

Hairy black holes and null circular geodesics

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Einstein-matter theories in which hairy black-hole configurations have been found are studied. We prove that the nontrivial behavior of the hair must extend *beyond* the null circular orbit (the photonsphere) of the corresponding spacetime. We further conjecture that the region above the photonsphere contains at least 50% of the total hair's mass. We support this conjecture with analytical and numerical results.

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The influential “no-hair conjecture” of Wheeler [1] has played a key role in the development of black-hole physics [2,3]. This conjecture suggests that black holes are fundamental objects in general relativity, Einstein's theory of gravity—they should be described by only a few parameters, very much like atoms in quantum mechanics.

The no-hair conjecture was motivated by earlier uniqueness theorems on black-hole solutions of the Einstein vacuum theory and the Einstein-Maxwell theory [4–8]. According to these uniqueness theorems, all stationary solutions of the Einstein-Maxwell equations are uniquely described by only three conserved parameters which are associated with a Gauss-like law: mass, charge, and angular momentum.

The belief in the no-hair conjecture was based on a simple physical picture according to which all matter fields left in the exterior of a newly born black hole would eventually be radiated away to infinity or be swallowed by the black hole itself (except when those fields were associated with conserved charges). In accord with this logic, early no-hair theorems indeed excluded scalar [9], massive vector [10], and spinor [11] fields from the exterior of stationary black holes.

However, the interplay between particle physics and general relativity in the following years has led to the somewhat surprising discovery of various types of “hairy” black holes, the first of which were the “colored black holes” [12]. These are black-hole solutions of the Einstein-Yang-Mills (EYM) theory that require for their complete specification not only the value of the mass but also an additional integer n , which counts the number of nodes of the Yang-Mills field outside the horizon. Remarkably, this integer is not associated with any conserved charge.

Soon after this discovery, a variety of hairy black-hole solutions equipped with different types of exterior fields have been discovered [13–24]. These include the Einstein-Skyrme, Einstein-non-Abelian-Proca, Einstein-Yang-Mills-Higgs, and Einstein-Yang-Mills-Dilaton hairy black holes.

It has become clear [3] that the *nonlinear* character of the matter fields mentioned above plays a key role in the construction of these hairy black-hole configurations.

Núñez *et al.* [3] have presented a nice heuristic picture according to which it is the self-interaction between the part of the field in a region *near* the black-hole horizon (a loosely defined region from which the hair tends to be swallowed by the black hole) and the part of the field in a region relatively *far* from the black hole (a region from which the hair tends to be radiated away to infinity) which is responsible, together with gravity, for the existence of stationary black-hole solutions with exterior matter fields (hair). The nonlinear (self-interaction) character of the fields thus plays an essential role in binding together the hair in these two regions in such a way that the “near-horizon” hair does not collapse into the black hole while the “far-region” hair does not escape to infinity.

Thus, according to the heuristic picture of [3], the nontrivial (nonlinear) behavior of the matter fields which constitute the hair is expected to extend into some loosely defined far-region well above the black-hole horizon. But is it possible to provide a more explicit characterization of the hair's length?

Here we turn our attention to another important characteristic of black-hole spacetimes: null geodesics. Geodesic motions provide important information on the structure of the spacetime geometry. Among the different kinds of geodesic motion, circular geodesics are especially important [25,26]. In particular, the null circular orbit (also known as the “photon orbit” or photonsphere) is the boundary between two qualitatively different regions in the exterior of a black hole: No stationary spherically symmetric configurations made of test particles (with no self-interactions) can exist below this orbit [27]. Gravity is simply too strong there. Relating this property of the null circular geodesic to our former discussion on hairy black holes, we conjecture that the “near region” (the region from which the hair tends to be sucked into the black hole) extends at least up to the height of the photon orbit.

The aim of this paper is to prove a theorem which supports this conjecture. We shall show that the nontrivial behavior of the hair indeed extends into the region above the photonsphere. More explicitly, we shall show that the asymptotic behavior of the exterior fields cannot start before the null circular orbit is crossed.

The first clue for the important role played by the photonsphere in determining the effective length of the hair could have been elicited from the nice theorem proved in [3]. There it was shown that the nontrivial behavior of some suitably defined pressure function $\mathcal{E}(r) \equiv e^{-\delta} r^4 p$ (see details below) must extend beyond $3/2$ the horizon radius. But what is so special in the location $r = \frac{3}{2} r_H$?

Here we point out that the null circular orbit of the (bare) Schwarzschild spacetime is actually located exactly at $\frac{3}{2} r_H$. This location is not expected to change much for hairy black holes with “thin” hair (these are characterized by $m_{\text{hair}} \ll M$, where M is the total Arnowitt-Deser-Misner (ADM) mass of the spacetime and $m_{\text{hair}} \equiv M - \frac{1}{2} r_H$ is the mass of the hair which resides outside the horizon). This suggests that, for black holes with thin hair, the nontrivial (nonasymptotic) behavior of the fields extends beyond the photonsphere. Here we shall show that this is actually a generic property of hairy black-hole configurations, regardless of the amount of hair.

We consider static spherically symmetric asymptotically flat spacetimes. The line element may take the following form in Schwarzschild coordinates [3,28]:

$$ds^2 = -e^{-2\delta} \mu dt^2 + \mu^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (1)$$

where the metric functions $\delta(r)$ and $\mu(r) \equiv 1 - 2m(r)/r$ depend only on the Schwarzschild areal coordinate r . Asymptotic flatness requires that as $r \rightarrow \infty$,

$$\mu(r) \rightarrow 1 \quad \text{and} \quad \delta(r) \rightarrow 0, \quad (2)$$

and a regular event horizon at $r = r_H$ requires [3]

$$\mu(r_H) = 0 \quad \text{and} \quad \delta(r_H) < \infty. \quad (3)$$

Taking $T_t^t = -\rho$, $T_r^r = p$, and $T_\theta^\theta = T_\phi^\phi = p_T$, where ρ , p , and p_T are identified as the energy density, radial pressure, and tangential pressure, respectively, [29], the Einstein equations $G_\nu^\mu = 8\pi T_\nu^\mu$ read

$$\mu' = -8\pi r \rho + (1 - \mu)/r, \quad (4)$$

and

$$\delta' = -4\pi r(\rho + p)/\mu, \quad (5)$$

where the prime stands for differentiation with respect to r . (We use natural units in which $G = c = 1$.)

The mass $m(r)$ contained within a sphere of radius r is given by

$$m(r) = \frac{1}{2} r_H + \int_{r_H}^r 4\pi r'^2 \rho(r') dr', \quad (6)$$

where $m(r_H) = r_H/2$ is the horizon mass.

The conservation equation, $T_{\nu;\mu}^\mu = 0$, has only one non-trivial component [3],

$$T_{r;\mu}^\mu = 0. \quad (7)$$

Substituting Eqs. (4) and (5) in Eq. (7), one finds for the pressure gradient,

$$p'(r) = \frac{1}{2\mu r} [(3\mu - 1 - 8\pi r^2 p)(\rho + p) + 2\mu T - 8\mu p], \quad (8)$$

where $T = -\rho + p + 2p_T$ is the trace of the energy-momentum tensor. Below we shall analyze the behavior of the function $P(r) \equiv r^4 p(r)$, whose derivative is given by

$$P'(r) = \frac{r^3}{2\mu} [(3\mu - 1 - 8\pi r^2 p)(\rho + p) + 2\mu T]. \quad (9)$$

When analyzing the coupled Einstein-matter system, one usually imposes some energy conditions on the matter fields. We shall assume that the hair outside the horizon satisfies the following conditions:

- (1) The weak energy condition. This means that the energy density ρ is positive semidefinite and that it bounds the pressures. In particular, $|p| \leq \rho$. This implies the inequality,

$$\rho + p \geq 0. \quad (10)$$

- (2) The trace of the energy-momentum tensor plays a central role in determining the spacetime geometry of static configurations [29]. It is usually assumed to satisfy the relation $p + 2p_T \leq \rho$ (see [29] and references therein), which implies

$$T \leq 0. \quad (11)$$

- (3) The energy density ρ goes to zero faster than r^{-4} . This requirement is the natural way to impose the condition that there are no extra conserved charges (besides the ADM mass) defined at asymptotic infinity associated with the matter fields [3]. (We recall that the charges defined at spatial infinity, like the electric charge of the Reissner-Nordström solution in Einstein-Maxwell theory, are associated with the $\rho \sim r^{-4}$ asymptotic behavior.) We therefore have the boundary condition,

$$P(r \rightarrow \infty) \rightarrow 0. \quad (12)$$

It should be emphasized that in all Einstein-matter theories in which hair has been found, these conditions are indeed satisfied (see details in [3]).

We shall next examine the behavior of the function $P(r)$ in the vicinity of the black-hole horizon. Regularity of the horizon imposes the requirement (see [3,30] for details):

$$-p(r_H) = \rho(r_H) < (8\pi r_H^2)^{-1}, \quad (13)$$

the last inequality being valid for nonextremal black holes. Substituting Eqs. (3), (10), (11), and (13) into Eq. (9), one finds

$$P(r \rightarrow r_H) \leq 0 \quad \text{and} \quad P'(r \rightarrow r_H) < 0, \quad (14)$$

in the vicinity of the black-hole horizon.

We shall now prove that the asymptotic behavior of the pressure function $P(r)$, as characterized by Eq. (12), can start only above the photonsphere. We shall follow the analyses of [25,26] in order to compute the location $r = r_\gamma$ of the null circular geodesic for a black-hole spacetime described by the line element (1). The Lagrangian describing the geodesics in the spacetime (1) is given by

$$2\mathcal{L} = -e^{-2\delta} \mu \dot{t}^2 + \mu^{-1} \dot{r}^2 + r^2 \dot{\phi}^2, \quad (15)$$

where a dot denotes a derivative with respect to proper time. The generalized momenta derived from this Lagrangian are given by [25,26]

$$p_t = -e^{-2\delta} \mu \dot{t} \equiv -E = \text{const}, \quad (16)$$

$$p_\phi = r^2 \dot{\phi} \equiv L = \text{const}, \quad (17)$$

and

$$p_r = \mu^{-1} \dot{r}. \quad (18)$$

The Lagrangian is independent of both t and ϕ . This implies that E and L are constants of the motion. The Hamiltonian of the system is given by [25,26] $\mathcal{H} = p_t \dot{t} + p_r \dot{r} + p_\phi \dot{\phi} - \mathcal{L}$, which implies

$$2\mathcal{H} = -E\dot{t} + L\dot{\phi} + \mu^{-1} \dot{r}^2 = \epsilon = \text{const}, \quad (19)$$

where $\epsilon = 0$ for null geodesics and $\epsilon = 1$ for timelike geodesics. Substituting Eqs. (16) and (17) into (19), one finds

$$\dot{r}^2 = \mu \left[\frac{E^2}{e^{-2\delta} \mu} - \frac{L^2}{r^2} - \epsilon \right]. \quad (20)$$

Circular geodesics are characterized by $\dot{r}^2 = (\dot{r}^2)' = 0$ [25,26]. This implies the relations

$$\begin{aligned} E^2 &= \frac{2e^{-4\delta} \mu^2}{2e^{-2\delta} \mu - r(e^{-2\delta} \mu)'}, \\ L^2 &= \frac{r^3 (e^{-2\delta} \mu)'}{2e^{-2\delta} \mu - r(e^{-2\delta} \mu)'}, \end{aligned} \quad (21)$$

for timelike geodesics. The requirement that the energy E be real enforces the inequality,

$$2e^{-2\delta} \mu - r(e^{-2\delta} \mu)' > 0. \quad (22)$$

From (20) one finds that the radius $r = r_\gamma$ of the null circular geodesic satisfies the relation,

$$r_\gamma = \frac{2e^{-2\delta} \mu}{(e^{-2\delta} \mu)'}. \quad (23)$$

Substituting the Einstein equations (4) and (5) into Eqs. (22) and (23), one finds

$$3\mu - 1 - 8\pi r^2 p \geq 0 \quad (24)$$

in the spacetime region where circular geodesics are allowed to exist. The equality sign corresponds to the limiting case of the null circular geodesic. In case there are several such zeroes, the photonsphere corresponds to the innermost one [31].

Finally, substituting Eq. (24) into (9), one finds the surprisingly simple relation,

$$P'(r_\gamma) = r_\gamma^3 T \leq 0, \quad (25)$$

where the last inequality follows from (11). The spacetime region between the horizon and the photonsphere in which circular geodesics are excluded (we refer to this region as the ‘‘no-circling zone’’) is characterized by $3\mu - 1 - 8\pi r^2 p < 0$. This implies

$$P'(r < r_\gamma) \leq 0. \quad (26)$$

Thus, Eqs. (14) and (26) imply that $P(r)$ is a nonpositive and decreasing function at least up to the point where the photonsphere is crossed. If we define $r = r_{\text{hair}}$ to be the point at which $|P(r)|$ has a local maximum [Eqs. (12) and (14) together imply that such a point must exist and that it must be crossed before the trivial asymptotic behavior (12) dominates], then our analysis reveals the lower bound,

$$r_{\text{hair}} \geq r_\gamma. \quad (27)$$

Thus, the nontrivial (nonasymptotic) behavior of the hair must extend *beyond* the photonsphere [32]. Note that our pressure function $P(r) \equiv r^4 p$ is different from the function $\mathcal{E}(r) \equiv e^{-\delta} r^4 p$ considered in [3]. Thus, the definitions of the length of the hair adopted in these papers are different from each other.

An interesting quantity which characterizes the spatial distribution of the hair is given by the dimensionless ratio $m_{\text{hair}}^+ / m_{\text{hair}}^-$, where

$$m_{\text{hair}}^+ \equiv M - m(r_\gamma) \quad (28)$$

is the mass of the hair which resides above the photonsphere, and

$$m_{\text{hair}}^- \equiv m(r_\gamma) - m(r_H) \quad (29)$$

is the mass of the hair which is contained between the horizon and the photonsphere. (Here M is the total ADM mass of the spacetime.) The result (27) suggests (but obviously does not prove) that a considerable fraction of the hair’s mass resides above the photonsphere. This raises the following question: Is there some fundamental lower bound on the ratio $m_{\text{hair}}^+ / m_{\text{hair}}^-$ for hairy black holes?

To answer this interesting question, we shall first examine the limiting case of the linear Maxwell field outside the Reissner-Nordström (RN) black-hole solution. Of course, this is not a case where a genuine hair is present since an additional conserved charge is needed in order to complete the specification of the solution. Nevertheless, for this

black-hole solution (like in any real hairy solution) the region exterior to the horizon is characterized by a nonzero energy density. For the Maxwell field, one has $-p(r) = \rho(r) = Q^2/8\pi r^4$ [30], which yields $m(r) = M - Q^2/2r$ for the mass function and $r_\gamma = \frac{1}{2}[3M + (9M^2 - 8Q^2)^{1/2}]$ for the location of the photonsphere; see Eqs. (6) and (24), respectively. Here M and Q are the total mass and electric charge of the spacetime, respectively. Substituting these relations into Eqs. (28) and (29) with $r_H = M + (M^2 - Q^2)^{1/2}$, one finds the ratio,

$$m_{\text{RN}}^+/m_{\text{RN}}^- = \frac{1}{r_\gamma/r_H - 1} \geq 1. \quad (30)$$

(The case $m_{\text{RN}}^+/m_{\text{RN}}^- = 1$ corresponds to the extremal black-hole solution with $Q = M$.)

The result (30) for the marginal case of a linear hair leads us to conjecture that genuine hairy black holes always satisfy the lower bound,

$$m_{\text{hair}}^+/m_{\text{hair}}^- \geq 1. \quad (31)$$

In other words, the region above the photonsphere always contains at least 50% of the total hair's mass.

There is one family of hairy black-hole configurations for which the suggested bound (31) can be tested analytically: the Einstein-Yang-Mills hairy black-hole solutions. The EYM equations can be solved analytically in the limit of large black holes (for which $m_{\text{hair}} \ll r_H$); see [19] for details. The hair of the $n = 1$ solution is then described by

Eq. (4.8) of [19], which yields the ratio $m_{\text{hair}}^+/m_{\text{hair}}^- = 2.08$. Thus, large EYM black holes indeed respect the conjectured bound (31).

We have also performed some numerical studies in order to put the conjectured bound (31) into test. The models we have considered include the Einstein-Yang-Mills, Einstein-Skyrme, Einstein-non-Abelian-Proca, Einstein-Yang-Mills-Higgs, and Einstein-Yang-Mills-Dilaton systems. We have found that all these hairy black-hole solutions indeed conform to the suggested bound (31). These studies will be reported elsewhere [33].

In summary, in this paper we have analyzed the non-trivial spatial behavior of the matter fields outside hairy black holes. In particular, we have proved a theorem which reveals the important role played by the null circular geodesic (the photonsphere) in the context of hairy black-hole configurations. According to this theorem, the nontrivial structure of the hair must extend *above* the photonsphere of the corresponding spacetime.

Furthermore, motivated by this theorem, we have put forward a conjecture according to which the region above the null circular geodesic contains at least 50% of the total hair's mass. This conjecture is supported by numerical computations for a variety of hairy black-hole configurations [33].

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- [1] R. Ruffini and J. A. Wheeler, *Phys. Today* **24**, 30 (1971); B. Carter, in *Black Holes, Proceedings of 1972 Session of Ecole d'ete de Physique Theorique*, edited by C. De Witt and B. S. De Witt (Gordon and Breach, New York, 1973).
- [2] J. D. Bekenstein, *Phys. Rev. D* **7**, 2333 (1973); *Phys. Today* **33**, 24 (1980); *Phys. Rev. D* **51**, R6608 (1995).
- [3] D. Núñez, H. Quevedo, and D. Sudarsky, *Phys. Rev. Lett.* **76**, 571 (1996).
- [4] W. Israel, *Phys. Rev.* **164**, 1776 (1967); *Commun. Math. Phys.* **8**, 245 (1968).
- [5] B. Carter, *Phys. Rev. Lett.* **26**, 331 (1971).
- [6] S. W. Hawking, *Commun. Math. Phys.* **25**, 152 (1972).
- [7] D. C. Robinson, *Phys. Rev. D* **10**, 458 (1974); *Phys. Rev. Lett.* **34**, 905 (1975).
- [8] J. Ispert, *Phys. Rev. Lett.* **27**, 529 (1971).
- [9] J. E. Chase, *Commun. Math. Phys.* **19**, 276 (1970); J. D. Bekenstein, *Phys. Rev. Lett.* **28**, 452 (1972); C. Teitelboim, *Lett. Nuovo Cimento* **3**, 326 (1972).
- [10] J. D. Bekenstein, *Phys. Rev. D* **5**, 1239 (1972); **5**, 2403 (1972); M. Heusler, *J. Math. Phys. (N.Y.)* **33**, 3497 (1992); D. Sudarsky, *Classical Quantum Gravity* **12**, 579 (1995).
- [11] J. Hartle, *Phys. Rev. D* **3**, 2938 (1971); C. Teitelboim, *Lett. Nuovo Cimento* **3**, 397 (1972).
- [12] P. Bizoń, *Phys. Rev. Lett.* **64**, 2844 (1990); M. S. Volkov and D. V. Gal'tsov, *Sov. J. Nucl. Phys.* **51**, 1171 (1990); H. P. Kuenzle and A. K. M. Masood-ul-Alam, *J. Math. Phys. (N.Y.)* **31**, 928 (1990).
- [13] G. Lavrelashvili and D. Maison, *Nucl. Phys.* **B410**, 407 (1993).
- [14] P. Bizoń and T. Chamaj, *Phys. Lett. B* **297**, 55 (1992); M. Heusler, S. Droz, and N. Straumann, *Phys. Lett. B* **268**, 371 (1991); **271**, 61 (1991); **285**, 21 (1992).
- [15] B. R. Greene, S. D. Mathur, and C. O'Neill, *Phys. Rev. D* **47**, 2242 (1993); T. Torii, K. Maeda, and T. Tachizawa, *Phys. Rev. D* **51**, 1510 (1995).
- [16] N. Straumann and Z.-H. Zhou, *Phys. Lett. B* **243**, 33 (1990).
- [17] P. Bizoń and R. M. Wald, *Phys. Lett. B* **267**, 173 (1991).
- [18] N. E. Mavromatos and E. Winstanley, *Phys. Rev. D* **53**, 3190 (1996).
- [19] M. S. Volkov and D. V. Gal'tsov, *Phys. Rep.* **319**, 1 (1999).
- [20] P. Bizoń and T. Chmaj, *Phys. Rev. D* **61**, 067501 (2000).
- [21] G. V. Lavrelashvili and D. Maison, *Phys. Lett. B* **343**, 214 (1995).
- [22] M. S. Volkov, O. Brodbeck, G. V. Lavrelashvili, and N. Straumann, *Phys. Lett. B* **349**, 438 (1995).
- [23] P. Bizoń, *Phys. Lett. B* **259**, 53 (1991).

- [24] J. D. Bekenstein, [arXiv:gr-qc/9605059](#).
- [25] S. Chandrasekhar, *The Mathematical Theory of Black Holes* (Oxford University Press, New York, 1983).
- [26] V. Cardoso, A. S. Miranda, E. Berti, H. Witek, and V. T. Zanchin, *Phys. Rev. D* **79**, 064016 (2009).
- [27] The circular null geodesic is actually the limiting case of circular timelike geodesics (that is, the innermost circular orbit [25,26]); see details below.
- [28] S. Hod, *Phys. Lett. B* **657**, 255 (2007); S. Hod, *Classical Quantum Gravity* **24**, 6019 (2007); *Phys. Lett. B* **661**, 175 (2008).
- [29] H. Bondi, *Mon. Not. R. Astron. Soc.* **259**, 365 (1992).
- [30] A. E. Mayo and J. D. Bekenstein, *Phys. Rev. D* **54**, 5059 (1996); N. E. Mavromatos, [arXiv:gr-qc/9606008](#).
- [31] Note that Eqs. (3) and (13) imply $3\mu - 1 - 8\pi r^2 p < 0$ at the black-hole horizon. On the other hand, Eqs. (2) and (12) imply $3\mu - 1 - 8\pi r^2 p \rightarrow 2$ as $r \rightarrow \infty$. Thus, there must be some intermediate point at which $3\mu - 1 - 8\pi r^2 p = 0$. This corresponds to the location of the null circular geodesic.
- [32] We emphasize that our analysis reveals that $P(r)$ is a nonpositive and decreasing function in the interval $[r_H, r_{\text{hair}}]$, with $r_{\text{hair}} \geq r_\gamma$.
- [33] S. Hod (unpublished).