PHYSICAL REVIEW D **84,** 124046 (2011)

Massive cosmologies

G. D'Amico,[1] C. de Rham,[2,3] S. Dubovsky,[1] G. Gabadadze,[1] D. Pirtskhalava,[1] and A. J. Tolley[3]

[1]*Center for Cosmology and Particle Physics, Department of Physics, New York University, New York, New York, 10003, USA*

[2]*Départment de Physique Théorique and Center for Astroparticle Physics, Université de Genève, 24 Quai E. Ansermet, CH-1211 Genève*

[3]*Department of Physics, Case Western Reserve University, Euclid Ave, Cleveland, Ohio, 44106, USA*

(Received 18 September 2011; published 23 December 2011)

We explore the cosmological solutions of a recently proposed extension of general relativity with a Lorentz-invariant mass term. We show that the same constraint that removes the Boulware-Deser ghost in this theory also prohibits the existence of homogeneous and isotropic cosmological solutions. Nevertheless, within domains of the size of inverse graviton mass we find approximately homogeneous and isotropic solutions that can well describe the past and present of the Universe. At energy densities above a certain crossover value, these solutions approximate the standard Friedmann-Robertson-Walker evolution with great accuracy. As the Universe evolves and density drops below the crossover value the inhomogeneities become more and more pronounced. In the low-density regime each domain of the size of the inverse graviton mass has essentially non-Friedmann-Robertson-Walker. cosmology. This scenario imposes an upper bound on the graviton mass, which we roughly estimate to be an order of magnitude below the present-day value of the Hubble parameter. The bound becomes especially restrictive if one utilizes an exact self-accelerated solution that this theory offers. Although the above are robust predictions of massive gravity with an explicit mass term, we point out that if the mass parameter emerges from some additional scalar field condensation, the constraint no longer forbids the homogeneous and isotropic cosmologies. In the latter case, there will exist an extra light scalar field at cosmological scales, which is screened by the Vainshtein mechanism at shorter distances.

DOI: 10.1103/PhysRevD.84.124046 PACS numbers: 04.50.Kd, 98.80.−k

I. INTRODUCTION AND SUMMARY

The purpose of this work is to study the cosmology of general relativity (GR) with an explicit Lorentz-invariant mass term (massive gravity or massive GR). Historically, it has been difficult to construct a nonlinear theory of massive gravity that would describe no more than the 5 degrees of freedom, required for the massive spin-2 state by the representations of the Poincaré group. This situation was recently transformed by the proposal in Ref. [1] of a theory of massive gravity with 5° of freedom.

This theory was shown to be free of the sixth degree of freedom (the so-called Boulware-Deser (BD) ghost [2]) to all orders in the decoupling limit (DL) in Refs. [1,3], where it was also shown to be ghost-free away from the DL up to and including quartic order in nonlinearities [1]. These arguments were recently generalized in [4] to a complete nonlinear proof of the absence of ghosts, away from the decoupling limit, in the Arnowitt-Deser-Misner/Hamiltonian formalism. The same result can be reached in both the Stückelberg (see Ref. [5]) and helicity formalisms (see Ref. [6]). In each language, there exists a constraint that eliminates 1° of freedom which otherwise would have been the BD ghost.

In this work we will show that the very same constraint that eliminates the BD ghost in massive gravity [1], also forbids homogeneous and isotropic cosmological solutions [Friedmann-Robertson-Walker (FRW) cosmologies]. For solutions with FRW symmetries, the all-orders constraint can straightforwardly be seen to prohibit any time evolution, leaving Minkowski space as the only vacuum solution which is consistent with homogeneity and isotropy.

This result raises the question: How could the non-FRW cosmologies of massive gravity recover the FRW solutions of GR in the massless limit? For this one should rely upon the Vainshtein mechanism by which massive GR is expected to recover GR in the $m \to 0$ limit [7]. Although the original theory in which the Vainshtein mechanism was proposed contains a BD ghost, the mechanism itself seems to be universal, and has been established in other models where the BD ghost is not present [8–11]. Moreover, it was shown that the mechanism is operative for spherically symmetric solutions in the massive GR theories discussed here, at least for a certain choice of the three *a priori* free parameters of the theory (i.e., the graviton mass and two arbitrary constants), [12,13].

Assuming that the Vainshtein mechanism is at work, one would expect to find in massive GR cosmological solutions that are more and more homogeneous and isotropic as the value of the graviton mass is taken to zero. If this is the case, then the fact that massive gravity leads to non-FRW solutions will not immediately rule it out via observations, but rather just place a constraint on the magnitude of the mass of the graviton, to be consistent with known constraints on homogeneity and isotropy.

To see a close connection between the Vainshtein mechanism and cosmology, consider matter of constant density ρ stored in a sphere of radius R. The Vainshtein radius of such a source is

$$r_* = \left(\frac{r_g}{m^2}\right)^{1/3} = \left(\frac{\rho}{3M_{\rm Pl}^2 m^2}\right)^{1/3} R, \tag{1}$$

where $r_g = 2MG_N$ is the gravitational radius of the source of mass $M = \frac{4}{3}\pi R^3 \rho$, and $G_N = (8\pi M_{\rm Pl}^2)^{-1}$. Furthermore, it is useful to introduce a notion of a crossover energy density,

$$\rho_{\rm co} \equiv 3M_{\rm Pl}^2 m^2. \tag{2}$$

From (1) we conclude that any source with density above the crossover value (2), is characterized by the Vainshtein radius that is greater than the size of the source itself. For such sources gravity is close to that described by GR at distance scales $\ll r_*$ (the Vainshtein regime), however, deviates significantly from GR at distance scales $\gtrsim r_*$ (the vDVZ regime) [14]. Furthermore, sources with density below the crossover $\rho_{\rm co}$, are always in the vDVZ regime and their gravity differs significantly from GR.

Let us now apply these observations to cosmology. Suppose we took a snapshot of a Universe at a certain stage of its evolution when the matter in it had an average energy density ρ (averaged, say, at scales greater than the Hubble scale H^{-1} at that epoch; here we suppose that inflation, or an alternative early Universe framework, prepared such a state). Let us look into a $1/m$-size domain. An arbitrary Hubble patch in this domain (i.e., a patch enclosed by a sphere of radius $H^{-1} = (\rho/3M_{\rm Pl}^2)^{-1/2}$), that is far enough from the edges of the domain, is well within the Vainshtein regime as long as $\rho \gg \rho_{\rm co}$. Then, cosmology within such Hubble patches can be approximated by the standard FRW metric of GR with small corrections. Hence, for $\rho \gg \rho_{\rm co}$, and well within each $1/m$-size domain, the early Universe in massive GR would evolve as it does in GR, with some small corrections that vanish in the $m \to 0$ limit. Such an expansion in each of these Hubble patches will last until the size of the patch, H^{-1}, approaches the scale $\sim 1/m$, or equivalently, until ρ dilutes down to density of the order of $\rho_{\rm co}$. At scales larger than $1/m$ gravitational interactions are expected to be screened.

The same arguments should hold for the radiation dominated epoch, in which case $\rho \sim T^4$ (T being temperature) should be compared with $\rho_{\rm co}$.

Requiring that the graviton mass be less then the Hubble parameter today, $m < H_0$, we find that $\rho_{\rm co} < \rho_c$, where ρ_c, is the present-day value of the critical density in the Universe. If so, then according to the above-described scenario, the cosmological evolution of the early Universe ($\rho \gg \rho_{\rm co}$) within each $1/m$-size domain will mimic the FRW expansion with some accuracy. However, this will change significantly at densities $\rho \sim \rho_{\rm co}$. As long as the graviton mass m is sufficiently small, the observational tests of such cosmologies of massive GR would impose an upper bound on m. We estimate this bound to be approximately an order of magnitude smaller than $H_0 \simeq 10^{-33}$ eV.

In general, the mass term introduces an effective stress tensor in the Einstein equation. The backreaction of this term should be negligible in the Vainshtein regime, becoming dominant in the vDVZ regime. While generically we expect the above to be the case, interestingly enough, we find one particular exact solution for which the backreaction is described by a perfect fluid with the equation of state of dark energy, and the magnitude of the energy density/pressure set by $m^2 M_{\rm Pl}^2$. This behavior is similar to the self-accelerated solutions of massive GR first found in Ref. [15] in the DL or its extension in Ref. [16], and to exact self-accelerated solutions obtained in Refs. [12,17,18].

If one utilizes this particular solution, then by the end of the Vainshtein regime the Universe may become dominated by the self-accelerated solution. However, the latter is not regular at spatial infinity, and could only exist as a transient solution in space and time, matched upon the low-density inhomogeneous solution at larger scales.[1] Whether such a matching is possible, is not shown here; in principle the evolution could just bypass this solution and transition directly to a low-density regime. Putting the question about the matching aside, however, the bound on the graviton mass becomes especially restrictive if the expansion of the Universe is described by the self-accelerated solution (see discussions in Sec. III).

In this paper we consider the cosmological evolution both in the Vainshtein and vDVZ regimes. In the Vainshtein regime, the metric to which the matter couples is homogeneous and isotropic with some small corrections, but it is the Stückelberg sector that carries all the inhomogeneities. Moreover, in this regime, and for the self-accelerated solution, we will show the existence of the backreaction of the mass term that is small and mimics dark energy. What is not shown is that there is a matching between the Vainshtein (with or without self-acceleration) and vDVZ regimes.[2]

The theory of Ref. [1] is the only potentially viable classical theory of Lorentz-invariant massive GR with 5 helicity states. Its cosmology is unusual, and this paper is a first attempt at unfolding peculiarities of such a theory in a cosmological setup. Therefore, the majority of this paper is qualitative in character, where we emphasize certain universal aspects and set up a general framework in which

[1] Note that the solution of Ref. [17] exhibit singularities at finite values of the coordinates, and hence, should be matched to other solutions before reaching those points.

[2] To address this issue, one could consider a possibility that the matter/radiation that is being expelled from the bulk of the $1/m$-size domains, which are densely packed and adjacent to each other, gets accumulated near the boundaries of the domains. If the domains are well separated, or there is only one domain, density near the edge will be suppressed due to screening of gravity and free streaming. Different scenaria are determined by different initial conditions and need more detailed studies.

such cosmologies can further be studied in details. A number of particular exact cosmological solutions are discussed in the appendices.

II. MASSIVE GR AND COSMOLOGY

For massive GR the action is a functional of the metric $g_{\mu\nu}(x)$, and four spurious scalar fields $\phi^a(x)$, $a = 0, 1, 2, 3$; the latter are introduced to give a manifestly diffeomorphism invariant description [19,20]. One defines a covariant tensor $H_{\mu\nu}$ as follows:

$$g_{\mu\nu} = \partial_\mu \phi^a \partial_\nu \phi^b \eta_{ab} + H_{\mu\nu}, \tag{3}$$

where $\eta_{ab} = \mathrm{diag}(-1, 1, 1, 1)$. The first term on the right-hand side (r.h.s.) is nothing but the Minkowski metric in the coordinate system defined by ϕ^a's. Hence, gravity in this formulation is described by the tensor $H_{\mu\nu}$ propagating on Minkowski space. In the unitary gauge all the four scalars $\phi^a(x)$ are frozen and equal to the corresponding space-time coordinates, $\phi^a(x) = x^\mu \delta^a_\mu$. However, often it is helpful to use a nonunitary gauge in which $\phi^a(x)$'s are allowed to fluctuate.

A covariant Lagrangian density for massive GR can be written as follows:

$$\mathcal{L} = \frac{M_{\rm Pl}^2}{2}\sqrt{-g}\left(R - \frac{m^2}{4}\mathcal{U}(g, H)\right), \tag{4}$$

where $\mathcal{U}$ includes the mass, and nonderivative interaction terms for $H_{\mu\nu}$ and $g_{\mu\nu}$.

A necessary condition for the theory to be ghost-free in the DL is that the potential $\sqrt{-g}\mathcal{U}(g, H)$ be a total derivative upon the field substitution $h_{\mu\nu} \equiv g_{\mu\nu} - \eta_{\mu\nu} = 0$, $\phi^a = \delta^a_\mu x^\mu - \eta^{a\mu}\partial_\mu \pi$ [3]. With this substitution, the potential $\sqrt{-g}\mathcal{U}(g, H)$ becomes a function of $\Pi_{\mu\nu} \equiv \partial_\mu \partial_\nu \pi$ and its various contractions.

For instance, the following expression composed of $\Pi_{\mu\nu}$ is a total derivative

$$\mathcal{L}^{(2)}_{\rm der}(\Pi) \equiv [\Pi]^2 - [\Pi^2], \tag{5}$$

where we use the notations $[\Pi] \equiv \mathrm{tr}\Pi^\mu_\nu$, $[\Pi]^2 \equiv (\mathrm{tr}\Pi^\mu_\nu)^2$, while $[\Pi^2] \equiv \mathrm{tr}\Pi^\mu_\nu \Pi^\nu_\alpha$.

Then, as argued in [1], the Lagrangian for massive GR that is automatically ghost-free to all orders in the DL is obtained by replacing the matrix elements Π^μ_ν in the total derivative term (5) by the matrix elements of a tensor $\mathcal{K}^\mu_\nu$, defined as follows:

$$\mathcal{K}^\mu_\nu(g, H) = \delta^\mu_\nu - \sqrt{\partial^\mu \phi^a \partial_\nu \phi^b \eta_{ab}}. \tag{6}$$

Here, the indices on $\mathcal{K}$ should be lowered and raised by $g_{\mu\nu}$ and its inverse, respectively. This procedure defines the mass term (along with the interaction potential in the Lagrangian density) in massive GR

$$\mathcal{L} = \frac{M_{\rm Pl}^2}{2}\sqrt{-g}[R - m^2(\mathcal{K}^\mu_\nu \mathcal{K}^\nu_\mu - (\mathcal{K}^\alpha_\alpha)^2)]. \tag{7}$$

The matter and other fields are coupled to $g_{\mu\nu}$ as in GR. The above expression has no free parameters once the graviton mass is fixed. In general, however, there exist other polynomial terms in $\mathcal{K}$ with similar properties. These terms can be constructed straightforwardly by using the procedure outlined in Ref. [1]. In any dimensions there are only a finite number of total derivative combinations, made of Π [11]. They are all captured by the recurrence relation [3]:

$$\mathcal{L}^{(n)}_{\rm der} = -\sum_{m=1}^{n}(-1)^m \frac{(n-1)!}{(n-m)!}[\Pi^m]\mathcal{L}^{(n-m)}_{\rm der}, \tag{8}$$

with $\mathcal{L}^{(0)}_{\rm der} = 1$ and $\mathcal{L}^{(1)}_{\rm der} = [\Pi]$. This also guarantees that the sequence terminates, i.e., $\mathcal{L}^{(n)}_{\rm der} \equiv 0$, for any $n \geq 5$ in four dimensions. The list of all nonzero total derivative terms starting with the quadratic one reads as,

$$\mathcal{L}^{(2)}_{\rm der}(\Pi) = [\Pi]^2 - [\Pi^2], \tag{9}$$

$$\mathcal{L}^{(3)}_{\rm der}(\Pi) = [\Pi]^3 - 3[\Pi][\Pi^2] + 2[\Pi^3], \tag{10}$$

$$\mathcal{L}^{(4)}_{\rm der}(\Pi) = [\Pi]^4 - 6[\Pi^2][\Pi]^2 + 8[\Pi^3][\Pi] + 3[\Pi^2]^2 - 6[\Pi^4]. \tag{11}$$

One can use the method of Ref. [1] to obtain the two other polynomials in $\mathcal{K}$ to be included in massive GR. For this, we replace in (9)–(11) the matrix elements Π^μ_ν by the matrix elements $\mathcal{K}^\mu_\nu$ defined in (6). As a result of this procedure, we get the Lagrangian density [1]:

$$\mathcal{L} = \frac{M_{\rm Pl}^2}{2}\sqrt{-g}(R + m^2(\mathcal{L}^{(2)}_{\rm der}(\mathcal{K}) + \alpha_3 \mathcal{L}^{(3)}_{\rm der}(\mathcal{K}) + \alpha_4 \mathcal{L}^{(4)}_{\rm der}(\mathcal{K}))). \tag{12}$$

Since all terms in (8) with $n \geq 5$ vanish identically, by construction all terms $\mathcal{L}^{(n)}_{\rm der}$ with $n \geq 5$ in (12) are also zero. Hence, the most general Lagrangian density (12) has three free parameters, m, α_3 and α_4.

As it is straightforward to see, Minkowski space is a vacuum with $\phi^a = x^a$, and the spectrum of the theory (12) contains a graviton of mass m; the graviton also has additional nonlinear interactions specified by the action at hand.

A. Proof of the absence of FRW cosmologies

Let us begin by considering homogeneous and isotropic solutions to the theory (12). There exists a coordinate system in which the most general ansatz consistent with these symmetries reads as follows:

$$ds^2 = -dt^2 + a^2(t)d\vec{x}^2, \qquad \phi^0 = f(t), \qquad \phi^i = x^i. \tag{13}$$

Here and in the following we are assuming a flat three-dimensional metric.[3] Plugging these expressions for the metric and scalar fields into (12), and setting for simplicity $\alpha_3 = \alpha_4 = 0$, one obtains the following Lagrangian for a and f:

$$\mathcal{L} = 3M_{\rm Pl}^2(-a\dot{a}^2 - m^2|\dot{f}|(a^3 - a^2) + m^2(2a^3 - 3a^2 + a)), \tag{14}$$

where overdot denotes the time derivative ∂_0. We emphasize that the quantity $\dot{f}$ appears in the Lagrangian only linearly. The same remains true if we keep nonzero α_3 and α_4—it is just the special structure of the terms $\mathcal{L}_{\rm der}^{(n)}(\mathcal{K})$, $n = 2, 3, 4$ in (12), that ensures that $\dot{f}$ enters only linearly! This is a consequence of the fact that in the decoupling limit the equations of motion of this theory have no more than two time derivatives acting on the helicity-0 field, in particular, (and on any field in general) [3]. Away from the decoupling limit this is related to the constraint that was found in Refs. [1,4,5]. Here we see the constraint for the FRW metric to all orders, by taking variation of (14) with respect to (w.r.t.) f:

$$m^2\partial_0(a^3 - a^2) = 0. \tag{15}$$

This constraint makes time evolution of the scale factor impossible. As we have noted above, keeping the $\mathcal{K}^3$ and $\mathcal{K}^4$ terms in (12) can only modify the polynomial function of a on which ∂_0 acts in (15). Therefore, there are no nontrivial homogeneous and isotropic solutions in the theory of massive GR, defined by (12).

It is also instructive to show the absence of FRW solutions in the unitary gauge, for which $\phi^a = \delta^a_\mu x^\mu$, and no f field appears in the action to begin with. In this gauge, the most general homogeneous and isotropic ansatz involves the lapse function $N(t)$,

$$ds^2 = -N^2(t)dt^2 + a^2(t)d\vec{x}^2, \tag{16}$$

and the Lagrangian (12) with $\alpha_3 = \alpha_4 = 0$ reads

$$\mathcal{L} = 3M_{\rm Pl}^2\left(-\frac{a\dot{a}^2}{N} - m^2(a^3 - a^2) + m^2N(2a^3 - 3a^2 + a)\right). \tag{17}$$

As can be straightforwardly verified, the condition (15) in this case arises as the requirement of consistency of the equations of motion for the two fields, a and N in (17). More specifically, one can obtain (15) by taking the difference between the time-derivative of the equation of motion for N and the equation of motion for a. Technically, this is so because the second term on the r.h.s. of (17) has no factors of N in it and the constraint arises as the direct result of the Bianchi identity of GR.

We briefly note that the homogeneous and isotropic solutions would not be forbidden if the mass term were not an explicit constant, but instead emerged as a vacuum expectation value of some field-dependent function; i.e., if we replaced $m^2 \to m^2(\sigma)$ in (12), where σ is a scalar field that also has its own kinetic and potential terms. Then, variation w.r.t. f would give rise to a constraint

$$\partial_0(m^2(\sigma)(a^3 - a)) = 0, \tag{18}$$

that relates time evolution of the scale factor to that of the σ field, but it does not forbid homogeneous and isotropic solutions. Hence, the absence of the homogeneous and isotropic solutions is an intrinsic property of massive GR with an explicit mass term, as in (12). By this property it could potentially be distinguished observationally from the theory with a dynamical mass $m^2(\sigma)$. Moreover, for the latter theory one should expect the presence of an additional massless (or very light) scalar at cosmological distances, which is hidden by the Vainshtein mechanism at shorter scales. One example of this is when $m^2 \to m^2 \exp(\sigma/M_{\rm Pl})$, for which the DL theory (with the kinetic term for σ) reduces to a theory with two Galileons coupled to the tensor field.

The above-described properties of massive GR are similar to those of a peculiar scalar field theory (the so-called Cuscuton), defined by the following Lagrangian [22]:

$$\mathcal{L} = \mu^2\sqrt{-g}\sqrt{|g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi|}, \tag{19}$$

where μ is some dimensionful constant. Assuming $\phi = \phi(t)$, and the homogeneous and isotropic FRW metric (13), the scalar field equation reduces to a constraint, similar to (15)

$$\partial_\mu\left(\sqrt{-g}\frac{g^{\mu\nu}\partial_\nu\phi}{\sqrt{|g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi|}}\right) = 0 \Rightarrow \partial_0 a^3 = 0. \tag{20}$$

Therefore, the theory (19) does not possess homogeneous and isotropic cosmological solutions in full analogy to massive GR described above. Pursuing this analogy further, the homogeneous and isotropic cosmological solutions would be permitted if we were to promote the parameter μ into a field-dependent function $\mu^2 \to \mu^2(\sigma)$. In this case the constraint would read as $\partial_0(\mu^2(\sigma)a^3) = 0$; the latter links the time evolution of the scale factor to that of σ, but it does not forbid homogeneous and isotropic cosmological solutions.

We note that the equations of motion of the theory (19) are invariant under the replacement $\phi \to E(\phi)$, where E is an arbitrary differentiable function. Hence, by finding a particular solution one immediately generates an infinite

[3]Homogeneity here is thought as a shift symmetry of Cartesian coordinates, $\vec{x} \to \vec{x} + \vec{c}$. This, however, does not exclude open FRW universes, in which case an exact solution was found in Ref. [21], after the first version of this paper was posted on the arXiv.

number of solutions. Whether a somewhat similar invariance exists in the equations of motion of massive GR is not obvious.

Last but not least, we complement the present section with the discussion of the degravitating solution of massive GR (for degravitation see [23], for the corresponding decoupling-limit solution, see [15]). Although the constraint (15) forbids the time-evolving FRW solutions, it still allows for a static Minkowski metric—even in the presence of a Cosmological Constant. To see this, we note that the action (17) in the presence of the vacuum energy density $\mathcal{E}$ has an additional contribution of the form $-Na^3\mathcal{E}$. In this case one can determine the value of the scale factor from the N-equation, which for a static homogeneous field reduces to

$$2a^3 - 3a^2 + a = a^3\frac{\mathcal{E}}{\rho_{\rm co}}, \tag{21}$$

where, as before, $\rho_{\rm co} = 3M_{\rm Pl}^2 m^2$. The value of N is determined from the equation of motion for a. The solution to this equation exists if the following inequality is satisfied:

$$\mathcal{E} \geq -\frac{\rho_{\rm co}}{4}. \tag{22}$$

Hence, the degravitation works for an arbitrarily large positive vacuum energy density. If we were to include also an arbitrary α_3 and α_4, degravitation could have been achieved for arbitrary $\mathcal{E}$. However, we note that fluctuations on the degravitated background exhibit the Vainshtein mechanism at the scale determined by the degravitated vacuum energy—the larger the degravitated energy, the smaller is the corresponding Vainshtein radius [15]. Because of this, there is an unscreened fifth force, and we cannot be living today on such a background. Hence, the degravitation could have only taken place in a far past after which the Universe must have transitioned to a different background (see more in [15]). Note that the screening solution exists for a broad class of external sources, not just for a Cosmological Constant.

III. COSMOLOGY AT HIGH DENSITIES: $\rho \gg \rho_{\rm co}$

We begin by thinking of the Universe filled with pressureless dust of density $\rho \gg \rho_{\rm co}$, with $\rho_{\rm co}$ defined in (2). As was discussed in Sec. I, cosmological evolution of such a universe can very well be approximated by the standard FRW metric of GR. This is so because an arbitrary Hubble patch enclosed by a sphere of radius $H^{-1} = (\rho/3M_{\rm Pl}^2)^{-1/2}$, is well within its Vainshtein radius, $H^{-1} \ll r_*$. Hence, the early Universe in massive GR would evolve as it does in GR, with some small corrections. These corrections, for any observer in such a universe, can be estimated as some positive power of the ratio $(m/H) \ll 1$. On the other hand, we would expect the scalar fields, ϕ^a, to be in a nonperturbative (Vainshtein) regime at these scales, and yet, their stress tensor should be subdominant to the matter/radiation stress tensor that drives the FRW expansion. That this is so is a necessary condition for the self-consistency of the solution. Below, we will calculate the expressions for the scalar fields ϕ^a in an FRW background, and discuss its backreaction.

The most general spherically symmetric solution, including the four scalars ϕ^a, can always be put in the following form:

$$\begin{aligned} ds^2 &= -dt^2 + C(t,r)dtdr + A^2(t,r)[dr^2 + r^2 d\Omega^2], \\ \phi^0 &= f(t,r), \\ \phi^i &= g(t,r)\frac{x^i}{r} \end{aligned} \tag{23}$$

The advantage of the latter form of the metric is that it is easier to compare to the standard FRW, while the appropriate ϕ^a fields can be treated separately.

The Einstein equation, obtained by varying (12) with $\alpha_3 = \alpha_4 = 0$ with respect to the metric, reads as follows:

$$G_{\mu\nu} = m^2 T^{(K)}_{\mu\nu} + \frac{1}{M_{\rm Pl}^2} T^{(m)}_{\mu\nu}, \tag{24}$$

where $T^{(K)}_{\mu\nu}$ is the effective stress tensor due to the mass term in (12), while $T^{(m)}_{\mu\nu}$ denotes the stress energy tensor of standard matter. Taking a covariant derivative of the above equation leads to the Bianchi constraint, $m^2\nabla^\mu T^{(K)}_{\mu\nu} = 0$, which is just the equation of motion obtained by varying the action w.r.t. ϕ^a,

$$\frac{\delta S}{\delta \phi^a} = 0. \tag{25}$$

As discussed above, we will be neglecting at the zeroth order the $m^2 T^{(K)}_{\mu\nu}$ term, as well as the (possible) r dependence in the metric (23). Hence, the zeroth order solution for the metric will be the standard FRW solution, with $A(r,t) = a(t)$ and $C(t,r) = 0$ corresponding to the matter content encoded in $T^{(m)}_{\mu\nu}$. For the scalars ϕ^a, instead, we have to solve the full equations (25) in the background FRW metric just defined, since these are already proportional to m^2. For this, we can rewrite the potential part of the massive GR action (7) using the following identity,

$$\begin{aligned} \mathcal{K}^\mu_\nu \mathcal{K}^\nu_\mu - (\mathcal{K}^\alpha_\alpha)^2 &= -12 + 6{\rm tr}\sqrt{g^{-1}\Sigma} + {\rm tr} g^{-1}\Sigma \\ &\quad - \left({\rm tr}\sqrt{g^{-1}\Sigma}\right)^2, \end{aligned} \tag{26}$$

where the matrix $(g^{-1}\Sigma)^\mu_\nu$ is defined as follows:

$$\begin{aligned} (g^{-1}\Sigma)^\mu_\nu &= g^{\mu\alpha}\partial_\alpha\phi^a\partial_\nu\phi^b\eta_{ab} \equiv g^{\mu\alpha}\Sigma_{\alpha\nu} \\ &= \begin{pmatrix} \dot f^2 - \dot g^2 & \dot f f' - \dot g g' & 0 & 0 \\ \frac{\dot g g' - \dot f f'}{a^2} & \frac{-f'^2 + g'^2}{a^2} & 0 & 0 \\ 0 & 0 & \frac{g^2}{a^2 r^2} & 0 \\ 0 & 0 & 0 & \frac{g^2}{a^2 r^2} \end{pmatrix}. \end{aligned} \tag{27}$$

Evaluating the eigenvalues of the latter matrix, and varying the resulting action w.r.t the two fields $f(t, r)$ and $g(t, r)$, one obtains the corresponding equations of motion,

$$\partial_t\left[\frac{y}{\sqrt{X}}\left(\dot{f} + \mu\frac{g'}{a}\right) + \mu a^2 r^2 g'\right] - \partial_r\left[\frac{y}{a\sqrt{X}}\left(\frac{f'}{a} + \mu\dot{g}\right) + \mu a^2 r^2 \dot{g}\right] = 0, \quad (28)$$

$$\partial_t\left[\frac{y}{\sqrt{X}}\left(\dot{g} + \mu\frac{f'}{a}\right) + \mu a^2 r^2 f'\right] - \partial_r\left[\frac{y}{a\sqrt{X}}\left(\frac{g'}{a} + \mu\dot{f}\right) + \mu a^2 r^2 \dot{f}\right] - 6a^2 r + 2ag + 2a^2 r\sqrt{X} = 0 \quad (29)$$

where dot denotes time derivative and prime denotes derivative w.r.t. r, and we have introduced the following notation:

$$y \equiv 2ga^2 r - 3a^3 r^2,$$
$$X \equiv \left(\dot{f} + \mu\frac{g'}{a}\right)^2 - \left(\dot{g} + \mu\frac{f'}{a}\right)^2, \quad (30)$$
$$\mu \equiv \mathrm{sgn}(\dot{f}g' - \dot{g}f').$$

By solving (28) and (29) for the Stückelberg fields, and calculating the effective stress-energy tensor $T^{(\mathcal{K})}_{\mu\nu}$ on the solution, one can evaluate the backreaction on the geometry from the presence of ϕ^a's. For the Vainshtein mechanism to be operative, two conditions should be met inside the Vainshtein radius: A1 the backreaction of the Stückelberg stress tensor should be negligible, so that the background evolution is described by the FRW geometry to a very high precision, and A2 the metric fluctuations should be those of GR to a very high precision. This would guarantee, that the scalar is successfully screened. Generically one would expect these properties to hold in the Vainshtein region as the stress tensor of the Stückelberg fields in multiplied by a small parameter m^2. Hence, one should anticipate significant departures for the standard GR results at scales of order $1/m$. Even though the metric is homogeneous and isotropic, the Stückelberg fields are not. Because of these inhomogeneous fields there is a physical center in each $1/m$-size domain. This center is not felt by matter coupled to the metric, but perturbations will be sensitive to it. This by itself restricts the value of m to be an order of magnitude smaller than H_0, or less.

For illustrative purposes, we give in Appendix A a particular exact solution, which satisfies the condition A1 in an interesting way. On that solution the stress tensor $T^{(\mathcal{K})}_{\mu\nu}$ *exactly* coincides with the stress tensor of dark energy with the energy density $\sim\rho_{\rm co}$, in spite of the fact that the Stückelberg fields are inhomogeneous! As a result, this solution can exist even when no external stress tensor is introduced. Hence, it is in a class of self-accelerated solutions. This solution can exist not only in the Vainshtein regime, but also outside of it.

What is however not clear is whether the fluctuations on this self-accelerated solution are close to those of GR (it is easy to show that subhorizon fluctuations are, but one needs to demonstrate it for larger scales as well, which requires some careful calculations). Until this is known the solution can only serve a demonstrational purpose showing the smallness of the backreaction in the Vainshtein regime. This is precisely how we regard this solution in the present work.

Putting the issues of perturbations aside, on the self-accelerated solution the value of the mass is related to that of the present-day Hubble parameter as $m^2 = CH_0^2$, where C is a free constant in the theory, which would depend on the parameters α_3 and α_4 (if we were to include them); without significant tunings of these parameters the theoretical value of C should not be assumed to be outside of the interval $C \sim (0.01 - 1)$. In this case, it should be possible to rule out such a scenario observationally (or at least to rule out a significant fraction of the parameter space for α_3, α_4), as at the present-day Hubble scales one would expect departures from the FRW evolution of the order of C.

IV. COSMOLOGY AT LOW DENSITIES: $\rho \ll \rho_{\rm co}$

After the energy density ρ drops below its crossover value (2), massive gravity enters the linear regime, as discussed in Sec. I. In this regime, no matter how small the graviton mass, the massive theory differs from the massless one by quantities of order $\mathcal{O}(m^0)$, thus exhibiting the vDVZ discontinuity [14]. Therefore, the cosmology described by the massive theory is expected to differ significantly from the conventional one. The purpose of this section is to study that cosmology. For this we first recall the status of linearized cosmology in GR. There are some subtleties in this, and we would like to emphasize those relevant for our discussions.

First, the matter/radiation stress tensor should be conserved in this approximation, $\partial^\mu T_{\mu\nu} = 0$. Then, if we were to choose a diagonal stress tensor, $T^\mu_\nu = \mathrm{diag}(-\rho, p, p, p)$, the conservation would impose ρ to be time-independent. To avoid this restriction, we will have to choose a coordinate system in which the stress tensor is not diagonal and takes the form:

$$T_{\mu\nu} = \begin{pmatrix} \rho & -H(\rho + p)x^i \\ -H(\rho + p)x^i & p\delta_{ij} \end{pmatrix}, \quad (31)$$

where H, ρ, p are arbitrary time-dependent functions. It is straightforward to check then that the condition

$\partial^\mu T_{\mu 0} = 0$ leads to the proper conservation equation, $\dot{\rho} + 3H(\rho + p) = 0$, as soon as H, ρ and p are interpreted as the Hubble parameter, density and pressure, respectively.[4]

Second, strictly speaking, linearized GR itself has no homogeneous and isotropic cosmology: indeed, assuming that $h_{\mu\nu}$ is a function of t only, the 00 component of the Einstein tensor vanishes, and the 00 component of the Einstein equation cannot be satisfied. The way out is to resort to the Fermi coordinate system in which the metric takes an inhomogeneous form (for recent discussions see, e.g., [25]):

$$ds^2 = -(1 - (\dot{H} + H^2)\mathbf{x}^2)dt^2 + (1 - \tfrac{1}{2}H^2\mathbf{x}^2)d\mathbf{x}^2 = (\eta_{\mu\nu} + h^{\rm FRW}_{\mu\nu})dx^\mu dx^\nu, \tag{32}$$

where the corrections to the above expression are suppressed by higher powers of $H^2\mathbf{x}^2$. As long as $H^2\mathbf{x}^2 \ll 1$, the above metric describes cosmology in any local patch as a small deviation from Minkowski space.

In GR there exists a coordinate transformation that brings (32), in the approximation considered, to a homogeneous and isotropic metric in a comoving coordinate system t_c, $\vec{x}_c$, in which the stress tensor has a conventional form, $T_{\mu\nu} = {\rm diag}(\rho(t_c),\ \delta_{ij}a^2(t_c)p(t_c))$. This coordinate transformation takes the form (see [11])

$$t_c = t - \frac{1}{2}H(t)\vec{x}^2, \qquad \vec{x}_c = \frac{\vec{x}}{a(t)}\left(1 + \frac{1}{4}H^2(t)\vec{x}^2\right), \tag{33}$$

however, the transformation itself is essentially nonlinear.[5]

Linearized massive gravity is not much different in that respect—it does not admit homogeneous and isotropic solutions either. We prove this by assuming the opposite and showing the contradiction. For this, consider the Fierz-Pauli Lagrangian to which any consistent Lorentz-invariant massive gravity should reduce at the linearized level [26],

$$\mathcal{L} = -\tfrac{1}{2}h^{\mu\nu}\mathcal{E}^{\rho\sigma}_{\mu\nu}h_{\rho\sigma} - \tfrac{1}{4}m^2(h^{\mu\nu}h_{\mu\nu} - h^2) + h^{\mu\nu}T_{\mu\nu}. \tag{34}$$

Here, $h_{\mu\nu} \equiv g_{\mu\nu} - \eta_{\mu\nu}$ and $\mathcal{E}$ denotes the linearized Einstein operator

$$\mathcal{E}^{\rho\sigma}_{\mu\nu}h_{\rho\sigma} = -\tfrac{1}{2}(\Box h_{\mu\nu} - \partial_\mu\partial^\alpha h_{\alpha\nu} - \partial_\nu\partial^\alpha h_{\alpha\mu} + \partial_\mu\partial_\nu h - \eta_{\mu\nu}\Box h + \eta_{\mu\nu}\partial_\alpha\partial_\beta h^{\alpha\beta}), \tag{35}$$

all indices are contracted with the flat metric, and the Planck mass has been set to one. Furthermore, applying ∂^μ to the equation of motion obtained from (34) and using the Bianchi identity, one gets a constraint,

$$\partial^\mu h_{\mu\nu} = \partial_\nu h. \tag{36}$$

In the unitary gauge the metric perturbation $h_{\mu\nu}$ represents a *physical* field, and requiring homogeneity makes all of its components space-independent,

$$\partial_i h_{\mu\nu} = 0. \tag{37}$$

With this assumption, the constraint (36) reduces to the following equations:

$$h_{0i} = {\rm const} \equiv c_i, \qquad h_{ii} = {\rm const} \equiv c. \tag{38}$$

The second of these equations implies that either a solution is trivial or else it cannot be isotropic. Hence, linearized massive GR has no nontrivial homogeneous and isotropic solutions. Moreover, it is straightforward to check that in general, inhomogeneity cannot be removed entirely into the longitudinal degrees of freedom.

Then, what is a snapshot of the Universe at $\rho \ll \rho_{\rm co}$? We will show below that it can be pictured as a collection of multiple domains, each of size $1/m$, such that well within a given domain, at scales $\ll 1/m$, cosmology deviates significantly from the conventional GR one. How these domains are glued together is a complicated question that is not addressed here. Nevertheless, at a scale much greater than $1/m$, when averaged over many domains enclosed by this scale, the Universe should look homogeneous and isotropic again.

What exactly is then a solution to the linearized massive GR in each domain? To get this solution, it is useful to introduce appropriately normalized Stückelberg fields:

$$h_{\mu\nu} = \bar{h}_{\mu\nu} + \partial_\mu V_\nu + \partial_\nu V_\mu = \bar{h}_{\mu\nu} + \frac{\partial_\mu A_\nu + \partial_\nu A_\mu}{m} + \frac{2\partial_\mu\partial_\nu\pi}{m^2}, \tag{39}$$

and consider scales that are much smaller than $1/m$. In this approximation, the Fierz-Pauli theory (excluding the totally decoupled vector mode) reduces to:

$$\mathcal{L} = -\tfrac{1}{2}\bar{h}^{\mu\nu}\mathcal{E}^{\rho\sigma}_{\mu\nu}\bar{h}_{\rho\sigma} - \bar{h}^{\mu\nu}(\partial_\mu\partial_\nu\pi - \eta_{\mu\nu}\Box\pi) + \bar{h}^{\mu\nu}T_{\mu\nu} + \mathcal{O}(m^2). \tag{40}$$

The latter can be diagonalized by the conformal shift $\bar{h}_{\mu\nu} = \tilde{h}_{\mu\nu} + \eta_{\mu\nu}\pi$, giving rise to the Lagrangian

$$\mathcal{L} = -\tfrac{1}{2}\tilde{h}^{\mu\nu}\mathcal{E}^{\rho\sigma}_{\mu\nu}\tilde{h}_{\rho\sigma} + \tfrac{3}{2}\pi\Box\pi + \tilde{h}^{\mu\nu}T_{\mu\nu} + \pi T + \mathcal{O}(m^2). \tag{41}$$

The equations of motion that follow from this Lagrangian are: the GR equations for $\tilde{h}_{\mu\nu}$, and a simple equation for π, $\Box\pi = -T/3$. The solution for $\tilde{h}_{\mu\nu}$, as argued above, is $h^{\rm FRW}_{\mu\nu}$ given in (32), while for π we obtain $\Box\pi_{\rm sol} = 2(\dot{H} + 2H^2)$. As a result, the physical metric is $h^{\rm Phys}_{\mu\nu} = h^{\rm FRW}_{\mu\nu} + \eta_{\mu\nu}\pi_{\rm sol}$, and the interval takes the form:

[4]For momentum conservation one should use the full covariant expression $\nabla^\mu T_{\mu i}$ that leads to the acceleration equation; in this case however, it is obtained at the *linear* order in $\vec{x}$ and requires correcting the stress tensor by $\mathcal{O}(H^2\mathbf{x}^2)$ quantities [24].

[5]This is consistent with the fact that the Friedmann equation in the comoving system relates the *square* of the perturbation to density, $(\dot{\delta a})^2 \sim G_N\rho$, where δa denotes a departure of the scale factor from the Minkowski space, $\delta a = a - 1$.

$$ds^2 = -(1 - \tfrac{1}{3}(2\dot{H} + H^2)\mathbf{x}^2)dt^2 + (1 + \tfrac{1}{6}(2\dot{H} + H^2)\mathbf{x}^2)d\mathbf{x}^2. \tag{42}$$

Note that the linearized Ricci curvature on the physical metric $h_{\mu\nu}^{\rm Phys}$ is zero (in the leading approximation), as it should be the case for all cosmologies due to (36). This metric deviates from its GR counterpart by the value of $\pi_{\rm sol}$, which is of the same order as $h_{\mu\nu}^{\rm FRW}$ itself. This is a cosmological manifestation of the vDVZ discontinuity.

At scales $\sim 1/m$ the mass terms neglected in (41) should be reinstated and used. However, at yet larger scales, which enclose a large number of domains, the Universe should look homogeneous, if we average over all enclosed domains. We notice that at scales $\gg 1/m$, all the derivative terms in the Lagrangian (34) should be neglected and only the mass terms should be kept. Then, the equation of motion takes the form:

$$m^2(h_{\mu\nu} - \eta_{\mu\nu}h) = 2\langle T_{\mu\nu}\rangle, \tag{43}$$

where $\langle T_{\mu\nu}\rangle$ is the stress tensor (31) averaged over many $1/m$-size domains. At such large scales gravity is screened. Depending on the initial conditions there may be a number of different scenaria of how one could match the large and small scale behavior to each other. If the $1/m$-size domains are densely packed and adjacent to each other, then the matter/radiation will get accumulated near the boundaries of the domains, as it is expelled from the bulk. However, if the domains are well separated, or there is only one domain, then density near the boundaries should be expected to be suppressed since gravity of the bulk material is screened and matter particles will free stream out of the domain. All these scenaria, and the initial conditions that could give rise to them, need separate detailed studies.

ACKNOWLEDGMENTS

We would like to thank M. Kleban and R. Scoccimarro for useful comments. The work of G. D'A is supported by the James Arthur Fellowship. C. d. R. is supported by the Swiss National Science Foundation. The work of S. D. is supported in part by the NSF under Grant No. PHY-1068438. S. D. thanks the hospitality of the Kavli Institute for Theoretical Physics at UC Santa Barbara (under NSF Grant No. PHY05-51164), while this work was being completed. G. G. is supported by the NSF under Grant No. PHY-0758032. The work of D. P. is supported by a Mark Leslie Graduate Assistantship. A. J. T. would like to thank the Université de Genève for hospitality while this work was being completed.

APPENDIX A: AN EXACT BACKGROUND SOLUTION

We will now show that it is possible to find a solution of the system (28) and (29)

$$\partial_t\left[\frac{y}{\sqrt{X}}\left(\dot{f} + \mu\frac{g'}{a}\right) + \mu a^2 r^2 g'\right] - \partial_r\left[\frac{y}{a\sqrt{X}}\left(\frac{f'}{a} + \mu\dot{g}\right) + \mu a^2 r^2 \dot{g}\right] = 0, \tag{A1}$$

$$\partial_t\left[\frac{y}{\sqrt{X}}\left(\dot{g} + \mu\frac{f'}{a}\right) + \mu a^2 r^2 f'\right] - \partial_r\left[\frac{y}{a\sqrt{X}}\left(\frac{g'}{a} + \mu\dot{f}\right) + \mu a^2 r^2 \dot{f}\right] - 6a^2 r + 2ag + 2a^2 r\sqrt{X} = 0 \tag{A2}$$

.

We notice that (A1) is identically satisfied for $y = 0$, or

$$g(t, r) = \tfrac{3}{2}a(t)r. \tag{A3}$$

For this ansatz Eq. (A2) reduces to the following equation for $f(t, r)$:

$$\sqrt{\left(\mu\dot{f} + \frac{3}{2}\right)^2 - \left(\frac{3}{2}Har + \mu\frac{f'}{a}\right)^2} = \mu\dot{f} + \frac{3}{2} - \mu Hrf', \tag{A4}$$

with $H = \dot{a}/a$ denoting the usual Hubble parameter.

The structure of Eq. (A4) guarantees that all square roots appearing in it are always well defined. Indeed, the above equation for f can be reduced to a simpler one by rewriting it as follows:

$$\sqrt{\left(\mu\dot{f} + \frac{3}{2} - \mu Hrf'\right)^2 - \frac{9}{4}H^2a^2r^2 - \frac{f'^2}{a^2} + 2Hr\dot{f}f' - H^2r^2f'^2} = \mu\dot{f} + \frac{3}{2} - \mu Hrf'. \tag{A5}$$

Squaring the latter equation puts it in the following form:

$$\frac{f'^2}{a^2}(1 + a^2H^2r^2) - 2Hr\dot{f}f' + \frac{9}{4}a^2H^2r^2 = 0. \tag{A6}$$

Obviously, the expression under the square root in (A5) is positive semidefinite for any solution of the squared equation. Moreover, for any solution of this equation, the quantity appearing on the r.h.s of (A5) is positive, which can be seen from writing it as

$$\mu\dot{f} + \frac{3}{2} - \mu Hrf' = \frac{2}{3a}\mu(\dot{f}g' - \dot{g}f') + \frac{3}{2},$$

and recalling the definition of μ, Eq. (30). Therefore, any solution of Eq. (A6) will solve Eq. (A4).

One solution of Eq. (A6) is

$$f(t, r) = \frac{9}{16T}\int^t \frac{d\tilde{t}}{a(\tilde{t})H(\tilde{t})} + a(t)T\left(1 + \frac{9r^2}{16T^2}\right), \tag{A7}$$

where T is an integration constant with dimensions of time, and the choice of the lower limit of integration corresponds to a constant shift of ϕ^0.

We are now in a position to compute the stress tensor of the ϕ^a fields and compare it with that of matter. It follows from (7), that the effective stress tensor is given by the following expression:

$$T^{(K)}_{\mu\nu} = \frac{1}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\Big[\sqrt{-g}(\mathcal{K}^2_{\alpha\beta} - \mathcal{K}^2)\Big] = \frac{1}{2}\Big[(12 + \Big(\mathrm{Tr}\sqrt{g^{-1}\Sigma} - 6)\mathrm{Tr}\sqrt{g^{-1}\Sigma} - g^{\alpha\beta}\Sigma_{\alpha\beta}\Big)g_{\mu\nu} + 2\Sigma_{\mu\nu} + \Big(3 - \mathrm{Tr}\sqrt{g^{-1}\Sigma}\Big) \times\Big(g_{\mu\alpha}\Big(\sqrt{g^{-1}\Sigma}\Big)^\alpha_\nu + g_{\nu\alpha}\Big(\sqrt{g^{-1}\Sigma}\Big)^\alpha_\mu\Big)\Big]. \tag{A8}$$

The nonzero components of $T^{(\mathcal{K})}_{\mu\nu}$ are therefore:

$$T^{(K)}_{00} = \frac{1}{2}\Big[-12 - 4\sqrt{X}\Big(\frac{g}{ar} - \frac{3}{2}\Big) - 2\frac{g}{ar}\Big(\frac{g}{ar} - 6\Big) + \frac{2}{\sqrt{X}}\Big(3 - 2\frac{g}{ar}\Big)\Big(\dot{g}^2 - \dot{f}^2 - \frac{\mu}{a}(\dot{f}g' - \dot{g}f')\Big)\Big],$$

$$T^{(K)}_{0r} = \frac{1}{\sqrt{X}}\Big(3 - 2\frac{g}{ar}\Big)(\dot{g}g' - \dot{f}f'),$$

$$T^{(K)}_{rr} = \frac{1}{2}a^2\Big[12 + 4\sqrt{X}\Big(\frac{g}{ar} - \frac{3}{2}\Big) + 2\frac{g}{ar}\Big(\frac{g}{ar} - 6\Big) + \frac{2}{\sqrt{X}}\Big(3 - 2\frac{g}{ar}\Big)\Big(\Big(\frac{g'}{a}\Big)^2 - \Big(\frac{f'}{a}\Big)^2 + \frac{\mu}{a}(\dot{f}g' - \dot{g}f')\Big)\Big],$$

and

$$T^{(K)}_{\theta\theta} = \frac{T^{(K)}_{\phi\phi}}{\sin^2\theta} = \frac{1}{2}a^2r^2\Big[12 + 4\sqrt{X}\Big(\frac{g}{ar} - \frac{3}{2}\Big) + 2\frac{g}{ar}\Big(\frac{g}{ar} - 6\Big) + 2\frac{\mu}{a}(\dot{f}g' - \dot{g}f') + 2\Big(\frac{g}{ar}\Big)^2 + 2\frac{g}{ar}\Big(3 - \sqrt{X} - 2\frac{g}{ar}\Big)\Big].$$

Remarkably enough, as we show below, for the solution at hand the inhomogeneities of the Stückelberg fields completely fall out from the expression for the effective stress tensor $T^{(K)}_{\mu\nu}$.

Using the exact solution for $g(r, t)$, as well as the equation of motion for $f(r, t)$ (A6), $T^{(\mathcal{K})}_{\mu\nu}$ *exactly* reduces to the diagonal, cosmological-constant-type form—with the corresponding Hubble scale set by the value of the graviton mass,

$$T^{(\mathcal{K})\mu}{}_\nu = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}, \tag{A9}$$

$$\rho = -p = \frac{3}{4}m^2M^2_{\mathrm{Pl}} = \frac{1}{4}\rho_{co}.$$

Therefore, the backreaction from the Stückelberg fields is indeed negligible for the universe filled with matter or radiation with density significantly exceeding $\sim\rho_{\mathrm{co}}$.

The background solution we just found, consisting of the FRW metric plus Eqs. (A3) and (A7), is exact. Cosmology for this solution—at least at the background level—is therefore completely insensitive to the presence of inhomogeneity in the Stückelberg scalars. Even in the absence of any external sources, the geometry describes the homogeneous and isotropic self-acceleration of the Universe with the Hubble constant equal to $m/2$, all of the inhomogeneities of the solution being removed into the Stückelberg scalars. On the other hand, the space-dependence of the background value of ϕ^a can be probed by perturbations on the FRW metric.

More generally, we expect the theory to admit solutions with truly inhomogeneous geometry—with the metric being impossible to put in a homogeneous form by any coordinate transformations. For such solutions, we expect the backreaction of the Stückelberg fields to be negligible in the high-density regime, while becoming important for densities $\lesssim \rho_{\mathrm{co}}$. We emphasize again that the question of whether the evolution can continue on the self-accelerated solution, or alternatively should switch to the low-density vDVZ regime, remains open, as the matching between these two regimes is hard to study analytically.

APPENDIX B: ANISOTROPIC SOLUTIONS

In this Appendix, we show that there are cosmological solutions which are homogeneous but anisotropic.

1. Parity symmetric solutions

Let us start by considering the following diagonal Ansatz for the unitary gauge metric:

$$ds^2 = -N^2(\tau)d\tau^2 + a_1^2(\tau)dx^2 + a_2^2(\tau)dy^2 + a_3^2(\tau)dz^2, \tag{B1}$$

which is the most general homogeneous metric invariant under the discrete parity symmetry $\vec{x} \to -\vec{x}$. Before plugging this into the action, it is convenient to redefine the time variable as

$$dt = N(\tau)d\tau. \tag{B2}$$

This gauge transformation will excite one of the Stückelberg fields, which will now read:

$$\phi^0 = f(t), \qquad \phi^i = x^i. \tag{B3}$$

In this gauge we can immediately derive a constraint on the scale factors.

The matrix $\Sigma^\mu_\nu \equiv g^{\mu\alpha}\partial_\alpha\phi^a\partial_\nu\phi_a$ is diagonal, with the following eigenvalues:

$$\sigma_0 = \dot{f}^2, \qquad \sigma_1 = \frac{1}{a_1^2}, \qquad \sigma_2 = \frac{1}{a_2^2}, \qquad \sigma_3 = \frac{1}{a_3^2}. \tag{B4}$$

The mass term in the action therefore reads

$$S = \frac{M_{\rm Pl}^2}{2} m^2 \int d^4x[-12a_1a_2a_3 + 6(a_1a_2 + a_1a_3 + a_2a_3) - 2(a_1 + a_2 + a_3) + 2\dot{f}(3a_1a_2a_3 - a_1a_2 - a_1a_3 - a_2a_3)]. \tag{B5}$$

It is immediate to derive the following algebraic constraint on the scale factors, which is proportional to m^2 and it is valid independently of the matter Lagrangian coupled to the metric:

$$3a_1a_2a_3 - (a_1a_2 + a_1a_3 + a_2a_3) = k, \tag{B6}$$

where k is an integration constant. As a check of the calculations, we can easily see that there is no isotropic solution other than Minkowski space-time; for $a_1 = a_2 = a_3 = a$, we have $a^3 - a^2 = k$, which is exactly Eq. (15).

It is simple to show that we cannot have a solution for which all the scale factors grow at small times. In fact, requiring that $a_i \to 0$ as $t \to 0$, we see that we should set $k = 0$, because the whole left-hand side should vanish for small times. However, since the scale factors are positive, the equation cannot be satisfied. Thus, we cannot have a solution with all the scale factors growing with time. The only nontrivial solutions will have one direction contracting and the others expanding, or vice versa.

2. Axisymmetric solutions

We now discuss the most general homogeneous and axisymmetric solution. The most general anisotropic ansatz can be written as

$$\phi^0 = f(t) + b_jx^j, \tag{B7}$$

$$\phi^i = A^i{}_jx^j + c^i(t), \tag{B8}$$

$$ds^2 = -dt^2 + a_1^2(t)dx^2 + a_2^2(t)dy^2 + a_3^2(t)dz^2, \tag{B9}$$

where b_j is a constant vector and $A^i{}_j$ is a constant matrix.

If we impose axial symmetry around the third direction, our ansatz should satisfy

$$b_i = (0, 0, q), \qquad A^i{}_j = \mathrm{diag}(1, 1, B), \qquad c^i(t) = (0, 0, c(t)) \tag{B10}$$

$$a_1(t) = a_2(t) = a(t), \qquad a_3(t) = b(t). \tag{B11}$$

The matrix $\Sigma^\alpha_\beta = g^{\alpha\lambda}\partial_\lambda\phi^a\partial_\nu\phi^b\eta_{ab}$ is block diagonal and can be easily diagonalized. Its eigenvalues are given by

$$\sigma_1^2 = \sigma_2^2 = \frac{1}{a^2}, \qquad \sigma_3^2 = \frac{g + \sqrt{g^2 - h}}{b}, \qquad \sigma_4^2 = \frac{g - \sqrt{g^2 - h}}{b}, \tag{B12}$$

where

$$g \equiv \frac{1}{2}[B^2 - q^2 + b^2(\dot{f}^2 - \dot{c}^2)], \qquad h \equiv b^2(q\dot{c} - B\dot{f})^2. \tag{B13}$$

We can derive two constraints by varying the action with respect to f and c. After some algebraic manipulations, we arrive at

$$\sqrt{\frac{B + q - \mu b(\dot{f} + \dot{c})}{B - q - \mu b(\dot{f} - \dot{c})}} = \frac{a^2(B + q) - \mu k_-}{b(3a^2 - 2a)}, \tag{B14}$$

$$b^2 = \frac{[a^2(B - q) - \mu k_+][a^2(B + q) - \mu k_-]}{(3a^2 - 2a)^2}, \tag{B15}$$

where $\mu = \mathrm{sgn}(q\dot{c} - B\dot{f})$.

If we require that both a and b vanish when $t \to 0$, we must choose $k_+ = k_- = 0$. Now, from the second constraint, we find $B^2 > q^2$. In order to have a positive b, the only possible choice would be $B + q < 0 \Rightarrow B < 0$. We can prove that this is an inconsistent solution by looking at the square root. First, we notice that the symmetry $\phi^a \to -\phi^a$ allows us to study only the case in which $B + q - \mu b(\dot{f} + \dot{c}) > 0$. So, we would like to have

$$\mu b(\dot{f} + \dot{c}) < B + q < 0, \qquad \mu b(\dot{f} - \dot{c}) < B - q < 0. \tag{B16}$$

Now, these conditions have no solution. In fact, they imply

$$\mu\dot{f} < 0, \tag{B17}$$

which is absurd. Indeed, we also see from (B16) that $|\dot{f}| > |\dot{c}|$, which along with $|B| > |q|$ and $B < 0$ implies that $\mu = \mathrm{sgn}\dot{f}$, so necessarily $\mu\dot{f} > 0$. In conclusion, also in this case we cannot describe a Universe which expands starting from a small initial volume at early times.

[1] C. de Rham, G. Gabadadze, and A. J. Tolley, Phys. Rev. Lett. **106**, 231101 (2011).
[2] D. G. Boulware and S. Deser, Phys. Rev. D **6**, 3368 (1972).
[3] C. de Rham and G. Gabadadze, Phys. Rev. D **82**, 044020 (2010).
[4] S. F. Hassan and R. A. Rosen, arXiv:1106.3344.
[5] C. de Rham, G. Gabadadze, and A. J. Tolley, arXiv:1107.3820.
[6] C. de Rham, G. Gabadadze, and A. J. Tolley, arXiv:1108.4521.
[7] A. I. Vainshtein, Phys. Lett. **39B**, 393 (1972).
[8] C. Deffayet, G. R. Dvali, G. Gabadadze, and A. I. Vainshtein, Phys. Rev. D **65**, 044026 (2002).
[9] M. A. Luty, M. Porrati, and R. Rattazzi, J. High Energy Phys. 09 (2003) 029.
[10] A. Nicolis and R. Rattazzi, J. High Energy Phys. 06 (2004) 059.
[11] A. Nicolis, R. Rattazzi, and E. Trincherini, Phys. Rev. D **79**, 064036 (2009).
[12] K. Koyama, G. Niz, and G. Tasinato, Phys. Rev. Lett. **107**, 131101 (2011); Phys. Rev. D **84**, 064033 (2011).
[13] G. Chkareuli and D. Pirtskhalava, arXiv:1105.1783.
[14] H. van Dam and M. J. G. Veltman, Nucl. Phys. **B22**, 397 (1970); V. I. Zakharov, Pis'ma Zh. Eksp. Teor. Fiz. **12**, 447 (1970) [JETP Lett. **12**, 312 (1970)].
[15] C. de Rham, G. Gabadadze, L. Heisenberg, and D. Pirtskhalava, Phys. Rev. D **83**, 103516 (2011).
[16] C. de Rham and L. Heisenberg, Phys. Rev. D **84**, 043503 (2011).
[17] T. M. Nieuwenhuizen, Phys. Rev. D **84**, 024038 (2011).
[18] A. H. Chamseddine and M. S. Volkov, Phys. Lett. B **704**, 652 (2011).
[19] N. Arkani-Hamed, H. Georgi, and M. D. Schwartz, Ann. Phys. (N.Y.) **305**, 96 (2003).
[20] S. L. Dubovsky, J. High Energy Phys. 10 (2004) 076.
[21] A. E. Gumrukcuoglu, C. Lin, and S. Mukohyama, arXiv:1109.3845.
[22] N. Afshordi, D. J. H. Chung, and G. Geshnizjani, Phys. Rev. D **75**, 083513 (2007).
[23] N. Arkani-Hamed, S. Dimopoulos, G. Dvali, and G. Gabadadze, arXiv:hep-th/0209227; G. Dvali, S. Hofmann, and J. Khoury, Phys. Rev. D **76**, 084006 (2007).
[24] P. Creminelli, G. D'Amico, J. Norena, L. Senatore, and F. Vernizzi, J. Cosmol. Astropart. Phys. 03 (2010) 027.
[25] D. Baumann, A. Nicolis, L. Senatore, and M. Zaldarriaga, arXiv:1004.2488.
[26] M. Fierz and W. Pauli, Proc. R. Soc. A **173**, 211 (1939).