# Interior structure of rotating black holes. II. Uncharged black holes

Andrew J. S. Hamilton*

*JILA and Department of Astrophysical & Planetary Sciences, Box 440, University of Colorado, Boulder, Colorado 80309, USA*

(Received 6 April 2011; published 28 December 2011)

A solution is obtained for the interior structure of an uncharged rotating black hole that accretes a collisionless fluid. The solution is conformally stationary, axisymmetric, and conformally separable, possessing a conformal Killing tensor. The solution holds approximately if the accretion rate is small but finite, becoming more accurate as the accretion rate tends to zero. Hyper-relativistic counter-streaming between collisionless ingoing and outgoing streams drives inflation at (just above) the inner horizon, followed by collapse. As ingoing and outgoing streams approach the inner horizon, they focus into twin narrow beams directed along the ingoing and outgoing principal null directions, regardless of the initial angular motions of the streams. The radial energy-momentum of the counter-streaming beams gravitationally accelerates the streams even faster along the principal directions, leading to exponential growth in the streaming density and pressure, and in the Weyl curvature and mass function. At exponentially large density and curvature, inflation stalls, and the spacetime collapses. As the spacetime collapses, the angular motions of the freely-falling streams grow. When the angular motion has become comparable to the radial motion, which happens when the conformal factor has shrunk to an exponentially tiny scale, conformal separability breaks down, and the solution fails. The condition of conformal separability prescribes the form of the ingoing and outgoing accretion flows incident on the inner horizon. The dominant radial part of the solution holds provided that the densities of ingoing and outgoing streams incident on the inner horizon are uniform, independent of latitude; that is, the accretion flow is ‘‘monopole.’’ The subdominant angular part of the solution requires a special nonradial pattern of angular motion of streams incident on the inner horizon. The prescribed angular pattern cannot be achieved if the collisionless streams fall freely from outside the horizon, so the streams must be considered as delivered *ad hoc* to just above the inner horizon.

DOI: 10.1103/PhysRevD.84.124056 PACS numbers: 04.20.−q

## I. INTRODUCTION

‘‘No satisfactory interior solutions are known’’ [[1], Sec. 20.5] for rotating black holes. The purpose of this paper is to present nonlinear, dynamical solutions for the interior structure of a rotating black hole in the special case where it slowly accretes a collisionless fluid in a ‘‘conformally separable’’ fashion. A companion paper [2], hereafter Paper 3, extends the solutions to charged rotating black holes. A concise derivation of the principal results of the present paper are given in Paper 1 [3]. A Mathematica notebook containing many details of the calculations is at [4].

Unlike the Schwarzschild [5,6] geometry, the Kerr [7] geometry contains an inner horizon as well as an outer horizon. In the classic analytically extended Kerr solution, the inner horizon is a gateway to delightful but unrealistic pathologies, including wormholes, white holes, timelike singularities, and closed timelike curves [8].

Sadly, the Kerr geometry fails at the inner horizon, because it is subject to the mass inflation instability discovered by Poisson & Israel (1990) [9]. The inflationary instability is the nonlinear realization of the infinite blueshift at the inner horizon first pointed out by Penrose (1968) [10]. Most studies of the inflationary instability have focused on spherical charged black holes (see [11] for a review). The first investigation of inflation inside rotating black holes was that of Barrabès, Israel & Poisson [12], who showed that when two light sheets pass through each other, a mass parameter defined by the product of the expansions along the two light bundles inflates. Ori [13,14] and Brady and collaborators [15,16] (see also references in [17]) considered inflation driven by a Price tail of ingoing and outgoing gravitational waves in the late time collapse of a rotating black hole. Reference [18] found an exact mass inflation solution for a rotating black hole in $1 + 2$ dimensions.

Mass inflation is a generic classical mechanism that requires one essential ingredient to precipitate it: a source of ingoing (positive energy) and outgoing (negative energy) streams near the inner horizon that can stream relativistically through each other [11] (in this context, positive and negative energy refer to minus the sign of the covariant $t$-component $p_t$ of the momentum of a particle in a tetrad frame aligned with the principal frame, not to the conserved energy associated with translation invariance of the coordinate time $t$). Even the tiniest sources of ingoing and outgoing streams suffice to trigger inflation. As shown by [11,19], in spherical charged black holes, the smaller the streams, the more rapidly inflation exponentiates. This is a

*Andrew.Hamilton@colorado.edu

 

fierce and uncommon kind of instability, where the smaller the trigger, the more violent the reaction. The sensitivity of inflation to the smallest influence suggests that it would be difficult to avoid in a real black hole.

Inflation is not particular about the origin of the ingoing and outgoing streams needed to trigger it: anything will do. Most of the literature has considered the situation where inflation is ignited by a Price [20,21] tail of radiation generated during the initial collapse of the black hole. In real astronomical black holes, however, ongoing accretion of baryons and dark matter from outside probably soon overwhelms the initial Price tails. To illustrate how easy it is for accretion to generate both ingoing and outgoing streams at the inner horizon, consider a massive neutral dark matter particle that is slowly moving at infinity (energy per unit mass $E = 1$) in the equatorial plane of a Kerr black hole of mass $M_\bullet$ and angular momentum parameter $a$. The dark matter particle can free-fall into the black hole if its specific angular momentum $L$ lies in the interval (the following comes from solutions of the Hamilton-Jacobi equation derived in Sec. IVA)

$$-2M_\bullet\left(1+\sqrt{1+a/M_\bullet}\right) < L < 2M_\bullet\left(1+\sqrt{1-a/M_\bullet}\right). \tag{1}$$

The dark matter particle will become ingoing or outgoing at the inner horizon depending on whether its specific angular momentum $L$ is lesser or greater than $L_0$ given by

$$L_0 = \frac{2a}{1+\sqrt{1-a^2/M_\bullet^2}}, \tag{2}$$

which lies within the range (1) (hitting the upper limit of (1) in the case of an extremal black hole, $a = M_\bullet$). In fact distant, slowly moving dark matter particles can become either ingoing or outgoing at the inner horizon provided that the inclination of their orbit to the equatorial plane is less than a value that varies from 90° (i.e. all inclinations are allowed) at $a = 0$, to 62° at $a = 0.96M_\bullet$, to $\sin^{-1}\sqrt{1/3} \approx 35°$ at the extremal limit $a = M_\bullet$. Thus cold dark matter particles falling from afar provide a natural continuing source of both ingoing and outgoing collisionless matter near the inner horizon.

In black holes that continue to accrete, the generic outcome following inflation in spherical charged black holes is collapse to a spacelike singularity [11,22,23].

Even if there were no accretion from outside, quantum mechanical pair creation would provide a source of ingoing and outgoing radiation near the inner horizon. Before the mechanism of mass inflation was discovered, [24] showed that electromagnetic pair creation in a spherical charged black hole would destroy the inner horizon. They suggested that pair creation, probably by electromagnetic rather than gravitational processes in view of the much greater strength of electromagnetism, would also destroy the inner horizon of a rotating black hole. [25] considered a 2-dimensional dilaton model of gravity simple enough to allow a fully self-consistent treatment of pair creation and its nonlinear back-reaction on the spacetime of a charged black hole. They found that pair creation led to mass inflation near the inner horizon, followed by collapse to a spacelike singularity.

The inflationary instability means that the analytic extension of the Kerr geometry from the inner horizon inward never occurs in real black holes. Papers such as [26] that focus on the ring singularity of the Kerr geometry, while of interest in exploring how theories beyond general relativity might remove singularities, do not apply to real rotating black holes.

Why does inflation take place at the inner but not outer horizon? The essential ingredient of inflation is the simultaneous presence of both ingoing and outgoing streams. Ingoing and outgoing streams tend to move towards the inner horizon, amplifying their counter-streaming. By contrast, outgoing streams tend to move away from the outer horizon, deamplifying any counter-streaming. Classically, all matter at the outer horizon is ingoing: no outgoing matter can pass through the outer horizon. Outgoing modes will however be excited quantum mechanically near the outer horizon, leading to Hawking radiation. There have been various speculations that quantum effects could cause a quantum phase transition at the outer horizon [27–29], or prevent the outer horizon from forming in the first place [30]. The present paper assumes that the outer horizon is not subject to a quantum instability.

To arrive at a solution, this paper builds on physical insight gained from inflation in charged spherical black holes [11]. Two fundamental features of inflation point the way forward. Firstly, inflation is ignited by hyper-relativistic counter-streaming between ingoing and outgoing beams just above the inner horizon. As shown in Sec. V, regardless of their source or of their initial orbital parameters, near the inner horizon the counter-streaming beams become highly focused along the ingoing and outgoing principal null directions. This has the consequence that the energy-momentum tensor of the beams takes a predictable form, Sec. VII. The Kerr geometry is separable [31], and the alignment of the inflationary energy-momentum along the principal directions suggests that the geometry could continue to be separable during inflation.

The second key fact is that inflation in spherical charged black holes produces a geometry that looks like a step-function, being close to the electrovac (Reissner-Nordström) solution above the inner horizon, then exponentiating to super-Planckian curvature and density over a tiny scale of length and time. This suggests generalizing the usual separability condition (30) to a more general condition (46) that departs, at least initially, by an amount that is tiny, but with finite derivatives. Unless one were specifically looking for steplike solutions, one would not think to consider such a generalization.

The strategy of this paper is to seek the simplest inflationary solution for a rotating black hole. I assume that the black hole is accreting at a tiny (infinitesimal) rate, so that the Kerr solution applies accurately down to just above the inner horizon. This assumption is a good approximation for an astronomical black hole during most of its lifetime, since the accretion time scale is typically far longer than the characteristic light crossing time of a black hole. The approximation of small accretion rate breaks down during the initial collapse of the black hole from a stellar core, or during rare instances of high accretion, such as a black hole merger.

For simplicity, I assume further that the geometry is axisymmetric, and possesses conformal time-translation symmetry (self-similarity), even though the accretion flow in real black holes is unlikely to be axisymmetric or self-similar. One might imagine that the assumption of conformal time-translation invariance would, in the limit of infinitesimal accretion rate, be equivalent to the assumption of stationarity, but this is false. As explained in Sec. 4.4 of [11], and elaborated further in Sec. III of the present paper, the stationary approximation is equivalent to the assumption of symmetrically equal ingoing and outgoing streams at the inner horizon, whereas in reality the initial conditions of the accretion flow will generically lead to unequal streams near the inner horizon. The stationary approximation was first introduced to inflation by [32–34], who called it the homogeneous approximation because the time direction $t$ is spacelike inside the horizon. The present paper follows the convention of [[35], p. 203] in referring to time-translation symmetry as stationarity, even when the time direction is spacelike rather than timelike.

I call the combination of conformal time-translation invariance coupled with the limit of small accretion rates "conformal stationarity," to distinguish it from strict stationarity.

Motivated by the argument of the paragraph above beginning "To arrive at," I pursue the hypothesis that the spacetime is conformally separable. By conformal separability is meant the conditions (29) on the parameters of the line-element (3) and electromagnetic potential (24) that emerge from requiring that the equations of motion of massless particles be Hamilton-Jacobi separable, Sec. IV A. Whereas strict stationarity requires that the conformal factor take the separable form (30), conformal separability does not. Conformal separability implies the existence of a conformal Killing tensor, Sec. VI.

Finally, I assume that the streams that ignite and then drive inflation are freely-falling and collisionless. The present paper restricts to uncharged streams, while a companion paper [2], Paper 3, generalizes to charged streams. The assumption of collisionless flow is likely to break down when center-of-mass energies between ingoing and outgoing particles exceed the Planck energy, but for simplicity I neglect any collisional processes.

## II. LINE ELEMENT

Choose coordinates $x^\mu \equiv \{x, t, y, \phi\}$ in which $t$ is the conformal time with respect to which the spacetime is conformally time-translation symmetric (see Sec. III), $\phi$ is the azimuthal angle with respect to which the spacetime is axisymmetric, and $x$ and $y$ are radial and angular coordinates. Appendix A shows that under the conditions of conformal stationarity, axisymmetry, and conformal separability assumed in this paper, the line-element may be taken to be

$$ds^2 = \rho^2\left[\frac{dx^2}{\Delta_x} - \frac{\Delta_x}{\sigma^4}(dt - \omega_y d\phi)^2 + \frac{dy^2}{\Delta_y} + \frac{\Delta_y}{\sigma^4}(d\phi - \omega_x dt)^2\right], \tag{3}$$

where

$$\sigma \equiv \sqrt{1 - \omega_x \omega_y}. \tag{4}$$

The line-element is essentially the Hamilton-Jacobi and Schrödinger separable line-element given by [[31], eq. (1)], except that the conformal factor $\rho$ is left arbitrary, consistent with the weaker assumptions of conformal stationarity and conformal separability made here, as opposed to the stronger assumptions of strict stationarity and separability made by [31].

Thanks to the invariant character of the coordinates $t$ and $\phi$, the metric coefficients $g_{tt}$, $g_{t\phi}$, and $g_{\phi\phi}$ all have a gauge-invariant significance. The determinant of the $2\times 2$ submatrix of $t$-$\phi$ coefficents defines the radial and angular horizon functions $\Delta_x$ and $\Delta_y$:

$$g_{tt}g_{\phi\phi} - g_{t\phi}^2 = -\frac{\rho^4}{\sigma^4}\Delta_x\Delta_y. \tag{5}$$

Horizons occur when one or other of the horizon functions $\Delta_x$ and $\Delta_y$ vanish. In physically relevant cases, horizons occur in the radial direction, where $\Delta_x$ vanishes. The focus of this paper is inflation, which occurs in a region just above the inner horizon, where the radial horizon function $\Delta_x$ is negative and tending to zero. In this region the radial coordinate $x$ is timelike, while the time coordinate $t$ is spacelike. It is natural to choose the sign of the timelike radial coordinate $x$ so that it increases inward, the direction of advancing proper time.

Through the identity

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \eta_{mn}e^m{}_\mu e^n{}_\nu dx^\mu dx^\nu, \tag{6}$$

the line-element (3) encodes not only a metric $g_{\mu\nu}$, but a complete inverse vierbein $e^m{}_\mu$, corresponding vierbein $e_m{}^\mu$, and orthonormal tetrad $\{\gamma_x, \gamma_t, \gamma_y, \gamma_\phi\}$, satisfying $\gamma_m \cdot \gamma_n \equiv \eta_{mn}$ with $\eta_{mn}$ the Minkowski metric. The tetrad defined by the line-element (3) aligns with the principal null tetrad, as will become evident from the fact that the Weyl tensor is diagonal in the tetrad frame (it has only spin-0 components, Eq. (155)). It is convenient to choose the time axis of the tetrad to lie in the radial $x$-direction, since that direction is timelike near the inner horizon. Explicitly, the inverse vierbein $e^m{}_\mu$ is

$$e^m{}_\mu = \rho \begin{pmatrix} \frac{1}{\sqrt{-\Delta_x}} & 0 & 0 & 0 \\ 0 & \frac{\sqrt{-\Delta_x}}{\sigma^2} & 0 & -\frac{\omega_y\sqrt{-\Delta_x}}{\sigma^2} \\ 0 & 0 & \frac{1}{\sqrt{\Delta_y}} & 0 \\ 0 & -\frac{\omega_x\sqrt{\Delta_y}}{\sigma^2} & 0 & \frac{\sqrt{\Delta_y}}{\sigma^2} \end{pmatrix}, \quad (7)$$

while the corresponding vierbein $e_m{}^\mu$ is

$$e_m{}^\mu = \frac{1}{\rho} \begin{pmatrix} \sqrt{-\Delta_x} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{-\Delta_x}} & 0 & \frac{\omega_x}{\sqrt{-\Delta_x}} \\ 0 & 0 & \sqrt{\Delta_y} & 0 \\ 0 & \frac{\omega_y}{\sqrt{\Delta_y}} & 0 & \frac{1}{\sqrt{\Delta_y}} \end{pmatrix}. \quad (8)$$

The convention in this paper and its companions is that dummy tetrad-frame indices are Latin, while dummy coordinate-frame indices are Greek. The tetrad-frame directed derivative is denoted $\partial_m$, not to be confused with the coordinate-frame partial derivative $\partial/\partial x^\mu$. Tetrad-frame and coordinate-frame derivatives are related by

$$\partial_m \equiv e_m{}^\mu \frac{\partial}{\partial x^\mu}. \quad (9)$$

Whereas coordinate-frame derivatives $\partial/\partial x^\mu$ commute, tetrad-frame derivatives $\partial_m$ do not.

## III. CONFORMAL STATIONARITY

A fundamental simplifying assumption made in this paper is that the spacetime is conformally stationary, by which is meant that the spacetime is conformally time-translation invariant (that is, self-similar), and that the accretion rate is asymptotically tiny. Conformal stationarity generalizes the stationary (or homogeneous) approximation of [32–34], which required equal ingoing and outgoing streams at the inner horizon, to the realistic case of unequal ingoing and outgoing streams.

One might imagine that in the limit of asymptotically small accretion rate, the spacetime would automatically become stationary, $\partial/\partial t \equiv 0$, but that is false. A characteristic of inflation is that the smaller the accretion rate, the more rapidly inflation exponentiates [11]. Even in the limit of infinitesimal accretion rate, the counter-streaming energy of ingoing and outgoing streams grows exponentially huge during inflation. The stationary assumption sets to zero some quantities that, although initially infinitesimal, nevertheless grow huge. As discussed in Sec. 4.4 of [11] and Sec. X A of the present paper, the stationary assumption is equivalent to assuming equal ingoing and outgoing streams near the inner horizon. In reality, the relative fluxes of streams near the inner horizon depend on boundary conditions that generically do not lead to equal streams.

In place of stationarity, a consistent approach is to impose conformal time-translation invariance, also known as self-similarity. Conformal time-translation symmetry allows the conformal factor $\rho$ in the line-element (3) to include a time-dependent factor,

$$\rho = e^{vt}\hat{\rho}(x, y), \quad (10)$$

where the dimensionless factor $\hat{\rho}(x, y)$ is a function only of radius $x$ and angle $y$ (but not $\phi$, given axisymmetry). Equation (10) says that if the conformal time $t$ increases by an interval $\Delta t$, then the spacetime expands by a factor $e^{v\Delta t}$. The expansion is conformal, meaning that the spacetime keeps the same shape as it expands. The coefficient $v$ can be thought of as a dimensionless measure of the rate at which the black hole is expanding. The conformal time $t$ is dimensionless, and there is a gauge freedom in the choice of its scaling. A natural gauge choice is to match the change $dt$ in the conformal time at some fixed conformal position well outside the outer horizon to the change $dt_{\rm KN}$ in Kerr-Newman time measured in the natural units $c = G = M_\bullet = 1$ of the Kerr-Newman black hole:

$$dt = \frac{dt_{\rm KN}}{M_\bullet}. \quad (11)$$

With that gauge choice, the accretion rate $v$ is just equal to the dimensionless rate $\dot{M}_\bullet$ at which the mass $M_\bullet$ of the black hole increases, as measured by a distant observer:

$$v = \frac{\partial \ln\rho}{\partial t} = \frac{\partial \ln M_\bullet}{\partial t} = \frac{\partial M_\bullet}{\partial t_{\rm KN}} = \dot{M}_\bullet. \quad (12)$$

The accretion rate $v$ is constant, so the mass of the black hole increases linearly with time as measured by a distant observer, $M_\bullet = vt_{\rm KN}$.

The limit of small accretion rate is attained when

$$v \to 0. \quad (13)$$

It might seem that the small accretion rate limit (13) would be equivalent to stationarity, but that is false. Whereas stationarity sets the accretion rate $v$ to zero at the outset, conformal stationarity takes the limit (13) *after* completing all requisite calculations, not before.

In self-similar spacetimes, all quantites are proportional to some power of the time-dependent conformal factor $e^{vt}$, and that power can be determined by dimensional analysis. The metric coefficients $g_{\mu\nu}$, inverse vierbein $e^m{}_\mu$, vierbein $e_m{}^\mu$, tetrad-frame connections $\Gamma_{klm}$, tetrad-frame Riemann tensor $R_{klmn}$, tetrad-frame electromagnetic potential $A_m$, tetrad-frame electromagnetic field $F_{mn}$, and tetrad-frame electromagnetic current $j_m$, have respective dimensions

$$\begin{aligned} &g_{\mu\nu} \propto \rho^2, \qquad e^m{}_\mu \propto \rho, \qquad e_m{}^\mu \propto \rho^{-1}, \\ &\Gamma_{klm} \propto \rho^{-1}, \qquad R_{klmn} \propto \rho^{-2}, \qquad A_m \propto \rho^0, \\ &F_{mn} \propto \rho^{-1}, \qquad j_m \propto \rho^{-2}. \end{aligned} \tag{14}$$

A quantity that is scale-free, such as the electromagnetic potential $A_m$, is said to be dimensionless.

The form of the time-dependent factor $e^{vt}$ in the conformal factor (10) can be regarded as following from the fact that $\partial_m \rho$ must be dimensionless, which in turn requires that $\partial \ln\rho/\partial t$ must be dimensionless, hence independent of $t$.

## IV. CONFORMAL SEPARABILITY

Conformal separability is the proposition that the Hamilton-Jacobi equations of motion are separable for massless particles, but not necessarily for massive particles. Operationally, the proposition requires that the left-hand side, but not the right-hand side, of the Hamilton-Jacobi Eq. (27) be separable. A good part of Sec. IV A and IV B below overlaps ground that is familiar since the work of [31]. These subsections are nevertheless needed to establish notation, to highlight where conformal stationarity and conformal separability differ from full stationarity and separability, and to provide the basis for the discussion in Sec. V and for the derivation of the energy-momentum tensor of collisionless streams in Sec. VII. Section IV C covers new ground, showing that in the inflationary regime of interest here, even though the spacetime is only conformally separable, the equations of motion of massive particles are nevertheless Hamilton-Jacobi separable to an excellent approximation. Physically, massive particles are hyper-relativistic during inflation and collapse, and their trajectories are approximated accurately by those of massless particles.

### A. Hamilton-Jacobi separation

The equation of motion of a particle may be derived from Hamilton's equations, which express the derivative with respect to affine parameter $\lambda$ along the path of the particle of its coordinates $x^\mu$ and associated generalized conjugate momenta $\pi_\mu$ in terms of its Hamiltonian $H(x^\mu, \pi_\mu)$:

$$\frac{d\pi_\mu}{d\lambda} = -\frac{\partial H}{\partial x^\mu}, \qquad \frac{dx^\mu}{d\lambda} = \frac{\partial H}{\partial \pi_\mu}. \tag{15}$$

The Hamiltonian of a test particle of rest mass $m$ and charge $q$ moving in a prescribed background with metric $g_{\mu\nu}$ and electromagnetic potential $A_\mu$ is

$$H = \frac{1}{2} g^{\mu\nu}(\pi_\mu - qA_\mu)(\pi_\nu - qA_\nu). \tag{16}$$

The Hamilton-Jacobi method equates the Hamiltonian to minus the partial derivative of the action $S$ with respect to affine parameter, $H = -\partial S/\partial\lambda$, and replaces the generalized momenta with the partial derivatives of the action with respect to coordinates, $\pi_\mu = \partial S/\partial x^\mu$:

$$\frac{1}{2} g^{\mu\nu}\left(\frac{\partial S}{\partial x^\mu} - qA_\mu\right)\left(\frac{\partial S}{\partial x^\nu} - qA_\nu\right) = -\frac{\partial S}{\partial \lambda}. \tag{17}$$

One integral of motion, associated with conservation of the rest mass $m$ of the particle, follows from the fact that the Hamiltonian does not depend explicitly on the affine parameter. This implies that the Hamiltonian is itself a constant of motion:

$$H = -\frac{\partial S}{\partial \lambda} = -\frac{1}{2} m^2. \tag{18}$$

The normalization (18) of the Hamiltonian in terms of the rest mass $m$ is equivalent to choosing the affine parameter $\lambda$ to be related to the proper time $\tau$ along the path of the particle by

$$d\lambda = \frac{d\tau}{m}. \tag{19}$$

Two more integrals of motion follow from conformal time-translation symmetry, and axisymmetry. Conformal time-translation symmetry implies that the generalized momentum $\pi_t$ conjugate to conformal time $t$ satisfies the equation of motion

$$\frac{d\pi_t}{d\lambda} = -\frac{\partial H}{\partial t} = 2vH = -m^2 v, \tag{20}$$

where the factor of $-2v$ comes from the time-dependent conformal factor $e^{-2vt}$ in the inverse metric $g^{\mu\nu}$ in the Hamiltonian (16). Equation (20) integrates to

$$\pi_t = -E - mv\tau. \tag{21}$$

In the small accretion rate limit $v \to 0$, this reduces to the usual conservation of energy, $\pi_t = -E$. The concern expressed in Sec. III that some small quantities grow large during inflation does not apply here, because, as found in Sec. VIII F, Eq. (110), the proper time $\tau$ experienced by a particle during inflation and collapse is always tiny, in the conformally stationary limit (this is checked explicitly at the end of Sec. IV C).

The two integrals of motion associated with conformal stationarity and axisymmetry correspond to conservation of energy $E$ and azimuthal angular momentum $L$,

$$\pi_t = \frac{\partial S}{\partial t} = -E, \qquad \pi_\phi = \frac{\partial S}{\partial \phi} = L. \tag{22}$$

Write the covariant tetrad-frame momentum $p_k$ of a particle in terms of a set of Hamilton-Jacobi parameters $P_k$,

$$p_k \equiv \frac{1}{\rho}\left\{\frac{P_x}{\sqrt{-\Delta_x}}, \frac{P_t}{\sqrt{-\Delta_x}}, \frac{P_y}{\sqrt{\Delta_y}}, \frac{P_\phi}{\sqrt{\Delta_y}}\right\}, \tag{23}$$

and the covariant tetrad-frame electromagnetic potential $A_k$ in terms of a set of Hamilton-Jacobi potentials $\mathcal{A}_k$,

$$A_k \equiv \frac{1}{\rho}\left\{\frac{\mathcal{A}_x}{\sqrt{-\Delta_x}}, \frac{\mathcal{A}_t}{\sqrt{-\Delta_x}}, \frac{\mathcal{A}_y}{\sqrt{\Delta_y}}, \frac{\mathcal{A}_\phi}{\sqrt{\Delta_y}}\right\}. \tag{24}$$

In Paper 3 [2] it will be found that $\mathcal{A}_t$ is related to the enclosed electric charge within radius $x$, while $\mathcal{A}_\phi$ is related to the enclosed magnetic charge above latitude $y$. If magnetic charge does not exist, then $\mathcal{A}_\phi$ should vanish, but $\mathcal{A}_\phi$ is retained here to bring out the symmetry. The contravariant coordinate momenta $dx^\kappa/d\lambda = e_k{}^\kappa p^k$ are related to the Hamilton-Jacobi parameters $P_k$ by

$$\frac{dx^\kappa}{d\lambda} = \frac{1}{\rho^2}\left\{-P_x, \frac{P_t}{-\Delta_x} + \frac{\omega_y P_\phi}{\Delta_y}, P_y, \frac{\omega_x P_t}{-\Delta_x} + \frac{P_\phi}{\Delta_y}\right\}. \tag{25}$$

The tetrad-frame momenta $p_k$ are related to the generalized momenta $\pi_\kappa$ by $p_k = e_k{}^\kappa \pi_\kappa - q\mathcal{A}_k$, which implies that the Hamilton-Jacobi parameters $P_k$ are related to the canonical momenta $\pi_\kappa$ by

$$P_x \equiv -\Delta_x \pi_x - q\mathcal{A}_x, \tag{26a}$$

$$P_t \equiv \pi_t + \pi_\phi \omega_x - q\mathcal{A}_t, \tag{26b}$$

$$P_y \equiv \Delta_y \pi_y - q\mathcal{A}_y, \tag{26c}$$

$$P_\phi \equiv \pi_\phi + \pi_t \omega_y - q\mathcal{A}_\phi. \tag{26d}$$

In terms of the parameters $P_k$, the Hamilton-Jacobi Eq. (17) is

$$\frac{P_x^2 - P_t^2}{\Delta_x} + \frac{P_y^2 + P_\phi^2}{\Delta_y} = -m^2\rho^2. \tag{27}$$

Separation of variables of the Hamilton-Jacobi equation proceeds by postulating that the action $S$ separates as (Eqs. (28)–(30) below together constitute assumption III of [31], that the Hamilton-Jacobi equation separates "in the simplest possible way")

$$S = \frac{1}{2}m^2\lambda - Et + L\phi + S_x(x) + S_y(y), \tag{28}$$

where $S_x(x)$ and $S_y(y)$ are, respectively, functions only of $x$ and $y$. The left-hand side of the Hamilton-Jacobi Eq. (27) is separable for arbitrary values of the constants $E$, $L$, and $q$ provided that (cf. Appendix A)

$$\begin{array}{lllll} \omega_x, & \Delta_x, & \mathcal{A}_x, & \mathcal{A}_t & \text{are functions of } x \text{ only,} \\ \omega_y, & \Delta_y, & \mathcal{A}_y, & \mathcal{A}_\phi & \text{are functions of } y \text{ only.} \end{array} \tag{29}$$

These are the conditions of conformal separability adopted in this paper. In the remainder of this paper, the black hole will be taken to be neutral, so that the electromagnetic potential $A_m$ is identically zero. The case of a charged black hole is addressed in Paper 3.

### B. Strict separability

Full, or strict, separability, as opposed to just conformal separability, would require that not only the left-hand side but also the right-hand side of the Hamilton-Jacobi Eq. (27) separates. This would require that the conformal factor $\rho$ separates, as (this is Eq. (43) of [31])

$$\rho^2 = \rho_s^2 = \rho_x^2 + \rho_y^2, \tag{30}$$

where $\rho_x$ is a function only of the radial coordinate $x$, and $\rho_y$ is a function only of the angular coordinate $y$. Equation (30) holds during the electrovac phase prior to inflation, and it also holds during early inflation, when the conformal factor $\rho$ remains at its electrovac value, but its radial derivatives $\partial\rho/\partial x$ and $\partial^2\rho/\partial x^2$ are becoming large. Equation (30) breaks down as the conformal factor begins $\rho$ to shrink from its electrovac value, presaging collapse.

If $\rho$ separates as Eq. (30), then the Hamilton-Jacobi Eq. (27) separates as

$$-\left(\frac{P_x^2 - P_t^2}{\Delta_x} + m^2\rho_x^2\right) = \frac{P_y^2 + P_\phi^2}{\Delta_y} + m^2\rho_y^2 = \mathcal{K}, \tag{31}$$

with $\mathcal{K}$ a separation constant. The separated Hamilton-Jacobi Eqs. (31) imply that

$$P_x = \pm\sqrt{P_t^2 - (\mathcal{K} + m^2\rho_x^2)\Delta_x}, \tag{32a}$$

$$P_y = \pm\sqrt{-P_\phi^2 + (\mathcal{K} - m^2\rho_y^2)\Delta_y}. \tag{32b}$$

The trajectory of a freely-falling particle follows from integrating $dy/dx = -P_y/P_x$, equivalent to the implicit equation

$$-\frac{dx}{P_x} = \frac{dy}{P_y}. \tag{33}$$

The time and azimuthal coordinates $t$ and $\phi$ along the trajectory are then obtained by quadratures:

$$dt = \frac{P_t dx}{P_x\Delta_x} + \frac{\omega_y P_\phi dy}{P_y \Delta_y}, \qquad d\phi = \frac{\omega_x P_t dx}{P_x\Delta_x} + \frac{P_\phi dy}{P_y\Delta_y}. \tag{34}$$

### C. Conformal separability

A feature of inflation, discussed in the next section, Sec. V, is that ingoing and outgoing streams of particles move hyper-relativistically relative to each other and to the no-going tetrad frame. The streams remain hyper-relativistic throughout inflation and subsequent collapse. One should not be too surprised that the trajectories of hyper-relativistic massive particles would be hardly distinguishable from those of massless particles. This subsection shows that Hamilton-Jacobi separability holds to an excellent approximation for massive as well as massless particles. The arguments are confirmed formally by showing that the difference (39) between the tetrad-frame momentum predicted by the Hamilton-Jacobi solution and the true momentum, integrated over the path of a particle during inflation and collapse, is adequately small.

The Hamilton-Jacobi Eq. (27) can be separated as

$$P_x^2 = P_t^2 - [\mathcal{K} + m^2(\rho^2 - \rho_y^2)]\Delta_x, \tag{35a}$$

$$P_y^2 = -P_\phi^2 + (\mathcal{K} - m^2\rho_y^2)\Delta_y, \tag{35b}$$

where $\mathcal{K}$ is the same separation constant as before. The condition of separability is that the right-hand side of Eq. (35a) is a function only of $x$, while the right-hand side of Eq. (35b) is a function only of $y$. The latter follows from the conformal separability conditions (29) and the condition that $\rho_y$ is a function only of $y$, provided that $\pi_t$ is treated as a constant.

During the electrovac and early inflationary phases, $\rho^2$ equals its separable electrovac value $\rho_s^2$, and Eq. (35a) simplifies to its electrovac form

$$P_x^2 = P_t^2 - (\mathcal{K} + m^2\rho_x^2)\Delta_x, \tag{36}$$

whose right-hand side is a function only of $x$, consistent with separability. During inflation and early collapse, the radial horizon function is tiny, $|\Delta_x| \ll 1$, so Eq. (35a) simplifies to

$$P_x^2 = P_t^2, \tag{37}$$

whose right-hand side is again a function only of $x$, consistent with separability. Once $|\Delta_x| \gtrsim 1$, Eq. (37) no longer holds, but during collapse the conformal factor $\rho$ shrinks to a tiny value, with the net result that $\rho^2|\Delta_x| \ll 1$, so that Eq. (35a) simplifies to

$$P_x^2 = P_t^2 - (\mathcal{K} - m^2\rho_y^2)\Delta_x. \tag{38}$$

The term proportional to $\rho_y^2$ on the right-hand side of Eq. (38) depends on $y$, apparently destroying separability. However, a feature of inflation and collapse, which will be discovered in Sec. VIII F, is that the coordinates $x$ and $y$ of a freely-falling particle remain frozen at their inner horizon values throughout inflation and collapse. Thus $\rho_y$ remains constant along the trajectory of any particle, and the right-hand side of Eq. (38) can be considered to be a function only of $x$, again consistent with separability of Eq. (35a).

Equation (38) shows that during inflation and collapse the trajectory of a particle of rest mass $m$ is accurately approximated by that of a massless particle with separation constant $\mathcal{K}_0 = \mathcal{K} - m^2\rho_y^2$.

That Eqs. (35) provide accurate expressions for $P_x$ and $P_y$ for massive particles can be confirmed by considering the total derivative $dp_k/dx$ of the tetrad-frame momentum $p_k$, Eq. (23), of a neutral particle of rest mass $m$ along its trajectory. If the parameters $P_x$ and $P_y$ are taken to be given by Eqs. (35), with the parameters $P_t$ and $P_\phi$ from Eqs. (26) and $\pi_t$ constant (as opposed to from Eq. (21)), then the total derivative of the tetrad-frame momentum is

$$\begin{aligned}\frac{dp_k}{d\lambda} &= \frac{m^2}{\rho^3}\left(\frac{P_y}{2}\frac{\partial(\rho^2 - \rho_y^2)}{\partial y} + \frac{\rho^2 \upsilon \omega_y P_\phi}{\Delta_y}\right) \\ &\quad \times \left\{\frac{\sqrt{-\Delta_x}}{P_x}, 0, \frac{\sqrt{\Delta_y}}{P_y}, 0\right\} + \frac{m^2 \upsilon}{\rho} \\ &\quad \times \left\{\frac{P_t}{P_x\sqrt{-\Delta_x}}, \frac{1}{\sqrt{-\Delta_x}}, -\frac{P_\phi \omega_y}{P_y\sqrt{\Delta_y}}, \frac{\omega_y}{\sqrt{\Delta_y}}\right\}. \end{aligned} \tag{39}$$

The right-hand side of Eq. (39), which would be zero if the motion were exactly geodesic (and is in fact zero for massless particles, $m = 0$), is nonzero because the solution (35) for $P_x$ and $P_y$ is not exact, for massive particles. The second of the two terms on the right-hand side of Eq. (39) arises from approximating $\pi_t$ as a constant, and would disappear if $\pi_t$ were set to the more accurate value (21), and $\tau$ were replaced by its value as a function of respectively $x$ and $y$ in respectively $P_t$ and $P_\phi$.

The deviation between the momentum $p_k$ predicted by Eqs. (35) and the true momentum can be obtained by integrating Eq. (39) over the path of a particle during inflation and collapse. As shown in Appendix D, the ratio $\Delta p_k/p_k$ of the deviation $\Delta p_k \equiv \int(dp_k/d\lambda)d\lambda$ to the momentum $p_k$ itself is of order $\sim\upsilon^2$, Eq. (D7), which may be considered adequately small. This confirms the accuracy of the Hamilton-Jacobi approximation (35).

## V. FOCUSING ALONG THE PRINCIPAL DIRECTIONS

A central feature of inflation, demonstrated in this section, is that as ingoing and outgoing streams approach the inner horizon, they see the opposite stream narrow into an increasingly intense, blueshifted beam focused along the opposite principal null direction. The fact that near the inner horizon the energy-momenta of the streams becomes highly focused along the principal null directions regardless of the initial conditions of the streams is what motivates the idea that the spacetime, which is separable in the Kerr-Newman geometry, may continue to be separable during inflation.

A particle is said to be ingoing if $P_t$, Eq. (26b), is negative, outgoing if $P_t$ is positive. Outside the outer horizon, $P_t$ is necessarily negative (ingoing) while $P_x$ can be either negative or positive. At the outer horizon, $P_t$ and $P_x$ are equal in magnitude, and continuous across the horizon. Inside the outer horizon, $P_t$ and $P_x$ switch roles: $P_x$ is necessarily negative (given that the sign of the timelike radial coordinate $x$ is being chosen so that it increases as proper time advances, Eq. (25)), while $P_t$ can be either negative (ingoing) or positive (outgoing). A particle falling from outside the horizon necessarily has negative $P_t$ as long as it is outside the horizon, but its $P_t$ can change sign inside the horizon, if its angular momentum and/or charge are sufficiently large with the same sign as the black hole, as exampled in the Introduction in the paragraph containing Eq. (1).

A particle at rest in the tetrad frame has by definition tetrad-frame momentum $p^k = m\{1, 0, 0, 0\}$, hence its Hamilton-Jacobi parameters $P_k$, Eq. (23), are

$$P_k = -m\rho\sqrt{-\Delta_x}\{1, 0, 0, 0\}. \tag{40}$$

The tetrad rest frame defines a special frame, the no-going frame, where $P_t = 0$, at the boundary between ingoing and outgoing.

The tetrad-frame 4-momentum $p^k$ of a freely-falling particle, as seen in the no-going tetrad rest frame, is given by Eq. (23). Near the inner horizon, where $\Delta_x \to -0$, the no-going observer sees both ingoing and outgoing streams become hugely blueshifted and focused along the ingoing and outgoing principal null directions:

$$p^k \to \frac{-P_x}{\rho\sqrt{-\Delta_x}}\{1, \pm 1, 0, 0\}. \tag{41}$$

Instead of the no-going observer, consider an ingoing or outgoing observer, of mass $m'$, with Hamilton-Jacobi parameters $P'_k$. Near the inner horizon, where $\Delta_x \to -0$, the ingoing or outgoing observer see particles in the opposite stream with hugely blueshifted 4-momentum $p^k$ (the following is Eq. (41) appropriately Lorentz-boosted in the radial direction),

$$p^k \to \frac{2P'_x P_x}{m'\rho^2(-\Delta_x)}\{1, \pm 1, 0, 0\}, \tag{42}$$

in which the $\pm$ sign is $+$ for ingoing observers and $-$ for outgoing observers. An ingoing observer see a piercing beam of outgoing particles coming from the direction towards the black hole, focused along the outgoing principal null direction $\{1, 1, 0, 0\}$. An outgoing observer see a similarly intense beam of ingoing particles falling from the direction away from the black hole, focused along the ingoing principal null direction $\{1, -1, 0, 0\}$.

It is only particles on the opposing stream that appear highly beamed: particles in an observer's own stream appear normal, not beamed. The no-going observer is exceptional in being skewered from both directions, albeit with the square root of the energy and blueshift that an ingoing or outgoing observer experiences.

## VI. KILLING TENSOR

Separability is associated with the existence of a Killing tensor, and conformal separability is associated with the existence of a conformal Killing tensor.

### A. Electrovac Killing tensor

The separated Hamilton-Jacobi Eq. (31) can be written

$$K^{mn} p_m p_n = \mathcal{K}, \tag{43}$$

where $p_m$ is the covariant tetrad-frame momentum (23), and $K^{mn}$ is the tetrad-frame Killing tensor

$$K^{mn} = \mathrm{diag}(\rho_y^2, -\rho_y^2, \rho_x^2, \rho_x^2). \tag{44}$$

The Killing tensor $K^{mn}$ satisfies Killing's equation

$$D_{(k}K_{mn)} = 0, \tag{45}$$

where $D_k$ denotes covariant differentiation, and parentheses denote symmetrization over enclosed indices.

### B. Early inflationary Killing tensor

The separation of the conformal factor $\rho$ as Eq. (30) leads to the usual electrovac solutions [8,31], but not to inflation. To admit inflation, it is necessary to go beyond Eq. (30). This section proposes a modification of the conformal factor, and shows that there is an associated Killing tensor during early inflation.

In spherically symmetric models, and in the limit of slow accretion, inflation has a step-function character at the inner horizon. This suggests generalizing the separation of the conformal factor $\rho$ by allowing it to depart infinitesimally from Eq. (30), but with finite derivatives in the radial $x$ direction. Specifically, the modified conformal factor is

$$\rho = \rho_s e^{\upsilon t - \xi}, \tag{46}$$

where $\rho_s$ is the usual separable conformal factor, Eq. (30), and $\xi$, a function of $x$ and $y$ (not $t$ or $\phi$), is negligibly small, but with finite radial derivatives satisfying the hierarchy of inequalities

$$0 \approx \xi \ll \frac{\partial \xi}{\partial x} \ll \frac{\partial^2 \xi}{\partial x^2} \quad \text{(early inflation)}, \tag{47}$$

and with negligible angular derivatives,

$$0 \approx \frac{\partial \xi}{\partial y} \approx \frac{\partial^2 \xi}{\partial y^2}. \tag{48}$$

The inequalities (47) and (48) mean that $\xi$ is mainly a function of the radial coordinate $x$, its derivatives with respect to the angular coordinate $y$ being small. The kind of function $\rho$ that Eq. (46) describes is one that takes a sharp turn, like a step function, in the radial direction. In Sec. VIII C, it will be found that the initial conditions in the electrovac phase lead to a function $\xi$ that satisfies the conditions (47) and (48) during early inflation.

The separation (46) of the conformal factor $\rho$ indeed proves to admit a Killing tensor, satisfying Killing's Eq. (45), subject to the conditions (47) and (48). The tetrad-frame Killing tensor $K^{mn}$ associated with the separation (46) is

$$K^{mn} = \mathrm{diag}(\rho_y^2, -\rho_y^2, \rho^2 - \rho_y^2, \rho^2 - \rho_y^2), \tag{49}$$

in which, to linear order in the small parameters $\upsilon$ and $\xi$,

$$\rho^2 - \rho_y^2 = \rho_x^2 + 2\rho_s^2(\upsilon t - \xi). \tag{50}$$

Since $\upsilon$ and $\xi$ are negligibly small, it might appear that the modified Killing tensor given by Eq. (49) is identical to the original tensor, Eq. (44). Indeed, $\upsilon$ can be set to zero in Eq. (50) without further delay. However, Killing's Eq. (45) involves derivatives of the Killing tensor, which bring in non-negligible derivatives of $\xi$. Thus the modified Killing tensor (49) differs nontrivially from the original (44). The Killing tensor (49) satisfies Killing's Eq. (45) provided that not only $\upsilon$ and $\xi$ are negligible, but also $\partial\xi/\partial y$ is negligible, but $\partial\xi/\partial x$ may be large, consistent with conditions (47) and (48).

The Killing tensor (49) applied to the tetrad-frame momentum $p_k$ given by Eqs. (23) and (35) gives

$$K^{mn} p_m p_n = \mathcal{K}, \tag{51}$$

confirming that Eqs. (35) constitute a valid separation of the Hamilton-Jacobi equations for massive particles during early inflation.

### C. Conformal Killing tensor

The previous two subsections, Secs. VI A and VI B, showed that the spacetime possesses a Killing tensor in the electrovac and early inflationary stages, but not later. However, the spacetime possesses a conformal Killing tensor at all times, from electrovac through inflation and collapse. During early inflation, the inflationary exponent $\xi$ remains negligibly small while its radial derivatives grow large, conditions (47), but during later inflation and collapse the inflationary exponent $\xi$ grows huge.

The traceless part $\hat{K}^{mn}$ of the Killing tensor (49),

$$\hat{K}^{mn} = K^{mn} - \frac{1}{4}\eta^{mn} K^k_k = \frac{1}{2}\rho^2 \mathrm{diag}(1, -1, 1, 1), \tag{52}$$

is a conformal Killing tensor [[1], Sec. 35.3], satisfying

$$D_{(k}\hat{K}_{mn)} - \frac{1}{3}\eta_{(km} D^l \hat{K}_{n)l} = 0. \tag{53}$$

The tensor $\hat{K}^{mn}$ satisfies the condition (53) to be a conformal Killing tensor provided that the conformal separability conditions (29) hold, but without any restriction on the conformal factor $\rho$, and, in particular, without any restriction on the inflationary exponent $\xi$.

In Sec. VIII D it will be found that during inflation and collapse the horizon function $\Delta_x$ remains a function of radius $x$, as required by the conformal separability conditions (29), only provided that the inflationary exponent $\xi$ is purely radial:

$$\xi \quad \text{is a function of } x \text{ only.} \tag{54}$$

## VII. COLLISIONLESS FREELY-FALLING STREAMS

The essential ingredient that triggers mass inflation is the presence near the inner horizon of ingoing and outgoing streams that can stream relativistically through each other. This paper adopts a general collisionless fluid as the source of energy-momentum that ignites and then drives inflation. In astronomical black holes, streams near the inner horizon will typically originate from accretion of baryons and cold dark matter. A combination of collisions and magnetohydrodynamic processes [36,37] are likely to keep baryons, electrons, and photons tightly coupled above the inner horizon, forcing them into a common ingoing or outgoing stream before inflation ignites. Dark matter, and also gravitational waves, which should behave like a collisionless fluid of gravitons in the high-frequency limit, can occupy the opposing stream, and stream relativistically through the baryonic stream without collisions, driving inflation. In the limit of small accretion rate considered here, the geometry above the inner horizon is accurately approximated by the electrovac ($\Lambda$-Kerr-Newman) solution, so the precise behavior of the gas there is irrelevant. The strategy adopted in this paper and its companions is to seek solutions that hold from just above the inner horizon inward.

The approximation of a collisionless fluid will probably break down when center-of-mass collision energies between ingoing and outgoing particles exceed the Planck energy. Such super-Planckian collisional processes, though probably important, are neglected in the present paper.

It should be commented that general relativistic numerical treatments that model the energy-momentum as a single fluid with wave speed less than the speed of light are not satisfactory near the inner horizon, since such a fluid cannot support relativistic counter-streaming, and therefore artificially suppresses the inflation that would occur if even the tiniest admixture of an oppositely going fluid were admitted.

### A. Occupation number

The distribution of particles in a collisionless fluid is described by a scalar occupation number $f(x^\mu, \boldsymbol{p})$ that specifies the number $dN$ of particles at position $x^\mu$ with tetrad-frame momentum $p^m \equiv \{p^x, \boldsymbol{p}\}$ in a Lorentz-invariant 6-dimensional volume of phase-space,

$$dN = f(x^\mu, \boldsymbol{p}) \frac{d^3x d^3p}{(2\pi\hbar)^3}. \tag{55}$$

Here $d^3x$ denotes the proper tetrad-frame 3-volume element measured by an observer at rest in the tetrad frame, not a coordinate-frame 3-volume element. The collisionless Boltzmann equation is

$$\frac{df(x^\mu, \boldsymbol{p})}{d\lambda} = p^m \partial_m f + \frac{dp^m}{d\lambda}\frac{\partial f}{\partial p_m} = 0, \tag{56}$$

where the affine parameter is $d\lambda \equiv d\tau/m$, with $\tau$ the proper time of an observer at rest in the tetrad frame. The Boltzmann Eq. (56) asserts that the occupation number $f(x^\mu, \boldsymbol{p})$ is constant along phase-space trajectories.

The tetrad-frame momentum 3-volume element $d^3p$ is related to the scalar 4-volume element $d^4p$ by

$$(2\pi\hbar)\delta_D(p^kp_k + m^2)\frac{d^4p}{(2\pi\hbar)^4} = \frac{d^3p}{2p^x(2\pi\hbar)^3}, \tag{57}$$

where the Dirac delta-function enforces conservation of rest mass $m$. The Lorentz-invariant tetrad-frame momentum volume element from Eq. (57) translates into a 4-volume element of parameters $P_m$ with the Jacobian from the relation (23) between $p_m$ and $P_m$:

$$\frac{d^3p}{2p^x(2\pi\hbar)^3} = (2\pi\hbar)\delta_D\left(\frac{P_x^2 - P_t^2}{\rho^2\Delta_x} + \frac{P_y^2 + P_\phi^2}{\rho^2\Delta_y} + m^2\right) \times \frac{dP^4}{(2\pi\hbar)^4\rho^4\Delta_x\Delta_y}. \tag{58}$$

This in turn translates into an element of orbital constants of motion, the energy $E \equiv -\pi_t$, angular momentum $L \equiv \pi_\phi$, and separation constant $\mathcal{K}$, with the Jacobian calculated from Eqs. (26b), (26d), and (35):

$$\frac{d^3p}{2p^x(2\pi\hbar)^3} = \frac{\sigma^2}{\rho^2P_xP_y}\frac{dEdLd\mathcal{K}}{4(2\pi\hbar)^3}. \tag{59}$$

### B. Number current

The number density and flux of particles at any position form a tetrad-frame 4-vector $n^k$,

$$n^k = \int p^kf(x^\mu, \boldsymbol{p})\frac{d^3p}{2p^x(2\pi\hbar)^3} = \int p^kf(x^\mu, \boldsymbol{p})\frac{\sigma^2}{\rho^2P_xP_y}\frac{dEdLd\mathcal{K}}{4(2\pi\hbar)^3}. \tag{60}$$

Covariant number conservation follows from the collisionless Boltzmann Eq. (56),

$$D_kn^k = \int p^kD_kf\frac{d^3-p}{2p^x(2\pi\hbar)^3} = \int\frac{df}{d\lambda}\frac{d^3p}{2p^x(2\pi\hbar)^3} = 0. \tag{61}$$

For a single stream with fixed constants of motion $E$, $L$, and $\mathcal{K}$, it follows from Eq. (60) and the constancy of the occupation number $f$ along phase-space trajectories, Eq. (56), that the number current $n^k$ along the stream varies as

$$n^k = Np^k, \qquad N \propto \frac{\sigma^2}{\rho^2P_xP_y}. \tag{62}$$

The denominators $P_x$ and $P_y$ would vanish where orbits turned around in radius $x$ or angle $y$, and there would be cusps in the number current at such points. However, this never happens in the inflationary zone of interest in the present paper, because the radius $x$ is a timelike coordinate, so $P_x$ never changes sign, while $P_y$, given to an excellent approximation by Eq. (35b), remains essentially constant along any stream throughout inflation and collapse.

For massive particles, $P_x$ and $P_y$ along a stream are approximated accurately by Eqs. (35). The accuracy of the approximations (35) can be checked by seeing how closely the covariant divergence $D_kn^k$ that they predict vanishes. With $P_x$ and $P_y$ given by Eqs. (35), along with $P_t$ and $P_\phi$ from Eqs. (26) and $\pi_t$ constant (as opposed satisfying Eq. (21)), the covariant divergence of the single-stream number current $n^k$ given by Eq. (62) is

$$D_kn^k = \frac{m^2N}{\rho^2P_x^2}\left[\Delta_x\left(\frac{P_y}{2}\frac{\partial(\rho^2-\rho_y^2)}{\partial y} + \frac{\rho^2 v\omega_yP_\phi}{\Delta_y}\right) - \rho^2vP_t\right]. \tag{63}$$

As expected, the divergence vanishes identically for massless particles, $m = 0$, but not for massive particles, because Hamilton-Jacobi separation is exact for massless particles, but not quite exact for massive particles. Since

$$\frac{d\ln N}{d\lambda} = \frac{1}{N}D_kn^k - D_kp^k, \tag{64}$$

and the momentum $p^k$ has already been checked, from Eq. (39), to be given accurately by expressions (35) for $P_x$ and $P_y$, the deviation between the predicted and true logarithmic density $\ln N$ can be estimated by integrating Eq. (63) over the path of a particle during inflation and collapse. As shown in Appendix D, the resulting deviation $\Delta\ln N \equiv \int(d\ln N/d\lambda)d\lambda$ is of order $v^2$, Eq. (D8), which may be considered adequately small. This again confirms the accuracy of the Hamilton-Jacobi approximation (35), and the consequent expressions (62) for the number density $N$ and number current $n^k$.

### C. Energy-momentum

The tetrad-frame energy-momentum density $T_{kl}$ of a system of freely-falling particles is

$$T_{kl} = \int p_kp_lf(x^\mu, \boldsymbol{p})\frac{d^3p}{2p^x(2\pi\hbar)^3}. \tag{65}$$

Covariant energy-momentum conservation follows from the collisionless Boltzmann Eq. (56),

$$D^kT_{kl} = \int p_kp_lD^kf\frac{d^3p}{2p^x(2\pi\hbar)^3} = \int p_l\frac{df}{d\lambda}\frac{d^3p}{2p^x(2\pi\hbar)^3} = 0. \tag{66}$$

For a single stream with fixed constants of motion $E$, $L$, and $\mathcal{K}$, the energy-momentum is

$$T_{kl} = n_kp_l = Np_kp_l, \tag{67}$$

where the number current $n^k$ and number density $N$ are given by Eqs. (62). Covariant energy-momentum conservation follows from number conservation and the geodesic equation,

$$D^k T_{kl} = D^k n_k p_l = p_l D^k n_k + N \frac{dp_l}{d\lambda} = 0. \tag{68}$$

In accordance with the expression (23) for the tetrad-frame momentum $p_k$ and the proportionality (62) for $N$, the energy-momentum of a single stream is

$$T_{kl} = \frac{N P_k P_l}{\rho^2 \sqrt{|\Delta_k \Delta_l|}} \propto \frac{\sigma^2 P_k P_l}{\rho^4 P_x P_y \sqrt{|\Delta_k \Delta_l|}}, \tag{69}$$

where

$$\Delta_k \equiv \begin{cases} \Delta_x & \text{for } m = x, t, \\ \Delta_y & \text{for } m = y, \phi. \end{cases} \tag{70}$$

The behavior of the collisionless energy-momentum will be explored in detail in Sec. VIII, but qualitative features of the behavior are already apparent from Eq. (69). As a stream approaches the inner horizon, $\Delta_x \to -0$, the radial components of its energy-momentum grow large, because of the inverse factors of the radial horizon function in Eq. (69). During inflation, the behavior of the energy-momentum continues to be dominated by the radial horizon function $\Delta_x$, which is driven to an exponentially tiny value, causing the radial components of the energy-momentum to grow exponentially huge. During collapse, the conformal factor $\rho$ shrinks, amplifying all components of the energy-momentum.

The trace of the energy-momentum tensor of a stream of particles of rest mass $m$,

$$T^k_k = N p^k p_k = -N m^2, \tag{71}$$

can always be treated as negligibly small. During inflation, the trace is negligible because the incident accretion flow is negligibly small, in the conformally stationary limit. As inflation develops and collapse begins, the density $N \propto \rho^{-2}$ increases as the conformal factor $\rho$ shrinks, but the individual components of the energy-momentum increase more rapidly, as $T_{kl} \propto \rho^{-4}$, Eq. (69), so the trace is never significant.

## VIII. INFLATIONARY SOLUTIONS

This section presents the inflationary solutions that emerge from Einstein's equations with a collisionless source under the conditions of conformal stationarity, axisymmetry, and conformal separability assumed in this paper.

Above the inner horizon, the $\Lambda$-Kerr-Newman electrovac geometry, Sec. VIII B, provides the background in which inflation ignites, Sec. VIII C. As inflation develops, it back-reacts on the geometry, driving the conformal factor $\rho$ and the radial horizon function $\Delta_x$ from their electrovac forms. The equations governing the evolution of the conformal factor and radial and angular horizon functions are obtained in Sec. VIII D by separation of variables in two components of the Einstein tensor that have negligible collisionless source. The equations are solved to obtain the evolution of the conformal factor and radial horizon function in Sec. VIII E. The implications for the trajectories and densities of freely-falling streams are presented in Sec. VIII F. The solutions for the conformal factor and horizon functions are inserted into the remaining 8 Einstein components in Sec. VIII G, and in Sec. VIII H it is shown that the 8 Einstein components fit the form of the energy-momentum tensor of two collisionless streams, one ingoing and one outgoing. The solutions indicate that during collapse the angular ($y$ and $\phi$) motion of the collisionless streams grows, and would begin to dominate once the radial horizon function is no longer small, $|\Delta_x| \gtrsim 1$. The last two subsections, Secs. VIII I and VIII J address the effect of the angular motions to higher order, showing that, while the earlier results remain robust as long as angular motions are small, $|\Delta_x| \ll 1$, the solutions fail when the angular motions become important, $|\Delta_x| \gtrsim 1$.

### A. Charged black hole

The case of a charged black hole is deferred to Paper 3, because the inclusion of electromagnetic currents, fields, and energy-momenta adds a whole extra layer of complexity to the solutions. Of course, a charged black hole is physically less interesting than the simpler case of an uncharged black hole, which is the focus of the present paper.

Nevertheless, much of the results of this section and the next, Secs. VIII and IX, carry over essentially unchanged to the case of a charged black hole. In particular, the expressions (88) and (124) for the Einstein tensor are unchanged in the presence of an electromagnetic field produced by a collisionless source, and the solution, Sec. VIII E, for the evolution of the horizon function and conformal factor is unchanged, the only difference being that the derivative $\Delta'_x$ of the electrovac radial horizon function at the inner horizon is altered by the presence of charge (because the electrovac horizon function (77a) depends on charge), effectively changing the boundary conditions of the solution. The various differences between the charged and uncharged cases are detailed in Paper 3.

To avoid repetition of the same results in Paper 3, the spacetime that provides the boundary conditions for the inflationary solution is referred to hereafter as "electrovac" rather than just "vacuum."

### B. Electrovac initial conditions

Electrovac solutions [1,31], of which the physically relevant solutions are Kerr-Newman with a cosmological constant $\Lambda$, provide the boundary conditions for the inflationary solutions.

The electrovac solutions are strictly stationary, satisfying $v = 0$, and strictly separable, so the conformal factor $\rho$ is separable, Eq. (30). Solution of the Einstein equations, Appendix C, leads to the standard results

$$\rho_s^2 = \rho_x^2 + \rho_y^2 = \frac{\sigma^2}{(f_0 + f_1\omega_x)(f_1 + f_0\omega_y)}, \tag{72a}$$

$$\rho_x = \sqrt{\frac{g_0 - g_1\omega_x}{(f_0g_1 + f_1g_0)(f_0 + f_1\omega_x)}},$$
$$\rho_y = \sqrt{\frac{g_1 - g_0\omega_y}{(f_0g_1 + f_1g_0)(f_1 + f_0\omega_y)}} \tag{72b}$$

and

$$\frac{d\omega_x}{dx} = 2\sqrt{(f_0 + f_1\omega_x)(g_0 - g_1\omega_x)}, \tag{73a}$$
$$\frac{d\omega_y}{dy} = 2\sqrt{(f_1 + f_0\omega_y)(g_1 - g_0\omega_y)}, \tag{73b}$$

where $f_0$, $f_1$, $g_0$, and $g_1$ are constants set by boundary conditions. The sign of the square root for $d\omega_x/dx$ is the same as that for $\rho_x$, while the sign of the square root for $d\omega_y/dy$ is the same as that for $\rho_y$. Inflation leaves the separable factor $\rho_s$ in the conformal factor $\rho$, Eq. (10), and the vierbein coefficients $\omega_x$ and $\omega_y$ unchanged from their electrovac values.

For $\Lambda$-Kerr-Newman, the constants $f_0$, $f_1$, $g_0$, and $g_1$ are

$$f_0 = 0, \qquad f_1 = a^{-1/2}, \qquad g_0 = a^{3/2}, \qquad g_1 = a^{5/2}, \tag{74}$$

where $a$ is the usual angular momentum parameter. The conformal factor and vierbein coefficients are given by

$$\rho_x = r = a\cot(ax), \qquad \rho_y = a\cos\theta = -ay, \tag{75}$$

$$\omega_x = \frac{a}{R^2}, \qquad \omega_y = a\sin^2\theta,$$
$$\sigma \equiv 1 - \omega_x\omega_y = \frac{\rho_s}{R}, \qquad R \equiv \sqrt{r^2 + a^2}. \tag{76}$$

The radial and angular horizon functions $\Delta_x$ and $\Delta_y$ are

$$\Delta_x = \frac{1}{R^2}\left(1 - \frac{2M_\bullet r}{R^2} + \frac{Q_\bullet^2 + \mathcal{Q}_\bullet^2}{R^2} - \frac{\Lambda r^2}{3}\right), \tag{77a}$$
$$\Delta_y = \sin^2\theta\left(1 + \frac{\Lambda a^2\cos^2\theta}{3}\right), \tag{77b}$$

where $M_\bullet$ is the black hole's mass, $Q_\bullet$ and $\mathcal{Q}_\bullet$ are its electric and magnetic charge, and $\Lambda$ is the cosmological constant. Inflation modifies the radial horizon function $\Delta_x$, but leaves the angular horizon function $\Delta_y$, along with $\omega_x$ and $\omega_y$, unchanged.

The full conformal factor $\rho$, Eq. (46), involves an additional time-dependent factor of $e^{vt}$ (as well as an inflationary factor $e^{-\xi}$ that equals unity away from the inner horizon). The parameters $M_\bullet$, $Q_\bullet$, $\mathcal{Q}_\bullet$, $a$ of the black hole coincide with the actual mass, charge, and specific angular momentum of the black hole at conformal time $t = 0$, and increase linearly with proper external time $t_{KN}$. Physically, the cosmological constant $\Lambda$ should not increase with time, but since it becomes completely overwhelmed during inflation by other exponentially growing energy-momenta, it can be included consistently in the description of the spacetime outside the inflationary regime (see the remarks in the paragraph following Eq. (90)).

### C. Ignition

This subsection outlines the behavior of the inflationary exponent $\xi$ during the earliest phase of inflation, when the geometry is still electrovac, and collisionless streams are approaching the inner horizon. A more precise treatment that is valid throughout electrovac, inflation, and collapse starts in the next subsection, Sec. VIII D.

As discussed in Sec. V, collisionless streams become highly focused along the ingoing and outgoing principal null directions as they approach the inner horizon, causing the radial components $T_{xx}$, $T_{xt}$, and $T_{tt}$ of their energy-momentum to grow large. The Einstein combination $G_{xx} + G_{tt}$, which has zero electrovac source, is

$$\rho^2(G_{xx} + G_{tt}) = -2\Delta_x\left[\frac{\partial^2 - \xi}{\partial x^2} + \left(\frac{\partial\xi}{\partial x}\right)^2 - 2\frac{\partial\ln(\rho_s/\sigma)}{\partial x}\frac{\partial\xi}{\partial x}\right] - \frac{2v^2}{\Delta_x}. \tag{78}$$

The dominant term in this expression is the one proportional to the second derivative $\partial^2\xi/\partial x^2$ of the inflationary exponent. Equating the Einstein component $G_{xx} + G_{tt}$ to the energy-momentum $8\pi(T_{xx} + T_{tt})$ of collisionless streams, Eq. (69), yields to leading order in $1/\Delta_x$

$$\frac{\partial^2\xi}{\partial x^2} = \frac{8\pi\sum NP_x^2}{\Delta_x^2}, \tag{79}$$

where the sum is over collisionless streams. Because collisionless streams are hyper-relativistic near the inner horizon, the Hamilton-Jacobi parameters of every stream satisfy $P_t^2 = P_x^2$ to an excellent approximation. The parameters $N$ and $P_x$ of each component of the collisionless streams are sensibly constant as a function of radius $x$ near the inner horizon preceding inflation and during early inflation (but may be a function of angle $y$), and thus $\sum NP_x^2$ is sensibly constant. The sum over streams $\sum NP_x^2$ is of the order of the accretion rate $v$.

In the situation of small accretion rate considered in this paper, the electrovac geometry provides an excellent approximation down to just above the inner horizon. Near the inner horizon, where $\Delta_x \to -0$, the horizon function $\Delta_x$ may be approximated by

$$\Delta_x = (x - x_{\mathrm{in}})\Delta'_x, \tag{80}$$

where $\Delta'_x \equiv d\Delta_x/dx|_{x_{\mathrm{in}}}$ is the (positive) derivative of the electrovac horizon function at the electrovac inner horizon at $x = x_{\mathrm{in}}$. The derivative $\Delta'_x$ is nonzero provided that the black hole is nonextremal, as should be true for any astronomically realistic black hole. Introduce the quantity $u$, a small positive parameter of order the accretion rate, $u \sim v$, defined by

$$u \equiv \frac{8\pi \sum NP_x^2}{\Delta'_x}\bigg|_{x_{\mathrm{in}}}, \tag{81}$$

evaluated at the inner horizon $x_{\mathrm{in}}$. Later, Sec. X A, the overall accretion rates of ingoing and outgoing streams on to the inner horizon will be found to be proportional, respectively, to the combinations $u \mp v$. Potentially $u$ could be a function of angle $y$, but in Sec. VIII D it will be found that separability continues to hold as inflation develops only if $u$ is independent of angle $y$. In terms of $u$, the second derivative (79) of $\xi$ is

$$\frac{\partial^2 \xi}{\partial x^2} = \frac{u/\Delta'_x}{(x_{\mathrm{in}} - x)^2}. \tag{82}$$

Integrating Eq. (82) gives

$$\frac{\partial \xi}{\partial x} = \frac{u/\Delta'_x}{x_{\mathrm{in}} - x} = -\frac{u}{\Delta_x}, \tag{83}$$

where the constant of integration, established well outside the inner horizon, has been dropped because it is tiny compared to the retained term, which is diverging at the inner horizon $x \to x_{\mathrm{in}}$. Integrating Eq. (83) in turn yields

$$\xi = -\frac{u}{\Delta'_x} \ln\left(\frac{x_{\mathrm{in}} - x}{x_{\mathrm{in}}}\right), \tag{84}$$

where the constant of integration follows from requiring that $\xi$ starts at zero well outside the inner horizon, where $x \to 0$. Equation (84) shows that the inflationary exponent $\xi$ is the product of a small factor $u \sim v$ and a term that diverges logarithmically at the inner horizon. Thus $\xi$ remains small even while its derivatives are becoming large. The inflationary exponent $\xi$ fulfills the conditions (47) and (48) anticipated in Sec. VI B.

### D. Equations governing evolution of the horizon function and conformal factor

Inflation alters the spacetime geometry by changing the radial horizon function $\Delta_x$ and conformal factor $\rho$ from their initial electrovac forms. Equations governing the evolution of the horizon function and conformal factor are obtained from the Einstein equations for the two diagonal components $G_{xx} - G_{tt}$ and $G_{yy} + G_{\phi\phi}$. At least initially, these two components have negligible collisionless source in the conformally stationary limit. The angular component $G_{yy} + G_{\phi\phi}$ has negligible collisionless source because the flow incident on the inner horizon has negligible energy-momentum, and inflation amplifies only radial components, not angular components. The component $G_{xx} - G_{tt}$ has negligible collisionless source because the trace of the energy-momentum of a collisionless source is always negligible, Eq. (71).

In Sec. VIII J a nonvanishing collisionless source for $G_{xx} - G_{tt}$ and $G_{yy} + G_{\phi\phi}$ will be taken into account, and it will be found that the results of the present subsection are robust.

Since the collisionless source is negligible for these two Einstein components, it is natural to seek homogeneous solutions of Eqs. (88) by separation. To achieve the desired separation, introduce $U_x$ and $U_y$ defined by

$$U_x \equiv -\frac{\partial \xi}{\partial x}\Delta_x, \tag{85a}$$

$$U_y \equiv \frac{\partial \xi}{\partial y}\Delta_y. \tag{85b}$$

Initially, in the electrovac phase just above the inner horizon, $U_x$ equals the small parameter $u$ defined by Eq. (81),

$$U_x = u, \tag{86}$$

and $U_y$ is similarly small. As will be seen in Sec. VIII E, $U_x$ is driven by inflation to large values, but $U_y$ remains always small. Further, define $X_x$, $X_y$, $Y_x$, and $Y_y$ by

$$X_x \equiv \frac{\partial U_x}{\partial x} + 2\frac{U_x^2 - v^2}{\Delta_x}, \tag{87a}$$

$$X_y \equiv \frac{\partial U_y}{\partial y} - 2\frac{U_y^2 + v^2\omega_y^2}{\Delta_y}, \tag{87b}$$

$$Y_x \equiv \frac{d\Delta_x}{dx} + 3U_x - \Delta_x \frac{d}{dx}\ln\left[(f_0 + f_1\omega_x)\frac{d\omega_x}{dx}\right], \tag{87c}$$

$$Y_y \equiv \frac{d\Delta_y}{dy} - 3U_y - \Delta_y \frac{d}{dy}\ln\left[(f_1 + f_0\omega_y)\frac{d\omega_y}{dy}\right]. \tag{87d}$$

In terms of $U_x$, $U_y$, $X_x$, $X_y$, $Y_x$, and $Y_y$, the Einstein components $G_{xx} - G_{tt}$ and $G_{yy} + G_{\phi\phi}$ are

$$\rho^2(G_{xx} - G_{tt}) = \frac{1}{\sigma^2}\left(Y_x \frac{d\ln\omega_x}{dx} - Y_y \frac{d\ln\omega_y}{dy}\right) - 2X_x + Y_x \frac{d}{dx}\ln\left(\frac{f_0 + f_1\omega_x}{\omega_x}\right) + X_y - \frac{\partial Y_y}{\partial y} + Y_y \frac{d}{dy}\ln\left[\frac{\omega_y(f_1 + f_0\omega_y)}{d\omega_y/dy}\right]$$
$$+ U_x \frac{\partial}{\partial x}\ln[\sigma^2(f_0 + f_1\omega_x)] - U_y \frac{\partial}{\partial y}\ln\left[\frac{(g_1 - g_0\omega_y)}{\sigma^2}\frac{d\omega_y}{dy}\right], \tag{88a}$$
$$\rho^2(G_{yy} + G_{\phi\phi}) = \frac{1}{\sigma^2}\left(Y_x \frac{d\ln\omega_x}{dx} - Y_y \frac{d\ln\omega_y}{dy}\right) - 2X_y - Y_y \frac{d}{dy}\ln\left(\frac{f_1 + f_0\omega_y}{\omega_y}\right) + X_x + \frac{\partial Y_x}{\partial x} - Y_x \frac{d}{dx}\ln\left[\frac{\omega_x(f_0 + f_1\omega_x)}{d\omega_x/dx}\right]$$
$$+ U_y \frac{\partial}{\partial y}\ln[\sigma^2(f_1 + f_0\omega_y)] - U_x \frac{\partial}{\partial x}\ln\left[\frac{(g_0 - g_1\omega_x)}{\sigma^2}\frac{d\omega_x}{dx}\right]. \tag{88b}$$

Homogeneous solutions of these equations can be found by supposing that $U_x$, $X_x$, and $Y_x$ are all functions of radius $x$, while $U_y$, $X_y$, and $Y_y$ are all functions of angle $y$, and by separating each of the equations as

$$\frac{1}{\sigma^2}\left(\frac{f_0h_0 + h_2\omega_x + f_1h_1\omega_x^2}{\omega_x} - \frac{f_1h_1 + h_2\omega_y + f_0h_0\omega_y^2}{\omega_y}\right) - \frac{f_0h_0 + h_3\omega_x}{\omega_x} + \frac{f_1h_1 + h_3\omega_y}{\omega_y} = 0, \tag{89}$$

for some constants $h_0$, $h_1$, $h_2$, and $h_3$. If one attempts to separate Eqs. (88) exactly, then the attempt fails unless $U_x$ and $U_y$ are identically zero, which is the usual electrovac case. But if $U_x$ is taken to be small but finite, then separation succeeds, and inflation emerges. If $U_x$ and $U_y$ on the second lines of Eqs. (88) are treated as negligibly small, then separating the first lines of each of Eqs. (88) according to the pattern of Eq. (89) leads to the homogeneous solutions

$$X_x = 0, \tag{90a}$$
$$X_y = 0, \tag{90b}$$
$$Y_x = \frac{(f_0 + f_1\omega_x)(h_0 + h_1\omega_x)}{d\omega_x/dx}, \tag{90c}$$
$$Y_y = \frac{(f_1 + f_0\omega_y)(h_1 + h_0\omega_y)}{d\omega_y/dy}. \tag{90d}$$

If $U_x = U_y = 0$, then solution of the differential equations (87c) and (87d) with the homogenous solutions (90c) and (90d) for $Y_x$ and $Y_y$, yields, subject to appropriate boundary conditions, the radial and angular horizon functions $\Delta_x$ and $\Delta_y$ of the Kerr line-element. The separable solutions generalize to other electrovac spacetimes by admitting appropriate sources for $Y_x$ and $Y_y$, Appendix C. The solutions with a static radial electromagnetic field have a contribution $G^e_{mn} = [(Q_\bullet^2 + \mathcal{Q}_\bullet^2)/(\rho^2\rho_s^2)]$ diag$(1, -1, 1, 1)$, and those with a cosmological constant have $G^\Lambda_{mn} = -(\rho_s^2/\rho^2)\Lambda\eta_{mn}$. These electrovac contributions cease to describe a static radial electromagnetic field or cosmological constant when $\rho \neq \rho_s$, but this happens only from the onset of collapse, by which time any electrovac contribution is overwhelmed by the collisionless energy-momentum, so the failure is unimportant (see §IV F of Paper 3 for a more precise treatment).

Solution of the angular behavior during inflation is immediate. The vanishing, Eq. (90b), of $X_y$ defined by Eq. (87b) implies that $U_y$, which is initially negligible in the conformally stationary limit $v \to 0$, remains negligible throughout, and may be set to zero

$$U_y = 0. \tag{91}$$

The expression (87d) for $Y_y$ governing the angular horizon function $\Delta_y$ is then unchanged from its electrovac form, and the angular horizon function $\Delta_y$ thus retains its electrovac form during inflation and collapse.

Down to just above the inner horizon, solution of the radial Eqs. (87a), (87c), (90a), and (90c) for $X_x$ and $Y_x$ leads to the usual electrovac form of the radial horizon function $\Delta_x$. But near the inner horizon, $\Delta_x \to -0$, the term proportional to $1/\Delta_x$ on the right-hand side of Eq. (87a) for $X_x$ starts to diverge, presaging inflation. In the vicinity of the inner horizon, where $\Delta_x$ is near zero, expression (87c) for $Y_x$ simplifies to

$$Y_x = \frac{d\Delta_x}{dx} + 3U_x. \tag{92}$$

It will be found in the next subsection, Sec. VIII E, that the radius $x$ remains frozen at its inner horizon value $x_{\rm in}$ throughout inflation and subsequent collapse. Consequently $\omega_x$, hence $Y_x$, are also frozen at their inner horizon values during inflation and collapse. Thus $Y_x$ is given by its electrovac value incident on the iner horizon, $Y_x = \Delta'_x$ where $\Delta'_x \equiv d\Delta_x/dx|_{x_{\rm in}}$ is the derivative of the electrovac horizon function at the inner horizon $x = x_{\rm in}$. It follows that in the vicinity of the inner horizon the equations (87a), (87c), (90a), and (90c) governing the evolution of $U_x$ and $\Delta_x$ reduce to

$$\frac{\partial U_x}{\partial x} + 2\frac{U_x^2 - v^2}{\Delta_x} = 0, \tag{93a}$$
$$\frac{d\Delta_x}{dx} + 3U_x = \Delta'_x. \tag{93b}$$

During the electrovac and earliest phase of inflation, when not only $U_x$ but also $\partial U_x/\partial x$ are negligible, which is true when $|\Delta_x| \gtrsim u$, the Einstein components (88) separate without requiring that $U_x$, which at this early stage equals $u$, be a function only of $x$. As inflation progresses however, continued separability, which requires that $\Delta_x$ be a function only of $x$, requires also that $U_x$ be a function only of $x$. It follows that the inflationary exponent $\xi$, given initially by Eq. (84), must also be a function only of $x$.

Below, Eqs. (124), it will be found that the energy-momenta of ingoing and outgoing collisionless streams are proportional, respectively, to $U_x \mp v$. If one or other stream vanished, then $U_x^2 - v^2$ would vanish, so Eq. (93a) would have no diverging term, and there would be no inflation. This is consistent with the physical argument that inflation requires the simultaneous presence of both ingoing and outgoing streams at the inner horizon.

Equation (93a) indicates an instability only at the inner horizon, not the outer horizon. As streams approach the inner horizon, $U_x$ is driven away from zero because the horizon function is negative and tending to zero, $\Delta_x \to -0$. By contrast, as infalling streams approach the outer horizon, the horizon function is positive and tending to zero, $\Delta_x \to 0$, causing $U_x$ to decay rather than grow.

### E. Evolution of the horizon function and conformal factor during inflation and collapse

This subsection integrates Eqs. (93), along with Eq. (85a), to determine the evolution of the radial horizon function $\Delta_x$ and conformal factor $\rho$ during inflation and collapse.

The separation of the Einstein components (88) in the previous subsection, Sec. VIII D, was premised on $U_x$ (and $U_y$) being negligibly small. However, the separation continues to remain valid during inflation and collapse when $U_x$ grows huge. The reason for this is that the dominant terms in the Einstein components (88) during inflation and collapse are of order $U_x^2/\Delta_x$, coming from the expression (87a) for $X_x$. Thus, once $U_x$ ceases to be negligible, the condition for the validity of the separation becomes $U_x \ll U_x^2/|\Delta_x|$, or equivalently $|\Delta_x| \ll U_x$. Thus the condition for the validity of the separation of the Einstein components (88) is

$$\text{either } U_x \ll 1 \quad \text{or} \quad |\Delta_x| \ll U_x. \tag{94}$$

It will be found in Sec. VIII I that the conformally separable Einstein equations cease to be satisfied with a collisionless source once the angular motion of the collisionless streams becomes large, which happens when $|\Delta_x| \gtrsim 1$. At this point $U_x$ is exponentially huge, Eq. (107). Thus condition (94) remains well satisfied throughout inflation and collapse.

The fact that the separation of the Einstein components (88) remains valid even when $U_x$ grows large, subject only to the condition (94), is confirmed in Sec. VIII J, where the equations are solved to the next higher order in $\Delta_x/U_x$.

Integrating Eq. (85a) for $U_x$ and using Eq. (93a), gives the inflationary exponent $\xi$ as a function of $U_x$:

$$\xi = -\int \frac{U_x dx}{\Delta_x} = \int \frac{U_x dU_x}{2(U_x^2 - v^2)} = \frac{1}{4} \ln\left(\frac{U_x^2 - v^2}{u^2 - v^2}\right), \tag{95}$$

the constant of integration coming from $\xi = 0$ at $U_x = u$. Equivalently, the inflationary part $e^{-\xi}$ of the conformal factor $\rho$ is

$$e^{-\xi} = \left(\frac{u^2 - v^2}{U_x^2 - v^2}\right)^{1/4}. \tag{96}$$

Inverting Eq. (96) gives $U_x$ in terms of $\xi$:

$$U_x = \sqrt{v^2 + (u^2 - v^2)e^{4\xi}}. \tag{97}$$

Equation (93b) divided by Eq. (93a) yields an equation for the horizon function $\Delta_x$:

$$\frac{d\ln\Delta_x}{dU_x} = \frac{3U_x - \Delta_x'}{2(U_x^2 - v^2)}. \tag{98}$$

Equation (98) integrates to

$$\Delta_x = -\left(\frac{U_x^2 - v^2}{u^2 - v^2}\right)^{3/4} \left[\frac{(U_x + v)(u - v)}{(U_x - v)(u + v)}\right]^{\Delta_x'/(4v)}, \tag{99}$$

where the constant of integration is established by $\Delta_x \sim -1$ at $U_x = u$. The precise constant is not important since near $U_x \approx u$, the second factor on the right-hand side of Eq. (99) is a number near unity taken to a large power $\Delta_x'/(4v)$. The conformal factor $\rho$ starts to depart significantly from its electrovac value $\rho_s$ when $\xi$ departs appreciably from zero, which occurs when $U_x$ is a factor somewhat greater than unity times $u$. At this point the horizon function is of order

$$\Delta_x \sim e^{-1/v}, \tag{100}$$

which is exponentially tiny.

The horizon function goes through an extremum, $d\Delta_x/dU_x = 0$, where, according to Eq. (98),

$$U_x = \frac{\Delta_x'}{3}, \tag{101}$$

which is of order unity. At horizon extremum, the inflationary exponent $\xi$ is

$$\xi = \frac{1}{2} \ln\left(\frac{\Delta_x'}{3\sqrt{u^2 - v^2}}\right), \tag{102}$$

and consequently the inflationary part of the conformal factor $\rho$ is

$$e^{-\xi} = \left( \frac{3\sqrt{u^2 - v^2}}{\Delta'_x} \right)^{1/2} \sim v^{1/2}, \tag{103}$$

which is becoming small. The horizon function at its extremum is

$$\Delta_x = -\left( \frac{e\Delta'_x}{3\sqrt{u^2 - v^2}} \right)^{3/2} \left( \frac{u - v}{u + v} \right)^{\Delta'_x/(4v)} \sim e^{-1/v}, \tag{104}$$

which is exponentially tiny.

The value of the radial coordinate $x$ can be found by integrating Eq. (93a),

$$x - x_{\rm in} = -\int \frac{\Delta_x dU_x}{2(U_x^2 - v^2)}. \tag{105}$$

The integral can be expressed analytically as an incomplete beta-function, but the expression is not useful. Physically, Eq. (105) says that the radius $x$ is frozen at its inner horizon value $x_{\rm in}$ during inflation and collapse, where $U_x$ is growing, while $\Delta_x$ remains small. The radial coordinate $x$ remains frozen even while the conformal factor $\rho \propto e^{-\xi}$ is shrinking.

After the horizon function $\Delta_x$ goes through its extremum, it starts increasing in absolute value as $U_x^{3/2}$ according to Eq. (99), or equivalently as $e^{3\xi}$ according to Eq. (97):

$$\Delta_x \approx U_x^{3/2} \Delta_0 \approx e^{3\xi} \Delta_0, \qquad \Delta_0 \equiv -\left( \frac{u - v}{u + v} \right)^{\Delta'_x/4v}, \tag{106}$$

where factors of order unity have been dropped compared to the exponentially huge factor $\Delta_0$. In Sec. VIII I it will be found that the conformally separable Einstein equations can no longer be solved with a collisionless source once $|\Delta_x| \gtrsim 1$, the point at which the motion of freely-falling ingoing and outgoing streams become predominantly angular rather than radial. Equation (106) implies that at this point $U_x$ is

$$U_x \approx |\Delta_0^{-2/3}| \quad \text{at } |\Delta_x| \approx 1, \tag{107}$$

which is exponentially huge. In particular, the condition (94) for the validity of the separation of Einstein components is well satisfied throughout inflation and collapse.

During collapse, $v$ and $\Delta'_x$ in Eqs. (93) may be neglected, and the equations yield a simplified equation for the evolution of the radius $x$,

$$\frac{d \ln(\Delta_x / U_x)}{dx} = -\frac{U_x}{\Delta_x}, \tag{108}$$

which integrates to

$$x - x_{\rm in} = -\frac{\Delta_x}{U_x}, \tag{109}$$

the constant of integration coming from $x = x_{\rm in}$ at $\Delta_x = 0$. Since $|\Delta_x|/U_x \ll 1$ throughout collapse, the radial coordinate $x$ remains frozen at its inner horizon value $x_{\rm in}$ to high precision through inflation and collapse.

### F. Trajectory and density of a freely-falling stream

In the previous subsection, Sec. VIII E, it was found that inflation and collapse takes place over an extremely narrow zone in (conformal) radial coordinate $x$ about the inner horizon value $x_{\rm in}$. Consequently the radial coordinate $x$ attached to a freely-falling stream remains essentially frozen at its inner horizon value throughout inflation and collapse. The radial coordinate remains frozen even while the conformal factor $\rho$ is collapsing to an exponentially tiny scale.

The proper time $\tau$ that elapses on the stream satisfies, Eq. (25),

$$\frac{d\tau}{dx} = m\frac{d\lambda}{dx} = -\frac{m\rho^2}{P_x}. \tag{110}$$

The right-hand side of Eq. (110) is approximately constant during early inflation, while the conformal factor $\rho$ is still close to its separable value $\rho_{\rm s}$, but then decreases as the conformal factor shrinks. Thus very little proper time elapses on a stream during the entire of inflation and collapse. During collapse, the proper time $\tau$ is even more frozen than the radial coordinate $x$, which itself is frozen.

The angular coordinate $y$ along the trajectory of the stream satisfies, Eq. (33),

$$\frac{dy}{dx} = -\frac{P_y}{P_x}. \tag{111}$$

The right-hand side of Eq. (111) is a number of order unity as long as $|\Delta_x| \ll 1$. and then becomes less than unity when $|\Delta_x| \gtrsim 1$. Thus the angular coordinate $y$ is also frozen along the trajectory of a stream. It follows that quantities such as $\omega_x$ and $\omega_y$, hence $\sigma$, are likewise frozen along the trajectory of a stream.

The conformal time $t$ and azimuthal angle $\phi$ coordinates along the trajectory are given by Eqs. (34), which can be written

$$\frac{dt}{dx} = \frac{1}{P_x}\left( \frac{P_t}{\Delta_x} - \frac{\omega_y P_\phi}{\Delta_y} \right), \qquad \frac{d\phi}{dx} = \frac{1}{P_x}\left( \frac{\omega_y P_t}{\Delta_x} - \frac{P_\phi}{\Delta_y} \right). \tag{112}$$

As long as $|\Delta_x| \ll 1$, the conformal time coordinate $t$ along an ingoing (+) or outgoing (−) stream is given by

$$t = \pm \int \frac{dx}{\Delta_x} = \mp \int \frac{dU_x}{2(U_x^2 - v^2)} = \pm \frac{1}{4v} \ln\left[ \frac{(U_x + v)(u - v)}{(U_x - v)(u + v)} \right], \tag{113}$$

the constant of integration being established by $t = 0$ at $U_x = u$. Consequently the time part $e^{vt}$ of the conformal factor $\rho$ is

$$e^{vt} = \left[\frac{(U_x + v)(u - v)}{(U_x - v)(u + v)}\right]^{\pm 1/4}. \tag{114}$$

Unlike $x$ and $y$, the coordinate $t$ is not frozen along the trajectory of the particle. Rather $e^{vt}$, Eq. (114), varies by a factor of order unity as $U_x$ increases from $u$ to some large value. Once $U_x$ increases to some value much larger than $u$ (such as $U_x \sim 1$, since $u \ll 1$), the time coordinate $t$ freezes out. Once $|\Delta_x| \gtrsim 1$, the relation $t = \pm \int dx/\Delta_x$ fails, but by that time $t$ is already frozen out, so in practice Eq. (114) is valid accurately throughout inflation and collapse.

The no-going tetrad frame has the special property that $P_t = 0$. In the no-going frame the potentially large term proportional to $P_t/\Delta_x$ in Eq. (113) vanishes, and $t$, like $x$, remains frozen throughout inflation and collapse. Physically, the factor $e^{vt}$ in Eq. (114) is the expansion factor of the black hole at the time that particles were accreted relative to the time that no-going particles were accreted. Equation (114) says that outgoing particles were accreted to the past of when no-going particles were accreted, when the black hole was smaller, while ingoing particles were accreted to the future of when no-going particles were accreted, when the black hole was larger.

The furthest into the past that ingoing particles see (or into the future that outgoing particles see) is at the end of inflation when the geometry is collapsing. At this point the ratio of the sizes of the black hole at the times the ingoing (+) and outgoing (−) particles were accreted is, from Eq. (114) with large $U_x$,

$$\frac{\rho^+_{\text{accrete}}}{\rho^-_{\text{accrete}}} = \left(\frac{u + v}{u - v}\right)^{1/2}, \tag{115}$$

which is a number of order unity or a few. Equation (115) shows that what happens deep inside the black hole depends only on the finite past and future of the black hole, not on what happens at the initial moments of collapse, nor on the indefinite future.

The density $N$ along a stream varies according to proportionality (62), which given that $\sigma$ is frozen is

$$N \propto \frac{1}{\rho^2 P_x P_y}. \tag{116}$$

The parameters $P_x$ and $P_y$ are given by Eqs. (35), which are exact for massless particles, and whose accuracy for massive particles was established in Sec. IV C.

As long as $|\Delta_x| \ll 1$, the Hamilton-Jacobi parameters $P_x$ and $P_y$ are constant along the trajectory of a stream, and the density $N^\pm$ of an ingoing (+) or outgoing (−) stream simplifies to

$$N^\pm \propto \frac{1}{\rho^2} \propto e^{2(\xi - vt)} \propto U_x \mp v, \tag{117}$$

where the inflationary exponent $\xi$ and conformal time $t$ have been eliminated in favor of $U_x$ using Eqs. (96) and (114). In this regime, the tetrad-frame momentum $p^\pm_k$ is hyperrelativistic, and focused along the radial direction,

$$\begin{aligned} p^\pm_k &\propto \frac{1}{\rho}\left\{-\frac{1}{\sqrt{-\Delta_x}}, \mp\frac{1}{\sqrt{-\Delta_x}}, \mu_y, \mu_\phi\right\}, \\ \mu_k &\equiv \frac{P_k}{|P_t|\sqrt{\Delta_y}}. \end{aligned} \tag{118}$$

Note that $\mu_k$, which are constant along the trajectory of the stream, are generically of order unity.

It will be found in Sec. VIII J that the conformally separable Einstein equations cease to be satisfied by the energy-momentum of collisionless streams once the angular motions of the streams begin to exceed their radial motions, which happens when $|\Delta_x| \gtrsim 1$. To see how this happens, it is necessary to (attempt to) follow the behavior of collisionless streams into this regime. For $|\Delta_x| \gtrsim 1$, as long as the spacetime remains conformally separable, the Hamilton-Jacobi parameters $P_t$, $P_y$, and $P_\phi$ all remain constant, but $P_x$ is no longer constant, varying in accordance with Eq. (38), which can be written (the argument of the square root in the following equation is positive since $\Delta_x$ is negative)

$$\frac{P_x}{P_t} = \pm\sqrt{1 - (\mu_y^2 + \mu_\phi^2)\Delta_x}, \tag{119}$$

where $\mu_y^2 + \mu_\phi^2$, a constant along the trajectory of the stream, is

$$\mu_y^2 + \mu_\phi^2 = \frac{\mathcal{K} - m^2\rho_y^2}{P_t^2}. \tag{120}$$

Putting together the dependence on $\rho$, Eq. (117), and $P_x$, Eq. (119), yields the density $N^\pm$ along an ingoing ( + ) or outgoing ( − ) stream,

$$N^\pm \propto \frac{U_x \mp v}{\sqrt{1 - (\mu_y^2 + \mu_\phi^2)\Delta_x}}. \tag{121}$$

The corresponding tetrad-frame momentum $p^\pm_k$ is

$$p_k^\pm \propto \frac{1}{\rho}\left\{-\frac{\sqrt{1-(\mu_y^2+\mu_\phi^2)\Delta_x}}{\sqrt{-\Delta_x}}, \mp\frac{1}{\sqrt{-\Delta_x}}, \mu_y, \mu_\phi\right\}, \tag{122}$$

whose angular components $p_y$ and $p_\phi$ exceed the radial component $p_t$ once $|\Delta_x| \gtrsim 1$.

### G. Einstein tensor

The two Einstein components (88) led to equations for the evolution of the conformal factor and radial horizon function during inflation and collapse. The remaining 8 components of the tetrad-frame Einstein tensor $G_{kl}$, which have zero electrovac source, may be written in terms of the expressions (85) for $U_x$ and $U_y$, and (87) for $X_x$, $X_y$, $Y_x$ and $Y_y$:

$$\rho^2\left(\frac{G_{xx}+G_{tt}}{2} \pm G_{xt}\right) = (U_x \mp v)\left[\frac{Y_x \pm v}{-\Delta_x} - \frac{d}{dx}\ln\left(\frac{d\omega_x}{dx}\right)\right] + X_x, \tag{123a}$$

$$\rho^2(G_{xy} \pm G_{ty}) = -\frac{1}{\sqrt{-\Delta_x\Delta_y}}(U_x \mp v)\left(\Delta_y\frac{\partial\ln\rho_s^2}{\partial y} - 2U_y\right) - \sqrt{\frac{-\Delta_x}{\Delta_y}}\left(U_y\frac{\partial\ln\rho_s^2}{\partial x} \pm \frac{v\omega_y}{\sigma^2}\frac{d\omega_y}{dy}\right), \tag{123b}$$

$$\rho^2(G_{x\phi} \pm G_{t\phi}) = \pm\frac{1}{\sqrt{-\Delta_x\Delta_y}}(U_x \mp v)\left(\frac{\Delta_y}{\sigma^2}\frac{d\omega_x}{dx} \mp 2v\omega_y\right) \mp \sqrt{\frac{-\Delta_x}{\Delta_y}}\left(\frac{U_y}{\sigma^2}\frac{d\omega_y}{dy} \mp v\omega_y\frac{\partial\ln\rho_s^2}{\partial x}\right), \tag{123c}$$

$$\rho^2\left(\frac{G_{yy}-G_{\phi\phi}}{2} \pm iG_{y\phi}\right) = (U_y \mp iv\omega_y)\left[\frac{-Y_y \pm iv\omega_y}{\Delta_y} - \frac{d}{dy}\ln\left(\frac{d\omega_y}{dy}\right)\right] + X_y \mp iv\frac{d\omega_y}{dy}. \tag{123d}$$

Most of the terms in Eqs. (123) are negligible in the conformally stationary limit $v \to 0$. Terms are negligible because: (a) they are proportional to one of $X_x$ or $X_y$, which vanish; (b) they are proportional to $U_y$, which remains negligibly small in the conformally stationary limit; or (c) they are proportional to $v$, and not proportional to inverse factors of $\Delta_x$, so remain negligible in the conformally stationary limit $v \to 0$. In addition, the term proportional to $d\ln(d\omega_x/dx)/dx$ inside square brackets in Eq. (123a) may be neglected. This term might potentially become important when $|\Delta_x| \gtrsim 1$, but it will be found below, Eq. (143), that this term in any case disappears when the Einstein equations are solved to next order in $\Delta_x/U_x$.

With all negligible and subdominant terms discarded, Eqs. (123) simplify in the conformally stationary limit to

$$\rho^2\left(\frac{G_{xx}+G_{tt}}{2} \pm G_{xt}\right) = \frac{1}{-\Delta_x}(U_x \mp v)(\Delta_x' \pm v), \tag{124a}$$

$$\rho^2(G_{xy} \pm G_{ty}) = -\frac{1}{\sqrt{-\Delta_x\Delta_y}}(U_x \mp v)\Delta_y\frac{\partial\ln\rho_s^2}{\partial y}, \tag{124b}$$

$$\rho^2(G_{x\phi} \pm G_{t\phi}) = \pm\frac{1}{\sqrt{-\Delta_x\Delta_y}}(U_x \mp v)\left(\frac{\Delta_y\omega_x'}{\sigma^2} \mp 2v\omega_y\right), \tag{124c}$$

$$\rho^2\left(\frac{G_{yy}-G_{\phi\phi}}{2} \pm iG_{y\phi}\right) = 0, \tag{124d}$$

in which $\omega_x' \equiv d\omega_x/dx|_{x_{\rm in}}$ and $\Delta_x' \equiv d\Delta_x/dx|_{x_{\rm in}}$ are the derivatives of $\omega_x$ and the electrovac horizon function $\Delta_x$ at the inner horizon $x = x_{\rm in}$. The $\Delta_x'$ in Eq. (124a) comes from $Y_x = \Delta_x'$, Eq. (93). The $\omega_x'$ in Eq. (124c) comes from replacing $d\omega_x/dx$, which remains frozen through inflation and collapse, by its inner horizon value $\omega_x'$, as determined by Eq. (73a).

Should not the factor $\Delta_x' \pm v$ on the right-hand side of Eq. (124a) be replaced by $\Delta_x'$ in the conformally stationary limit $v \to 0$, and likewise the factor $\Delta_y\omega_x'/\sigma^2 \mp 2v\omega_y$ on the right-hand side of Eq. (124c) be replaced by $\Delta_y\omega_x'/\sigma^2$? No. Each equation describes not one but two Einstein components, and the full expressions given are needed to capture both accurately. In the first case, the difference of Eqs. (124a) gives $\rho^2 G_{xt} = v(U_x - \Delta_x')/(-\Delta_x)$, and in the second case the sum of Eqs. (124c) gives $\rho^2 G_{x\phi} = -v(\Delta_y\omega_x'/\sigma^2 + 2\omega_y U_x)/\sqrt{-\Delta_x\Delta_y}$.

### H. Streaming energy-momenta

The form (124) of the Einstein components, derived under the conditions of conformal stationarity and conformal separability, fits to the form of the energy-momentum tensor $T_{kl}$ of a collisionless fluid consisting of two streams, one ingoing (+) and one outgoing (−):

$$T_{kl} = T^+_{kl} + T^-_{kl}. \tag{125}$$

The energy-momenta of the ingoing (positive energy, $p_t < 0$) and outgoing (negative energy, $p_t > 0$) streams are

$$T^\pm_{kl} = N^\pm p^\pm_k p^\pm_l, \tag{126}$$

with densities

$$N^\pm = \frac{1}{16\pi}(U_x \mp v)(\Delta'_x \pm v), \tag{127}$$

and tetrad-frame momenta $p^\pm_k$

$$p^\pm_k = \frac{1}{\rho}\left\{-\frac{1}{\sqrt{-\Delta_x}}, \mp\frac{1}{\sqrt{-\Delta_x}}, \frac{1}{\sqrt{\Delta_y}} \times\left(\frac{\Delta_y \partial \ln\rho_s^2/\partial y}{\Delta'_x \pm v}\right), \mp\frac{1}{\sqrt{\Delta_y}}\left(\frac{\Delta_y \omega'_x/\sigma^2 \mp 2v\omega_y}{\Delta'_x \pm v}\right)\right\}. \tag{128}$$

Eqs. (127) and (128) are defined up to an arbitrary normalization that leaves $T_{kl}$ constant: the tetrad-frame momentum $p_k$ and density $N$ can be multiplied by some constant $\alpha$ and $\alpha^{-2}$ respectively. The corresponding Hamilton-Jacobi parameters $P^\pm_k$, Eq. (23), of the collisionless streams are

$$P^\pm_k = \left\{-1, \mp 1, \frac{\Delta_y \partial \ln\rho_s^2/\partial y}{\Delta'_x \pm v}, \mp\frac{\Delta_y \omega'_x/\sigma^2 \mp 2v\omega_y}{\Delta'_x \pm v}\right\}, \tag{129}$$

again up to an arbitrary normalization factor $\alpha$. Eqs. (127) and (128) agree with the behavior (117) and (118) of collisionless streams in the regime where the radial horizon function is small,

$$|\Delta_x| \ll 1. \tag{130}$$

Eqs. (127) and (128) do not describe correctly the behavior (121) and (122) of collisionless streams when the horizon function approaches and exceeds unity, $|\Delta_x| \gtrsim 1$. Moreover the tetrad-frame momentum (128) predicts that the purely angular components of the collisionless energy-momentum grow, becoming dominant when $|\Delta_x| \gtrsim 1$, whereas it has been assumed from Sec. VIII D up to the present point that the angular components are negligible. The next two subsections, Secs. VIII I and VIII J, address these issues.

### I. Angular energy-momenta imposed by conformal separability

As long as conformal separability holds, the angular components $G_{yy} - G_{\phi\phi}$ and $G_{y\phi}$ of the Einstein tensor must necessarily vanish, so that the $2 \times 2$ angular submatrix of the Einstein tensor must be isotropic (proportional to the $2 \times 2$ unit matrix). The reason for this is that the expressions (123d) for those Einstein components are functions only of angle $y$, so the only kind of source of energy-momentum that they admit is one that depends only on $y$. By contrast, the densities $N^\pm$, Eq. (127), of the collisionless streams arising from inflation depend strongly on radius through $U_x$. The only way that the collisionless streams can source the angular components of the Einstein tensor is that their angular energy-momentum tensor must be isotropic.

Of course, as long as the angular components are subdominant, which is true when $|\Delta_x| \ll 1$, there is no need for the angular components of the Einstein equations to be satisfied accurately, because the subdominant components have no effect on the remaining Einstein equations. This expectation is confirmed explicitly in the next subsection, Sec. VIII J.

Nevertheless, as long as $|\Delta_x| \ll 1$, an isotropic collisionless angular energy-momentum tensor can be accomplished, by allowing not just one ingoing and one outgoing stream, but several streams $a$, with densities

$$N^\pm_a \propto N^\pm, \tag{131}$$

that sum to the totals prescribed by Eq. (127)

$$\sum_a N^+_a = N^+, \qquad \sum_a N^-_a = N^-. \tag{132}$$

Denote the means and mean squares of the angular components $\mu_y$ and $\mu_\phi$ of the tetrad-frame momenta of the streams, Eq. (118), by

$$\langle\mu^\pm_y\rangle \equiv \frac{\sum_a N^\pm_a \mu_{a,y} a}{N^\pm}, \qquad \langle\mu^\pm_\phi\rangle \equiv \frac{\sum_a N^\pm_a \mu_{a,\phi}}{N^\pm}, \tag{133a}$$

$$\langle\mu^2_y\rangle \equiv \frac{\sum_a N_a \mu^2_{a,y}}{N^+ + N^-}, \qquad \langle\mu^2_\phi\rangle \equiv \frac{\sum_a N_a \mu^2_{a,\phi}}{N^+ + N^-},$$

$$\langle\mu_y \mu_\phi\rangle \equiv \frac{\sum_a N_a \mu_{a,y}\mu_{a,\phi}}{N^+ + N^-}, \tag{133b}$$

in which for the means the sum is over either the ingoing or outgoing stream, while for the mean squares the sum is over both ingoing and outgoing streams combined. The angular components $\mu_y$ and $\mu_\phi$ must average to the values prescribed by Eq. (128),

$$\langle\mu_y^\pm\rangle = \frac{1}{\sqrt{\Delta_y}}\left(\frac{\Delta_y \partial \ln\rho_s^2/\partial y}{\Delta_x' \pm v}\right), \tag{134a}$$

$$\langle\mu_\phi^\pm\rangle = \mp\frac{1}{\sqrt{\Delta_y}}\left(\frac{\Delta_y \omega_x'/\sigma^2 \mp 2v\omega_y}{\Delta_x' \pm v}\right). \tag{134b}$$

The condition of angular isotropy requires that the mean squares of the angular components are equal, and the mean of their product is zero,

$$\langle\mu_y^2\rangle = \langle\mu_\phi^2\rangle = \mu^2, \qquad \langle\mu_y\mu_\phi\rangle = 0, \tag{135}$$

for some $\mu^2$. The angular components of the collisionless energy-momentum then form the isotropic matrix

$$\rho^2 T_{yy} = \rho^2 T_{\phi\phi} = (N^+ + N^-)\mu^2 = \frac{U_x\Delta_x' - v^2}{8\pi}\mu^2,$$
$$\rho^2 T_{y\phi} = 0. \tag{136}$$

The Schwarz inequality, which states that the mean square of a distribution must exceed the squared mean, requires that

$$\mu^2 \geq \max\left[\left(\frac{N^+\langle\mu_y^+\rangle + N^-\langle\mu_y^-\rangle}{N^+ + N^-}\right)^2, \left(\frac{N^+\langle\mu_\phi^+\rangle + N^-\langle\mu_\phi^-\rangle}{N^+ + N^-}\right)^2\right]. \tag{137}$$

While contrived, there is no difficulty to construct a distribution of collisionless streams with the required mean momenta (134) and the conditions (135) and (137) on the mean squared momenta.

On the other hand, the collisionless energy-momentum cannot be contrived to have isotropic angular components once $|\Delta_x| \gtrsim 1$. In this regime, as long as conformal separability holds, the densities $N^\pm$ and tetrad-frame momenta of streams are given by Eqs. (121) and (122). The condition that the angular components be isotropic requires that, generalizing (135),

$$\sum_a N_a \frac{\mu_{a,y}^2}{\sqrt{1-(\mu_{a,y}^2+\mu_{a,\phi}^2)\Delta_x}} = \sum_a N_a \frac{\mu_{a,\phi}^2}{\sqrt{1-(\mu_{a,y}^2+\mu_{a,\phi}^2)\Delta_x}}. \tag{138}$$

If it happened that the means (134) of $\mu_y$ and $\mu_\phi$ were the same, then the isotropy condition (138) could be satisfied for all values of the horizon function $\Delta_x$. But generically (e.g. for Kerr) the mean momenta differ, and the isotropy condition (138) on the squared momenta cannot hold for all $\Delta_x$.

### J. Inflation and collapse to next order

In Sec. VIII D, equations governing the evolution of the horizon function and conformal factor were derived from the assumption that the Einstein components $G_{xx} - G_{tt}$ and $G_{yy} + G_{\phi\phi}$ had negligible collisionless source. It is true that the trace of the collisionless energy-momentum always remains negligible, Eq. (71). However, in Sec. VIII H it was found that the angular components of the collisionless energy-momentum, though initially negligible, grow, becoming dominant when $|\Delta_x| \gtrsim 1$. In this subsection, the angular components are taken into account, which involves taking the Einstein equations to next order in $\Delta_x/U_x$. It is found that the earlier results are robust as long as $|\Delta_x| \ll 1$, but fail when angular motions become important, $|\Delta_x| \gtrsim 1$.

A collisionless source for the Einstein components $G_{xx} - G_{tt}$ and $G_{yy} + G_{\phi\phi}$ can be accommodated by admitting source terms for the quantities $X_x$ and $Y_x$ defined by Eqs. (87a) and (87c), in much the same way that electrovac sources (a radial electromagnetic field and a cosmological constant) could be accommodated by admitting source terms for $Y_x$ and $Y_y$. The sources for $X_x$ and $Y_x$ are conveniently written in terms of two arbitrary functions $F_X$ and $F_Y$:

$$X_x = -U_x\left(\frac{\partial \ln(\rho_s/\sigma^2)}{\partial x} + F_X\right), \tag{139a}$$

$$Y_x = \Delta_x\left(\frac{\partial \ln\rho_s}{\partial x} + \frac{2(f_0 g_1 + f_1 g_0)}{d\omega_x/dx} - F_Y\right). \tag{139b}$$

During early inflation, when both $U_x$ and $\Delta_x$ are negligibly small, both sets of source terms are negligible. The source terms are of order $\Delta_x/U_x$ compared to the principal terms in $X_x$ and $Y_x$, and remain subdominant through inflation and collapse. The contributions $\partial\ln(\rho_s/\sigma^2)/\partial x$ and $\partial\ln\rho_s/\partial x$ to the source terms depend on angle $y$ as well as radius $x$, breaking separability, but the breakdown is unimportant as long as condition (94) holds, which is well satisfied through inflation and collapse.

It is convenient to define new quantities $\tilde{X}_x$ and $\tilde{Y}_x$ that concatenate $X_x$ and $Y_x$ with their source terms:

$$\tilde{X}_x \equiv \frac{\partial U_x}{\partial x} + 2\frac{U_x^2 - v^2}{\Delta_x} + U_x\left(\frac{\partial \ln(\rho_s/\sigma^2)}{\partial x} + F_X\right), \tag{140a}$$

$$\tilde{Y}_x \equiv \frac{\partial \Delta_x}{\partial x} + 3U_x + \Delta_x\left(\frac{\partial \ln(\rho_s^3/\sigma^4)}{\partial x} + F_Y\right). \tag{140b}$$

The equations governing the evolution of $U_x$ and $\Delta_x$ are then

$$\tilde{X}_x = 0, \tag{141a}$$

$$\tilde{Y}_x = \Delta_x', \tag{141b}$$

generalizing Eqs. (93). Eqs. (141) constitute the evolution equations for $U_x$ and $\Delta_x$ taken to next order in $\Delta_x/U_x$. As long the condition (94) for the validity of the earlier evolution equations holds, the evolution of $U_x$ and $\Delta_x$ is essentially unchanged, and all the results of Sec. VIII E carry through unchanged. This is as expected physically: the evolution of the conformal factor and horizon function should be essentially unaffected by subdominant contributions to the energy-momentum.

Expressions for the Einstein components $G_{xx} - G_{tt}$ and $G_{yy} + G_{\phi\phi}$ in terms of the higher order $\tilde{X}_x$ and $\tilde{Y}_x$ defined by Eqs. (139a) and (139b) are

$$\rho^2(G_{xx} - G_{tt}) = -2\tilde{X}_x - 2\tilde{Y}_x \frac{\partial \ln\rho_s}{\partial x} + X_y - \frac{\partial Y_y}{\partial y} - Y_y \frac{\partial}{\partial y} \ln\left(\frac{\rho_s^2}{\sigma^4}\frac{d\omega_y}{dy}\right) + 2U_x F_X + 2\Delta_x\left[\frac{1}{4}\left(\frac{1}{\sigma^2}\frac{d\omega_y}{dy}\right)^2 + F_Y \frac{\partial \ln\rho_s}{\partial x}\right] + U_y \frac{\partial}{\partial y} \ln\left[\frac{\rho_s^{10}}{\sigma^{20}}\left(\frac{d\omega_y}{dy}\right)^3\right], \tag{142a}$$

$$\rho^2(G_{yy} + G_{\phi\phi}) = -2X_y + 2Y_y \frac{\partial \ln\rho_s}{\partial y} + \tilde{X}_x + \frac{\partial \tilde{Y}_x}{\partial x} + \tilde{Y}_x\left[\frac{\partial \ln(\rho_s/\sigma^2)}{\partial x} - F_Y\right] + U_x(-F_X + 3F_Y) + \Delta_x\left[\frac{3}{4}\left(\frac{1}{\sigma^2}\frac{d\omega_y}{dy}\right)^2 - \frac{\partial F_Y}{\partial x} + F_Y^2 + F_Y \frac{\partial \ln(\rho^2/\sigma^2)}{\partial x}\right] - 2U_y \frac{\partial \ln(\rho_s/\sigma^2)}{\partial y}, \tag{142b}$$

while the Einstein component (123a) becomes

$$\rho^2\left(\frac{G_{xx} + G_{tt}}{2} \pm G_{xt}\right) = (U_x \mp v)\left(\frac{\tilde{Y}_x \pm v}{-\Delta_x} - F_X + F_Y\right) + \tilde{X}_x \mp v\left(\frac{\partial \ln(\rho_s/\sigma^2)}{\partial x} + F_X\right), \tag{143}$$

with expressions (123b)–(123d) remaining unchanged.

During collapse, when $U_x$ is large, the dominant terms contributing to the right-hand sides of Eqs. (142) are those proportional to $U_x$. Expressions for the functions $F_X$ and $F_Y$ follow from requiring that the $U_x$-dependent terms on the right- hand sides of Eqs. (142) are, in accordance with Einstein's equations, equal to $8\pi$ times the corresponding collisionless energy-momenta. For $|\Delta_x| \ll 1$, the relevant collisionless energy-momenta are the isotropic angular components (136), and a negligible collisionless trace, $T_k^k = 0$. The resulting functions $F_X$ and $F_Y$ are

$$F_X = F_Y = \left(\Delta_x' - \frac{v^2}{U_x}\right)\mu^2. \tag{144}$$

The functions are of order unity, and thus, as anticipated, their inclusion has negligible effect on the evolution Eqs. (141) as long as condition (94) is true, which as shown in Sec. VIII E is well satisfied through inflation and collapse.

If the functions $F_X$ and $F_Y$ from Eqs. (144) are inserted into the right-hand side of Eq. (143), then the result agrees with the earlier expression (123a) except that the subdominant term proportional to $d\ln(d\omega_x/dx)/dx$ in the earlier expression disappears. The net result is that, when the conformally stationary limit is taken, the same set of expressions (124) is obtained for the Einstein components as found previously. This confirms that taking into account the purely angular components of the collisionless energy-momentum has essentially no effect on the solution, as long $|\Delta_x| \ll 1$.

On the other hand, as already discussed in Sec. VIII I, once $|\Delta_x| \gtrsim 1$, generically (e.g. for Kerr) the angular components of the collisionless energy-momentum can no longer be arranged to be isotropic, Eq. (138), so the conformally separable Einstein equations cannot be satisfied with a collisionless source. Even if it could be arranged that the angular components were isotropic, once $|\Delta_x| \gtrsim 1$ the situation is further complicated by the fact that $d\omega_x/dx$, which up to this point has been frozen at its electrovac value, must also change in order to satisfy the Einstein equation for $G_{t\phi}$. Further discussion is unwarranted in this paper.

In conclusion, the solution found earlier in this section, up to Sec. VIII H, holds as long as $|\Delta_x| \ll 1$, but breaks down when the angular motions of the collisionless streams exceed their radial motions, which happens when $|\Delta_x| \gtrsim 1$. What happens after angular motions become important is undetermined.

## IX. MASS AND CURVATURE

### A. Mass inflation

The term mass inflation comes from the fact that in charged spherically symmetric models of black holes, the interior mass, or Misner-Sharp [38] mass, inflates exponentially. In spherical black holes, the interior mass $M$ can be defined by

$$\frac{2M}{r} - 1 \equiv -\partial^m r \partial_m r, \tag{145}$$

where $r$ is the circumferential radius, which plays the role of a conformal factor, and $\partial_m$ is the directed derivative in any tetrad frame. An analogous scalar quantity in the rotating black holes considered in the present paper is

$$\frac{2M}{\rho} - 1 \equiv \mathcal{M} = \mathcal{M}_x + \mathcal{M}_y = -\partial^m \rho \partial_m \rho, \tag{146}$$

where $\mathcal{M}_x$ and $\mathcal{M}_y$ are dimensionless radial and angular mass parameters defined by

$$\mathcal{M}_x \equiv (\partial_x \rho)^2 - (\partial_t \rho)^2 = -\frac{1}{\Delta_x}\left[\left(\frac{\partial \ln\rho}{\partial x}\Delta_x\right)^2 - v^2\right], \tag{147a}$$

$$\mathcal{M}_y \equiv -(\partial_y \rho)^2 - (\partial_\phi \rho)^2 = -\frac{1}{\Delta_y}\left[\left(\frac{\partial \ln\rho}{\partial y}\Delta_y\right)^2 + v^2\omega_y^2\right]. \tag{147b}$$

The mass parameter proposed by [12] for rotating black holes is the radial mass parameter $\mathcal{M}_x$. In the Kerr-Newman geometry ($\Lambda = 0$), the interior mass $M$ defined by Eq. (146) goes over to the black hole mass $M_\bullet$ far from the black hole,

$$M \to M_\bullet \quad \text{as } r \to \infty. \tag{148}$$

In terms of the quantities $U_x$ and $U_y$ defined by Eqs. (85), the dimensionless mass parameters are

$$\mathcal{M}_x = -\frac{1}{\Delta_x}\left[\left(\frac{\partial \ln\rho_s}{\partial x}\Delta_x + U_x\right)^2 - v^2\right], \tag{149a}$$

$$\mathcal{M}_y = -\frac{1}{\Delta_y}\left[\left(\frac{\partial \ln\rho_s}{\partial y}\Delta_y - U_y\right)^2 + v^2\omega_y^2\right]. \tag{149b}$$

During the electrovac phase prior to inflation, the radial mass parameter $\mathcal{M}_x$ is proportional to the horizon function $\Delta_x$,

$$\mathcal{M}_x = -\left(\frac{\partial \ln\rho_s}{\partial x}\right)^2 \Delta_x, \tag{150}$$

which approaches some small value near the inner horizon $\Delta_x \to -0$. The radial mass parameter $\mathcal{M}_x$ given by Eq. (149a) reaches a minimum, signaling the start of inflation, when (here $U_x$ equals its initial value $u$)

$$\frac{\partial \ln\rho_s}{\partial x}\Delta_x = \sqrt{u^2 - v^2}. \tag{151}$$

At this point the streaming energy and pressure in the no-going tetrad frame are comparable to unity in natural black hole units, $c = G = M_\bullet = 1$, while the mass parameter is small, of order $\mathcal{M}_x \sim v$. Once inflation gets going, the mass parameter is

$$\mathcal{M}_x \approx \frac{U_x^2 - v^2}{-\Delta_x}, \tag{152}$$

which can be recognized as the principal term driving the evolution of $U_x$, Eq. (93a). During inflation the radial mass parameter increases exponentially.

During inflation and collapse, the ratio of the mass parameter $\mathcal{M}_x$ to the streaming energy $T_{xx}$ in the no-going frame is

$$\frac{\mathcal{M}_x}{8\pi T_{xx}} = \rho^2 \frac{U_x^2 - v^2}{U_x \Delta_x'} = \rho_s^2 \frac{\sqrt{u^2 - v^2}}{\Delta_x'}\sqrt{1 - \frac{v^2}{U_x^2}}, \tag{153}$$

which remains always of order $v$, increasing mildly as $U_x$ increases from $u$ to some large value.

After the horizon function has gone through its extremum, it increases in absolute value as $\Delta_x \propto U_x^{3/2}$, so the dimensionless mass parameter varies as $\mathcal{M}_x \propto U_x^{1/2}$, or equivalently as $\mathcal{M}_x \propto 1/\rho$. The interior mass $M$ defined by Eq. (146) thus goes to a (huge) constant during collapse, consistent with behavior of the interior mass during collapse in charged spherical black holes, Sec. 4.3 of [11].

### B. Weyl curvature

The only nonvanishing component of the Weyl tensor is the complex spin-0 component. The fact that only the spin-0 component is nonzero defines the spacetime as Petrov type D. Subject to the conditions of conformal time invariance (not necessarily conformal stationarity) and conformal separability assumed in this paper, the polar (real) part of spin-0 Weyl component, Eqs. (B5)–(B7), may be written in terms of the quantities $U_y$, $X_y$, and $Y_y$ defined by Eqs. (85b), (87b), and (87d), as well as the mass parameter $\mathcal{M}$ defined by Eq. (146), as

$$\rho^2 C^{(p)} = \rho^2\left(\frac{1}{6}G_{xx} - \frac{1}{6}G_{tt} + \frac{1}{12}G_{yy} + \frac{1}{12}G_{\phi\phi}\right) - \frac{1}{2}\mathcal{M} - \frac{3}{4}\left(\frac{1}{\sigma^2}\frac{d\omega_y}{dy}\right)^2\Delta_x - \frac{3}{4}\left(\frac{1}{\sigma^2}\frac{d\omega_x}{dx}\right)^2\Delta_y + \frac{1}{2}X_y + \frac{1}{2}\frac{\partial Y_y}{\partial y} + \frac{1}{2}Y_y\frac{\partial}{\partial y}\ln\left(\frac{1}{\sigma^4}\frac{d\omega_y}{dy}\right) + \frac{1}{2}U_y\frac{\partial}{\partial y}\ln\left[\frac{\rho_s^4}{\sigma^8}\left(\frac{d\omega_y}{dy}\right)^3\right], \tag{154}$$

which is valid throughout the electrovac, inflationary, and collapse regimes. Since $U_y$ and $X_y$ are negligible in the conformally stationary limit, the polar spin-0 Weyl component reduces to

$$\rho^2 C^{(p)} = \rho^2\left(\frac{1}{6}G_{xx} - \frac{1}{6}G_{tt} + \frac{1}{12}G_{yy} + \frac{1}{12}G_{\phi\phi}\right) - \frac{1}{2}\mathcal{M} - \frac{3}{4}\left(\frac{1}{\sigma^2}\frac{d\omega_y}{dy}\right)^2\Delta_x - \frac{3}{4}\left(\frac{1}{\sigma^2}\frac{d\omega_x}{dx}\right)^2\Delta_y + \frac{1}{2}\frac{\partial Y_y}{\partial y} + \frac{1}{2}Y_y\frac{\partial}{\partial y}\ln\left(\frac{1}{\sigma^4}\frac{d\omega_y}{dy}\right). \tag{155}$$

I have not been able to find any enlightening expression for the axial (imaginary) spin-0 component of Weyl tensor, beyond that already given as Eq. (B8).

During inflation and collapse, the Weyl curvature is dominated by the mass term,

$$\rho^2 C \approx -\frac{1}{2}\mathcal{M}. \tag{156}$$

## X. BOUNDARY CONDITIONS

In Sec. VIII it was found that a conformally stationary, conformally separable solution exists for the interior structure of a rotating black hole, from electrovac through inflation to collapse, in which the Einstein equations are sourced by the energy-momentum of ingoing and outgoing collisionless streams. Because the accretion rate is asymptotically tiny, the streams have negligible energy-momentum above the inner horizon, and therefore have no effect on the geometry above the inner horizon. From the point of view of boundary conditions, what is important is the form of the collisionless streams incident on the inner horizon.

### A. Density of collisionless streams incident on the inner horizon

The most important boundary conditions are those on the radial ($x$-$t$) components of the collisionless energy-momentum incident on the inner horizon. The radial components of the energy-momentum of ingoing (+) and outgoing (−) collisionless during inflation and collapse are, Eqs. (127) and (128),

$$T_{xx}^{\pm} = \pm T_{xt}^{\pm} = T_{tt}^{\pm} = \frac{N^{\pm}}{\rho^2|\Delta_x|}. \tag{157}$$

The initial values of these components are set by the densities $N^{\pm}$, Eq. (127), of the ingoing and outgoing collisionless streams incident on the inner horizon, which are, since $U_x = u$ initially,

$$N^{\pm} = \frac{1}{16\pi}(u \mp v)(\Delta_x' \pm v). \tag{158}$$

The densities $N^{\pm}$ of incident ingoing and outgoing streams are proportional to $u \mp v$ respectively. Both ingoing and outgoing streams must be present for inflation to occur, so both densities must be strictly positive. Moreover, at least classically, the black hole must expand as it accretes, so $v$ must be positive. Thus the accretion rates $u$ and $v$, both of which are to be considered small, must satisfy

$$u > v > 0. \tag{159}$$

Positive $v$ implies that in the no-going tetrad frame the density of the outgoing stream exceeds that of the ingoing stream during inflation and collapse. Equivalently, the center-of-mass frame is outgoing. An outgoing density that exceeds the ingoing density ensures that the black hole's angular momentum, which in self-similar solutions is determined by the angular momentum of the accreted streams, is positive. The case $v = 0$ corresponds to equal ingoing and outgoing streams, which is the stationary (or homogeneous) approximation first applied to inflation by [32–34].

Equation (158) prescribes that the densities $N^{\pm}$ of incident ingoing and outgoing streams must be uniform, independent of latitude $y$. In other words, the accretion flow must be monopole in density. It makes physical sense that the condition of conformal separability would require a high degree of symmetry of the incident accretion flow.

The small accretion rate parameters $u$ and $v$ completely characterize the inflationary solution. The evolution of the inflationary exponent $\xi$ and horizon function $\Delta_x$ found in Sec. VIII E is determined entirely by these parameters, along with the derivative $\Delta_x'$ of the electrovac horizon function at the inner horizon.

The positivity of the ingoing and outgoing densities $N^{\pm}$, Eq. (158), requires also that $\Delta_x' \pm v$ be positive. This means that the inflationary solutions do not apply to extremal black holes, whose inner horizons coincide with their outer horizons, and for which $\Delta_x' = 0$.

### B. Angular motion of collisionless streams incident on the inner horizon

The angular components of the momenta of the collisionless streams are subdominant during inflation and collapse. As seen in § VIII J, the evolution of the inflationary exponent $\xi$ and horizon function $\Delta_x$, and of the radial components of the collisionless energy-momentum, are unaffected by angular motions until angular motions become important, at which point the solution fails, which happens when the geometry has collapsed to exponentially tiny scale. As far as the radial solution is concerned, a sufficient condition for conformal separability to hold is the condition (158) on the density of the accretion flow. If however the Einstein equations are required to hold also for the subdominant radial-angular components of the collisionless energy-momentum, which is a more stringent constraint on conformal separability, then the angular components of the tetrad-frame momenta of the ingoing and outgoing collisionless streams must be as given by Eq. (128). The corresponding Hamilton-Jacobi parameters $P_k^{\pm}$ are given by Eq. (129). The condition (129) prescribes that the angular components of the Hamilton-Jacobi parameters $P_k^{\pm}$ do not vanish, but rather vary in the given fashion with latitude $y$.

Equation (129) requires that the collisionless streams have some net motion in the angular $y$ direction. One might perhaps have anticipated that conformally stationary flow might require that each particle in the collisionless stream fall along a surface of constant latitude $y$, which would happen if $P_y$ for each particle were identically zero. Trajectories at constant $y$ occur when $P_y$ is not only zero but also an extremum with respect to variation of $x$ or $y$ at fixed constants of motion $E$, $L$, $\mathcal{K}$. In fact Eq. (129) shows that the required $P_y$ is not zero.

### C. The angular boundary conditions cannot be achieved with collisionless streams accreted from outside the outer horizon

Can the relations (129) governing the angular motions of streams incident on the inner horizon be accomplished by collisionless streams that accrete from outside the horizon? As will now be shown, the answer is no.

Equation (129) shows that the ratio $P_t^{\pm}/P_\phi^{\pm}$ incident on the inner horizon for each of the ingoing (+) and outgoing (−) streams is

$$\frac{P_t^{\pm}}{P_\phi^{\pm}} = \frac{\Delta_x' \pm v}{\Delta_y \omega_x'/\sigma^2 \mp 2v\omega_y}. \tag{160}$$

If the ingoing and outgoing streams contain multiple components, then the ratio (160) is a density-weighted average. If one insists that every particle originate from outside the outer horizon, then $P_t$ must be negative at the outer horizon for every particle. The relations (26) with vanishing electromagnetic potential (the case of a charged black hole is deferred to Paper 3) then imply that $P_t$ and $P_\phi$ of the particle at the inner horizon must satisfy the inequality

$$P_t \leq P_\phi \frac{\omega_{x,\mathrm{in}} - \omega_{x,\mathrm{out}}}{1 - \omega_{x,\mathrm{out}}\omega_{y,\mathrm{in}}}, \tag{161}$$

where the subscripts in and out denote values, respectively, at the inner and outer horizon. The condition (161) excludes half of the $P_t$-$P_\phi$ plane. As illustrated in Fig. 1, the right-hand sides of Eqs. (160) and (161) have different dependences on latitude. While the inequality (161) may be satisfied at any latitude by either the ingoing or the outgoing stream, the inequality cannot be satisfied simultaneously by both ingoing and outgoing streams at that latitude.

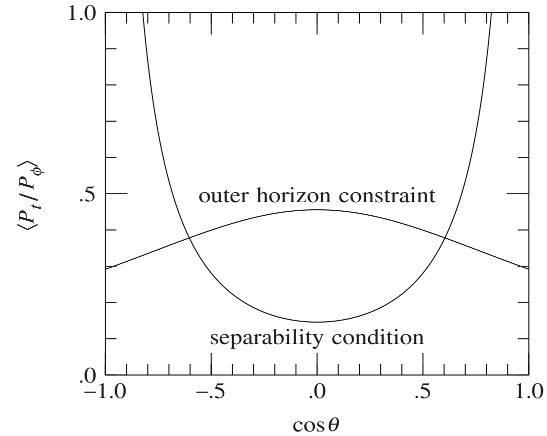


FIG. 1. Comparison of the condition on the average ratio of $P_t$ to $P_\phi$ at the inner horizon required by conformal separability, Eq. (160), to the constraint on the same ratio imposed by the condition that particles free-fall from outside the outer horizon, Eq. (161), for the case of an uncharged black hole with angular momentum parameter $a = 0.96M_\bullet$. In the equatorial region $\cos\theta \sim 0$, outgoing particles (positive $P_t$ and $P_\phi$) satisfy the constraint but ingoing particles (negative $P_t$ and $P_\phi$) do not. Conversely in the polar regions $|\cos\theta| \sim 1$, ingoing particles satisfy the constraint but outgoing particles do not.

### D. Dispersion of angular motions incident on the inner horizon

The purely angular components of the collisionless energy-momentum are sub-sub-dominant. During inflation, the angular components are negligible because the incident accretion flow is by assumption tiny, and inflation does not amplify the angular components of the momenta of the streams. During collapse, the angular components of the momentum grow faster than the radial components, eventually becoming comparable to the radial components, at which point the solution fails. As argued in Sec. VIII I, if the Einstein equations are required to hold also for the sub-sub-dominant angular components of the collisionless energy-momentum, then conformal separability requires that the angular energy-momentum be isotropic (proportional to the unit $2 \times 2$ matrix). If the angular energy-momentum is initially isotropic, then the behavior of freely-falling streams ensures that it will remain so during inflation and collapse.

An isotropic angular energy-momentum incident on the inner horizon can be contrived by allowing multiple ingoing and outgoing streams whose angular components of momentum satisfy condition (129) in the mean, but are isotropic in their mean squares, which can be arranged.

It is apparent that the required boundary conditions become more special and more contrived as the condition of conformal separability is tightened. To satisfy the radial components of the Einstein equations, only condition (158) on the incident densities $N^{\pm}$ is required. To satisfy the subdominant radial-angular components of the Einstein equations, condition (129) on the angular components of the momenta is required. To satisfy the sub-sub-dominant angular components of the Einstein equations, the angular energy-momentum tensor must be isotropic.

## XI. CONCLUSIONS

This paper has presented details of a new set of conformally stationary, axisymmetric, conformally separable solutions for the interior structure of a rotating uncharged black hole that accretes a collisionless fluid. The solutions are generalized to charged black holes in Paper 3 [2]. A concise derivation appears in Paper 1 [3].

The solutions confirm explicitly, for the special case of conformal separability, that inflation develops in rotating black holes as anticipated by [12].

Conformal stationarity means that the black hole accretes steadily at an asymptotically tiny rate, in such a way that the spacetime expands conformally (self-similarly), without change of shape. The solutions presented should be a good approximation, in the sense of perturbation theory, when the accretion rate is small, and should become increasingly accurate as the accretion rate tends to zero.

The most important equations in this paper are Eqs. (88) for two of the diagonal Einstein components, which have negligible collisionless source of energy-momentum. These equations hold over the entire regime of interest, from electrovac through inflation and collapse. The equations can be separated, and their solution seamlessly yields both electrovac and inflationary solutions. The equations recover several known physical features of inflation: inflation occurs at the inner horizon but not at the outer horizon; inflation requires the simultaneous presence of both ingoing and outgoing streams; and the smaller the accretion rate, the more violently inflation exponentiates.

A central feature of separable electrovac ($\Lambda$-Kerr-Newman) solutions is that as ingoing ($p_t < 0$) and outgoing ($p_t > 0$) collisionless streams approach the inner horizon, they concentrate into narrow, intense beams focused along the ingoing and outgoing principal null directions, regardless of the initial angular motion of the streams. If there were no back-reaction on the electrovac geometry, then the streams would exceed the speed of light relative to each other and fall through distinct ingoing and outgoing inner horizons into two causally separated regions of spacetime. In reality, the energy and pressure of the counter-streaming ingoing and outgoing beams builds to the point that it becomes a significant source of gravity, however tiny the accretion rate. The gravity of the streaming energy-momentum acts so as to accelerate the ingoing and outgoing beams even faster through each other along the principal null directions. The result is the inflationary instability, in which the energy-momentum of the hyper-relativistically counter-streaming beams grows, along with the Weyl curvature and mass parameter, to exponentially huge values, while the radial horizon function decreases exponentially.

At an exponentially tiny value, the horizon function goes through a minimum (in absolute value), whereupon the spacetime collapses, the conformal factor shrinking to an exponentially tiny scale. This is consistent with the argument of [11] that inflation leads to collapse, not a null singularity, if the black hole continues to accrete, as is ensured in the present solutions by the presumption of conformal time-translation invariance (self-similarity).

During collapse, the angular motion of the infalling streams grows (the tetrad is chosen to align with the principal frame, so the angular directions are orthogonal to the principal null directions). The solutions presented here break down when the angular motion of the streams becomes comparable to the radial motion, but this occurs only when the spacetime has collapsed to an exponentially tiny scale. That the solutions fail when angular motions become large is consistent with the physical idea that conformal separability can persist only so long as the energy-momentum is focused along the radial direction. The result is also consistent with the slowly rotating black hole solutions of [17].

Until angular motions become important, the radial components of the collisionless energy-momentum dominate the angular components, satisfying the hierarchy of inequalities

$$|T_{xx}| \gg |T_{xy}| \gg |T_{yy}|, \tag{162}$$

where $x$ signifies radial ($x$-$t$) directions, and $y$ angular ($y$-$\phi$) directions. The radial Einstein equations are unaffected by the subdominant radial-angular Einstein equations, which in turn are unaffected by the sub-sub-dominant angular Einstein equations.

Conformal separability imposes a corresponding hierarchy of boundary conditions on the collisionless ingoing and outgoing streams incident on the inner horizon. The indispensible boundary condition is set by requiring that the radial Einstein equations be satisfied. Conformal separability requires that the accretion flow be monopole in the sense that the densities of ingoing and outgoing streams must be independent of latitude, Eq. (158). It makes physical sense that conformal separability would require this high degree of symmetry on the incident accretion flow. If the accretion rate were different at different latitudes, then the spacetime would collapse faster where the accretion rate is higher, destroying the symmetry.

The evolution of the conformal factor and radial horizon function during inflation and collapse is determined by two small constant parameters $u$ and $v$ set by the incident densities of ingoing and outgoing streams, which are proportional to $u \mp v$. Positivity of both densities, coupled with the requirement that the black hole expand as it accretes, imposes $u > v > 0$. The case $v = 0$ corresponds to the stationary (homogeneous) approximation of [32–34].

If the subdominant radial-angular Einstein equations are required to be satisfied, then conformal separability requires that the angular components of the momenta of the incident ingoing and outgoing streams have Hamilton-Jacobi parameters satisfying Eq. (129).

A limitation of the required angular motions is that they cannot be accomplished by collisionless streams that fall freely from outside the outer horizon. Particles that fall from outside are necessarily ingoing at the outer horizon. This condition excludes half the phase space available to freely-falling particles, and makes it impossible to fulfill the required angular conditions at the inner horizon. Thus, if the angular conditions are imposed, then the ingoing and outgoing streams must be regarded as being delivered *ad hoc* to just above the inner horizon.

If the sub-sub-dominant purely angular Einstein equations are required to be satisfied, then conformal separability requires that the mean squares of the angular components of the collisionless momenta be isotropic, which can be contrived.

## ACKNOWLEDGMENTS

I thank Gavin Polhemus for numerous conversations that contributed materially to the development of the ideas herein, and Carlos Herdeiro for bringing attention to the possibility of a conformal Killing tensor. This work was supported by NSF AST-0708607.

## APPENDIX A: REDUCTION OF THE LINE-ELEMENT

This Appendix shows how the assumptions of conformally stationarity, axisymmetry, and conformal separability lead to the line-element (3) and associated vierbein (8) adopted in the text.

By conformal separability is meant conditions on the vierbein $e_m{}^\mu$ and electromagnetic potential $A_m$ that follow from assuming that the action $S$ governing the free motion of neutral or charged particles separates as a sum of terms each depending only on a single coordinate, Eq. (28), and that the left-hand side of the resulting Hamilton-Jacobi equation, after multiplication by an arbitrary overall factor, separates ''in the simplest possible way,'' assumption III of [31], as a sum of terms each depending only on either radius $x$ or angle $y$.

Conformal separability differs from separability in that the resulting Hamilton-Jacobi equation separates exactly only for massless particles, $m = 0$. Consequently the spacetime has a conformal Killing tensor.

The assumptions of conformal stationarity and axisymmetry imply that the canonical momenta $\pi_t$ and $\pi_\phi$ are constants, Eqs. (22). The assumption of conformal separability implies that the canonical momenta $\pi_x = \partial S/\partial x$ and $\pi_y = \partial S/\partial y$ are functions only of $x$ and $y$ respectively.

Let $\hat{e}_m{}^\mu$ and $\hat{A}_m$ denote the vierbein coefficients and tetrad-frame electromagnetic potential with an overall conformal factor $\rho$ factored out:

$$\hat{e}_m{}^\mu \equiv \rho e_m{}^\mu, \qquad \hat{A}_m \equiv \rho A_m. \tag{A1}$$

The conformal factor $\rho$ here could be any arbitrary function of all the coordinates. In terms of the scaled vierbein $\hat{e}_m{}^\mu$ and electromagnetic potential $\hat{A}_m$, the Hamilton-Jacobi Eq. (17) is

$$\eta^{mn}(\hat{e}_m{}^\mu \pi_\mu - q\hat{A}_m)(\hat{e}_n{}^\nu \pi_\nu - q\hat{A}_n) = -m^2\rho^2. \tag{A2}$$

Following assumption III of [31], assume that the left-hand side of Eq. (A2) separates in the simplest possible way as a sum of terms each of which depends only on $x$ or only on $y$. Since $\pi_t$ and $\pi_\phi$ are constants, while $\pi_x$ and $\pi_y$ are, respectively, functions of $x$ and $y$, the assumption implies that

$$\text{for each } m, \begin{cases} \text{either} & \hat{e}_m{}^\mu \text{ for all } \mu, \text{ and } \hat{A}_m, \text{ are functions of } x \text{ only, and } \hat{e}_m{}^y = 0, \\ \text{or} & \hat{e}_m{}^\mu \text{ for all } \mu, \text{ and } \hat{A}_m, \text{ are functions of } y \text{ only, and } \hat{e}_m{}^x = 0. \end{cases} \tag{A3}$$

The case that matches the $\Lambda$-Kerr-Newman geometry, which provides the boundary conditions for the inflationary solutions considered in this paper and its companions, is

$$\text{the } \begin{Bmatrix} \text{top} \\ \text{bottom} \end{Bmatrix} \text{condition of (A3) holds for } \begin{Bmatrix} m = x & \text{and} & t \\ m = y & \text{and} & \phi \end{Bmatrix}. \tag{A4}$$

Thus conformal separability ''in the simplest possible way'' consistent with $\Lambda$-Kerr-Newman requires that

$$\hat{e}_x{}^y = \hat{e}_t{}^y = \hat{e}_y{}^x = \hat{e}_\phi{}^x = 0. \tag{A5}$$

Given the conformal separability conditions (A5), the vierbein coefficients $\hat{e}_t{}^x$ and $\hat{e}_\phi{}^y$ can be transformed to zero by a tetrad gauge transformation consisting of a Lorentz boost by velocity $\hat{e}_t{}^x/\hat{e}_x{}^x$ between tetrad axes $\gamma_x$ and $\gamma_t$, and a (commuting) spatial rotation by angle $\tan^{-1}(\hat{e}_\phi{}^y/\hat{e}_y{}^y)$ between tetrad axes $\gamma_y$ and $\gamma_\phi$. Thus without loss of generality,

$$\hat{e}_t{}^x = \hat{e}_\phi{}^y = 0. \tag{A6}$$

The gauge conditions (A6) having been effected, the vierbein coefficients $\hat{e}_x{}^t$, $\hat{e}_y{}^t$, $\hat{e}_x{}^\phi$, and $\hat{e}_y{}^\phi$ can be eliminated by coordinate gauge transformations $t \to t'$ and $\phi \to \phi'$ defined by

$$dt = dt' + \frac{\hat{e}_x{}^t}{\hat{e}_x{}^x} dx + \frac{\hat{e}_y{}^t}{\hat{e}_y{}^y} dy, \qquad d\phi = d\phi' + \frac{\hat{e}_x{}^\phi}{\hat{e}_x{}^x} dx + \frac{\hat{e}_y{}^\phi}{\hat{e}_y{}^y} dy. \tag{A7}$$

Eqs. (A7) are integrable because $\hat{e}_x{}^\mu$ and $\hat{e}_y{}^\mu$ are, respectively, functions of $x$ and $y$ only. The transformations (A7) of $t$ and $\phi$ are admissible because they preserve the Killing vectors $\partial/\partial t$ and $\partial/\partial \phi$,

$$\frac{\partial}{\partial t}\bigg|_{x,y,\phi} = \frac{\partial}{\partial t'}\bigg|_{x,y,\phi}, \qquad \frac{\partial}{\partial \phi}\bigg|_{x,t,y} = \frac{\partial}{\partial \phi'}\bigg|_{x,t,y}. \tag{A8}$$

Thus without loss of generality

$$\hat{e}_x{}^t = \hat{e}_y{}^t = \hat{e}_x{}^\phi = \hat{e}_y{}^\phi = 0. \tag{A9}$$

Finally, coordinate transformations of the $x$ and $y$ coordinates

$$x \to x', \qquad y \to y', \tag{A10}$$

can be chosen such that $\hat{e}_x{}^x$ and $\hat{e}_y{}^y$ satisfy

$$\hat{e}_x{}^x \hat{e}_t{}^t = \hat{e}_y{}^y \hat{e}_\phi{}^\phi = 1. \tag{A11}$$

The conformal separability conditions (A3) with (A4), which imply conditions (A5), coupled with the gauge conditions (A6), (A9), and (A11), lead to the line-element (3) and vierbein (8) adopted in this paper.

## APPENDIX B: TETRAD-FRAME CONNECTIONS, EINSTEIN AND WEYL TENSORS

This appendix gives expressions for the tetrad-frame connections, and Einstein and Weyl tensors, in the case that the conformal separability conditions (29) hold, and the conformal factor $\rho$ is any arbitrary function not only of $x$ and $y$, but also of $t$ and $\phi$. There are 18 nonvanishing tetrad-frame connections $\Gamma_{klm}$ (counting $\Gamma_{klm} = -\Gamma_{lkm}$ as one), comprising 8 distinct connections,

$$\Gamma_{yxy} = \Gamma_{\phi x\phi} = \partial_x \ln\rho, \tag{B1a}$$
$$\Gamma_{txx} = \Gamma_{yty} = \Gamma_{\phi t\phi} = \partial_t \ln\rho, \tag{B1b}$$
$$\Gamma_{yxx} = \Gamma_{tyt} = \partial_y \ln\rho, \tag{B1c}$$
$$\Gamma_{\phi xx} = \Gamma_{t\phi t} = \Gamma_{y\phi y} = \partial_\phi \ln\rho, \tag{B1d}$$
$$\Gamma_{xt\phi} = \Gamma_{x\phi t} = \Gamma_{\phi tx} = \frac{\sqrt{\Delta_y}}{2\rho\sigma^2}\frac{d\omega_x}{dx}, \tag{B1e}$$
$$\Gamma_{yt\phi} = \Gamma_{y\phi t} = \Gamma_{t\phi y} = \frac{\sqrt{-\Delta_x}}{2\rho\sigma^2}\frac{d\omega_y}{dy}, \tag{B1f}$$
$$\Gamma_{txt} = \partial_x \ln\rho + \frac{\sigma^2}{\rho}\frac{\partial\sqrt{-\Delta_x}/\sigma^2}{\partial x}, \tag{B1g}$$
$$\Gamma_{\phi y\phi} = \partial_y \ln\rho + \frac{\sigma^2}{\rho}\frac{\partial\sqrt{\Delta_y}/\sigma^2}{\partial y}, \tag{B1h}$$
$$\Gamma_{xty} = \Gamma_{xyt} = \Gamma_{tyx} = \Gamma_{xy\phi} = \Gamma_{x\phi y} = \Gamma_{y\phi x} = 0. \tag{B1i}$$

Since the tetrad-frame connections $\Gamma_{klm}$ constitute a set of 4 bivectors, it is possible to combine the connections into complex combinations, but no additional insightful result emerges from those combinations.

The 10 tetrad-frame components $G_{kl}$ of the Einstein tensor satisfy

$$\frac{2}{\rho}D_k\partial_l\rho + G_{kl} + \eta_{kl}\left(\frac{1}{3}R - \frac{1}{\rho^2}\partial_m\rho\partial^m\rho\right) = \begin{cases} 4C_x - 2C_y + \Gamma^2_{xt\phi} + \Gamma^2_{yt\phi} & xx \\ -4C_x + 2C_y - \Gamma^2_{xt\phi} + \Gamma^2_{yt\phi} & tt \\ -4C_y + 2C_x + \Gamma^2_{xt\phi} + \Gamma^2_{yt\phi} & yy \\ -4C_y + 2C_x - \Gamma^2_{xt\phi} + \Gamma^2_{yt\phi} & \phi\phi \\ -6\Gamma_{xt\phi}\Gamma_{yt\phi} & xy \\ -2(W_x - W_y) & t\phi \\ 0 & xt,\ x\phi,\ yt,\ y\phi \end{cases} \tag{B2}$$

where $R$ is the Ricci scalar, and $C_x$, $C_y$, $W_x$, and $W_y$ are related to the Weyl tensor as described immediately below. The Weyl tensor can be thought of as a matrix with bivector indices, and as such has a natural complex structure [39], embodied in the complexified, self-dual Weyl tensor $\tilde{C}_{klmn}$ defined in terms of the usual Weyl tensor $C_{klmn}$ by

$$\tilde{C}_{klmn} \equiv \frac{1}{4}\left(\delta^p_k\delta^q_l + \frac{i}{2}\varepsilon_{kl}{}^{pq}\right)\left(\delta^r_m\delta^s_n + \frac{i}{2}\varepsilon_{mn}{}^{rs}\right)C_{pqrs}, \tag{B3}$$

where $\varepsilon_{klmn}$ is the totally antisymmetric tensor, normalized here to $\varepsilon^{klmn} = [klmn]$ in an orthonormal tetrad frame. The distinct components of the complexified Weyl tensor form a $3 \times 3$ traceless, symmetric, complex matrix, whose 5 distinct complex components form objects of spin 0, $\pm1$, and $\pm2$. The spin-0 part of the Weyl tensor is

$$C = \tilde{C}_{xtxt} = -\frac{1}{2}\tilde{C}_{xyxy} = -\frac{1}{2}\tilde{C}_{xtx\phi}. \tag{B4}$$

The real and imaginary parts of the spin-0 Weyl tensor are commonly called its polar ($p$) and axial ($a$) parts, the polar part remaining unchanged, while the axial part changes sign, under a flip $\gamma_\phi \to -\gamma_\phi$ of the azimuthal tetrad axis:

$$C = C^{(p)} + iC^{(a)}. \tag{B5}$$

In the present case, the polar spin-0 component of the Weyl tensor is

$$C^{(p)} = C_x + C_y, \tag{B6}$$

where

$$\rho^2 C_x = \frac{\sigma^2}{12}\frac{\partial}{\partial x}\left[\sigma^2 \frac{\partial(\Delta_x/\sigma^4)}{\partial x}\right] - \frac{1}{6}\left(\frac{1}{\sigma^2}\frac{d\omega_x}{dx}\right)^2 \Delta_y, \tag{B7a}$$

$$\rho^2 C_y = \frac{\sigma^2}{12}\frac{\partial}{\partial y}\left[\sigma^2 \frac{\partial(\Delta_y/\sigma^4)}{\partial y}\right] - \frac{1}{6}\left(\frac{1}{\sigma^2}\frac{d\omega_y}{dy}\right)^2 \Delta_x, \tag{B7b}$$

while the axial spin-0 component is given by

$$\rho^2 C^{(a)} = \frac{\sigma^2}{4}\left[\frac{d\omega_y}{dy}\frac{\partial(\Delta_x/\sigma^4)}{\partial x} - \frac{d\omega_x}{dx}\frac{\partial(\Delta_y/\sigma^4)}{\partial y}\right]. \tag{B8}$$

The only other nonvanishing component of the Weyl tensor is the spin-1 component

$$\tilde{C}_{xtx\phi} = W_x + W_y, \tag{B9}$$

where

$$\rho^2 W_x = \frac{\sqrt{-\Delta_x \Delta_y}}{4}\frac{\partial}{\partial x}\left(\frac{1}{\sigma^2}\frac{d\omega_x}{dx}\right), \tag{B10a}$$

$$\rho^2 W_y = \frac{\sqrt{-\Delta_x \Delta_y}}{4}\frac{\partial}{\partial y}\left(\frac{1}{\sigma^2}\frac{d\omega_y}{dy}\right). \tag{B10b}$$

In the spacetimes presented in this paper and its companions, the spin-1 component (B9) always vanishes.

## APPENDIX C: ELECTROVAC SOLUTIONS

The standard electrovac solutions can be derived from the assumptions of strict stationarity, axisymmetry, and strict separability as follows. As shown by [31], the line-element takes the form (3) with a separable conformal factor $\rho_s$, Eq. (30).

Given strict stationary ($\upsilon = 0$) and separability, the Einstein component $G_{xy}$, which has zero electrovac source, is

$$\rho_s^2 G_{xy} = -\frac{3}{2}\sqrt{-\Delta_x \Delta_y}\frac{\partial^2 \ln(\rho_s^2/\sigma^2)}{\partial x \partial y}. \tag{C1}$$

Homogeneous solution of this equation implies the form (72) of the conformal factor $\rho_s$. At this point the constants $g_0$ and $g_1$ can be adjusted arbitrarily without affecting either $\rho_x$ or $\rho_y$: the overall normalization of $g_0$ and $g_1$ is cancelled by the normalizing factor of $1/\sqrt{f_0 g_1 + f_1 g_0}$ in $\rho_x$ and $\rho_y$, and the relative sizes of $g_0$ and $g_1$ can be changed by adjusting an arbitrary constant in the split between $\rho_x^2$ and $\rho_y^2$.

The Einstein component $G_{t\phi}$, which also has zero electrovac source, is

$$\rho_s^2 G_{t\phi} = -\frac{\sqrt{-\Delta_x \Delta_y}}{2\rho_s^2}\left[\frac{\partial}{\partial x}\left(\frac{\rho_s^2}{\sigma^2}\frac{d\omega_x}{dx}\right) - \frac{\partial}{\partial y}\left(\frac{\rho_s^2}{\sigma^2}\frac{d\omega_y}{dy}\right)\right], \tag{C2}$$

which, given the expression (72) for the conformal factor $\rho_s$, reduces to

$$\rho_s^2 G_{t\phi} = \frac{1}{2}\frac{\sqrt{-\Delta_x \Delta_y}}{\sigma^2}\left[\frac{d\omega_x}{dx}\frac{d\ln(f_0 + f_1\omega_x)}{dx} - \frac{d\omega_y}{dy}\frac{d\ln(f_1 + f_0\omega_x)}{dy}\right]. \tag{C3}$$

Homogeneous solution of this equation can be accomplished by separation of variables, setting each of the two terms inside square brackets, the first of which is a function only of $x$, while the second is a function only of $y$, to the same separation constant $2f_2$. The result is

$$\frac{d\omega_x}{dx} = 2\sqrt{(f_0 + f_1\omega_x)\left[g_0 + \frac{1}{f_0}(f_1 g_0 + f_2)\omega_x\right]}, \tag{C4a}$$

$$\frac{d\omega_y}{dy} = 2\sqrt{(f_1 + f_0\omega_y)\left[g_1 + \frac{1}{f_1}(f_0 g_1 + f_2)\omega_y\right]}, \tag{C4b}$$

for some constants $g_0$ and $g_1$, which can be taken without loss of generality to equal those in the conformal factor (72).

The Einstein components $G_{xx} + G_{tt}$ and $G_{yy} - G_{\phi\phi}$, which too have zero electrovac source, are

$$\rho_s^2(G_{xx} + G_{tt}) = -\frac{2\Delta_x}{\sigma^2}\left[\rho_s\frac{\partial}{\partial x}\left(\sigma^2\frac{\partial(1/\rho_s)}{\partial x}\right) + \frac{1}{4\sigma^2}\left(\frac{d\omega_y}{dy}\right)^2\right], \tag{C5a}$$

$$\rho_s^2(G_{yy} - G_{\phi\phi}) = \frac{2\Delta_y}{\sigma^2}\left[\rho_s\frac{\partial}{\partial y}\left(\sigma^2\frac{\partial(1/\rho_s)}{\partial y}\right) + \frac{1}{4\sigma^2}\left(\frac{d\omega_x}{dx}\right)^2\right], \tag{C5b}$$

which with the conformal factor $\rho_s$ given by Eq. (72) and $d\omega_x/dx$ and $d\omega_y/dy$ given by Eqs. (C4) reduce to

$$\rho_s^2(G_{xx} + G_{tt}) = -2\Delta_x\frac{(f_2 + f_0 g_1 + f_1 g_0)(f_0 + f_1\omega_x)^2}{f_0 f_1 \sigma^4}, \tag{C6a}$$

$$\rho_s^2(G_{yy} - G_{\phi\phi}) = 2\Delta_y\frac{(f_2 + f_0 g_1 + f_1 g_0)(f_1 + f_0\omega_y)^2}{f_0 f_1 \sigma^4}. \tag{C6b}$$

These vanish provided that the constant $f_2$ satisfies

$$f_2 = -(f_0 g_1 + f_1 g_0). \tag{C7}$$

Inserting this value into Eqs. (C4) yields Eqs. (73).

The four off-diagonal Einstein components $G_{xt}$, $G_{yt}$, $G_{x\phi}$, and $G_{y\phi}$ vanish identically.

Solution of the Einstein equations for the remaining two Einstein components $G_{xx} - G_{tt}$ and $G_{yy} + G_{\phi\phi}$ was already discussed in Sec. VIII D. Inserting the homogeneous solutions (90c) and (90d) into the defining Eqs. (87c) and (87d) for $Y_x$ and $Y_y$, and setting $U_x = U_y = 0$ (which eliminates inflation), yields differential equations for the radial and angular horizon functions $\Delta_x$ and $\Delta_y$, solution of which, with appropriate boundary conditions, recovers the Kerr solution.

Solutions including the energy-momentum of a static electromagnetic field fall out with little extra work. With appropriate boundary conditions, this is the Kerr-Newman solution. Electrovac solutions have $G_{tt} = -G_{xx}$ and $G_{\phi\phi} = G_{yy}$. Electrovac solutions with $G_{yy} = G_{xx}$, as is true for a static radial electromagnetic field, are found by taking the difference of Eqs. (88) and separating that difference in the pattern of Eq. (89). The solution is a sum of a homogeneous solution and a particular solution

$$Y_x = \frac{2(Q_\bullet^2 + \mathcal{Q}_\bullet^2)(f_0 + f_1\omega_x)^2}{d\omega_x/dx}, \qquad Y_y = 0. \tag{C8}$$

Inserting Eqs. (C8) into the Einstein expressions (88) yields Einstein components that have precisely the form $G_{mn} = [(Q_\bullet^2 + \mathcal{Q}_\bullet^2)/\rho_s^4]\mathrm{diag}(1, -1, 1, 1)$ of the tetrad-frame energy-momentum tensor of a static radial electromagnetic field.

Similarly, solutions including vacuum energy, which has $G_{yy} = -G_{xx}$, can be found by separating the sum of Eqs. (88) in the pattern of Eq. (89). A particular solution is

$$Y_x = \frac{2\Lambda}{f_1^2 d\omega_x/dx}, \qquad Y_\phi = \frac{2\Lambda\omega_y^2}{f_1^2 d\omega_x/dx}. \tag{C9}$$

Inserting Eqs. (C9) into the Einstein expressions (88) yields Einstein components that have precisely the form of a cosmological constant, $G_{mn} = -\Lambda\eta_{mn}$.

Solving Eqs. (87c) and (87d) with $Y_x$ and $Y_y$ given by a sum of the homogeneous, Eqs. (90c) and (90d), electromagnetic, Eq. (C8), and vacuum, Eq. (C9), contributions, yields the general electrovac solution for the radial and angular horizon functions $\Delta_x$ and $\Delta_y$,

$$\Delta_x = (f_0 + f_1\omega_x)\left[(k_0 + k_1\omega_x) - \frac{2M_\bullet\sqrt{(f_0 + f_1\omega_x)(g_0 - g_1\omega_x)}}{(f_0 g_1 + f_1 g_0)^{3/2}} + \frac{(Q_\bullet^2 + \mathcal{Q}_\bullet^2)(f_0 + f_1\omega_x)}{f_0 g_1 + f_1 g_0}\right] - \frac{\Lambda(g_0 - g_1\omega_x)}{3f_1(f_0 g_1 + f_1 g_0)^2}, \tag{C10a}$$

$$\Delta_y = (f_1 + f_0\omega_y)\left[(k_1 + k_0\omega_y) - \frac{2\mathcal{N}_\bullet\sqrt{(f_1 + f_0\omega_y)(g_1 - g_0\omega_y)}}{(f_0 g_1 + f_1 g_0)^{3/2}}\right] + \frac{\Lambda\omega_y(g_1 - g_0\omega_y)}{3f_1(f_0 g_1 + f_1 g_0)^2}, \tag{C10b}$$

where $k_0$ and $k_1$ are arbitrarily adjustable constants arising from the freedom of choice in the constants $h_0$ and $h_1$ of the homogeneous solution. The constant $M_\bullet$ in the expression (C10a) for $\Delta_x$ is the black hole's mass. The constant $\mathcal{N}_\bullet$ in the expression (C10b) for $\Delta_y$ is the NUT parameter [1,40–42], which is to the mass $M_\bullet$ as magnetic charge $\mathcal{Q}_\bullet$ is to electric charge $Q_\bullet$.

The electrovac Weyl tensor (B5) has only a spin-0 component, and is

$$C = -\frac{1}{(\rho_x - i\rho_y)^3}\left(M_\bullet + i\mathcal{N}_\bullet + \frac{Q_\bullet^2 + \mathcal{Q}_\bullet^2}{\rho_x + i\rho_y}\right), \tag{C11}$$

with $\rho_x$ and $\rho_y$ given by Eqs. (72b).

## APPENDIX D: INTEGRALS ALONG THE PATH OF A PARTICLE

This appendix derives conditions under which integrals along the path of a particle can be deemed small, in the conformally stationary limit of small accretion rate. The results confirm that the tetrad-frame momentum $p_k$ and the density $N$ predicted by the Hamilton-Jacobi equations for a massive particle are accurate, by demonstrating that the integrals of Eqs. (39) and (63) are adequately small.

Consider an integral of the form

$$I = \int |\Delta_x|^\alpha (\rho/\rho_s)^\beta d\lambda. \tag{D1}$$

Along the path of a particle, $d\lambda = (\rho^2/|P_x|)dx$, and $\rho/\rho_s = e^{vt-\xi} = [(u \mp v)/(U_x \mp v)]^{1/2}$ from Eqs. (96) and (114). The integration interval $dx$ may be converted to either $d\Delta_x$ or $dU_x$ using either of Eqs. (93). The integral (D1) thus becomes

$$I = \int \frac{|\Delta_x|^{1+\alpha}}{|P_x|}\left(\frac{u \mp v}{U_x \mp v}\right)^{1+\beta/2} \frac{d\ln|\Delta_x|}{\Delta_x' - 3U_x} = \int \frac{|\Delta_x|^{1+\alpha}}{|P_x|}\left(\frac{u \mp v}{U_x \mp v}\right)^{1+\beta/2} \frac{dU_x}{2(U_x^2 - v^2)}. \tag{D2}$$

The horizon function $\Delta_x$ is given as a function of $U_x$ by Eq. (99).

Inflation ignites when $|\Delta_x| \sim v$, and the integral (D1) may be counted from this point. At the beginning of inflation, before $|\Delta_x|$ has been driven to an exponentially small value, $U_x$ changes little from its initial value of $u$, and then the middle expression of Eqs. (D2) yields

$$I \approx \frac{|\Delta_x|^{1+\alpha}}{(1+\alpha)|P_x|\Delta'_x} \sim v^{1+\alpha}. \tag{D3}$$

This integral is small in the conformally stationary limit $v \to 0$ provided that $\alpha > -1$, but as to whether an integral in an actual situation can be judged small will depend on additional factors of $v$ that the integral may be multiplied by.

By the time that $U_x$ has reached of order unity times its initial value $u$, the horizon function $|\Delta_x|$ has become exponentially small. In this regime further contribution to the integral is exponentially small provided that

$$\alpha > -1. \tag{D4}$$

During collapse, both $|\Delta_x|$ and $U_x$ increase exponentially ($|\Delta_x|$ from an exponentially small starting point). As long as $|\Delta_x| \lesssim 1$, further contribution to the integral remains exponentially small provided that

$$\beta > -4. \tag{D5}$$

At the end of collapse, there is a regime where $|\Delta_x| \gtrsim 1$ and $|P_x| \sim \sqrt{-\Delta_x}$ but $|\Delta_x| \ll U_x$, while $U_x$ is exponentially huge. Here further contribution to the integral remains exponentially small provided that

$$\text{either} \quad \beta \geq -3 + 2\alpha \quad \text{or} \quad \beta > -4. \tag{D6}$$

In the case of Eq. (39) for $dp_k/d\lambda$ there are two terms. The first has $\alpha = 0$ and $\beta = -3$. The second has $\alpha = 0$ and $\beta = -1$, except that the $x$ component has $\alpha = -1/2$ and $\beta = -1$ when $|\Delta_x| \lesssim 1$. Both terms have prefactors proportional to $v$ during early inflation (including the term proportional to $\partial(\rho^2 - \rho_y^2)\partial y$). For both terms, the main contribution to the integral is from early inflation, Eq. (D3), which, multiplied by the $v$ prefactor yields $\sim v^2$ for the first term, and $\sim v^{3/2}$ for the second. The ratio of the integral of $dp_k/d\lambda$ to $p_k$ itself is, since $p_x \propto 1/\sqrt{|\Delta_x|}$ when $|\Delta_x| \lesssim 1$,

$$\frac{\Delta p_k}{p_k} \equiv \frac{\int dp_k/d\lambda d\lambda}{p_k} \sim v^2. \tag{D7}$$

Both terms satisfy the conditions (D4)–(D6) so the contributions to the integrals after early inflation are exponentially small.

In the case of Eq. (63) for $D_k n^k$, the exponents are $\alpha = 0$ and $\beta = -2$, and the prefactor is proportional to $v$ during early inflation. Again, the main contribution to the integral is during early inflation, giving

$$\Delta \ln N \equiv \frac{\int dp_k/d\lambda d\lambda}{p_k} \sim v^2. \tag{D8}$$

Again the conditions (D4)–(D6) are satisfied, so the contribution to the integral after early inflation is exponentially small.

---


[1] Hans Stephani, Dietrich Kramer, Malcolm MacCallum, Cornelius Hoenselaers, and Eduard Herlt, *Exact Solutions of Einstein's Field Equations* (Cambridge University Press, Cambridge, England, 2003), 2nd ed..
[2] Andrew J. S. Hamilton, following Article, Phys. Rev. D **84**, 124057 (2011).
[3] Andrew J. S. Hamilton and Gavin Polhemus, preceding Article, Phys. Rev. D **84**, 124055 (2011).
[4] Andrew J. S. Hamilton, Mathematica notebook on rotation inflationary spacetimes. http://jila.colorado.edu/~ajsh/rotatinginflationary/rotatinginflationary.nb, 2011.
[5] Karl Schwarzschild, *Sitzungsberichte der Preussische Akademie der Wissenschaften zu Berlin, Klasse für Mathematik, Physik, und Technik* (1916), Vol. 1916, pp. 189–196.
[6] Karl Schwarzschild, Gen. Relativ. Gravit. **35**, 951 (2003). English translation of [5].
[7] Roy P. Kerr, Phys. Rev. Lett. **11**, 237 (1963).
[8] Brandon Carter, Phys. Rev. **174**, 1559 (1968).
[9] E. Poisson and W. Israel, Phys. Rev. D **41**, 1796 (1990).
[10] Roger Penrose, in *Battelle Rencontres: 1967 Lectures in Mathematics and Physics*, edited by Cécile de Witt-Morette and John A. Wheeler (W. A. Benjamin, New York, 1968), pp. 121–235.
[11] Andrew J. S. Hamilton and Pedro P. Avelino, Phys. Rep. **495**, 1 (2010).
[12] C. Barrabès, W. Israel, and E. Poisson, Classical Quantum Gravity **7**, L273 (1990).
[13] Amos Ori, Phys. Rev. Lett. **68**, 2117 (1992).
[14] Amos Ori, Phys. Rev. Lett. **83**, 5423 (1999).
[15] Patrick R. Brady and Chris M. Chambers, Phys. Rev. D **51**, 4177 (1995).
[16] Patrick R. Brady, Serge Droz, and Sharon M. Morsink, Phys. Rev. D **58**, 084034 (1998).
[17] Andrew J. S. Hamilton, Classical Quantum Gravity **26**, 165006 (2009).
[18] J. S. F. Chan, K. C. K. Chan, and Robert B. Mann, Phys. Rev. D **54**, 1535 (1996).
[19] Andrew J. S. Hamilton and Scott E. Pollack, Phys. Rev. D **71**, 084032 (2005).
[20] Richard H. Price, Phys. Rev. **5**, 2419 (1972).
[21] Mihalis Dafermos and Igor Rodnianski, Inventiones Mathematicae **162**, 381 (2005).
[22] Lior M. Burko, Phys. Rev. D **66**, 024046 (2002).
[23] Lior M. Burko, Phys. Rev. Lett. **90**, 121101 (2003).

[24] Igor D. Novikov and Alexei A. Starobinskii, JETP **51**, 1 (1980).
[25] Andrei V. Frolov, Kristjan R. Kristjansson, and Larus Thorlacius, Phys. Rev. D **73**, 124036 (2006).
[26] C. Angulo Santacruz, D. Batic, and M. Nowakowski, J. Math. Phys. (N.Y.) **51**, 082504 (2010).
[27] G. Chapline, E. Hohlfeld, R. B. Laughlin, and D. I. Santiago, Int. J. Mod. Phys. **A18**, 3587 (2003).
[28] Pawel O. Mazur and Emil Mottola, Proc. Natl. Acad. Sci. U.S.A. **101**, 9545 (2004).
[29] Aharon Davidson and Ilya Gurwich, Int. J. Mod. Phys. D **19**, 2345 (2010).
[30] Carlos Barcelo, Stefano Liberati, Sebastiano Sonego, and Matt Visser, Phys. Rev. D **77**, 044032 (2008).
[31] Brandon Carter, Commun. Math. Phys. **10**, 280 (1968).
[32] Lior M. Burko, Ann. Isr. Phys. Soc. **13**, 212 (1997).
[33] Lior M. Burko, Phys. Rev. D **58**, 084013 (1998).
[34] Lior M. Burko, Phys. Rev. D **59**, 024011 (1999).
[35] Sean M. Carroll, *Spacetime and Geometry: An Introduction to General Relativity* (Addison-Wesley, Reading, MA, 2004).
[36] Steven A Balbus and John F. Hawley, Rev. Mod. Phys. **70**, 1 (1998).
[37] Steven A. Balbus, Annu. Rev. Astron. Astrophys. **41**, 555 (2003).
[38] Charles W. Misner and David H. Sharp, Phys. Rev. **136**, B571 (1964).
[39] Chris Doran and Anthony Lasenby, *Geometric Algebra for Physicists* (Cambridge University Press, Cambridge, England, 2003).
[40] A. H. Taub, Ann. Math. **53**, 472 (1951).
[41] E. T. Newman, L. A. Tamburino, and T. Unti, J. Math. Phys. (N.Y.) **4**, 915 (1963).
[42] Valeria Kagramanova, Jutta Kunz, Eva Hackmann, and Claus Lammerzahl, Phys. Rev. D **81**, 124044 (2010).