle during inflation and collapse, as long as $|\Delta_x| \ll 1$.

Maxwell's equations (29) show that the potentials $\mathcal{A}_x \pm \mathcal{A}_t$ are sourced, respectively, by ingoing and outgoing currents, and may thus be called ingoing and outgoing potentials. To bring out the dependence on the ingoing and outgoing potentials $\mathcal{A}_x \pm \mathcal{A}_t$, it is helpful to reexpress Maxwell's equations (29) in terms of a sum of ingoing and outgoing currents

$$j_k = j_k^+ + j_k^-, \qquad (41)$$

where for vanishing Z_k, Eq. (34),

$$j_x^\pm = \pm j_t^\pm = \frac{1}{8\pi\rho^3} \frac{v}{\sqrt{-\Delta_x}} (\mathcal{A}_x \pm \mathcal{A}_t)$$
$$\times \left[\pm \frac{\partial}{\partial x} \ln\left(\frac{1}{\sigma^2} \frac{d\omega_x}{dx} \right) - \frac{v\omega_y^2}{\Delta_y} \right], \qquad (42a)$$

$$j_y^\pm \equiv \mp \frac{1}{8\pi\rho^3} \frac{v\omega_y}{\sqrt{\Delta_y}} (\mathcal{A}_x \pm \mathcal{A}_t) \frac{1}{\sigma^2} \frac{d\omega_y}{dy}, \qquad (42b)$$

$$j_\phi^\pm \equiv - \frac{1}{8\pi\rho^3} \frac{v\omega_y}{\sqrt{\Delta_y}} (\mathcal{A}_x \pm \mathcal{A}_t) \frac{\partial}{\partial x} \ln\left(\frac{1}{\sigma^2} \frac{d\omega_x}{dx} \right). \qquad (42c)$$

Should not the term proportional to v inside square brackets on the right-hand side of Eq. (42a) be neglected compared to the dominant first term, in the conformally stationary limit $v \to 0$? No. The term is needed to ensure that the equations hold not only to leading radial order but also to subdominant angular order.

B. Collisionless source of electric current

Maxwell's equations (41) and (42) can be satisfied by currents from a sum of ingoing and outgoing collisionless streams,

$$j_k^\pm = q^\pm N^\pm p_k^\pm, \qquad (43)$$

with charge densities

$$q^\pm N^\pm = \frac{vQ^\pm}{32\pi\rho^2(f_0 g_1 + f_1 g_0)} \frac{d\omega_x}{dx}$$
$$\times \left[\frac{\partial}{\partial x} \ln\left(\frac{1}{\sigma^2} \frac{d\omega_x}{dx} \right) \mp \frac{v\omega_y^2}{\Delta_y} \right], \qquad (44)$$

and hyper-relativistic tetrad-frame momenta

$$p_{\bar{k}}^{\pm} = \frac{1}{\rho}\left\{ -\frac{1}{\sqrt{-\Delta_x}}, \mp\frac{1}{\sqrt{-\Delta_x}}, \frac{1}{\sqrt{\Delta_y}}\frac{\frac{1}{\sigma^2}\frac{d\omega_y}{dy}}{\frac{\partial}{\partial x}\ln(\frac{1}{\sigma^2}\frac{d\omega_x}{dx}) \mp \frac{v\omega_y^2}{\Delta_y}}, \right.$$

$$\left. \pm\frac{1}{\sqrt{\Delta_y}}\frac{\frac{\partial}{\partial x}\ln(\frac{1}{\sigma^2}\frac{d\omega_x}{dx})}{\frac{\partial}{\partial x}\ln(\frac{1}{\sigma^2}\frac{d\omega_x}{dx}) \mp \frac{v\omega_y^2}{\Delta_y}}\right\}. \tag{45}$$

The densities (44) and momenta (45) are defined up to arbitrary normalization factors such that their product is constant. The densities (44) and momenta (45) conform with the behavior of collisionless streams during inflation and collapse as long as

$$|\Delta_x| \ll 1, \tag{46}$$

which is to say before the angular motions of the streams become important. The Hamilton-Jacobi result (16) requires that the density along a stream evolve as

$$N \propto \frac{1}{\rho^2}, \tag{47}$$

since the Hamilton-Jacobi parameters P_x and P_y are constant and σ is frozen at its inner horizon value. The densities from Eq. (44) indeed satisfy the proportionality (47), since all other factors in the equation are constant along the path of the stream (including $Q^{\pm}$, as shown in the previous Sec. IV A). Similarly, the tetrad-frame momentum $p_{\bar{k}}^{\pm}$, Eq. (45), accords with the Hamilton-Jacobi form (7), with constant Hamilton-Jacobi parameters P_k along the path of the stream.

C. Only one stream can be charged

In Sec. II B, the condition $\mathcal{A}_t = \pm\mathcal{A}_x$, Eq. (14), emerged from requiring that the motions of charged particles be adequately described by the Hamilton-Jacobi equations, and in Sec. III C the same condition, Eq. (32), emerged from requiring that the angular components of the electromagnetic energy-momentum be isotropic, as conformal separability requires.

It has now been seen that the potentials $\mathcal{A}_x \pm \mathcal{A}_t$ are sourced, respectively, by ingoing and outgoing collisionless streams, Sec. IV B. Thus, the condition $\mathcal{A}_t = \pm\mathcal{A}_x$ requires that only one of the streams can be charged, and the other must be neutral. If the ingoing stream is charged, then $\mathcal{A}_t = \mathcal{A}_x$, while if the outgoing stream is charged, then $\mathcal{A}_t = -\mathcal{A}_x$.

If the ingoing stream is charged, then $Q_\bullet^+$ is nonzero, while if the outgoing stream is charged, then $Q_\bullet^-$ is nonzero.

In a real astronomical black hole, collisions and magnetohydrodynamic processes are likely to keep charged particles tightly coupled above the inner horizon, forcing them

into a common ingoing or outgoing stream before inflation ignites. Thus, the condition that only one stream be charged is physically realistic.

D. Einstein and energy-momentum tensors

For the solutions to be valid, Einstein's equations must also be satisfied. Equation (124) of Paper 2 gives 8 of the Einstein components in the conformally stationary limit (the remaining 2 components are considered in Sec. IV F):

$$\rho^2\left(\frac{G_{xx} + G_{tt}}{2} \pm G_{xt}\right) = \frac{U_x \mp v}{-\Delta_x}(\Delta_x' \pm v), \tag{48a}$$

$$\rho^2(G_{xy} \pm G_{ty}) = -\frac{U_x \mp v}{\sqrt{-\Delta_x\Delta_y}}\Delta_y\frac{\partial\ln\rho_s^2}{\partial y}, \tag{48b}$$

$$\rho^2(G_{x\phi} \pm G_{t\phi}) = \pm\frac{U_x \mp v}{\sqrt{-\Delta_x\Delta_y}}\left(\frac{\Delta_y}{\sigma^2}\frac{d\omega_x}{dx} \mp 2v\omega_y\right), \tag{48c}$$

$$\rho^2\left(\frac{G_{yy} - G_{\phi\phi}}{2} \pm iG_{y\phi}\right) = 0, \tag{48d}$$

in which $\Delta_x' \equiv d\Delta_x/dx|_{x_{\mathrm{in}}}$ is the (positive) derivative of the electrovac horizon function Δ_x at the inner horizon $x = x_{\mathrm{in}}$.

In the present situation, there are two sources of energy-momentum, electromagnetic and collisionless. Given the expressions (25) for the radial electric and magnetic fields E and B, and the solutions (36) for the potentials $\mathcal{A}_x \pm \mathcal{A}_t$, and given that only one of the ingoing or outgoing streams can be charged (which implies that $Q^+Q^- = 0$), Eq. (31) for the electromagnetic energy-momentum T_{kl}^{e} is

$$8\pi\rho^2\left(\frac{T_{xx}^{\mathrm{e}} + T_{tt}^{\mathrm{e}}}{2} \pm T_{xt}^{\mathrm{e}}\right) = \frac{U_x \mp v}{-\Delta_x}X^{\pm}\frac{v\omega_y}{\Delta_y}, \tag{49a}$$

$$8\pi\rho^2(T_{xy}^{\mathrm{e}} \pm T_{ty}^{\mathrm{e}}) = \pm\frac{U_x \mp v}{\sqrt{-\Delta_x\Delta_y}}X^{\pm}\frac{1}{\sigma^2}\frac{d\omega_y}{dy}, \tag{49b}$$

$$8\pi\rho^2(T_{x\phi}^{\mathrm{e}} \pm T_{t\phi}^{\mathrm{e}}) = \frac{U_x \mp v}{\sqrt{-\Delta_x\Delta_y}}X^{\pm}\frac{\partial}{\partial x}\ln\left(\frac{1}{\sigma^2}\frac{d\omega_x}{dx}\right), \tag{49c}$$

$$\frac{T_{yy}^{\mathrm{e}} - T_{\phi\phi}^{\mathrm{e}}}{2} \pm iT_{y\phi}^{\mathrm{e}} = 0, \tag{49d}$$

where

$$X^{\pm} \equiv \frac{v\omega_y}{u \mp v}\left[\frac{Q_\bullet^{\pm}}{4\rho_s(f_0g_1 + g_1g_0)}\frac{d\omega_x}{dx}\right]^2. \tag{50}$$

The total energy-momentum prescribed by the Einstein components (48), minus the electromagnetic energy-momentum (49), is

$$8\pi\rho^2\left(\frac{T_{xx}+T_{tt}}{2}\pm T_{xt}\right)=\frac{U_x\mp v}{-\Delta_x}\left(\Delta_x'\pm v-X^\pm\frac{v\omega_y}{\Delta_y}\right),\tag{51a}$$

$$8\pi\rho^2(T_{xy}\pm T_{ty})=-\frac{U_x\mp v}{\sqrt{-\Delta_x\Delta_y}}\left(\Delta_y\frac{\partial\ln\rho_s^2}{\partial y}\pm X^\pm\frac{1}{\sigma^2}\frac{d\omega_y}{dy}\right),\tag{51b}$$

$$8\pi\rho^2(T_{x\phi}\pm T_{t\phi})=\pm\frac{U_x\mp v}{\sqrt{-\Delta_x\Delta_y}}\left[\Delta_y\frac{1}{\sigma^2}\frac{d\omega_x}{dx}\mp 2v\omega_y\mp X^\pm\frac{\partial}{\partial x}\ln\left(\frac{1}{\sigma^2}\frac{d\omega_x}{dx}\right)\right],\tag{51c}$$

$$8\pi\rho^2\left(\frac{T_{yy}-T_{\phi\phi}}{2}\pm iT_{y\phi}\right)=0.\tag{51d}$$

For the solution to be consistent, the energy-momentum tensor given by Eq. (51) must be consistent with being sourced by collisionless streams. Indeed it is, as shown in the next subsection.

E. Collisionless source of energy-momentum

The energy-momentum tensor (51) coincides with that of a sum of ingoing (+) and outgoing (−) collisionless streams

$$T_{kl}=N^+p_k^+p_l^++N^-p_k^-p_l^-,\tag{52}$$

with number densities

$$N^\pm=\frac{1}{16\pi}(U_x\mp v)\left(\Delta_x'\pm v-X^\pm\frac{v\omega_y}{\Delta_y}\right),\tag{53}$$

and tetrad-frame momenta

$$p_k^\pm=\frac{1}{\rho}\left\{-\frac{1}{\sqrt{-\Delta_x}},\mp\frac{1}{\sqrt{-\Delta_x}},\frac{1}{\sqrt{\Delta_y}}\frac{\Delta_y\frac{\partial\ln\rho_s^2}{\partial y}\pm X^\pm\frac{1}{\sigma^2}\frac{d\omega_y}{dy}}{\Delta_x'\pm v-X^\pm\frac{v\omega_y}{\Delta_y}},\right.$$
$$\left.\mp\frac{1}{\sqrt{\Delta_y}}\frac{\Delta_y\frac{1}{\sigma^2}\frac{d\omega_x}{dx}\mp 2v\omega_y\mp X^\pm\frac{\partial}{\partial x}\ln(\frac{1}{\sigma^2}\frac{d\omega_x}{dx})}{\Delta_x'\pm v-X^\pm\frac{v\omega_y}{\Delta_y}}\right\},\tag{54}$$

as long as condition (46) on the horizon function Δ_x holds. The densities (53) and momenta (54) are defined up to arbitrary normalization factors such that the energy-momentum (52) is fixed. The ingoing and outgoing densities $N^\pm$, Eq. (53), conform to the Hamilton-Jacobi behavior (16), satisfying [see Eq. (117) of Paper 2]

$$N^\pm\propto\frac{1}{\rho^2}\propto U_x\mp v,\tag{55}$$

the remaining factors in Eq. (53) being constant along the path of a stream. Similarly, the tetrad-frame momentum $p_k^\pm$, Eq. (54), accords with the Hamilton-Jacobi form (7), with constant Hamilton-Jacobi parameters P_k along the path of the stream.

If the black hole were uncharged, then $X^\pm$ defined by Eq. (50) would vanish, and Eqs. (53) and (54) would reduce to Eqs. (127) and (128) of Paper 2. If the black hole's charge-to-mass ratio $Q_\bullet/M_\bullet$ is of order unity, then (one of, if only one stream is charged) $X^\pm$ is of order unity,

but if the black hole's charge is small, then $X^\pm$ too will be small. In any case, the charge of the black hole has little effect on the collisionless densities $N^\pm$, Eq. (53), the term proportional to $X^\pm$ being of order v compared to the principal term Δ_x'. This expresses the fact that the radial components of the energy-momentum are dominated by the streaming energy-momentum, not the electromagnetic energy-momentum. This in turn reflects the fact that the inflationary instability is fundamentally gravitational, not electromagnetic. However, if the black hole's charge-to-mass ratio is of order unity, then it has order unity effect on the angular components of the collisionless momenta $p_k^\pm$, Eq. (54).

The mean charge $\langle q^\pm\rangle$ per accreted particle is the ratio of the charge density $q^\pm N^\pm$, Eq. (44), to the number density $N^\pm$, Eq. (53):

$$\langle q^\pm\rangle\equiv\frac{q^\pm N^\pm}{N^\pm}$$
$$=e^{-vt}\frac{vQ_\bullet^\pm}{2\rho_s^2(f_0g_1+f_1g_0)}\frac{\frac{d\omega_x}{dx}\left[\frac{\partial}{\partial x}\ln(\frac{1}{\sigma^2}\frac{d\omega_x}{dx})\mp\frac{v\omega_y^2}{\Delta_y}\right]}{\Delta_x'\pm v-X^\pm\frac{v\omega_y}{\Delta_y}}.\tag{56}$$

The mean charge per particle (56) decreases with time as the black hole expands, consistent with dimensional analysis:

$$\langle q^\pm\rangle\propto e^{-vt}.\tag{57}$$

This contrasts with the number densities $N^\pm$, which are dimensionless, independent of conformal time t. Aside from the dependence on t, the remaining factors in the mean charge per particle (56) are just functions of latitude y, the dependence on radius x being frozen at its inner horizon value.

The angular components of the number-weighted momenta $p_k^\pm$ given by Eq. (54) differ from those of the charge-weighted momenta $p_k^\pm$ given by Eq. (45). The difference poses no great difficulty, but it does mean that the angular conditions on the current and the energy-momentum of the charged stream cannot be accomplished simultaneously with a single collisionless component. Two components to the charged stream would suffice. For example, one

component could be charged, fulfilling the conditions (44) and (45), and the other could be neutral, its number and momentum chosen such that, when added to those of the charged component, their sum fulfills the conditions (53) and (54). More generally, both components could be charged, the more highly charged stream providing much of the current, and the more lightly charge stream providing much of the number density.

F. Remaining Einstein components

The remaining 2 Einstein components, besides the 8 given as Eq. (48), are $G_{xx} - G_{tt}$ and $G_{yy} + G_{\phi\phi}$. These two components govern the evolution of the horizon function Δ_x and inflationary exponent ξ, as described in Sec. VIII D, E of Paper 2. Unlike other Einstein components, these 2 Einstein components do not grow during inflation, although they do grow during collapse when the conformal factor shrinks. They do not grow firstly because the trace of the collisionless energy-momentum remains negligible, and the trace of the electromagnetic energy-momentum is zero, and secondly because the combination $G_{yy} + G_{\phi\phi}$ of angular components of the energy-momentum is, for both collisionless and electromagnetic contributions, independent of the radial horizon function Δ_x, and therefore they do not grow large at the inner horizon where $\Delta_x \rightarrow -0$, unlike radial components which are proportional to inverse powers of Δ_x.

To dominant order, the evolution of the horizon function and inflationary exponent during inflation and collapse are unaffected by the collisionless and electromagnetic energy-momentum, except that the initial electrovac solution sets the derivative Δ'_x of the horizon function at the inner horizon.

The effect of the subdominant purely angular components of the energy-momentum may be taken into account by solving the Einstein equations for the 2 components to next higher order in Δ_x/U_x, as described in Sec. VIII J of Paper 2. Given the expressions (25) for the radial electric and magnetic fields E and B, and the solutions (36) for the potentials $\mathcal{A}_x \pm \mathcal{A}_t$, the relevant components (31a) of the electromagnetic energy-momentum T_{kl}^e are

$$8\pi\rho^2(T_{xx}^e - T_{tt}^e)$$

$$= 8\pi\rho^2(T_{yy}^e + T_{\phi\phi}^e)\frac{1}{2}\left[(U_x - v)\frac{X^+}{v\omega_y} + (U_x + v)\frac{X^-}{v\omega_y}\right]$$

$$\times \left[\left(\frac{\partial}{\partial x}\ln\left(\frac{1}{\sigma^2}\frac{d\omega_x}{dx}\right)\right)^2 + \left(\frac{1}{\sigma^2}\frac{d\omega_y}{dy}\right)^2\right]. \tag{58}$$

If the charge-to-mass ratio of the black hole is of order unity, so that one of $X^\pm$ is of order unity, then the electromagnetic energy-momentum (58), which is of order $\sim U_x/v$, is larger by of order $1/v$ than the corresponding collisionless energy-momentum given by Eq. (136) of Paper 2. Thus, the angular components of the electromagnetic energy-momentum, though still a subdominant

influence on the evolution of the horizon function and inflationary exponent during inflation and collapse, have a potentially larger influence than the angular components of the collisionless energy-momentum.

The electromagnetic energy-momentum components (58) can be taken into account in the Einstein equations by introducing source functions F_X and F_Y to the evolutionary equations for U_x and Δ_x, Eq. (139) of Paper 2. The relevant source functions are

$$F_X = F_Y = \frac{4\pi\rho^2(T_{yy}^e + T_{\phi\phi}^e)}{U_x}. \tag{59}$$

During early inflation, while U_x remains at its initial value of u, the source functions are just those of the electrovac spacetime, which are already taken into account in the solution. Once U_x has increased appreciably above u, the horizon function is already exponentially tiny, $|\Delta_x| \sim e^{-1/v}$. The condition that the sources F_X and F_Y have negligible influence on the evolution of U_x and Δ_x is that $U_x F_X \ll U_x^2/|\Delta_x|$, which for the source functions (59) is equivalent to $|\Delta_x|/U_x \ll 1/v$. This is well satisfied during inflation because $|\Delta_x|$ is exponentially tiny, and during collapse because $|\Delta_x|$ remains small and U_x grows exponentially large.

Eventually, the conformally separable solution breaks down, but not because of the electromagnetic energy-momentum (provided that only one of the ingoing and outgoing streams is charged). Rather, when the angular motion of the collisionless streams becomes comparable to their radial motion, which happens when $|\Delta_x| \gtrsim 1$ at the end of collapse, the conformally separable Einstein and Maxwell equations cease to be satisfied by the energy-momentum and current of freely falling collisionless streams.

V. BOUNDARY CONDITIONS

Maxwell's and Einstein's equations have prescribed the forms of the electric current (43)–(45) and energy-momentum (52)–(54) of the freely falling ingoing and outgoing collisionless streams during inflation and collapse. These forms are required by the condition of conformal separability. As in Paper 2, because the accretion rate is asymptotically tiny, the charge and energy-momentum of the collisionless streams are negligible above the inner horizon, so they have negligible effect on the geometry above the inner horizon. From the perspective of boundary conditions, what is important is the form of the collisionless streams incident on the inner horizon.

During inflation and collapse, the radial $(x - t)$ components of the momenta $p_k^\pm$ of the streams are of order $1/\sqrt{-\Delta_x}$ times the angular $(y - \phi)$ components, and dominate as long as $|\Delta_x| \ll 1$. The solution for the dominant radial components is essentially unaffected by the angular components. If only the dominant radial

Maxwell and Einstein equations are required to be satisfied, then only radial boundary conditions, Sec. VA, need be imposed. If the subdominant angular Maxwell and Einstein equations are required to be satisfied, then also angular boundary conditions, Sec. VB, must be imposed. If the sub-sub-dominant purely angular Einstein equations are further required to be satisfied, then yet further boundary conditions, Sec. VD, must be imposed.

A. Density and charge of collisionless streams incident on the inner horizon

The indispensable boundary conditions are those on the radial $(x - t)$ components of the collisionless current and energy-momentum incident on the inner horizon.

The ingoing $(+)$ and outgoing $(-)$ radial energy-momenta during inflation and collapse are, Eqs. (52)–(54),

$$T_{xx}^\pm = \pm T_{xt}^\pm = T_{xx}^\pm = \frac{N^\pm}{\rho^2 |\Delta_x|}. \tag{60}$$

The initial values of these components are set by the incident number densities $N^\pm$, Eq. (53), which are, since $U_x = u$ initially,

$$N^\pm = \frac{1}{16\pi}(u \mp v)\left(\Delta_x' \pm v - X^\pm \frac{v\omega_y}{\Delta_y}\right). \tag{61}$$

If the subdominant parts proportional to v are neglected, then the density (61) is uniform, independent of latitude, meaning that the required accretion flow is monopole. If the subdominant contributions proportional to v are taken into account, which is necessary if the boundary conditions are to be imposed to angular order, Sec. VB, then the part proportional to $X^\pm$, which arises from the presence of the electromagnetic field, introduces a small angular dependence of the incident densities $N^\pm$.

The ingoing $(+)$ and outgoing $(-)$ radial currents during inflation and collapse are, Eqs. (43)–(45),

$$j_x^\pm = \pm j_t^\pm = \frac{q^\pm N^\pm}{\rho\sqrt{-\Delta_x}}. \tag{62}$$

The initial values of the radial currents (62) are set by the incident charge densities $q^\pm N^\pm$, Eq. (44), which are

$$q^\pm N^\pm = \frac{vQ_\bullet^\pm}{32\pi\rho_s^2(f_0 g_1 + f_1 g_0)}\frac{d\omega_x}{dx}\left[\frac{\partial}{\partial x}\ln\left(\frac{1}{\sigma^2}\frac{d\omega_x}{dx}\right) \mp \frac{v\omega_y^2}{\Delta_y}\right]. \tag{63}$$

Unlike the number density $N^\pm$, the charge density $q^\pm N^\pm$ varies significantly with latitude. If the subdominant term proportional to v inside square brackets on the right-hand side of Eq. (63) is neglected, then the incident charge density is proportional to the radial electric field E,

$$q^\pm N^\pm = \frac{vE}{4\pi}. \tag{64}$$

The different angular behaviors of the number and charge and densities (61)–(63) mean that the mean charge per particle must vary with latitude, Eq. (56).

The charge density Eq. (63) relates the ingoing and outgoing cumulative black hole charges $Q_\bullet^\pm$ to the rates $q^\pm N^\pm$ at which ingoing and outgoing charges are accreted. This is a feature of conformally time-translation symmetric (self-similar) spacetimes, that cumulative properties are determined by the rate of their accretion.

B. Angular motions of collisionless streams incident on the inner horizon

The angular components of the momenta of the collisionless streams are subdominant during inflation and collapse. If the subdominant Einstein and Maxwell equations are to be satisfied, then angular boundary conditions must be imposed in addition to the radial boundary conditions.

Equation (54) specifies the required number-weighted tetrad-frame momentum $p_k^\pm$ of the ingoing and outgoing streams. The corresponding number-weighted Hamilton-Jacobi parameters $P_k^\pm$ are

$$P_k^\pm = \left\{-1, \mp 1, \frac{\Delta_y \frac{\partial \ln\rho_s^2}{\partial y} \pm X^\pm \frac{1}{\sigma^2}\frac{d\omega_y}{dy}}{\Delta_x' \pm v - X^\pm \frac{v\omega_y}{\Delta_y}}, \right. $$
$$\left. \mp \frac{\Delta_y \frac{1}{\sigma^2}\frac{d\omega_x}{dx} \mp 2v\omega_y \mp X^\pm \frac{\partial}{\partial x}\ln(\frac{1}{\sigma^2}\frac{d\omega_x}{dx})}{\Delta_x' \pm v - X^\pm \frac{v\omega_y}{\Delta_y}}\right\}. \tag{65}$$

Similarly, Eq. (45) specifies the required charge-weighted tetrad-frame momentum $p_k^\pm$ of whichever of the ingoing or outgoing streams is charged, and the corresponding charge-weighted Hamilton-Jacobi parameters $P_k^\pm$ are

$$P_k^\pm = \left\{-1, \mp 1, \frac{\frac{1}{\sigma^2}\frac{d\omega_y}{dy}}{\frac{\partial}{\partial x}\ln(\frac{1}{\sigma^2}\frac{d\omega_x}{dx}) \mp \frac{v\omega_y^2}{\Delta_y}}, \pm \frac{\frac{\partial}{\partial x}\ln(\frac{1}{\sigma^2}\frac{d\omega_x}{dx})}{\frac{\partial}{\partial x}\ln(\frac{1}{\sigma^2}\frac{d\omega_x}{dx}) \mp \frac{v\omega_y^2}{\Delta_y}}\right\}. \tag{66}$$

The angular components $P_y^\pm$ and $P_\phi^\pm$ vary with latitude y, but remain frozen at their inner horizon values along the path of a stream during inflation and collapse, as long as $|\Delta_x| \ll 1$, that is, until angular motions become important at the end of collapse, when $|\Delta_x| \gtrsim 1$.

The number- and charge-weighted angular motions (65) and (66) differ. To achieve both angular conditions requires that the charged stream contain more than one component. As remarked in Sec. IVE, both conditions (65) and (66) together can be accomplished with two (or more) components, a more highly charged component that produces most of the current, and a more lightly charged component that produces most of the number density.

C. Feasibility of the angular boundary conditions with collisionless streams accreted from outside the outer horizon

Can the boundary conditions (65) and (66) on the angular motions of incident ingoing and outgoing streams be accomplished by real collisionless streams? In Sec. X C of Paper 2 [1], it was shown that, in the case of an uncharged black hole, the conditions cannot be accomplished if the collisionless streams are required to be accreted from outside the outer horizon. In the present subsection, it is shown that electric charge alleviates the problem, but does not eliminate it.

If every particle is accreted from outside the outer horizon, then the Hamilton-Jacobi parameter P_t must necessarily be negative (ingoing) at the outer horizon for every particle. For a particle of charge q, the P_t and P_ϕ of the particle at the inner horizon must then satisfy the inequality, generalizing inequality (161) of Paper 2,

$$P_t \le P_\phi \frac{\omega_{x,\mathrm{in}} - \omega_{x,\mathrm{out}}}{1 - \omega_{x,\mathrm{out}}\omega_{y,\mathrm{in}}}$$
$$+ q\left(-\mathcal{A}_{t,\mathrm{in}} + \frac{1 - \omega_{x,\mathrm{in}}\omega_{y,\mathrm{in}}}{1 - \omega_{x,\mathrm{out}}\omega_{y,\mathrm{in}}} \mathcal{A}_{t,\mathrm{out}}\right), \quad (67)$$

where subscripts out and in denote values at the outer and inner horizons. If the particle has the same charge as the black hole, as is more likely since the black hole inherits its charge from the accreted streams, then the charge-dependent factor in condition (67) is positive, so that the allowed region of the $P_t - P_\phi$ plane includes a finite region around the origin $P_t = P_\phi = 0$. Physically, the black hole's charge repels the charged particle between the outer and inner horizon, increasing its P_t. This is sufficient to allow the ratio P_t/P_ϕ prescribed by either of Eqs. (65) and (66) to be accomplished at all latitudes.

If both ingoing and outgoing streams were permitted to be charged, with the same charge as the black hole, then condition (67) could be accomplished by both streams at all latitudes. If both streams are charged, however, the spacetime does not remain conformally separable because the diagonal angular components of the electromagnetic energy-momentum diverge near the inner horizon. Conformal separability begins to break down when the diverging component $\rho^2(T^{\mathrm{e}}_{yy} - T^{\mathrm{e}}_{\phi\phi})$, Eq. (31e), which is dimensionless, is of order unity, which happens when

$$|\Delta_x| \sim \frac{v^2 Q^+_\bullet Q^-_\bullet}{\rho^2}. \quad (68)$$

This is small, both because it is proportional to the square v^2 of the small accretion rate v, and because real black holes are likely to have small charge, so $Q^\pm_\bullet$ will be small. The black hole charge can, despite its smallness, affect charged particles because of the large charge-to-mass ratio of real particles, protons and electrons. Although

the value (68) of the horizon function Δ_x when conformal separability fails is small, it is nonetheless large compared to the exponentially tiny values to which the horizon function is driven during inflation.

If conformal separability is required to persist during inflation and collapse, then one or other of the ingoing and outgoing streams must be neutral. For the neutral stream, as discussed in Sec. X C of Paper 2, the condition (67) (with $q = 0$) cannot be satisfied simultaneously at all latitudes.

To summarize, if conformal separability to angular order is demanded only during the early part of inflation, then the required conditions on the incident angular motions of ingoing and outgoing streams can be accomplished by collisionless streams accreted entirely from outside the outer horizon, provided that both streams are charged, with the same charge as the black hole. However, if conformal separability is demanded throughout inflation and collapse, then one of the streams must be neutral, and the angular conditions on that neutral stream cannot be accomplished by a collisionless stream accreted entirely from outside the outer horizon.

D. Dispersion of angular motions incident on the inner horizon

The purely angular components of the energy-momentum tensor are sub-sub-dominant during inflation and collapse while $|\Delta_x| \ll 1$. Their effect can nevertheless be taken into account by solving the Einstein equations to next higher order in Δ_x/U_x, as described in Sec. IV F. If the Einstein equations are required to hold to this order, then the condition of conformal separability requires that the 2×2 angular submatrix of the energy-momentum tensor must be isotropic. The electromagnetic energy-momentum tensor (31) satisfies the condition of isotropy provided that condition (32) holds, which is true provided that only one of the ingoing or outgoing streams is charged. As discussed in Sec. X D of Paper 2, the condition of angular isotropy on the collisionless energy-momentum can also be contrived, by allowing multiple incident ingoing and outgoing streams whose angular components of momentum satisfy (65) in the mean, but are isotropic in their mean squares.

The more precisely the Einstein and Maxwell equations are required to be satisfied, the more special and contrived the conditions required by conformal separability become.

VI. CONCLUSIONS

The conformally stationary, axisymmetric, conformally separable solutions for the interior structure of rotating black holes found in Paper 2 [1] generalize to the case of charged rotating black holes. Maxwell's equations separate consistently with Einstein's equations. The collisionless fluid accreted by the black hole is permitted to be electrically charged, and the charge of the black hole is produced self-consistently by the accretion of charge. As in the

uncharged case, hyper-relativistic counterstreaming between ingoing and outgoing streams drives mass inflation at the inner horizon, followed by collapse.

The only anomaly is that conformal separability in charged black holes requires that only one of the ingoing or outgoing streams can be charged: the other stream must be neutral. If both streams are charged, then conformal separability holds during early inflation, but radially counterstreaming electric currents, in concert with the rotation of the black hole, generate angular electromagnetic fields that cause the nonisotropic diagonal angular component of the electromagnetic energy-momentum to diverge, destroying conformal separability. I suspect that the physical reason for the breakdown of conformal separability is that if both streams are charged, then they can exchange energy-momentum via the electromagnetic field that they mutually create. This breaks the $v \leftrightarrow -v$ symmetry between the ingoing and outgoing streams, which appears to be important to the existence of the solutions.

In practice, the condition that only one stream be charged is physically realistic, since collisions and magnetohydrodynamic processes are likely to keep charged particles tightly coupled above the inner horizon, forcing them into a common ingoing or outgoing stream before inflation ignites.

The most important equations in this paper are the Maxwell equations (27). These equations hold over the entire regime of interest, from electrovac through inflation and collapse. In concert with the Einstein equations (88) from Paper 2, their solution yields the full suite of both stationary, separable electrovac, and conformally stationary, conformally separable inflationary solutions.

The condition of conformal separability imposes a hierarchy of boundary conditions on the collisionless streams incident on the inner horizon. The indispensible boundary condition is on the dominant radial components of the collisionless energy-momentum and current. The radial conditions require that the incident number densities $N^{\pm}$, Eq. (61), of the ingoing and outgoing streams must be uniform with latitude (with a subdominant order v angular dependence arising from the electromagnetic field). By contrast, the incident charge densities $q^{\pm}N^{\pm}$, Eq. (63), must vary with latitude with, to leading order, the same angular dependence as the radial electric field E, Eq. (64). The different angular dependences of the number and charge densities $N^{\pm}$ and $q^{\pm}N^{\pm}$ imply that the incident mean charge per particle must vary with latitude, Eq. (56).

If the subdominant radial-angular components of the Einstein equations and angular components of the Maxwell equations are required to be satisfied, then conformal separability requires that the angular components of the number-weighted and charge-weighted momenta of the incident streams have Hamilton-Jacobi parameters P_k satisfying Eqs. (65) and (66), respectively. The number- and charge-weighted angular motions differ, implying that the conditions cannot be accomplished by a collisionless charged stream containing just one component. However, the angular conditions on a charged stream can be achieved with two (or more) components, a more highly charged component that produces most of the current, and a more lightly charged component that produces most of the number density.

In Paper 2, it was emphasized that the solutions had the limitation that the angular conditions on the incident ingoing and outgoing streams could not be achieved by collisionless streams that fall freely from outside the outer horizon. The present paper finds that the required angular conditions can be achieved by a charged stream, provided that the stream has the same sign charge as the black hole, but not by a neutral stream. As commented above, if both streams are charged, then conformal separability holds only during early inflation. If conformal separability is required to hold throughout inflation and collapse, then one of the streams must be neutral, and then the angular conditions on the incident neutral stream cannot be achieved by a stream that falls freely from outside the outer horizon.

If the sub-sub-dominant purely angular Einstein equations are required to be satisfied, then conformal separability requires that the angular energy-momentum tensor be isotropic (proportional to the unit 2×2 matrix). The electromagnetic energy-momentum is isotropic provided that only one of the ingoing or outgoing streams is charged, and the collisionless energy-momentum can be contrived to be isotropic.

ACKNOWLEDGMENTS

APPENDIX A: NON-VANISHING Z_x AND Z_t

The calculation of the evolution of the potentials $\mathcal{A}_x \pm \mathcal{A}_t$ in Sec. IV A was premised in part on the vanishing of Z_x and Z_t defined by Eq. (24). This Appendix examines what happens if Z_x and Z_t do not vanish.

The conclusion is that Z_x and Z_t must take a certain form (A5) in order that Maxwell's equations can continue to be satisfied by the current of collisionless streams. By adjusting the proportionality factor z in this form (A5), the angular current $j_\phi^{\pm}$ of the collisionless ingoing or outgoing stream can be adjusted to be an essentially arbitrary function of latitude y. This change relaxes the angular conditions on the current imposed by the assumption of conformal separability, but leaves all the conclusions of the main text unchanged. Only the azimuthal angular current $j_\phi^{\pm}$ is adjustable: the radial currents $j_x^{\pm}$ and $j_t^{\pm}$ are

scarcely affected, and the angular current $j_y^\pm$ is not affected at all.

1. Required form of Z_x and Z_t

Currents sourced by Maxwell's equations (29) with non-vanishing Z_x and Z_t must continue to fit the form of collisionless currents. For the radial currents $j_x \pm j_t$, this requires that the set of terms proportional to Z_t on the right-hand side of Eq. (29a) must be proportional to $\mathcal{A}_x \pm \mathcal{A}_t$ times an appropriate factor of the horizon function Δ_x (the subdominant third term inside square brackets is temporarily neglected in this Sec. A 1, but is reinstated thereafter):

$$\left(\frac{\partial}{\partial x} \mp \frac{v}{\Delta_x}\right)Z_t \propto \frac{\mathcal{A}_x \pm \mathcal{A}_t}{\Delta_x}. \qquad (A1)$$

Equation (A1) requires that

$$\frac{\partial Z_t}{\partial x} = -\frac{\mathcal{A}_x}{\mathcal{A}_t}\frac{v}{\Delta_x}Z_t, \qquad (A2)$$

which, given that $\mathcal{A}_x = \pm\mathcal{A}_t$, integrates to

$$Z_t \propto \mathcal{A}_x \pm \mathcal{A}_t, \qquad (A3)$$

whichever one of $\mathcal{A}_x \pm \mathcal{A}_t$ is nonvanishing. Equation (33) shows that $Z_x \pm Z_t$ is a source for the evolution of $\mathcal{A}_x \pm \mathcal{A}_t$. To ensure that the combination $\mathcal{A}_x \pm \mathcal{A}_t$ that vanishes continues to vanish as $|\Delta_x|$ decreases to exponentially tiny values, the corresponding source

$Z_x \pm Z_t$ must also vanish. Putting this condition together with (A3) requires that

$$Z_x = \pm Z_t \propto \mathcal{A}_x \pm \mathcal{A}_t, \qquad (A4)$$

for whichever one of $\mathcal{A}_x \pm \mathcal{A}_t$ is nonvanishing. The constraints (A4) on Z_x and Z_t are conveniently written

$$Z_x \pm Z_t = z\frac{v\omega_y}{\Delta_y}(\mathcal{A}_x \pm \mathcal{A}_t), \qquad (A5)$$

for some factor z, which could be an arbitrary function of angle y.

2. Evolution of the electromagnetic potential and enclosed charge

Inserting the result (A5) for $Z_x \pm Z_t$ into the evolutionary Eq. (33) for $\mathcal{A}_x \pm \mathcal{A}_t$ gives

$$\left(\frac{\partial}{\partial x} \pm \frac{v}{\Delta_x} - z\frac{v\omega_y}{\Delta_y}\right)\frac{\mathcal{A}_x \pm \mathcal{A}_t}{d\omega_x/dx} = 0. \qquad (A6)$$

The driving term proportional to v/Δ_x, which diverges at the inner horizon $\Delta_x \rightarrow -0$, dominates the term proportional to z in the conformally stationary limit. The z term changes the evolution of the potentials negligibly.

3. Maxwell's equations

Given Z_x and Z_t from Eq. (A5), Maxwell's equations (29) become Eq. (41) with, in the same format as Eq. (42),

$$j_x^\pm = \pm j_t^\pm \equiv \frac{1}{8\pi\rho^3}\frac{v}{\sqrt{-\Delta_x}}(\mathcal{A}_x \pm \mathcal{A}_t)\left\{\pm\frac{\partial}{\partial x}\ln\left(\frac{1}{\sigma^2}\frac{d\omega_x}{dx}\right) - \frac{v\omega_y^2}{\Delta_y} + z\frac{\omega_y}{\Delta_y}\left[\pm v - \Delta_x\left(\frac{\partial}{\partial x}\ln\left(\frac{1}{\sigma}\frac{d\omega_x}{dx}\right) + \frac{z}{2}\frac{v\omega_y}{\Delta_y}\right)\right]\right\}, \qquad (A7a)$$

$$j_\phi^\pm \equiv -\frac{1}{8\pi\rho^3}\frac{v\omega_y}{\sqrt{\Delta_y}}(\mathcal{A}_x \pm \mathcal{A}_t)\left[\frac{\partial}{\partial x}\ln\left(\frac{1}{\sigma^2}\frac{d\omega_x}{dx}\right) - z\left(\pm\frac{1}{\sigma^2}\frac{d\omega_x}{dx} - \frac{v\omega_y}{\Delta_y}\right)\right], \qquad (A7b)$$

while the expression (42b) for $j_y^\pm$ remains unchanged. The term proportional to Δ_x on the far right-hand side of Eq. (A7a) is negligible compared to the other terms for $|\Delta_x| \ll 1$, so the radial Maxwell equation (A7a) simplifies to

$$\begin{aligned}j_x^\pm &= \pm j_t^\pm \\ &= \frac{1}{8\pi\rho^3}\frac{v}{\sqrt{-\Delta_x}}(\mathcal{A}_x \pm \mathcal{A}_t) \\ &\times\left[\pm\frac{\partial}{\partial x}\ln\left(\frac{1}{\sigma^2}\frac{d\omega_x}{dx}\right) - \frac{v\omega_y^2}{\Delta_y} \pm z\frac{v\omega_y}{\Delta_y}\right].\end{aligned} \qquad (A8)$$

4. Collisionless source of electric current

Maxwell's equations (41) with currents given by Eqs. (A8), (42b), and (A7b) can be satisfied by currents from a sum of ingoing and outgoing collisionless streams, Eq. (43), with charge densities

$$\begin{aligned}q^\pm N^\pm &= -\frac{vQ^\pm}{32\pi\rho^2(f_0 g_1 + f_1 g_0)}\frac{d\omega_x}{dx} \\ &\times\left[\frac{\partial}{\partial x}\ln\left(\frac{1}{\sigma^2}\frac{d\omega_x}{dx}\right) \mp \frac{v\omega_y^2}{\Delta_y} + z\frac{v\omega_y}{\Delta_y}\right],\end{aligned} \qquad (A9)$$

and charge-weighted tetrad-frame momenta

$$p_k^{\pm} = \frac{1}{\rho}\left\{ -\frac{1}{\sqrt{-\Delta_x}}, \mp\frac{1}{\sqrt{-\Delta_x}}, \right.$$

$$\frac{1}{\sqrt{\Delta_y}}\frac{\frac{1}{\sigma^2}\frac{d\omega_y}{dy}}{\frac{\partial}{\partial x}\ln(\frac{1}{\sigma^2}\frac{d\omega_x}{dx}) \mp \frac{v\omega_y^2}{\Delta_y} + z\frac{v\omega_y}{\Delta_y}},$$

$$\left. \pm\frac{1}{\sqrt{\Delta_y}}\frac{\frac{\partial}{\partial x}\ln(\frac{1}{\sigma^2}\frac{d\omega_x}{dx}) - z(\pm\frac{1}{\sigma^2}\frac{d\omega_x}{dx} + \frac{v\omega_y}{\Delta_y})}{\frac{\partial}{\partial x}\ln(\frac{1}{\sigma^2}\frac{d\omega_x}{dx}) \mp \frac{v\omega_y^2}{\Delta_y} + z\frac{v\omega_y}{\Delta_y}} \right\}. \quad \text{(A10)}$$

The adjustable angle-dependent factor z makes only a subdominant order v change to the charge densities $q^{\pm}N^{\pm}$, Eq. (A9), but an order unity change to the azimuthal component $p_{\phi}^{\pm}$ of the momentum (A10). Adjusting z changes the azimuthal current $j_{\phi}^{\pm}$ arbitrarily, but leaves the other current components essentially unchanged.

It is worth commenting that z can be adjusted so that the azimuthal current $j_{\phi}^{\pm}$ is zero for whichever stream is charged. This in no way affects the conclusion that only one of the ingoing or outgoing streams can be charged. The divergence in the angular component (31e) of the electromagnetic energy-momentum tensor is driven by counterstreaming of radial, not angular, ingoing and outgoing currents, coupled to the rotation of the black hole. The only way to remove the divergence, as conformal separability requires, is to allow only one of the radial currents to be charged.

5. Energy-momenta

A finite $Z_x \pm Z_t$ affects the radial electric field E, Eq. (25), to subdominant order v. This small change propagates into corresponding components (31) of the electromagnetic energy-momentum tensor, which leads to a small change in the azimuthal component $p_{\phi}^{\pm}$ of the number-weighted tetrad-frame momenta of collisionless ingoing and outgoing streams. The small change amounts to changing

$$\frac{\partial}{\partial x}\ln\left(\frac{1}{\sigma^2}\frac{d\omega_x}{dx}\right) \rightarrow \frac{\partial}{\partial x}\ln\left(\frac{1}{\sigma^2}\frac{d\omega_x}{dx}\right) + z\frac{v\omega_y}{\Delta_y} \quad \text{(A11)}$$

in expressions (49c) and (58) for the electromagnetic energy-momentum components $T_{x\phi}^e \pm T_{t\phi}^e$ and $T_{xx}^e - T_{tt}^e$, and in Eq. (54) for the azimuthal component $p_{\phi}^{\pm}$ of the number-weighted tetrad-frame momenta.

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