Constraints and analytical solutions of $f(R)$ theories of gravity using Noether symmetries

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We perform a detailed study of the modified gravity $f(R)$ models in the light of the basic geometrical symmetries, namely Lie and Noether point symmetries, which serve to illustrate the phenomenological viability of the modified gravity paradigm as a serious alternative to the traditional scalar field approaches. In particular, we utilize a model-independent selection rule based on first integrals, due to Noether symmetries of the equations of motion, in order to identify the viability of $f(R)$ models in the context of flat Friedmann-Lemaître-Robertson-Walker cosmologies. The Lie/Noether point symmetries are computed for six modified gravity models that include also a cold dark matter component. As it is expected, we confirm that all the proposed modified gravity models admit the trivial first integral, namely energy conservation. We find that only the $f(R) = (R^b - 2\Lambda)^c$ model, which generalizes the concordance Λ cosmology, accommodates extra Lie/Noether point symmetries. For this $f(R)$ model the existence of nontrivial Noether (first) integrals can be used to determine the integrability of the model. Indeed within this context we solve the problem analytically and thus we provide for the first time the evolution of the main cosmological functions such as the scale factor of the universe and the Hubble expansion rate.

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I. INTRODUCTION

The comprehensive study carried out in recent years by the cosmologists has converged towards a cosmic expansion history that involves a spatially flat geometry and a recent accelerating expansion of the universe (see [1–8] and references therein). From a theoretical point of view, an easy way to explain this expansion is to consider an additional energy component with negative pressure, usually called dark energy, that dominates the universe at late times. In spite of that, the absence of a fundamental physical theory, regarding the mechanism inducing the cosmic acceleration, has given rise to a plethora of alternative cosmological scenarios. Most of them are based either on the existence of new fields in nature (dark energy) or in some modification of Einstein's general relativity (GR), with the present accelerating stage appearing as a sort of geometric effect (''geometrical'' dark energy).

The necessity to preserve Einstein's equations inspired cosmologists to conservatively invoke the simplest available hypothesis, namely, a cosmological constant, Λ (see [9–11] for reviews). Indeed the so-called spatially flat concordance ΛCDM model, which includes cold dark matter and a cosmological constant (Λ), fits accurately the current observational data and thus it is an excellent candidate model of the observed universe. Nevertheless, the identification of Λ with the quantum vacuum has brought another problem which is: *the estimate that the vacuum energy density should be 120 orders of magnitude larger than the measured* Λ *value*. This is the ''old'' cosmological constant problem [9]. The ''new'' problem [12] is related with the following question: *why is the vacuum density so similar to the matter density at the present time?*

Such problems have inspired many authors to propose alternative dark energy candidates (see [13] for review) such as $\Lambda(t)$ cosmologies, quintessence, k essence, vector fields, phantom dark energy, tachyons, and Chaplygin gas (see [9,14–30] and references therein). Naturally, in order to establish the evolution of the dark energy equation of state, a realistic form of $H(a)$ is required which should be constrained through a combination of independent dark energy probes.

On the other hand, there are other possibilities to explain the present accelerating stage. For instance, one may consider that the dynamical effects attributed to dark energy can be resembled by the effects of a nonstandard gravity theory. In other words, the present accelerating stage of the universe can be driven only by cold dark matter, under a modification of the nature of gravity. Such a reduction of the so-called dark sector is naturally obtained in the $f(R)$ gravity theories [31]. In the original nonstandard gravity models, one modifies the Einstein-Hilbert action with a general function $f(R)$ of the Ricci scalar R. The $f(R)$ approach is a relatively simple but still fundamental tool used to explain the accelerated expansion of the universe. A pioneering fundamental approach was proposed long ago, where $f(R) = R + mR^2$ [32]. Later on, the $f(R)$ models were further explored from different points of view in [33–35] and indeed a large number of functional forms of $f(R)$ gravity is currently available in the literature. It is interesting to mention here that subsequent investigations [35] confirmed that $1/R$ gravity is an unacceptable model because it fails to reproduce the correct cosmic expansion in the matter era.

In this paper, we wish to test some basic functional forms of $f(R)$ in light of the Lie/Noether point symmetries. The idea to use Noether symmetries in cosmological studies is not new and indeed a lot of attention has been paid in the literature (see [36–46]). Recently, we have proposed (see Basilakos *et al.* [47]) that the existence of Lie/Noether point symmetries can be used as a selection criterion in order to distinguish the functional form of the potential energy $V(\phi)$ of the dark energy models that adhere to general relativity (GR). In this work we would like to extend the paper of Basilakos *et al.* [47] by applying the same approach to $f(R)$ models. In particular, the scope of the current article is (a) to investigate which of the available $f(R)$ models admit extra Lie and Noether point symmetries, and (b) for these models to solve the system of the resulting field equations and derive analytically (for the first time to our knowledge) the main cosmological functions (the scale factor, the Hubble expansion rate, etc.). We would like to remind the reader that a fundamental approach to derive the Lie and Noether point symmetries for a given dynamical problem living in a Riemannian space has been published recently by Tsamparlis and Paliathanasis [48] (a similar analysis can be found in [49–55]).

The structure of the paper is as follows. The basic theoretical elements of the problem are presented in Sec. II, where we also introduce the basic Friedmann-Lemaître-Robertson-Walker (FLRW) cosmological equations in the framework of $f(R)$ models. The geometrical Lie/Noether point symmetries and their connections to the $f(R)$ models are discussed in Sec. III. In Sec. IV we provide analytical solutions for those $f(R)$ models which admit nontrivial Lie/Noether point symmetries. Finally, we draw our main conclusions in Sec. V.

II. COSMOLOGY WITH A MODIFIED GRAVITY

Consider the modified Einstein-Hilbert action:

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2k^2}f(R) + \mathcal{L}_m\right], \tag{1}$$

where $\mathcal{L}_m$ is the Lagrangian of dustlike ($p_m = 0$) matter and $k^2 = 8\pi G$. Now varying the action with respect to the metric[1] we arrive at

$$(1+f')G^{\mu}_{\nu} - g^{\mu\alpha}f_{R,\alpha;\nu} + \left[\frac{2\Box f' - (f - Rf')}{2}\right]\delta^{\mu}_{\nu} = k^2 T^{\mu}_{\nu}, \tag{2}$$

where the prime denotes derivative with respect to R, G^{μ}_{ν} is the Einstein tensor, and T^{μ}_{ν} is the energy-momentum tensor of matter. Based on the matter era we treat the expanding universe as a perfect fluid which includes only cold dark matter with 4-velocity U_μ. Thus, the energy-momentum tensor becomes $T^{\mu}_{\nu} = -p_m g^{\mu}_{\nu} + (\rho_m + p_m)U^{\mu}U_{\nu}$, where ρ_m and $p_m = 0$ are the energy density and pressure of the cosmic fluid, respectively. The Bianchi identity $\nabla^{\mu}T_{\mu\nu} = 0$ leads to the matter conservation law:

$$\dot{\rho}_m + 3H\rho_m = 0, \tag{3}$$

the solution of which is $\rho_m = \rho_{m0}a^{-3}$. Note that the overdot denotes derivative with respect to the cosmic time t, $a(t)$ is the scale factor, and $H \equiv \dot{a}/a$ is the Hubble parameter.

Now, in the context of a flat FLRW metric with Cartesian coordinates

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2), \tag{4}$$

the Einstein's tensor components are given by

$$G^0_0 = -3H^2, \qquad G^{\mu}_{\nu} = -\delta^{\mu}_{\nu}(2\dot{H} + 3H^2). \tag{5}$$

Inserting Eq. (5) into the modified Einstein's field equation (2), for comoving observers, we derive the modified Friedmann's equations:

$$3f'H^2 = k^2\rho_m + \frac{f'R - f}{2} - 3Hf''\dot{R} \tag{6}$$

$$2f'\dot{H} + 3f'H^2 = -2Hf''\dot{R} - (f'''\dot{R}^2 + f''\ddot{R}) - \frac{f - Rf'}{2}. \tag{7}$$

Also, the contraction of the Ricci tensor provides the Ricci scalar

$$R = g^{\mu\nu}R_{\mu\nu} = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) = 6(2H^2 + \dot{H}). \tag{8}$$

Of course, if we consider $f(R) = R$ then the field equation (2) boils down to the nominal Einstein's equations, a solution of which is the Einstein de Sitter model. On the other hand, the concordance Λ cosmology is fully recovered for $f(R) = R - 2\Lambda$.

From the current analysis it becomes clear that unlike the standard Friedmann equations in Einstein's GR the modified equations of motion (6) and (7) are complicated and thus it is difficult to solve analytically. However, the existence of nontrivial Noether (first) integrals can be used to simplify the system of differential equations (6) and (7) as well as to determine the integrability of the system (see Sec. IV).

A. The $f(R)$ functional forms

In order to solve the system of Eqs. (6) and (7), we need to know *a priori* the functional form of $f(R)$. Because of the absence of a physically well-motivated functional form for the $f(R)$ parameter, there are many theoretical speculations in the literature. Below we briefly present various $f(R)$ models whose free parameters, namely

[1]We use the metric, i.e., the Hilbert variational approach.

$(m, n, R_c) > (0, 0, 0)$, can be constrained from the current cosmological data.

(i) The power law model [33,56,57]:

$$f(R) = R - m/R^n. \tag{9}$$

(ii) The Amendola *et al.* [35] modified gravity model:

$$f(R) = R - mR_c(R/R_c)^p \tag{10}$$

with $0 < p < 1$.

(iii) The Hu and Sawicki [58] model:

$$f(R) = R - mR_c \frac{(R/R_c)^{2n}}{(R/R_c)^{2n} + 1}. \tag{11}$$

(iv) The Starobinsky [59] model:

$$f(R) = R - mR_c[1 - (1 + R^2/R_c^2)^{-n}]. \tag{12}$$

(v) The Tsujikawa [60] model:

$$f(R) = R - mR_c \tanh(R/R_c). \tag{13}$$

(vi) The generalization of the ΛCDM model (hereafter Λ_{bc}CDM model [61]):

$$f(R) = (R^b - 2\Lambda)^c, \tag{14}$$

where the product bc is of order of unity $\mathcal{O}(1)$ and $c \geq 1$. The latter inequality is due to the existence of the matter epoch.

Detailed analyses of these potentials exist in the literature, including their confrontation with the observational data (see [13] for extensive reviews). We would like to stress here that within the context of the metric formalism the above $f(R)$ cosmological models must obey simultaneously the same strong conditions (for an overall discussion see [13]). Briefly these are: (i) $f' > 0$ for $R \geq R_0 > 0$, where R_0 is the Ricci scalar at the present time. If the final attractor is a de Sitter point we need to have $f' > 0$ for $R \geq R_1 > 0$, where R_1 is the Ricci scalar at the de Sitter point, (ii) $f'' > 0$ for $R \geq R_0 > 0$, (iii) $f(R) \approx R - 2\Lambda$ for $R \gg R_0$, and finally (iv) $0 < \frac{Rf''}{f'}(r) < 1$ at $r = -\frac{Rf'}{f} = -2$.

Notice, that the power law $f(R)$ model fails with respect to condition (ii). The rest of the models satisfy all the above conditions and thus they provide predictions which are similar to those of the usual dark energy models, as far as the cosmic history (presence of the matter era, stability of cosmological perturbations, stability of the late de Sitter point, etc.) is concerned. Finally, in an appendix we discuss more $f(R)$ models which however do not satisfy the conditions (i)–(iv) [62].

III. MODIFIED GRAVITY VERSUS SYMMETRIES

In Basilakos *et al.* [47], we have proposed to use the Noether symmetry approach as a model-independent criterion, in order to classify the dark energy models that adhere to general relativity. The aim of this work is along the same lines, attempting to investigate the nontrivial Noether symmetries (first integrals of motion) by generalizing the methodology of Basilakos *et al.* [47] for modified gravity models (see Sec. II A). This can help us to understand better the theoretical basis of the $f(R)$ models as well as the variants from GR.

In the past decade, a large number of experiments have been proposed in order to constrain dark energy and study its evolution. Naturally, in order to establish the evolution of the dark energy (''geometrical'' in the current work) equation of state parameter a realistic form of $H(a)$ is required while the included free parameters must be constrained through a combination of independent DE probes (for example SNIa, BAOs, CMB, etc.). However, a weak point here is the fact that the majority of the $f(R)$ models which appeared in the literature are plagued with no clear physical basis and/or many free parameters. Because of the large number of free parameters many such models could fit the data. The proposed additional criterion of the Lie/Noether symmetry requirement is a physically meaningful geometric ansatz, which could be employed in order to select amongst the set of viable models those which satisfy this constraint. Practically for those $f(R)$ models which manage to survive from the comparison with the available cosmological data, our goal is to define a method that can further distinguish the $f(R)$ models on a more fundamental (e.g. geometrical) level and at the same time provide first integrals which can be used to integrate the modified Friedmann's equations.

According to the theory of general relativity, the space-time symmetries (Killing and homothetic vectors) via the Einstein's field equations, are also symmetries of the energy-momentum tensor. Because of the fact that the $f(R)$ models provide a natural generalization of GR, one would expect that the theories of modified gravity must inherit the symmetries of the space-time as the usual gravity (GR) does.

Furthermore, besides the geometric symmetries we have to consider the dynamical symmetries, which are the symmetries of the field equations (Lie symmetries). If the field equations are derived from a Lagrangian then there is a special class of Lie symmetries, the Noether symmetries,[2] which lead to conserved currents or, equivalently, to first integrals of the equations of motion. The Noether integrals are used to reduce the order of the field equations or even to solve them. Therefore a sound requirement which is possible to be made in Lagrangian theories is that they admit

[2]Note that the Noether symmetries are a subalgebra of the algebra defined by the Lie symmetries [48].

extra Noether symmetries. This assumption is model independent, because it is imposed after the field equations have been derived, therefore it does not lead to conflict with the geometric symmetries while, at the same time, serves the original purpose of a selection rule. Of course, it is possible that a different method could be assumed and select another subset of viable models. However, symmetry has always played a dominant role in physics and this gives an aesthetic and a physical priority to our proposal.

In the Lagrangian context, we can easily prove that the main field equations (6) and (7), described in Sec. II, can be produced by the following Lagrangian:

$$L = 6af'\dot{a}^2 + 6a^2 f''\dot{a}\dot{R} + a^3(f'R - f) \tag{15}$$

in the space of the variables $\{a, R\}$. Using Eq. (15) we obtain the Hamiltonian of the current dynamical system:

$$E = 6af'\dot{a}^2 + 6a^2 f''\dot{a}\dot{R} - a^3(f'R - f) \tag{16}$$

or

$$E = 6a^3\left[f'H^2 - \frac{(f'R - f)}{6} + \dot{R}Hf''\right]. \tag{17}$$

Combining the first equation of motion (6) with Eq. (17), we find

$$\rho_m = \frac{E}{2k^2}a^{-3}. \tag{18}$$

The latter equation together with $\rho_m = \rho_{m0}a^{-3}$ implies that

$$\rho_{m0} = \frac{E}{2k^2} \Rightarrow \Omega_m \rho_{cr,0} = \frac{E}{2k^2} \Rightarrow E = 6\Omega_m H_0^2, \tag{19}$$

where $\Omega_m = \rho_{m0}/\rho_{cr,0}$, $\rho_{cr,0} = 3H_0^2/k^2$ is the critical density at the present time, and H_0 is the Hubble constant.

We note that the current Lagrangian Eq. (15) is time independent implying that the dynamical system is autonomous; hence, the Hamiltonian E is conserved ($\partial_t E \equiv \frac{dE}{dt} = 0$). Therefore, all the $f(R)$ functions described in Sec. II A admit the trivial Noether symmetry, namely, energy conservation as they should.

Extra Lie and Noether symmetries

Here we briefly present only the main points of the method used to constrain the $f(R)$ models. In particular, let us assume a modified gravity $f(R)$ cosmological model which accommodates a late time ''accelerated'' expansion and it satisfies the strong conditions (i)–(iv) of Sec. II A. We pose here a similar question with that proposed in Basilakos *et al.* [47] for the dark energy models that adhere to GR. For the modified gravity, namely $f(R)$ that lives into a two-dimensional Riemannian space $\{a, R\}$ and which is embedded in the space-time, how many (if any) of the previously presented functional forms (see Sec. II A) can provide nontrivial Noether symmetries (or first integrals of motion)? As an example, if we find a modified gravity model (or a family of models) for which its $f(R)$ admits nontrivial first integrals of motion with respect to the other $f(R)$ cosmological models, then obviously this model contains an extra geometrical feature. Therefore, we can use this geometrical characteristic in order to classify this particular $f(R)$ cosmological model into a special category (see also [36,39–41,43,44]).

In order to compute the Lie/Noether point symmetries of equations of motion (6) and (7), we consider the Lagrangian[3] (15) as the sum of a kinetic energy and a conservative force field. The kinetic term defines a two-dimensional metric in the space of $\{a, R\}$. Following standard lines (see [47] and references therein), the two-dimensional metric takes the form

$$d\hat{s}^2 = 12af'da^2 + 12a^2 f''dadR \tag{20}$$

while the ''potential'' is

$$V(a, R) = -a^3(f'R - f). \tag{21}$$

The signature of the metric Eq. (20) is +1 and the Ricci scalar is computed to be $\hat{R} = 0$, therefore the space is the 2D Euclidean space.[4] Using the kinematic metric (20) we can utilize the plethora of results of differential geometry on collineations to produce the solution of the Lie/Noether point symmetry problem.

We recall that the special projective algebra of the of the Euclidean 2D metric (20) consists of the following vectors:

$$K^1 = a\partial_a - 3\frac{f'}{f''}\partial_R,$$

$$K^2 = \frac{1}{a}\partial_a - \frac{1}{a^2}\frac{f'}{f''}\partial_R,$$

$$K^3 = \frac{1}{a}\frac{1}{f''}\partial_R,$$

$$H^i = \frac{a}{2}\partial_a + \frac{1}{2}\frac{f'}{f''}\partial_R,$$

$$A^1 = f'\partial_a - \frac{1}{a}\frac{(f')^2}{f''}\partial_R,$$

$$A^2 = \frac{a}{f''}\partial_R,$$

$$A^3 = a\partial_a,$$

$$A^4 = \frac{f'}{f''}\partial_R,$$

$$P^1 = \frac{3}{2}a^2 f'\partial_a + \frac{3}{2}a\frac{(f')^2}{f''}\partial_R,$$

$$P^2 = \frac{3}{2}a^3\partial_a + \frac{3}{2}a^2\frac{f'}{f''}\partial_R$$

[3] In Appendix B we discuss the Noether symmetries in nonflat $f(R)$ models.

[4] For the traditional dark energy models the signature of the two-dimensional metric is −1 which means that the 2D space is Minkowski [47]. Also all two-dimensional Riemannian spaces are Einstein spaces implying that if $\hat{R} = 0$ the space is flat.

where **K** are Killing vectors ($\mathbf{K}^{2,3}$ are gradient), **H** is a gradient Homothetic vector, **A** are affine collineations, and **P** are special projective collineations. These are ten vectors whereas the projective algebra of the two-dimensional flat space consists of eight vectors [63]. It can be shown that the vectors K^1, H^i are a linear combination of the affine vectors $\mathbf{A}^I$, $I = 3, 4$.

Now we are looking for Noether symmetries beyond the standard one, ∂_t. Utilizing the potential Eq. (21) and theorems 1 and 2 of [48,55], we find that among the $f(R)$ models explored here (see Sec. II A), only the Λ_{bc}CDM model [61] with $(b, c) = (1, \frac{3}{2})$ admits extra Lie/Noether point symmetries. In particular, the Lie point symmetries are

$$X_{L_1} = A^3, \qquad X_{L_2} = (c_1 e^{\sqrt{m}t} + c_2 e^{-\sqrt{m}t})K^2, \tag{22}$$

where the quantity X_{L_2} is also Noether symmetry with gauge function

$$g_{L_2} = 9\sqrt{m}(c_1 e^{\sqrt{m}t} - c_2 e^{-\sqrt{m}t})a\sqrt{R - 2\Lambda},$$

where $c_{1,2}$ are constants and $m = 2\Lambda/3$. If we relax the condition of $c \geq 1$ [61], then we discover a second Λ_{bc}CDM model with $(b, c) = (1, \frac{7}{8})$ that accommodates two Lie point symmetries, the X_{L_1} and the Noether point symmetry

$$X_{L_3} = \left(\frac{c_1}{\sqrt{m}} e^{2\sqrt{m}t} - \frac{c_2}{\sqrt{m}} e^{-2\sqrt{m}t}\right)\partial_t + (c_1 e^{2\sqrt{m}t} + c_2 e^{-2\sqrt{m}t})H^i, \tag{23}$$

with gauge function

$$g_{L_3} = \frac{21}{4}\sqrt{m}(c_1 e^{2\sqrt{m}t} - c_2 e^{-2\sqrt{m}t})a^3(R - 2\Lambda)^{-(1/8)}.$$

We have to mention here that the $f(R) = (R - 2\Lambda)^{7/8}$ model does not satisfy the condition (ii), namely $f''(R) > 0$.

To conclude the discussion we would like to stress that the novelty in this work is the fact that among the current modified gravity models (see Sec. II A) only the Λ_{bc}CDM model [61], which generalizes the concordance ΛCDM model, admits extra Lie/Noether point symmetries. This implies that the Λ_{bc}CDM model can be clearly distinguished from the other modified gravity models. Interestingly enough, the existence of the extra Lie/Noether point symmetries puts even further theoretical constraints on the free parameters of the Λ_{bc}CDM model, $(b, c) = (1, \frac{3}{2})$ and $(b, c) = (1, \frac{7}{8})$. From now on, we focus on the latter $f(R)$ models and in the next section we provide for a first time (to our knowledge) analytical solutions.

IV. ANALYTICAL SOLUTIONS

Using the Noether symmetries and the associated Noether integrals we solve analytically differential Eqs. (6) and (7).

A. Λ_{bc}CDM model with $(b, c) = (1, \frac{3}{2})$

Inserting $f(R) = (R - 2\Lambda)^{3/2}$ into Eq. (15) we obtain

$$L = 9a\sqrt{R - 2\Lambda}\dot{a}^2 + \frac{9a^2}{2\sqrt{R - 2\Lambda}}\dot{a}\dot{R} + \frac{a^3}{2}(R + 4\Lambda)\sqrt{R - 2\Lambda}. \tag{24}$$

Changing now the variables from (a, R) to (x, y) via the relations

$$a = \left(\frac{9}{2}\right)^{-(1/3)}\sqrt{x}, \qquad R = 2\Lambda + \frac{y^2}{x}, \qquad (x, y) \neq (0, 0),$$

the Lagrangian (24) and the Hamiltonian (16) become

$$L = \dot{x}\dot{y} + V_0(y^3 + \bar{m}xy) \tag{25}$$

$$E = \dot{x}\dot{y} - V_0(y^3 + \bar{m}xy), \tag{26}$$

where $V_0 = 1/9$ and $\bar{m} = 6\Lambda$.

The equations of motion, using the Euler-Lagrange equations, in the new coordinate system are

$$\ddot{x} - 3V_0 y^2 - \bar{m}V_0 x = 0 \tag{27}$$

$$\ddot{y} - \bar{m}V_0 y = 0. \tag{28}$$

The Noether point symmetries (22) in the coordinate system $\{x, y\}$ become

$$X'_{L_2} = (c_1 e^{\omega t} + c_2 e^{-\omega t})\partial_y, \tag{29}$$

where $\omega = \sqrt{\bar{m}V_0} = \sqrt{2\Lambda/3}$ and the corresponding Noether integrals are

$$I_1 = e^{\omega t}\dot{y} - \omega e^{\omega t} y \tag{30}$$

$$I_2 = e^{-\omega t}\dot{y} + \omega e^{-\omega t} y. \tag{31}$$

From these we construct the time independent first integral

$$\Phi = I_1 I_2 = \dot{y}^2 - \omega^2 y^2. \tag{32}$$

The constants of integration are further constrained by the condition that at the singularity ($t = 0$), the scale factor has to be exactly zero, that is, $x(0) = 0$.

We consider the cases $\Phi = 0$ and $\Phi \neq 0$.

A. Case $\Phi = 0$.

We have the following subcases:

A.1. $I_1 = I_2 = 0$

The solution of the system of equations (27) and (28) is

$$x(t) = x_1 e^{\omega t} + x_2 e^{-\omega t}, \qquad y(t) = 0 \tag{33}$$

and the Hamiltonian constraint gives $E = 0$, where $x_{1,2}$ are constants. The singularity condition gives the constraint $x_1 = -x_2$. At late enough times the scale factor evolves as $a^2(t) \propto x(t) \propto x_1 e^{\omega t}$. However, this particular solution is ruled out because it violates $y(t) \neq 0$.

A.2. $I_1 = 0$ ($I_2 \neq 0$)

The solution of the system (27) and (28) is

$$y(t) = \frac{I_2}{2\omega} e^{\omega t} \tag{34}$$

$$x(t) = x_1' e^{\omega t} + x_2' e^{-\omega t} + \frac{I_2^2}{4\omega^2 \bar{m}} e^{2\omega t} \tag{35}$$

and the Hamiltonian constraint gives $E = -x_2' I_2 \omega$, where $x_{1,2}'$ are constants. The singularity condition gives the constraint

$$x_1' + x_2' + \frac{I_2^2}{4\omega^2 \bar{m}} = 0. \tag{36}$$

At late times the solution becomes $a^2(t) \propto x(t) \simeq \frac{I_2^2}{4\omega^2 \bar{m}} e^{2\omega t}$.

A.3. $I_2 = 0$ ($I_1 \neq 0$)

The solution of the system (27) and (28) is

$$y(t) = -\frac{I_1}{2\omega} e^{-\omega t} \tag{37}$$

$$x(t) = \bar{x}_1 e^{\omega t} + \bar{x}_2 e^{-\omega t} + \frac{I_1^2}{4\omega^2 \bar{m}} e^{-2\omega t} \tag{38}$$

and the Hamiltonian constraint gives $E = \bar{x}_1 I_1 \omega$, where $\bar{x}_{1,2}$ are constants. The singularity condition gives the constraint

$$\bar{x}_1 + \bar{x}_2 + \frac{I_1^2}{4\omega^2 \bar{m}} = 0.$$

This particular solution is not viable because in the matter era we have $e^{-\omega t} \sim 0$ implying that $y(t) \sim 0$.

B. Case $\Phi \neq 0$

In this case the $I_{1,2} \neq 0$. The general solution of the system (27) and (28) is

$$y(t) = \frac{I_2}{2\omega} e^{\omega t} - \frac{I_1}{2\omega} e^{-\omega t} \tag{39}$$

$$x(t) = x_{1G} e^{\omega t} + x_{2G} e^{-\omega t} + \frac{1}{4\bar{m}\omega^2}(I_2 e^{\omega t} + I_1 e^{-\omega t})^2 + \frac{\Phi}{\bar{m}\omega^2}. \tag{40}$$

The Hamiltonian constraint gives $E = \omega(x_{1G} I_1 - x_{2G} I_2)$, where $x_{1G,2G}$ are constants and the singularity condition results in the constraint

$$x_{1G} + x_{2G} + \frac{1}{4\bar{m}\omega^2}(I_1 + I_2)^2 + \frac{\Phi}{\bar{m}\omega^2} = 0.$$

Interestingly, one can show that the general solution includes a proper matter era in which $H(a) \propto a^{-3/2}$ (see Appendix C). Also, at late enough times the solution becomes $a^2(t) \propto x(t) \propto [I_2^2/(4\omega^2 \bar{m})] e^{2\omega t}$.

B. Λ_{bc}CDM model with $(b, c) = (1, \frac{7}{8})$

Despite the fact that the current $f(R)$ model is physically unacceptable due to $f''(R) < 0$, below we present its analytical solution for mathematical interest. In this case the Lagrangian Eq. (15) of the $f(R) = (R - 2\Lambda)^{7/8}$ model is written as

$$L = \frac{21a}{(R-2\Lambda)^{(1/8)}} \dot{a}^2 - \frac{21}{32} \frac{a^2}{(R-2\Lambda)^{(9/8)}} \dot{a}\dot{R} - \frac{1}{8} a^3 \frac{(R - 16\Lambda)}{(R-2\Lambda)^{(1/8)}}. \tag{41}$$

We introduce the new coordinates (u, v) by means of the transformations:

$$a = \left(\frac{21}{8}\right)^{-(1/3)} \sqrt{x}, \qquad R = 2\Lambda + \frac{x^4}{y^8}, \qquad (x, y) \neq (0, 0)$$

and

$$x = \frac{1}{\sqrt{2}} uv, \qquad y = \frac{1}{\sqrt{2}} \frac{u}{v}.$$

In the coordinates (u, v) the Lagrangian is

$$L = \frac{1}{2}\dot{u}^2 - \frac{1}{2}\frac{u^2}{v^2}\dot{v}^2 + V_0 \frac{m}{8} u^2 + 2V_0 \frac{v^{12}}{u^2}, \tag{42}$$

where $\bar{m} = -14\Lambda$, $V_0 = -\frac{1}{21}$, and the Hamiltonian

$$E = \frac{1}{2}\dot{u}^2 - \frac{1}{2}\frac{u^2}{v^2}\dot{v}^2 - V_0 \frac{m}{8} u^2 - 2V_0 \frac{v^{12}}{u^2}. \tag{43}$$

The Euler-Lagrange equations provide the following equations of motion:

$$\ddot{u} + \frac{u}{v^2}\dot{v}^2 - \frac{V_0 m}{4} u + 4V_0 \frac{v^{12}}{u^3} = 0 \tag{44}$$

$$\ddot{v} + \frac{2}{u}\dot{u}\dot{v} - \frac{1}{v}\dot{v}^2 + 24V_0 \frac{v^{13}}{u^4} = 0. \tag{45}$$

The Noether symmetries (23) become

$$X_{L_3}' = \left(\frac{c_1}{\lambda} e^{2\lambda t} - \frac{c_2}{\lambda} e^{-2\lambda t}\right)\partial_t + (c_1 e^{2\lambda t} + c_2 e^{-2\lambda t}) u \partial_u, \tag{46}$$

where $\lambda = \frac{1}{2}\sqrt{\bar{m}V_0} = \frac{1}{2}\sqrt{\frac{2}{3}\Lambda}$. The corresponding Noether integrals are

$$I_+ = \frac{1}{\lambda} e^{2\lambda t} E - e^{2\lambda t} u\dot{u} + \lambda e^{2\lambda t} u^2 \tag{47}$$

$$I_- = \frac{1}{\lambda} e^{-2\lambda t} E + e^{-2\lambda t} u\dot{u} + \lambda e^{-2\lambda t} u^2. \tag{48}$$

Following [53] we construct the following time independent integral using a combination of the first integrals (43), (47), and (48):

$$\phi = \frac{u^4}{v^2}\dot{v}^2 + 4V_0 v^{12}. \quad (49)$$

The first integral ϕ is called the Ermakov-Lewis invariant.[5] Using the Ermakov-Lewis invariant, the Hamiltonian (43) and Eq. (44) are written:

$$\frac{1}{2}\dot{u}^2 - V_0\frac{m}{8}u^2 - \frac{1}{2}\frac{\phi}{u^2} = E \quad (50)$$

$$\ddot{u} - \frac{V_0 m}{4}u + \frac{\phi}{u^3} = 0. \quad (51)$$

The solution of (51) has been given by Pinney [64] and it is the following:

$$u(t) = (u_1 e^{2\lambda t} + u_2 e^{-2\lambda t} + 2u_3)^{1/2}, \quad (52)$$

where u_{1-3} are constants such as

$$\phi = 4\lambda^2(u_3^2 - u_1 u_2). \quad (53)$$

From the Hamiltonian constraint (50) and the Noether integrals (47) and (48) we find

$$E = -2\lambda u_3, \qquad I_+ = 2\lambda u_2, \qquad I_- = 2\lambda u_1.$$

Replacing (52) in the Ermakov-Lewis invariant (49) and assuming $\phi \neq 0$ we find

$$v(t) = 2^{1/6}\phi^{1/12}e^{-A(t)}(4V_0 + e^{-12A(t)})^{-1/6}, \quad (54)$$

where

$$A(t) = \arctan\left[\frac{2\lambda}{\sqrt{\phi}}(u_1 e^{2\lambda t} + u_3)\right] + 4\lambda^2 u_1\sqrt{\phi}. \quad (55)$$

Then the solution is

$$x(t) = 2^{-1/3}\phi^{1/12}e^{-A(t)}(4V_0 + e^{-12A(t)})^{-1/6} \times (u_1 e^{2\lambda t} + u_2 e^{-2\lambda t} + 2u_3)^{1/2}, \quad (56)$$

where from the singularity condition $x(0) = 0$ we have the constraint $u_1 + u_2 + 2u_3 = 0$, or

$$2E - (I_+ + I_-) = 0. \quad (57)$$

At late enough time we find $A(t) \simeq A_0$, which implies $a^2(t) \propto x(t) \propto e^{\lambda t}$.

In the case where $\phi = 0$ equations (50) and (51) describe the hyperbolic oscillator and the solution is

$$u(t) = \sinh\lambda t, \qquad 2E = \lambda^2. \quad (58)$$

From the Ermakov-Lewis invariant we have

$$v(t) = \left(\frac{\lambda\sinh\lambda t}{\lambda v_1\sinh\lambda t - 12\sqrt{|V_0|}e^{-2\lambda t}}\right)^{1/6}, \quad (59)$$

where v_1 is a constant. In this case the solution is

$$x(t) = \frac{1}{\sqrt{2}}\left(\frac{\lambda\sinh^7\lambda t}{\lambda v_1\sinh\lambda t - 12\sqrt{|V_0|}e^{-2\lambda t}}\right)^{1/6}. \quad (60)$$

At late times the scale factor varies $a^2(t) \propto x(t) \propto e^{\lambda t}$.

V. CONCLUSIONS

In the literature the functional forms of $f(R)$ of the modified $f(R)$ gravity models are mainly defined on a phenomenological basis. In this article we use the Noether symmetry approach to constrain these models with the aim to utilize the existence of nontrivial Noether symmetries as a selection criterion that can distinguish the $f(R)$ models on a more fundamental (e.g., geometrical) level. Furthermore, the resulting Noether integrals can be used to provide analytical solutions.

In Basilakos *et al.* [47], we have utilized the Noether symmetry approach to study the dark energy (quintessence or phantom) models within the context of scalar field FLRW cosmology. Overall the combination of the work of Basilakos *et al.* [47] with the current article provides a complete investigation of the Noether symmetry approach in cosmological studies. From both works it becomes clear that the Noether symmetry approach could provide an efficient way to discriminate either the geometrical (modified gravity) dark energy models or the dark energy models that adhere to general relativity. This is possible via the geometrical symmetries of the FLRW space-time in which both GR gravity and modified gravity (or scalar field) live.

In the context of $f(R)$ models, following the general methodology of [48] (see also the references therein), the Noether symmetries are computed for six modified gravity models that contain also a dark matter component. The main results of the current paper can be summarized in the following statements (see Secs. III and IV):

(i) We verified that all the $f(R)$ models studied here admit the trivial first integral, namely energy conservation, as they should.

(ii) Among the six modified gravity models only the $f(R) = (R^b - 2\Lambda)^c$ Λ_{bc}CDM model with $(b, c) = (1, \frac{3}{2})$ provides a cosmic history which is similar to those of the usual dark energy models [see conditions (i)–(iv) in Sec. II A] and at the same time it admits extra integrals of motion. In general, we propose that the $f(R)$ models that simultaneously obey the conditions (i)–(iv), fit the cosmological data, and admit extra Noether symmetries (integral of motions) should be preferred along the hierarchy of modified gravity models. Of course, one has to test the $f(R) = (R - 2\Lambda)^{3/2}$ model against the

[5] An alternative way to compute the Ermakov-Lewis invariant is with the use of dynamical Noether symmetries [49]. The corresponding dynamical Noether symmetry is $X_D = u^2\dot{v}\partial_v$.

cosmological data (SNIa, BAOs, and CMB shift parameter). Such an analysis is in progress and will be published elsewhere. Therefore the Λ_{bc}CDM modified gravity model appears to be a promising candidate for describing the physical properties of ''geometrical'' dark energy. We argue that, although the Λ_{bc}CDM model [61] was phenomenologically selected in order to extend the concordance Λ cosmology, it appears from the current analysis that it has a strong geometrical basis.

(iii) Section IV provides for a first time (to our knowledge) analytical solutions in the light of the Λ_{bc}CDM model that include also a nonrelativistic matter (cold dark matter) component.

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APPENDIX A: ADDITIONAL $f(R)$ MODELS WHICH ADMIT EXTRA LIE/NOETHER POINT SYMMETRIES

From the mathematical point of view and for the completeness of the present study, we would like to give the form of all $f(R)$ functions which admit extra Lie/Noether point symmetries but do not pass the conditions (i)–(iv).

(i) If $f(R)$ is arbitrary we have the Lie point symmetries

$$X_{L_{1,2}} = l_1 \partial_t + l_2 a \partial_a$$

and the sole Noether point symmetry ∂_t with Noether integral (constant of motion) the Hamiltonian E.

(ii) If $f(R) \simeq R^{3/2}$ the dynamical system admits the extra Lie point symmetries

$$X_{L_3} = \frac{1}{a}\partial_a - \frac{2R}{a^2}\partial_R,$$
$$X_{L_4} = t\left(\frac{1}{a}\partial_a - \frac{2R}{a^2}\partial_R\right),$$
$$X_{L_5} = t\partial_t - 2R\partial_a$$

and the extra Noether point symmetries

$$X_{N_2} = \frac{1}{a}\partial_a - \frac{2R}{a^2}\partial_R,$$
$$X_{N_3} = t\frac{1}{a}\partial_a - t\frac{2R}{a^2}\partial_R,$$
$$X_{N_4} = 2t\partial_t + \frac{4}{3}a\partial_a - 4R\partial_R,$$

with corresponding Noether integrals

$$I_2 = \frac{d}{dt}(a\sqrt{R}),$$
$$I_3 = t\frac{d}{dt}(a\sqrt{R}) - a\sqrt{R},$$
$$I_4 = 2tE - 6a^2\dot{a}\sqrt{R} - 6\frac{a^3}{\sqrt{R}}\dot{R}.$$

(iii) If $f(R) \simeq R^{7/8}$ the dynamical system admits the extra Lie point symmetries

$$X_{L_6} = 2t\partial_t - 4R\partial_R,$$
$$X_{L_7} = t^2\partial_t + t\left(\frac{a}{2}\partial_a - 4R\partial_R\right)$$

and the extra Noether point symmetries

$$X_{N_5} = 2t\partial_t + \frac{a}{2}\partial_a - 4R\partial_R,$$
$$X_{N_6} = t^2\partial_t + t\left(\frac{a}{2}\partial_a - 4R\partial_R\right)$$

with corresponding Noether integrals

$$I_5 = 2tE - \frac{21}{8}\frac{d}{dt}(a^3R^{-(1/8)})$$
$$I_6 = t^2E - \frac{21}{8}t\frac{d}{dt}(a^3R^{-(1/8)}) + \frac{21}{8}a^3R^{-(1/8)}.$$

(iv) If $f(R) \simeq R^n$ (with $n \neq \frac{3}{2}, \frac{7}{8}$) the dynamical system admits the extra Lie point symmetry

$$X_{L_8} = -\frac{1}{2(n-1)}t\partial_t + \frac{1}{n-1}R\partial_R$$

and the extra Noether point symmetry

$$X_{N_7} = 2t\partial_t + (\tfrac{2}{3}a(2n-1)\partial_a - 4R\partial_R)$$

with Noether integral

$$I_7 = 2tE - 8na^2R^{n-1}\dot{a}(2-n) - 4na^3R^{n-2}\dot{R}(2n-1)(n-1).$$

Finally with the above analysis we would like to give the reader the opportunity to appreciate the fact that the Lie/Noether point symmetries provided in the current appendix can be seen as an extension of those found by Vakili [45].

APPENDIX B: NOETHER SYMMETRIES IN SPATIALLY NONFLAT $f(R)$ MODELS

In this Appendix we study further the Noether symmetries in nonflat $f(R)$ cosmological models. Briefly, in the context of a FRLW space-time the Lagrangian of the overall dynamical problem and the Ricci scalar are

$$L = 6f'a\dot{a}^2 + 6f''\dot{R}a^2\dot{a} + a^3(f'R - f) - 6Kaf'$$
$$R = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2 + K}{a^2}\right),$$

where K is the spatial curvature. Note that the two-dimensional metric is given by Eq. (20) while the ''potential'' in the Lagrangian takes the form $V(a, R) = -a^3(f'R - f) + Kaf'$. Based on the above equations and using the theoretical formulation presented in Sec. III, we find that the $f(R)$ models which admit nontrivial Noether symmetries are: $f(R) = (R - 2\Lambda)^{3/2}$, $f(R) = R^{3/2}$, and $f(R) = R^2$. Notice that the $f(R) = (R - 2\Lambda)^{7/8}$ does not accept an analytical solution.

In particular, inserting $f(R) = (R - 2\Lambda)^{3/2}$ into the Lagrangian and changing the variables from (a, R) to (x, y) [see Sec. IVA], we find

$$L = \dot{x}\dot{y} + V_0(y^3 + \bar{m}xy) - \bar{K}y \tag{B1}$$

$$E = \dot{x}\dot{y} - V_0(y^3 + \bar{m}xy) + \bar{K}y, \tag{B2}$$

where $\bar{K} = 3(6^{1/3}K)$. Therefore, the equations of motion are

$$\ddot{x} - 3V_0y^2 - \bar{m}V_0x + \bar{K} = 0 \qquad \ddot{y} - \bar{m}V_0y = 0.$$

The constant term $\bar{K}$ appearing into the first equation of motion is not expected to affect the Noether symmetries (or the integrals of motion). Indeed we find that the corresponding Noether symmetries coincide with those of the spatially flat $f(R) = (R - 2\Lambda)^{3/2}$ model [see Eqs. (22) and (30)–(32)]. However, in the case of $K \neq 0$ (or $\bar{K} \neq 0$) the analytical solution for the x variable is written as $x_K(t) \equiv x(t) + \frac{\bar{K}}{\omega^2}$, where $x(t)$ is the solution of the flat model $K = 0$ (see Sec. IVA). Note that the solution of the y variable remains unaltered [see Sec. IVA or Eq. (C4)]. As expected, in the spatially flat regime $K = 0$, the current equations reduce those equations of Sec. IVA.

Similarly, in the case of $f(R) = R^{3/2}$ and $f(R) = R^2$ the Noether symmetries can be found in Appendix A. Of course, we again confirm that all the proposed modified gravity models with $K \neq 0$ accommodate the trivial first integral $\partial_t E = 0$ (energy conservation).

APPENDIX C: TESTING THE ANALYTICAL SOLUTIONS

In this Appendix we would like to test the validity of our analytical solutions in the case of the $f(R) = (R - 2\Lambda)^{3/2}$ model. Below we investigate the behavior of the Hubble parameter in the matter dominated era. First of all inserting $\rho_m = \rho_{m0}a^{-3}$ [see our Eq. (18)] into the modified Friedmann equation [see Eq. (6)], we get

$$H^2 = k^2\frac{\rho_{m0}a^{-3}}{3f'} + \frac{f'R - f}{6f'} - \frac{Hf''\dot{R}}{f'}. \tag{C1}$$

Obviously, in order to reveal the evolution of the Hubble parameter in the matter era, in which the evolution of the matter density dominates the global dynamics, we have to understand the evolution of the first and the third term in Eq. (C1). Using $R = 2\Lambda + \frac{y^2}{x}$ (see the transformations in Sec. IVA) we have, after some simple algebra, that

$$k^2\frac{\rho_{m0}a^{-3}}{3f'} = 2k^2\frac{\rho_{m0}a^{-3}}{9(R - 2\Lambda)^{1/2}} = \frac{2k^2}{9}\rho_{m0}a^{-3}\left(\frac{x}{y^2}\right)^{1/2} \tag{C2}$$

$$\frac{f''\dot{R}}{f'} = \frac{\dot{R}}{2(R - 2\Lambda)} = \frac{d(y^2/x)/dt}{2(y^2/x)}. \tag{C3}$$

For the benefit of the reader we repeat here the general solution of the system:

$$y(t) = \frac{I_2}{2\omega}e^{\omega t} - \frac{I_1}{2\omega}e^{-\omega t} \tag{C4}$$

$$x(t) = x_{1G}e^{\omega t} + x_{2G}e^{-\omega t} + + \frac{1}{4\bar{m}\omega^2}(I_2e^{\omega t} + I_1e^{-\omega t})^2 + \frac{I_1I_2}{\bar{m}\omega^2}, \tag{C5}$$

where $I_{1,2} \neq 0$ are the Noether integrals.

Inserting the general solution into Eqs. (C2) and (C3) and using at the same time that $\mathrm{e}^{-\omega t} \sim 0$, we find

$$k^2\frac{\rho_{m0}a^{-3}}{3f'} = \frac{2k^2}{9}\rho_{m0}a^{-3}\left(\frac{x}{y^2}\right)^{1/2} \to \frac{2k^2\bar{m}}{9}\rho_{m0}a^{-3}$$
$$\frac{f''\dot{R}}{f'} = \frac{d(y^2/x)/dt}{2(y^2/x)} \sim \mathrm{e}^{-\omega t} \ll 1.$$

Obviously, inserting the above results into the modified Friedmann equation (C1), one can easily show that in the matter dominated era the evolution of the Hubble parameter tends to its nominal form, namely, $H(a) \to a^{-3/2}$.

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