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Collision of an object in the transition from adiabatic inspiral to plunge around a Kerr black hole

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An inspiraling object of mass μ around a Kerr black hole of mass $M(\gg \mu)$ experiences a continuous transition near the innermost stable circular orbit from adiabatic inspiral to plunge into the horizon as gravitational radiation extracts its energy and angular momentum. We investigate the collision of such an object with a generic counterpart around a Kerr black hole. We find that the angular momentum of the object is fine-tuned through gravitational radiation and that the high-velocity collision of the object with a generic counterpart naturally occurs around a nearly maximally rotating black hole. We also find that the center-of-mass energy can be far beyond the Planck energy for dark matter particles colliding around a stellar mass black hole and as high as 10^{58} erg for stellar mass compact objects colliding around a supermassive black hole, where the present transition formalism is well justified. Therefore, rapidly rotating black holes can accelerate objects inspiraling around them to energy high enough to be of great physical interest.

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I. INTRODUCTION

Bañados *et al.* [1] discovered that the center-of-mass (CM) energy can be arbitrarily high for the collision of two geodesic particles moving on the equatorial plane around a nearly maximally rotating Kerr black hole. The angular momentum of either of the particles must be artificially fine-tuned for such a striking event. This phenomenon is seen not only for Kerr black holes but also for Kerr-Newman black holes [2], exotic black holes [3], and naked singularities [4,5]. The analysis is extended to the collision of particles in nonequatorial motion for Kerr black holes [6], Kerr-Newman black holes [7], and accelerating and rotating black holes [8]. The general explanation of this phenomenon is proposed in Ref. [9]. This phenomenon is studied in the astrophysical contexts of dark matter particle annihilation [1,10,11], extreme mass-ratio inspirals (EMRIs), and accretion disks [6,12].

It is argued that the effects of gravitational radiation would constrain the maximum CM energy because the particle with the fine-tuned angular momentum can reach the horizon after it orbits around the black hole infinitely many times in infinitely long proper time [13,14]. On the other hand, the effects of conservative self-force bound the CM energy from above in the analogous system of spherical charged shells [15]. It is also argued [13,14] that the CM energy cannot be extremely high because the nondimensional spin of astrophysical black holes is bounded by Thorne's limit 0.998 [16]. However, it is not clear whether there is a universal bound strictly less than unity on the black hole spin, as Thorne's limit is thought to be dependent on the accretion flow models [17–19].

As for the fine-tuning problem, the present authors [12] proposed a scenario where the fine-tuning is realized in EMRIs. Since the ratio of the gravitational radiation time scale $t_{\rm GW}$ to the orbital period $t_{\rm orb}$ is given by $t_{\rm GW}/t_{\rm orb} \sim \eta^{-1}$, where $\eta = \mu/M$ is the mass ratio, the inspiral through gravitational radiation will be regarded as adiabatic if $\eta \ll 1$. Noting the circularization of the orbits in the post-Newtonian regime [20], we can assume that an inspiraling compact object adiabatically takes a circular orbit which is closer to the black hole as the object loses its energy and angular momentum through gravitational radiation. Once the compact object reaches the radius of the innermost stable circular orbit (ISCO), it begins to plunge into the black hole in the dynamical time scale. Thus, the compact object will eventually have the energy and angular momentum of the particle orbiting the ISCO. In the maximal rotation limit of the black hole, the fine-tuning of the angular momentum is realized for the ISCO particle. However, this scenario should be reconsidered carefully, when we take radiation reaction into account seriously. Although radiation reaction drives the inspiraling object inwardly, it also gives the object an inward radial velocity at the ISCO radius, implying that the energy and angular momentum of the compact object are no longer those of the ISCO particle. In such a situation, the formalism proposed by Ori and Thorne [21] to describe the transition from adiabatic inspiral to plunge into a Kerr black hole is quite useful. This formalism is extended to restore the consistency with the normalization of the four-velocity by Kesden [22].

In the present paper, we apply the Ori-Thorne-Kesden formalism for nearly maximally rotating black holes and

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estimate the CM energy for the collision of an object in the transition with a generic counterpart object. We find that the scenario proposed by the present authors [12] is justified: the fine-tuning of the angular momentum is realized by the object in the transition from inspiral to plunge through gravitational radiation and the CM energy for the collision can be significantly high. Under the condition for the Ori-Thorne-Kesden formalism to be justified, the CM energy can be much higher than the Planck energy for dark matter particles colliding around a stellar mass black hole and can be as high as 10^{58} erg for compact objects colliding around a supermassive black hole. As another application, based on the present framework, we discuss that radiation reaction gives subdominant contributions to the proposal that a nearly maximally rotating black hole may be overspun by plunging an object [23,24].

This paper is organized as follows. In Sec. II, we introduce the CM energy of two colliding particles around a Kerr black hole and its near-horizon limit. In Sec. III, we briefly review the Ori-Thorne-Kesden formalism of the transition from adiabatic inspiral to plunge. In Sec. IV, we apply the Ori-Thorne-Kesden formalism to nearly maximally rotating black holes. In Sec. V, based on the Ori-Thorne-Kesden formalism, we estimate the CM energy for the collision of a transition object with a generic counterpart. Section VI is devoted to the conclusion. In the Appendix, we revisit how the energy and the angular momentum radiated during the transition affect the spin of the final black hole in the merger with an inspiraling object. We use the units in which $c = G = 1$ and the abstract index notation of Wald [25].

II. CM ENERGY OF PARTICLES COLLIDING AROUND A KERR BLACK HOLE

The line element in the Kerr spacetime in the Boyer-Lindquist coordinates is given by [25,26]

$$ds^2 = -\left(1 - \frac{2Mr}{\rho^2}\right)dt^2 - \frac{4Mar\sin^2\theta}{\rho^2}d\phi dt + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \left(r^2 + a^2 + \frac{2Mra^2\sin^2\theta}{\rho^2}\right)\sin^2\theta d\phi^2, \quad (2.1)$$

where a and M are the spin and mass parameters, respectively, $\rho^2 = r^2 + a^2\cos^2\theta$, and $\Delta = r^2 - 2Mr + a^2$. If $0 \le a^2 \le M^2$, Δ vanishes at $r = r_\pm = M \pm \sqrt{M^2 - a^2}$, where $r = r_+$ and $r = r_-$ correspond to an event horizon and Cauchy horizon, respectively. Here, we denote $r_+ = r_H$. The surface gravity of the Kerr black hole is given by $\kappa_H = \sqrt{M^2 - a^2}/(r_H^2 + a^2)$. Thus, the black hole has a vanishing surface gravity and hence is extremal for the maximal rotation $a^2 = M^2$, while it is subextremal for the nonmaximal rotation $a^2 < M^2$. The angular velocity of the horizon is given by

$$\Omega_H = \frac{a}{r_H^2 + a^2}. \quad (2.2)$$

We can assume $a \ge 0$ without loss of generality.

Let particles 1 and 2 of rest masses μ_1 and μ_2 have four-momenta p_1^a and p_2^a at the same spacetime point, respectively. The CM energy $E_{\rm cm}$ of the two particles is then defined by

$$E_{\rm cm}^2 = -(p_1^a + p_2^a)(p_{1a} + p_{2a}) = \mu_1^2 + \mu_2^2 - 2g^{ab}p_{1a}p_{2b}. \quad (2.3)$$

The derivation of the expression for the CM energy of two general particles around a Kerr black hole is described in detail in the authors' previous papers [6,12]. We do not repeat it here but quote the formula for the particles moving on the equatorial plane, where $\theta = \pi/2$ and the Carter constant identically vanishes. Equation (3.2) of Ref. [6] then reduces to

$$E_{\rm cm}^2 = \mu_1^2 + \mu_2^2 + \frac{2}{r^2}\left[\frac{\mathcal{P}_1\mathcal{P}_2 - \sigma_{1r}\sqrt{\mathcal{R}_1}\sigma_{2r}\sqrt{\mathcal{R}_2}}{\Delta} - (L_1 - aE_1)(L_2 - aE_2)\right], \quad (2.4)$$

where $\sigma_{ir} = \mathrm{sgn}(p_i^r)$, $E_i = -p_{it}$, $L_i = p_{i\phi}$,

$$\mathcal{R}_i = \mathcal{R}_i(r) = \mathcal{P}_i(r)^2 - \Delta(r)[\mu_i^2 r^2 + (L_i - aE_i)^2], \quad (2.5)$$

$$\mathcal{P}_i = \mathcal{P}_i(r) = (r^2 + a^2)E_i - aL_i, \quad (2.6)$$

and $i = 1, 2$. Thus, the CM energy can be given in terms of μ_i, E_i, L_i, and r. If we assume that σ_{1r} and σ_{2r} are of the same sign, Eq. (2.4) for the near-horizon limit then reduces to

$$E_{\rm cm}^2 = \mu_1^2 + \mu_2^2 + \frac{1}{r_H^2}\left\{[\mu_1^2 r_H^2 + (L_1 - aE_1)^2] \times \frac{E_2 - \Omega_H L_2}{E_1 - \Omega_H L_1} + [\mu_2^2 r_H^2 + (L_2 - aE_2)^2] \times \frac{E_1 - \Omega_H L_1}{E_2 - \Omega_H L_2} - 2(L_1 - aE_1)(L_2 - aE_2)\right\}. \quad (2.7)$$

It is clear that the necessary condition for the CM energy to be arbitrarily high is that $(E - \Omega_H L)$ is arbitrarily close to zero for either of the two particles.

III. TRANSITION FROM ADIABATIC INSPIRAL TO PLUNGE

We here briefly review the formalism of the transition from adiabatic inspiral to plunge proposed by Ori and Thorne [21] and extended by Kesden [22].

A. Ori-Thorne formalism

The geodesic equation and the normalization of the four-velocity of a massive particle of rest mass μ in the Kerr spacetime are given by

$$\frac{d^2\tilde{r}}{d\tilde{\tau}^2} = -\frac{1}{2}\frac{\partial V}{\partial \tilde{r}}, \tag{3.1}$$

and

$$\left(\frac{d\tilde{r}}{d\tilde{\tau}}\right)^2 = \tilde{E}^2 - V, \tag{3.2}$$

respectively, where the effective potential $V = V(\tilde{r}, \tilde{E}, \tilde{L})$ is given by

$$V(\tilde{r}, \tilde{E}, \tilde{L}) = 1 - \frac{2}{\tilde{r}} + \frac{\tilde{L}^2 + \tilde{a}^2 - \tilde{E}^2\tilde{a}^2}{\tilde{r}^2} - \frac{2(\tilde{L} - \tilde{E}\,\tilde{a})^2}{\tilde{r}^3} \tag{3.3}$$

and we define nondimensional quantities $\tilde{r} = r/M$, $\tilde{t} = t/M$, $\tilde{a} = a/M$, $\tilde{\tau} = \tau/M$, $\tilde{E} = E/\mu$, and $\tilde{L} = L/(\mu M)$.

We first expand the effective potential in the Taylor series around the ISCO radius, energy, and angular momentum, i.e. $(\tilde{r}, \tilde{E}, \tilde{L}) = (\tilde{r}_{\rm ISCO}, \tilde{E}_{\rm ISCO}, \tilde{L}_{\rm ISCO})$, in terms of $R = \tilde{r} - \tilde{r}_{\rm ISCO}$, $\chi = \tilde{\Omega}^{-1}_{\rm ISCO}(\tilde{E} - \tilde{E}_{\rm ISCO})$, and $\xi = \tilde{L} - \tilde{L}_{\rm ISCO}$ up to $O(R^3, \chi, \xi)$, where $\Omega = (d\phi/dt) = \tilde{\Omega}/M$ is the angular velocity of a particle and $\tilde{\Omega}_{\rm ISCO}$ is $\tilde{\Omega}$ for a particle orbiting the ISCO. Then, the geodesic equation (3.1) becomes

$$\frac{d^2R}{d\tilde{\tau}^2} = -\alpha R^2 + \beta\xi - \frac{1}{2}\left(\tilde{\Omega}\frac{\partial^2 V}{\partial\tilde{E}\partial\tilde{r}}\right)_{\rm ISCO}(\chi - \xi) + \cdots, \tag{3.4}$$

where

$$\alpha = \frac{1}{4}\left(\frac{\partial^3 V}{\partial\tilde{r}^3}\right)_{\rm ISCO}, \qquad \beta = -\frac{1}{2}\left(\frac{\partial^2 V}{\partial\tilde{L}\partial\tilde{r}} + \tilde{\Omega}\frac{\partial^2 V}{\partial\tilde{E}\partial\tilde{r}}\right)_{\rm ISCO}, \tag{3.5}$$

and the subscript ISCO means the value estimated at $(\tilde{r}, \tilde{E}, \tilde{L}) = (\tilde{r}_{\rm ISCO}, \tilde{E}_{\rm ISCO}, \tilde{L}_{\rm ISCO})$.

To take radiation reaction into account, we introduce κ as follows:

$$\kappa \equiv -\left(\tilde{\Omega}^{-1}\eta^{-2}\frac{dE}{dt}\frac{d\tilde{t}}{d\tilde{\tau}}\right)_{\rm ISCO} = -\left(\tilde{\Omega}^{-1}\eta^{-1}\frac{d\tilde{E}}{d\tilde{t}}\frac{d\tilde{t}}{d\tilde{\tau}}\right)_{\rm ISCO}, \tag{3.6}$$

where $\eta \equiv \mu/M$ is the mass ratio. It is κ that drives the object in the radial direction. We assume that the loss of the object's energy is radiated away through gravitational radiation, i.e.

$$-\left(\frac{dE}{dt}\right) = \dot{E}_{\rm GW} = \frac{32}{5}\eta^2\tilde{\Omega}^{10/3}\dot{\mathcal{E}}, \tag{3.7}$$

where $\dot{\mathcal{E}}$ is the nondimensional correction factor to the Newtonian quadrupole formula for the gravitational wave luminosity [27]. $\dot{\mathcal{E}}$ contains all the relativistic effects including those from the spin of the black hole. κ is then rewritten as

$$\kappa = \frac{32}{5}\left(\tilde{\Omega}^{7/3}\frac{d\tilde{t}}{d\tilde{\tau}}\dot{\mathcal{E}}\right)_{\rm ISCO}. \tag{3.8}$$

For the neighboring circular orbits, we find $\delta E = \Omega\delta L$ (see e.g. Ref. [28]). It suggests $\chi = \xi$ and we can take radiation reaction into account from Eq. (3.7),

$$\chi = \xi = -\eta\kappa\tilde{\tau}. \tag{3.9}$$

In terms of the redefined variables, Eq. (3.4) yields

$$\ddot{X} = -X^2 - T, \tag{3.10}$$

where

$$\begin{aligned} R &= \eta^{2/5}R_0 X, & \tilde{\tau} &= \eta^{-1/5}\tau_0 T, \\ R_0 &= (\beta\kappa)^{2/5}\alpha^{-3/5}, & \tau_0 &= (\alpha\beta\kappa)^{-1/5}, \end{aligned} \tag{3.11}$$

and the dot in Eq. (3.10) denotes the derivative with respect to T.

Ori and Thorne numerically obtained a unique solution to Eq. (3.10) with the initial condition

$$X \approx \sqrt{-T} \tag{3.12}$$

as $T \to -\infty$. With this condition it is assumed that the object orbits circularly at the potential minimum, which moves inwardly adiabatically at early times. The solution is shown in Figs. 2 and 3 of Ref. [21]. This solution monotonically decreases with T and diverges to negative infinity at $T = T_{\rm div}$ [29] as

$$X \approx -\frac{6}{(T_{\rm div} - T)^2}, \tag{3.13}$$

where $T_{\rm div} \simeq 3.412$ [21,22]. This divergence should be regarded as the breakdown of the Taylor-series expansion at very large values of $|X|$.

The location of the ISCO, $\tilde{r} = \tilde{r}_{\rm ISCO}$, in terms of X is of course given by $X = 0$. The location of the horizon $\tilde{r} = \tilde{r}_H$ in terms of X is given by

$$X_H = \eta^{-2/5}\frac{\tilde{r}_H - \tilde{r}_{\rm ISCO}}{R_0}. \tag{3.14}$$

We denote the time T when the object crosses the ISCO radius as T_0, i.e. $X(T_0) = 0$, while the time T when it crosses the event horizon as T_H, i.e. $X(T_H) = X_H$. The numerical value of T_0 is given by $T_0 \simeq 0.72$ [22]. Clearly, $T_0 < T_H < T_{\rm div}$ holds.

B. Kesden's extension

Through the Taylor-series expansion around the ISCO particle, the normalization (3.2) becomes

$$\left(\frac{dR}{d\tilde{\tau}}\right)^2 = -\frac{2\alpha}{3}R^3 + 2\beta R\xi + \left(\frac{\partial V}{\partial \tilde{L}}\right)_{\rm ISCO}(\chi - \xi) - \left(\tilde{\Omega}\frac{\partial^2 V}{\partial \tilde{E}\partial \tilde{r}}\right)_{\rm ISCO}(\chi - \xi)R + \cdots . \tag{3.15}$$

Taking the same procedure as in the derivation of Eq. (3.10), we reduce Eq. (3.15) to

$$\dot{X}^2 = -\tfrac{2}{3}X^3 - 2XT. \tag{3.16}$$

It can be seen that the Ori-Thorne solution does not satisfy Eq. (3.16). Noting that the relation $\chi = \xi$ is required only for the quasicircular orbits, Kesden [22] introduces Y in Eq. (3.15) through

$$\chi - \xi = \eta^{6/5}(\chi - \xi)_0 Y, \quad (\chi - \xi)_0 = \alpha^{-4/5}(\beta\kappa)^{6/5}\left(\frac{\partial V}{\partial \tilde{L}}\right)^{-1}_{\rm ISCO}. \tag{3.17}$$

Then, we obtain

$$\dot{X}^2 = -\tfrac{2}{3}X^3 - 2XT + Y. \tag{3.18}$$

To restore the consistency between the equation of motion (3.10) and the normalization relation (3.18), Y must satisfy

$$\dot{Y} = 2X. \tag{3.19}$$

The solution for Y to Eq. (3.19) is numerically obtained and shown in Fig. 3 of Ref. [22]. In particular, the solution shows the asymptotic behaviors

$$Y \approx -\tfrac{4}{3}(-T)^{3/2} \tag{3.20}$$

for $T \to -\infty$, and

$$Y \approx -\frac{12}{T_{\rm div} - T}, \tag{3.21}$$

for $T \to T_{\rm div}$.

The energy or angular momentum of the object is now not conserved because the object no longer moves along a geodesic of the background geometry. The energy and angular momentum change as

$$\tilde{E} = \tilde{E}_{\rm ISCO} + \Delta\tilde{E}_{\rm tr} + \Delta\tilde{E}_{\rm norm}, \qquad \tilde{L} = \tilde{L}_{\rm ISCO} + \Delta\tilde{L}_{\rm tr}, \tag{3.22}$$

where

$$\Delta\tilde{E}_{\rm tr} = \tilde{\Omega}_{\rm ISCO}\Delta\tilde{L}_{\rm tr} = -\tilde{\Omega}_{\rm ISCO}\eta^{4/5}\kappa\tau_0 T, \quad \Delta\tilde{E}_{\rm norm} = \tilde{\Omega}_{\rm ISCO}\eta^{6/5}(\chi - \xi)_0 Y. \tag{3.23}$$

It should be noted that the correction to restore the normalization of the four-velocity may also be added to the angular momentum of the object. This ambiguity does not affect our conclusion in the present paper.

It is natural to take Eq. (3.17) into account also in Eq. (3.4). This implies

$$\ddot{X} = -X^2 - T + \epsilon Y, \tag{3.24}$$

where

$$\epsilon = \eta^{2/5}C, \quad C = -\frac{1}{2}\alpha^{-3/5}(\beta\kappa)^{2/5}\left[\tilde{\Omega}\frac{\partial^2 V}{\partial \tilde{E}\partial \tilde{r}}\left(\frac{\partial V}{\partial \tilde{L}}\right)^{-1}\right]_{\rm ISCO}.$$

Thus, Eq. (3.24) justifies the Ori-Thorne solution as a solution if $\epsilon \ll 1$.

The consistency of Eq. (3.24) with the normalization relation (3.18) of the four-velocity requires that Y must satisfy

$$\dot{Y} = 2X + 2\epsilon Y\dot{X}. \tag{3.25}$$

Equivalently, we can eliminate Y from Eq. (3.18) and obtain from Eq. (3.24)

$$\ddot{X} = -X^2 - T + \epsilon\left(\dot{X}^2 + \frac{2}{3}X^3 + 2XT\right). \tag{3.26}$$

Kesden numerically integrated Eqs. (3.24) and (3.25) simultaneously with different values of ϵ and the solutions are shown in Fig. 6 of Ref. [22]. As for the initial values for X, $\dot{X}$, and Y, $\dot{X} = \ddot{X} = 0$ is assumed in Eqs. (3.18) and (3.24) and X and Y are solved algebraically at some small value of T [30]. Since Eq. (3.12) no longer provides a proper asymptotic solution of Eqs. (3.24) and (3.25) for $\epsilon \neq 0$, this choice of the initial condition is one of the natural choices as the matching to the adiabatic inspiral phase at early times. We can see in Fig. 6 of Ref. [22] that the numerical solutions are very close to the Ori-Thorne solution ($\epsilon = 0$) for $\epsilon \ll 1$ and behave qualitatively similarly even for $\epsilon \sim 1$. Although one can still obtain numerical solutions to Eqs. (3.24) and (3.25) with this initial condition even if $\epsilon \gtrsim 1$, such numerical solutions may probably be invalid because higher-order terms in the Taylor-series expansions or higher-order terms in ϵ should not be negligible. We can here only assume that the Ori-Thorne-Kesden formalism is justified so that the numerical solutions for X and Y for $\epsilon = 0$, i.e., the Ori-Thorne solution to Eq. (3.10) for X together with Kesden's solution to Eq. (3.19) for Y, qualitatively give the right behaviors for $\epsilon \lesssim 1$. Hereafter, we restrict the analysis within the regime $\epsilon \lesssim 1$.

IV. MAXIMAL ROTATION LIMIT

In the maximal rotation limit $\delta = 1 - \tilde{a} \to 0$, the numerical results by the GREMLIN code can be fit by

$$\dot{\mathcal{E}} = A\delta^m, \tag{4.1}$$

where $A \simeq 1.80$ and $m \simeq 0.317$ [22]. There is another argument by Chrzanowski [31] which suggests $m \simeq 1/3$ [22].

We can obtain the dependence of the quantities on $\delta = 1 - \tilde{a} \ll 1$ as follows:

$$\tilde{r}_H \simeq 1 + (2\delta)^{1/2}, \qquad \tilde{\Omega}_H \simeq \frac{1}{2}[1 - (2\delta)^{1/2}] \tag{4.2}$$

for the black hole event horizon,

$$\tilde{r}_{\rm ISCO} \simeq 1 + (4\delta)^{1/3} + \tfrac{7}{8}(4\delta)^{2/3}, \tag{4.3}$$

$$\tilde{\Omega}_{\rm ISCO} \simeq \tfrac{1}{2}\left[1 - \tfrac{3}{4}(4\delta)^{1/3} - \tfrac{9}{32}(4\delta)^{2/3}\right], \tag{4.4}$$

$$\tilde{E}_{\rm ISCO} \simeq \frac{1}{\sqrt{3}}\left[1 + (4\delta)^{1/3} - \frac{5}{8}(4\delta)^{2/3}\right], \tag{4.5}$$

$$\tilde{L}_{\rm ISCO} \simeq \frac{2}{\sqrt{3}}\left[1 + (4\delta)^{1/3} + \frac{1}{8}(4\delta)^{2/3}\right], \tag{4.6}$$

$$\left(\frac{d\tilde{t}}{d\tilde{\tau}}\right)_{\rm ISCO} \simeq \frac{4}{\sqrt{3}}(4\delta)^{-1/3}\left[1 - \frac{3}{8}(4\delta)^{1/3} + \frac{7}{32}(4\delta)^{2/3}\right] \tag{4.7}$$

for the ISCO particle [12,32], and

$$\alpha \simeq 1 - 4(4\delta)^{1/3}, \tag{4.8}$$

$$\beta \simeq \frac{\sqrt{3}}{2}(4\delta)^{1/3}, \tag{4.9}$$

$$\left(\frac{\partial V}{\partial \tilde{L}}\right)_{\rm ISCO} \simeq \frac{4}{\sqrt{3}}(4\delta)^{1/3}, \tag{4.10}$$

$$\left(\frac{\partial^2 V}{\partial \tilde{E}\partial \tilde{r}}\right)_{\rm ISCO} \simeq -\frac{8}{\sqrt{3}}\left[1 - \frac{7}{2}(4\delta)^{1/3}\right] \tag{4.11}$$

from Eq. (3.3) for the derivatives of the effective potential for the ISCO particle. As for the dynamics driven by gravitational radiation, from Eqs. (3.8), (4.1), (4.4), and (4.7), we obtain

$$\kappa \simeq \frac{16}{5\sqrt{3}} A\delta^{m-1/3}, \tag{4.12}$$

and then R_0, τ_0, $(\chi - \xi)_0$, C, and ϵ are written in terms of δ and η as follows:

$$R_0 \simeq 2^{22/15} 5^{-2/5} A^{2/5} \delta^{2m/5}, \tag{4.13}$$

$$\tau_0 \simeq 2^{-11/15} 5^{1/5} A^{-1/5} \delta^{-m/5}, \tag{4.14}$$

$$(\chi - \xi)_0 \simeq 2^{26/15} 3^{1/2} 5^{-6/5} A^{6/5} \delta^{6m/5-1/3}, \tag{4.15}$$

$$C \simeq 2^{-1/5} 5^{-2/5} A^{2/5} \delta^{2m/5-1/3}, \tag{4.16}$$

$$\epsilon \simeq 2^{-1/5} 5^{-2/5} A^{2/5} \eta^{2/5} \delta^{2m/5-1/3}. \tag{4.17}$$

Since C is divergent as $\delta \to 0$ if $m < 5/6$, the parameter ϵ is divergent if we take the maximal rotation limit $\delta \to 0$ as the mass ratio η is kept constant. In this case, the Ori-Thorne-Kesden formalism may be invalid. Clearly, the two limits $\delta \to 0$ and $\eta \to 0$ cannot be taken independently. As δ is kept constant, we can always take the limit $\eta \to 0$, where the Ori-Thorne-Kesden formalism is justified. Note that Eq. (4.17) can be solved for η as follows:

$$\eta \simeq 5\sqrt{2} A^{-1} \delta^{5/6-m} \epsilon^{5/2}. \tag{4.18}$$

This implies that the present transition formalism is valid for a rather wide range of the mass ratio if we consider a reasonable value of the black hole spin.

The time varying parts of the energy and angular momentum of the object in the transition are given by

$$\begin{aligned}\Delta\tilde{E}_{\rm tr} &= \tilde{\Omega}_{\rm ISCO}\Delta\tilde{L}_{\rm tr} \\ &\simeq -2^{34/15} 3^{-1/2} 5^{-4/5} A^{4/5} \eta^{4/5} \delta^{4m/5-1/3} T \\ &\simeq -2^{8/3} 3^{-1/2} \delta^{1/3} \epsilon^2 T,\end{aligned} \tag{4.19}$$

$$\begin{aligned}\Delta\tilde{E}_{\rm norm} &\simeq 2^{11/15} 3^{1/2} 5^{-6/5} A^{6/5} \eta^{6/5} \delta^{6m/5-1/3} Y \\ &\simeq 2^{4/3} 3^{1/2} \delta^{2/3} \epsilon^3 Y.\end{aligned} \tag{4.20}$$

If we substituted $T = T_{\rm div}$, Y would diverge to negative infinity and hence $\Delta\tilde{E}_{\rm norm}$ would diverge. However, since we are interested in the CM energy of two objects colliding outside the event horizon, we should stop the calculation at $T = T_H$. Noting

$$X_H \simeq -2^{-4/5} 5^{2/5} A^{-2/5} \eta^{-2/5} \delta^{-2m/5+1/3} \simeq -\frac{1}{2\epsilon}, \tag{4.21}$$

we find $T_{\rm div} - T_H \approx 2\sqrt{3\epsilon}$ from Eq. (3.13). Hence, Eq. (3.21) implies $Y(T_H) \simeq -2\sqrt{3}\epsilon^{-1/2}$ and then

$$\Delta\tilde{E}_{\rm tr} \simeq -2^{8/3} 3^{-1/2} \delta^{1/3} \epsilon^2 T_H, \tag{4.22}$$

$$\Delta\tilde{E}_{\rm norm} \simeq -2^{7/3} 3 \delta^{2/3} \epsilon^{5/2}. \tag{4.23}$$

Therefore, the energy and angular momentum extracted through gravitational waves should be finite until the object plunges into the horizon.

V. CM ENERGY FOR THE COLLISION OF AN OBJECT IN THE TRANSITION

The CM energy can be directly calculated in terms of the four-velocities of the two colliding objects. Since the four-velocity can be uniquely expressed by $\tilde{r}$, $\tilde{E}$, and $\tilde{L}$ for the equatorial motion in the Kerr spacetime, we can use the formula for the CM energy in terms of $\tilde{E}$ and $\tilde{L}$ of each particle using their values at the moment of collision. The formula (2.7) implies that the CM energy can be arbitrarily high if the quantity $(E - \Omega_H L)$ is arbitrarily close to zero at the moment of collision. If we consider radiation reaction, this quantity is no longer conserved.

To examine whether the CM energy is bounded or not, we only have to see whether the quantity $(E - \Omega_H L)$ is vanishing or not in the limit to the event horizon. We can rewrite $(E - \Omega_H L)$ as follows:

$$\begin{aligned} E - \Omega_H L &= E - E_{\rm ISCO} - \Omega_{\rm ISCO}(L - L_{\rm ISCO}) \\ &\quad - (\Omega_H - \Omega_{\rm ISCO})(L - L_{\rm ISCO}) \\ &\quad + (E_{\rm ISCO} - \Omega_H L_{\rm ISCO}) \\ &= \mu[\Delta\tilde{E}_{\rm norm} - (\tilde{\Omega}_H - \tilde{\Omega}_{\rm ISCO})\Delta\tilde{L}_{\rm tr} \\ &\quad + (\tilde{E}_{\rm ISCO} - \tilde{\Omega}_H \tilde{L}_{\rm ISCO})], \end{aligned} \tag{5.1}$$

where we can find

$$\begin{aligned} -(\tilde{\Omega}_H - \tilde{\Omega}_{\rm ISCO})\Delta\tilde{L}_{\rm tr} &\simeq 2^{14/15}3^{1/2}5^{-4/5}A^{4/5}\eta^{4/5}\delta^{4m/5}T \\ &\simeq 2^{4/3}3^{1/2}\epsilon^2\delta^{2/3}T, \end{aligned} \tag{5.2}$$

$$\tilde{E}_{\rm ISCO} - \tilde{\Omega}_H \tilde{L}_{\rm ISCO} \simeq \frac{1}{\sqrt{3}}(2\delta)^{1/2}. \tag{5.3}$$

We can now compare Eqs. (4.23), (5.2), and (5.3) at $T = T_H$. In the limit $\delta \to 0$ as ϵ is kept constant, we find that $(\tilde{E}_{\rm ISCO} - \tilde{\Omega}_H \tilde{L}_{\rm ISCO})$ gives a dominant contribution. Therefore, the formula for the CM energy for the collision of an ISCO particle with a generic particle obtained in Ref. [6,12] is applicable. The result is the following:

$$E_{\rm cm} \simeq \frac{2^{1/4}}{3^{1/4}}\sqrt{\mu_1\mu_2}\frac{\sqrt{2\tilde{E}_2 - \tilde{L}_2}}{\delta^{1/4}}, \tag{5.4}$$

where we assume object 1 is in the transition while object 2 is a generic counterpart. Because of the condition $\epsilon \lesssim 1$, there appears a maximum value of the CM energy, which weakly depends on the mass ratio η_1:

$$\begin{aligned} E_{\rm cm} &\simeq \left(\frac{2}{3}\right)^{1/4}\left(\frac{A}{5\sqrt{2}}\right)^{-3/[2(5-6m)]}\sqrt{2\tilde{E}_2 - \tilde{L}_2} \\ &\quad \times \sqrt{M\mu_2}\,\eta_1^{(1-3m)/(5-6m)}\epsilon^{15/[4(5-6m)]}. \end{aligned} \tag{5.5}$$

Assuming $m = 1/3$, $A = 1.8$, $\tilde{E}_2 = 1$, and $\tilde{L}_2 = 0$, we find

$$E_{\rm cm} \simeq 2.6 \times 10^{30}\ {\rm GeV}\left(\frac{\mu_2}{100\,{\rm GeV}}\right)^{1/2}\left(\frac{M}{10M_\odot}\right)^{1/2}\epsilon^{5/4} \tag{5.6}$$

$$\simeq 4.6 \times 10^{58}\ {\rm erg}\left(\frac{\mu_2}{M_\odot}\right)^{1/2}\left(\frac{M}{10^8 M_\odot}\right)^{1/2}\epsilon^{5/4}, \tag{5.7}$$

and hence the maximum value realized for $\epsilon \simeq 1$ will not depend on the mass ratio of the object in the transition. If the object in the transition collides with a dark matter particle of mass 100 GeV around a stellar mass black hole, the CM energy can be much greater than the Planck energy. If the collision counterpart is a stellar mass compact object around a supermassive black hole, the CM energy can be as energetic as 10^{58} erg.

It is also interesting to see the CM energy for the collision at the ISCO radius. In this case, we cannot take the near-horizon limit before taking the maximal rotation limit. If we neglect radiation reaction, the particle orbiting the ISCO has a vanishing radial velocity by construction and we obtain [6,12]

$$E_{\rm cm} \simeq \frac{2^{2/3}}{3^{1/4}}\sqrt{\mu_1\mu_2}\frac{\sqrt{2\tilde{E}_2 - \tilde{L}_2}}{\delta^{1/6}}. \tag{5.8}$$

When we take radiation reaction into account, since the object in the transition has a nonvanishing radial velocity at the ISCO radius, it is not trivial whether or not the expression given by Eq. (5.8) is still valid. Indeed, substituting the expressions for $\tilde{E}$ and $\tilde{L}$ obtained in the previous section into the general equatorial formula (2.4) and evaluating it at the ISCO radius, we can find that the above expression gives the leading-order term in the limit $\delta \to 0$ as ϵ is kept constant. Then, the condition $\epsilon \lesssim 1$ taken into account, we estimate $E_{\rm cm}$ as follows:

$$\begin{aligned} E_{\rm cm} &\simeq 2^{2/3}3^{-1/4}\left(\frac{A}{5\sqrt{2}}\right)^{-1/(5-6m)}\sqrt{2\tilde{E}_2 - \tilde{L}_2} \\ &\quad \times \mu_1^{3(1-2m)/[2(5-6m)]}\mu_2^{1/2}M^{1/(5-6m)}\epsilon^{5/[2(5-6m)]}. \end{aligned} \tag{5.9}$$

Assuming $m = 1/3$, $A = 1.8$, $\tilde{E}_2 = 1$, and $\tilde{L}_2 = 0$, we find

$$\begin{aligned} E_{\rm cm} &\simeq 1.3 \times 10^{21}\ {\rm GeV}\left(\frac{\mu_1}{100\,{\rm GeV}}\right)^{1/6} \\ &\quad \times \left(\frac{\mu_2}{100\,{\rm GeV}}\right)^{1/2}\left(\frac{M}{10M_\odot}\right)^{1/3}\epsilon^{5/6}, \end{aligned} \tag{5.10}$$

$$\begin{aligned} &\simeq 2.3 \times 10^{57}\ {\rm erg}\left(\frac{\mu_1}{M_\odot}\right)^{1/6}\left(\frac{\mu_2}{M_\odot}\right)^{1/2} \\ &\quad \times \left(\frac{M}{10^8 M_\odot}\right)^{1/3}\epsilon^{5/6}. \end{aligned} \tag{5.11}$$

Thus the CM energy is lower than that for the near-horizon collision but still significantly high.

It should be noted that the fact that the maximum CM energy that can be reached within the present framework is extremely high suggests that the collision with reasonably high CM energy occurs rather frequently, although the precise estimate of its frequency is out of the scope of the present paper.

VI. CONCLUSIONS

When we consider the collision of two colliding particles around a nearly maximally rotating Kerr black hole, the CM energy of the particles can be arbitrarily high if gravitational radiation is neglected. Although the originally proposed scenario through direct collision from infinity needs an artificial fine-tuning of the angular momentum of either of the particles, it turns out that the fine-tuning is naturally realized for a particle orbiting the ISCO in the EMRI. We have studied this scenario with gravitational radiation reaction, where the object experiences a continuous transition from adiabatic inspiral to plunge into the horizon. Applying the Ori-Thorne-Kesden formalism of transition, we have found that it is gravitational radiation reaction that realizes the fine-tuning of the angular momentum and the expression for the CM energy is not affected. Then, we have discussed how high the CM energy can reach within the condition where the present transition formalism is well justified. We find that the CM energy can still be high enough to be of great physical interest. However, it should be noted that the present analysis incorporates some but not all the effects of self-force. A systematic approach is necessary to study the effects of conservative self-force on the problem of two-body collision around a rapidly rotating black hole.

Finally, we comment on the possibilities and difficulties of observing the consequences of the high-velocity collisions. It is shown in Ref. [14] that the Killing energy of the ejecta particle from the high-energy collision of two particles of rest mass m is at most $2m$. This can be explained by the effect of strong redshift. Thus, it is not expected that high-energy ejecta particles can be directly observed by a distant observer. On the other hand, it is suggested in e.g. Refs. [1,6,10–12] that some indirect signatures of the high-energy collision of particles near the black hole horizon might be observed by means of electromagnetic waves and/or gravitational waves. Further studies are necessary to reveal what observational signatures are expected.

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APPENDIX: FINAL SPIN OF THE BLACK HOLE IN THE MERGER

The present estimate also applies to the final spin after a rapidly spinning black hole swallows an inspiraling object. One can estimate the final spin of the black hole in the merger as follows:

$$\begin{aligned}\tilde{a}_f &= \frac{\tilde{a} + \eta(\tilde{L} + \Delta\tilde{L}_{\mathrm{tr}})}{[1 + \eta(\tilde{E}_{\mathrm{ISCO}} + \Delta E_{\mathrm{tr}} + \Delta E_{\mathrm{norm}})]^2} \\ &= \tilde{a} + \eta(\Delta\tilde{a}_{\mathrm{ISCO}} + \Delta\tilde{a}_{\mathrm{tr}} + \Delta\tilde{a}_{\mathrm{norm}}),\end{aligned} \tag{A1}$$

where

$$\Delta\tilde{a}_{\mathrm{ISCO}} = \tilde{L}_{\mathrm{ISCO}} - 2\tilde{E}_{\mathrm{ISCO}} \propto \delta^{2/3}, \tag{A2}$$

$$\Delta\tilde{a}_{\mathrm{tr}} = \Delta\tilde{L}_{\mathrm{tr}} - 2\Delta\tilde{E}_{tr} \propto -\delta^{2/3}\epsilon^2 T_H \propto -\delta^{2/3}\epsilon^2, \tag{A3}$$

$$\Delta\tilde{a}_{\mathrm{norm}} = -2\Delta\tilde{E}_{\mathrm{norm}} \propto -\delta^{2/3}\epsilon^3 Y_H \propto \delta^{2/3}\epsilon^{5/2}. \tag{A4}$$

Note that we should take $T = T_H$ for this estimate. We can see that $\Delta\tilde{a}_{\mathrm{tr}}$ and $\Delta\tilde{a}_{\mathrm{norm}}$ cannot dominate $\Delta\tilde{a}_{\mathrm{ISCO}}$ so that the effects of radiation reaction do not prevent or promote the overspinning of the black hole. The above estimate is slightly different from that in Ref. [22], where $\Delta\tilde{a}_{\mathrm{norm}}$ is estimated to be *negative* and *dominant* if $\delta \to 0$ as $\eta(>0)$ is kept constant. In the present analysis, since we assume $\epsilon \lesssim 1$, we obtain Eqs. (A2)–(A4), where all three are well controlled. If we further assume $\epsilon \ll 1$, we can see that $\Delta\tilde{a}_{\mathrm{ISCO}}$ is positive and dominant, $\Delta\tilde{a}_{\mathrm{tr}}$ is negative and subdominant and $\Delta\tilde{a}_{\mathrm{norm}}$ is *positive* and *subdominant*. Therefore, the effects of radiation reaction are subdominant within the transition formalism and this is consistent with the result in Ref. [23]. Although we will still need to consider the contribution of ingoing gravitational waves into the horizon, this would not change our conclusion. Thus, it is suggested that for $\epsilon \ll 1$, radiation reaction would not play an important role in an attempt to overspin a black hole by plunging an object and the conservative part of the self-force should be critical, which we have neglected in the present analysis.

[1] M. Bañados, J. Silk, and S. M. West, Phys. Rev. Lett. **103**, 111102 (2009).
[2] S. W. Wei, Y. X. Liu, H. Guo, and C. E. Fu, Phys. Rev. D **82**, 103005 (2010).
[3] S. W. Wei, Y. X. Liu, H. T. Li, and F. W. Chen, J. High Energy Phys. 12 (2010) 066.
[4] M. Patil and P. S. Joshi, arXiv:1103.1082.
[5] M. Patil, P. S. Joshi, K. i. Nakao, and M. Kimura, arXiv:1108.0288.
[6] T. Harada and M. Kimura, Phys. Rev. D **83**, 084041 (2011).
[7] C. Liu, S. Chen, and J. Jing, arXiv:1104.3225.
[8] W. Yao, S. Chen, C. Liu, and J. Jing, arXiv:1105.6156.
[9] O. B. Zaslavskii, Phys. Rev. D **84**, 024007 (2011).
[10] M. Bañados, B. Hassanain, J. Silk, and S. M. West, Phys. Rev. D **83**, 023004 (2011).
[11] A. J. Williams, Phys. Rev. D **83**, 123004 (2011).
[12] T. Harada and M. Kimura, Phys. Rev. D **83**, 024002 (2011).

[13] E. Berti, V. Cardoso, L. Gualtieri, F. Pretorius, and U. Sperhake, Phys. Rev. Lett. **103**, 239001 (2009).

[14] T. Jacobson and T. P. Sotiriou, Phys. Rev. Lett. **104**, 021101 (2010).

[15] M. Kimura, K. i. Nakao, and H. Tagoshi, Phys. Rev. D **83**, 044013 (2011).

[16] K. S. Thorne, Astrophys. J. **191**, 507 (1974).

[17] M. A. Abramowicz and J. P. Lasota, Acta Astronomica **30**, 35 (1980).

[18] R. Popham and C. F. Gammie, Astrophys. J. **504**, 419 (1998).

[19] A. Sadowski, M. Bursa, M. Abramowicz, W. Kluzniak, J. P. Lasota, R. Moderski, and M. Safarzadeh, Astron. Astrophys. **532**, A41 (2011).

[20] P. C. Peters and J. Mathews, Phys. Rev. **131**, 435 (1963).

[21] A. Ori and K. S. Thorne, Phys. Rev. D **62**, 124022 (2000).

[22] M. Kesden, Phys. Rev. D **83**, 104011 (2011).

[23] E. Barausse, V. Cardoso, and G. Khanna, Phys. Rev. Lett. **105**, 261102 (2010).

[24] T. Jacobson and T. P. Sotiriou, Phys. Rev. Lett. **103**, 141101 (2009); **103**, 209903(E) (2009).

[25] R. M. Wald, *General Relativity* (University of Chicago, Chicago, 1984).

[26] R. P. Kerr, Phys. Rev. Lett. **11**, 237 (1963).

[27] L. S. Finn and K. S. Thorne, Phys. Rev. D **62**, 124021 (2000).

[28] D. N. Page and K. S. Thorne, Astrophys. J. **191**, 499 (1974).

[29] We here denote the time of divergence as $T_{\rm div}$ rather than $T_{\rm plunge}$.

[30] M. Kesden (private communication).

[31] P. L. Chrzanowski, Phys. Rev. D **13**, 806 (1976).

[32] J. M. Bardeen, W. H. Press, and S. A. Teukolsky, Astrophys. J. **178**, 347 (1972).