

Flavor changing neutral currents, CP violation, and implications for some rare decays in a $SU(4)_L \otimes U(1)_X$ extension of the standard model

Alejandro Jaramillo and Luis A. Sánchez

Escuela de Física, Universidad Nacional de Colombia, A.A. 3840, Medellín, Colombia

(Received 11 May 2011; published 1 December 2011)

Extensions of the standard model with gauge symmetry $SU(3)_c \otimes SU(4)_L \otimes U(1)_X$ (3-4-1 extensions) where anomaly cancellation takes place between the fermion families (three-family models) predict the existence of two new heavy neutral gauge bosons which transmit flavor changing neutral currents at tree level. In this work, in the context of a three-family 3-4-1 extension which does not contain particles with exotic electric charges, we study the constraints coming from neutral meson mixing on the parameters of the extension associated to tree-level flavor changing neutral current effects. Taking into account experimental measurements of observables related to K and B meson mixing and including new CP -violating phases, we study the resulting bounds for angles and phases in the mixing matrix for the down-quark sector, as well as the implications of these bounds for the modifications in the amplitudes of the clean rare decays $K^+ \rightarrow \pi^+ \bar{\nu} \nu$, $K_L \rightarrow \pi^0 \nu \bar{\nu}$, $K_L \rightarrow \pi^0 l^+ l^-$ ($l = e, \mu$) and $B_{d/s} \rightarrow \mu^+ \mu^-$.

DOI: 10.1103/PhysRevD.84.115001

PACS numbers: 12.60.Cn, 13.20.Eb, 13.20.He

I. INTRODUCTION

Flavor violating couplings of ordinary fermions to extra neutral gauge bosons and to new scalar fields arise in many extensions of the standard model (SM). Two simple and interesting examples are the 3-3-1 model in which the SM gauge symmetry is enlarged to $SU(3)_c \otimes SU(3)_L \otimes U(1)_X$ [1–5], and the 3-4-1 extension where the enlargement is done to the gauge group $SU(3)_c \otimes SU(4)_L \otimes U(1)_X$ [6–10]. These extensions share the important feature of addressing the problem of the number of fermion families in nature in the sense that anomaly cancellation among the families (three-family models) takes place only if we have an equal number of left-handed triplets and antitriplets (in the 3-3-1 model) or an equal number of 4-plets and 4^* -plets (in the 3-4-1 extension), taking into account the color degree of freedom. As a consequence, the number of fermion families N_f must be divisible by the number of colors N_c of $SU(3)_c$. Moreover, since $SU(3)_c$ asymptotic freedom requires $N_c < 5$, it follows that $N_f = N_c = 3$. Cancellation of chiral anomalies among the families thus forces one family of quarks, in the weak basis, to transform differently from the other two under $SU(4)_L \otimes U(1)_X$. As a result, the chiral couplings of quarks to the new neutral gauge bosons are, in general, nonuniversal and, when rotating to the quark mass eigenstate basis, flavor changing neutral currents (FCNC) induced by fermion mixing arise.

In particular, the 3-4-1 extension predicts the existence of two heavy neutral gauge bosons Z' and Z'' which, in general, mix up with the ordinary Z boson of the SM. In contrast with the SM where FCNC processes occur only at the loop level, in the 3-4-1 extension these new gauge bosons can transmit FCNC at tree level and, consequently, the study of these processes can set stringent bounds on the Z' and Z'' masses and mixing. Moreover, since in general

each flavor couples to more than one Higgs 4-plet, FCNC coming from the scalar sector can also be present.

For simplicity, in this work we will restrict ourselves to 3-4-1 extensions without exotic electric charges in the fermion sector, that is, without electric charges different from $\pm 2/3$ and $\pm 1/3$ for exotic quarks and different from 0 and ± 1 for exotic leptons. The systematic analysis of the 3-4-1 gauge theory carried out in [8] has shown that the restriction to fermion field representations with only ordinary electric charges allows for eight different anomaly-free extensions. Four of them are three-family models and can be classified according to the values of the coefficients b and c which appear in the most general expression for the electric charge generator in $SU(4)_L \otimes U(1)_X$ [see Eq. (1) below]. The allowed simultaneous values for these coefficients are $b = c = 1$ and $b = 1, c = -2$. Two of the four three-family models belong to the $b = c = 1$ class; the other two belong to the $b = 1, c = -2$ class.

In 3-4-1 extensions without exotic electric charges the $Z - Z' - Z''$ mixing can be constrained to occur between Z and Z' only, which leaves $Z'' \equiv Z_3$ as a heavy mass eigenstate [7–10]. The diagonalization of the $Z - Z'$ mass matrix produces a light mass eigenstate Z_1 which can be identified as the neutral gauge boson of the SM, and a heavy Z_2 . After the breakdown of the 3-4-1 symmetry down to $SU(3)_c \otimes U(1)_Q$, and since we have one family of quarks transforming differently from the other two under the gauge group, one important difference between the aforementioned two classes of 3-4-1 extensions appears: even though in both classes the Z_1 current remains flavor diagonal, in the $b = c = 1$ class the new Z_2 gauge boson couples nonuniversally to ordinary quarks thus transmitting tree-level FCNC at low energies, while the couplings to Z_3 are universal and Z_3 couples only to exotic fermions. In the $b = 1, c = -2$ class, instead, it is the new Z_3 gauge

boson the responsible for this effect because couples non-universally to ordinary quarks, while the Z_2 current remains flavor diagonal and universal.

The study of the bounds on the Z_2 and Z_3 masses coming from electroweak precision data and from FCNC processes has shown that extensions for which $b = 1$, $c = -2$ are preferred in the sense that they give lower bounds on these masses smaller than the bounds predicted by extensions in the $b = c = 1$ class [9]. In fact, in the latter class a χ^2 fit to Z -pole observables and atomic parity violation data produces $m_{Z_2} \gtrsim 2$ TeV, a bound that is further increased to $m_{Z_2} \gtrsim 11$ TeV when the constraints coming from neutral meson mixing in the study of the tree-level FCNC effects are taken into account. In the $b = 1$, $c = -2$ class, instead, the same fit gives the lower bound $m_{Z_2} \gtrsim 0.8$ TeV and the analysis of the constraints arising from neutral meson mixing provides $m_{Z_3} \gtrsim 6.5$ TeV. This means that extensions in the $b = 1$, $c = -2$ class have a better chance to be tested at the LHC facility or further at the ILC. Here however we must point out that, as it has been noted repeatedly in the literature, the bounds from neutral meson mixing are always clouded by the lack of knowledge of the entries in the quark mixing matrices V_L^u and V_L^d involved in the new physics contribution, which forces to adopt well motivated but *ad hoc* textures for these matrices.

In this work, in the context of a 3-4-1 extension belonging to the $b = 1$, $c = -2$ class (the so-called “model F ” in Ref. [8]), we re-examine the issue of tree level FCNC transmitted by the new Z_3 gauge bosons but, due to the uncertainties associated with our ignorance of the quark mixing matrices, we will not search for bounds on the Z_3 mass, but rather we set this mass at fixed values in the range 1–5 TeV (to be justified below) and search for information about the size of angles and phases in the V_L^d mixing matrix. To this purpose we will use several well measured $\Delta F = 2$ ($F = S, B$) observables in the down sector, namely ΔM_K , $\Delta M_{d/s}$, ε_K , and $\sin\Phi_d$, associated to the $K_L - K_S$ and the $B_{d/s}^0 - \bar{B}_{d/s}^0$ mass differences, the Kaon CP -violation parameter and the $B_d^0 - \bar{B}_d^0$ mixing phase, respectively. We will also study the implications of the obtained bounds for the modifications in the amplitudes of the clean rare decays $K^+ \rightarrow \pi^+ \nu \bar{\nu}$, $K_L \rightarrow \pi^0 \nu \bar{\nu}$, $B_{d/s} \rightarrow \mu^+ \mu^-$ and $K_L \rightarrow \pi^0 l^+ l^-$ ($l = e, \mu$). In this context our mail goal will be to obtain upper and lower bounds for the corresponding branching ratios (BR) and to study their compatibility both with the experimental data and the SM predictions.

Another well-known FCNC rare process that deserves attention due to its sensitivity to new physics, is the radiative decay $b \rightarrow s\gamma$. In general, this decay receives contributions both from the new charged and neutral gauge bosons and from the new scalar fields and can be used to put limits on these sectors of the 3-4-1 extension. As can be seen from the 3-4-1 scalar structure in Eq. (3) below, and as it is done in the 3-3-1 model, the scalar contributions can

be accounted for by an effective two-Higgs-doublet model. This, however, demands the identification of the physical scalar fields and their couplings and, therefore, the diagonalization of the full scalar sector. It is then clear that the study of this decay requires a particular and dedicated analysis. Moreover, as in the SM, $b \rightarrow s\gamma$ is also a loop process in the 3-4-1 construction and since, as declared above, we are mainly interested in the analysis of the more stringent tree-level FCNC processes that receive contributions from Z_3 exchange, the detailed study of this decay in the context of the present SM extension will be postponed to a future work.

This paper is organized as follows. In Sec. II, we introduce the 3-4-1 extension to be considered and present its most important features for our purposes. In Secs. III and IV, we study the theoretical expression for the observables associated to neutral meson mixing and for the rare decays that will be analyzed, in such a way that the tree-level FCNC contributions of the extra neutral gauge boson enter as corrections to the SM expressions. In Sec. V, we numerically evaluate the theoretical expressions obtained in the two previous sections, from which some information on structure of the V_L^d mixing matrix can be obtained and the implications for the rare decays can be calculated. In the last section we summarize our results.

II. THE 3-4-1 EXTENSION

Let us begin by introducing the most relevant features of the 3-4-1 extension we are interested in. Several details of their phenomenology have been already worked out in [10].

The extension is based on the local $SU(3)_c \otimes SU(4)_L \otimes U(1)_X$ gauge symmetry which contains $SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$ as a subgroup, and belongs to the $b = 1$, $c = -2$ class, where b and c are parameters appearing in the most general expression for the electric charge generator in $SU(4)_L \otimes U(1)_X$

$$Q = aT_{3L} + \frac{1}{\sqrt{3}}bT_{8L} + \frac{1}{\sqrt{6}}cT_{15L} + XI_4, \quad (1)$$

where $T_{iL} = \lambda_{iL}/2$, with λ_{iL} the Gell-Mann matrices for $SU(4)_L$ normalized as $\text{Tr}(\lambda_i \lambda_j) = 2\delta_{ij}$, $I_4 = \text{diag}(1, 1, 1, 1)$ is the diagonal 4×4 unit matrix, and $a = 1$ gives the usual isospin of the electroweak interaction.

The anomaly-free fermion content of this extension has been discussed in Ref. [8] and is given in Table I where $i = 1, 2$ and $\alpha = 1, 2, 3$ are generation indexes. The numbers inside brackets match the $SU(3)_c$, $SU(4)_L$ and $U(1)_X$ quantum numbers, respectively. U_i and U_3 are exotic quarks of electric charge $2/3$, D_i and D_3 are also exotic quarks with electric charge $-1/3$, while E_α^- and N_α^0 are exotic leptons with electric charges -1 and 0 , respectively.

TABLE I. Anomaly-free fermion content.

$Q_{iL} = \begin{pmatrix} d_i \\ u_i \\ U_i \\ D_i \end{pmatrix}_L$	d_{iL}^c	u_{iL}^c	U_{iL}^c	D_{iL}^c
$[3, 4^*, \frac{1}{6}]$	$[3^*, 1, \frac{1}{3}]$	$[3^*, 1, -\frac{2}{3}]$	$[3^*, 1, -\frac{2}{3}]$	$[3^*, 1, \frac{1}{3}]$
$Q_{3L} = \begin{pmatrix} u_3 \\ d_3 \\ D_3 \\ U_3 \end{pmatrix}_L$	u_{3L}^c	d_{3L}^c	D_{3L}^c	U_{3L}^c
$[3, 4, \frac{1}{6}]$	$[3^*, 1, -\frac{2}{3}]$	$[3^*, 1, \frac{1}{3}]$	$[3^*, 1, \frac{1}{3}]$	$[3^*, 1, -\frac{2}{3}]$
$L_{\alpha L} = \begin{pmatrix} \nu_{e\alpha}^0 \\ e_{\alpha}^- \\ E_{\alpha}^- \\ N_{\alpha}^0 \end{pmatrix}_L$	$e_{\alpha L}^+$	$E_{\alpha L}^+$		
$[1, 4, -\frac{1}{2}]$	$[1, 1, 1]$	$[1, 1, 1]$		

The symmetry breaking occurs in three steps

$$\begin{aligned}
 & SU(3)_c \otimes SU(4)_L \otimes U(1)_{X \xrightarrow{V'}} SU(3)_c \otimes SU(3)_L \\
 & \otimes U(1)_{Z \xrightarrow{V}} SU(3)_c \otimes SU(2)_L \otimes U(1)_{Y \xrightarrow{V+V'}} SU(3)_c \\
 & \otimes U(1)_Q, \tag{2}
 \end{aligned}$$

a task that is done by the following four Higgs fields with vacuum expectation values (VEV) aligned as

$$\begin{aligned}
 \langle \phi_1^T \rangle &= \langle (\phi_1^0, \phi_1^+, \phi_1'^+, \phi_1^0) \rangle = (v, 0, 0, 0) \sim [1, 4^*, 1/2], \\
 \langle \phi_2^T \rangle &= \langle (\phi_2^-, \phi_2^0, \phi_2'^0, \phi_2^-) \rangle \\
 &= (0, v', 0, 0) \sim [1, 4^*, -1/2], \\
 \langle \phi_3^T \rangle &= \langle (\phi_3^-, \phi_3^0, \phi_3'^0, \phi_3^-) \rangle \\
 &= (0, 0, V, 0) \sim [1, 4^*, -1/2], \\
 \langle \phi_4^T \rangle &= \langle (\phi_4^0, \phi_4^+, \phi_4'^+, \phi_4^0) \rangle = (0, 0, 0, V') \sim [1, 4^*, 1/2]. \tag{3}
 \end{aligned}$$

This scalar structure consistently gives masses for all the gauge bosons and it is also enough to produce the observed fermion mass spectrum for the charged fermion sector (quarks and leptons) provided the hierarchy $V, V' \gg v, v' \simeq 174 \text{ GeV}$ is satisfied, where V' and V set the mass scales for exotic fields [10].

The gauge couplings g_4 and g_X , associated with the groups $SU(4)_L$ and $U(1)_X$, respectively, are defined through the covariant derivative for 4-plets as: $iD^\mu = i\partial^\mu - g_4 \lambda_{L\alpha} A_\alpha^\mu / 2 - g_X X B^\mu$. When the 3-4-1 symmetry is broken to the SM, we get the gauge matching conditions

$$g_4 = g, \quad \text{and} \quad \frac{1}{g'^2} = \frac{1}{g_2^2} + \frac{1}{g_X^2}, \tag{4}$$

where g and g' are the gauge coupling constants of the $SU(2)_L$ and $U(1)_Y$ groups of the SM, respectively.

Clearly, FCNC in this extension can arise from the mixing of ordinary and exotic fermions. These FCNC and violation of the unitarity of the Cabibbo-Kobayashi-Maskawa (CKM) mixing matrix can be avoided by the introduction of the following Z_2 discrete symmetry which assigns charges q_Z to the fields as [10]

$$\begin{aligned}
 q_Z(Q_{\alpha L}, u_{\alpha L}^c, d_{\alpha L}^c, L_{\alpha L}, e_{\alpha L}^+, \phi_1, \phi_2) &= 0, \\
 q_Z(U_{\alpha L}, D_{\alpha L}^c, E_{\alpha L}^+, \phi_3, \phi_4) &= 1, \tag{5}
 \end{aligned}$$

where $\alpha = 1, 2, 3$ is a family index as above.

The gauge invariance and the Z_2 symmetry allow for the following Yukawa Lagrangian in the quark sector

$$\begin{aligned}
 \mathcal{L}_Y^Q &= \sum_{i=1}^2 Q_{iL}^T C \left[\phi_2^* \sum_{\alpha=1}^3 h_{i\alpha}^u u_{\alpha L}^c + \phi_1^* \sum_{\alpha=1}^3 h_{i\alpha}^d d_{\alpha L}^c \right. \\
 &+ \left. \phi_3^* \sum_{\alpha=1}^3 h_{i\alpha}^U U_{\alpha L}^c + \phi_4^* \sum_{\alpha=1}^3 h_{i\alpha}^D D_{\alpha L}^c \right] \\
 &+ Q_{3L}^T C \left[\phi_1 \sum_{\alpha=1}^3 h_{3\alpha}^u u_{\alpha L}^c + \phi_2 \sum_{\alpha=1}^3 h_{3\alpha}^d d_{\alpha L}^c \right. \\
 &+ \left. \phi_4 \sum_{\alpha=1}^3 h_{3\alpha}^U U_{\alpha L}^c + \phi_3 \sum_{\alpha=1}^3 h_{3\alpha}^D D_{\alpha L}^c \right] + \text{H.c.}, \tag{6}
 \end{aligned}$$

where the h 's are Yukawa couplings and C is the charge conjugation operator. From this Lagrangian we get, for the up- and down-type quarks in the basis $(u_1, u_2, u_3, U_1, U_2, U_3)$ and $(d_1, d_2, d_3, D_1, D_2, D_3)$, respectively, 6×6 block diagonal mass matrices of the form

$$\begin{aligned}
 M_{uU} &= \begin{pmatrix} M_{3 \times 3}^u & 0 \\ 0 & M_{3 \times 3}^U \end{pmatrix} \quad \text{and} \\
 M_{dD} &= \begin{pmatrix} M_{3 \times 3}^d & 0 \\ 0 & M_{3 \times 3}^D \end{pmatrix}, \tag{7}
 \end{aligned}$$

where, for $V \sim V'$ and $v \sim v'$, the entries in the submatrices are

$$\begin{aligned}
 M_{\alpha\beta}^u &\simeq h_{\alpha\beta}^u v \quad \text{and} \quad M_{\alpha\beta}^U \simeq h_{\alpha\beta}^U V; \\
 M_{\alpha\beta}^d &\simeq h_{\alpha\beta}^d v \quad \text{and} \quad M_{\alpha\beta}^D \simeq h_{\alpha\beta}^D V. \tag{8}
 \end{aligned}$$

For the charged leptons we get the Yukawa terms

$$\mathcal{L}_Y^L = \sum_{\alpha=1}^3 \sum_{\beta=1}^3 L_{\beta L}^T C [\phi_2 h_{\alpha\beta}^e e_{\beta L}^+ + \phi_3 h_{\alpha\beta}^E E_{\beta L}^+] + \text{H.c.} \tag{9}$$

From this equation we find a block diagonal mass matrix in the basis $(e_1, e_2, e_3, E_1, E_2, E_3)$ given by

$$M_{eE} = \begin{pmatrix} M_{3 \times 3}^e & 0 \\ 0 & M_{3 \times 3}^E \end{pmatrix}, \quad (10)$$

where now the entries in the submatrices are

$$M_{\alpha\beta}^e = h_{\alpha\beta}^e v' \quad \text{and} \quad M_{\alpha\beta}^E = h_{\alpha\beta}^E V. \quad (11)$$

The mass matrices in (7) and (10) exhibit a simple mass splitting between ordinary and exotic charged fermions, and show that all the charged fermions in this extension acquire masses at the three level. Clearly, by a judicious tuning of the Yukawa couplings a consistent mass spectrum in the ordinary charged sector can be obtained. In the exotic charged sector all fermions acquire masses at the scale $V \sim V' \gg v_{\text{EW}} = 174$ GeV. We also remark that the tensor product form of the mass matrices M_{uU} and M_{dD} in (7) implies that they are diagonalized by unitary matrices which are themselves tensor products of unitary matrices. So, the discrete Z_2 symmetry also avoids violation of unitarity of the CKM mixing matrix. The neutral leptons $\nu_{e\alpha}^0$ and $N_{e\alpha}^0$ ($\alpha = 1, 2, 3$) remain massless as far as we use only the original fermion fields shown in Table I. However, their masses and mixing can be implemented by introducing one or more Weyl singlet states $N_{L,n}^0 \sim [1, 1, 0]$, $n = 1, 2, \dots$, without violating our assumptions, neither the anomaly constraint relations, because singlets with no X charges are as good as not being present as far as anomaly cancellation is concerned.

A look at the Yukawa Lagrangians in Eqs. (6) and (9) shows that each flavor couples to more than one Higgs 4-plet and, consequently, scalar mediated FCNC also arise. However, since all the processes we will consider in this work involve external light quarks and leptons, these contributions are suppressed by the small Yukawa couplings associated to these fields. Notice that, for these processes, this will also be the case even in the absence of the discrete Z_2 symmetry introduced in Eq. (5) because the Yukawa couplings will now be suppressed by small mixing angles. In the charged boson sector there are SM-like W^\pm gauge bosons that do not mix with the other charged bosons and acquire a squared mass $M_{W^\pm}^2 = (g_4^2/2)(v^2 + v'^2)$ so that, with $M_W = 80.399 \pm 0.023$ GeV [11], we have $\sqrt{v^2 + v'^2} \simeq v_{\text{EW}} = 174$ GeV. The ten remaining physical charged gauge bosons, namely: K^\pm , V^\pm , Y^\pm , $X^0(X'^0)$, and $K^0(K'^0)$, acquire masses at the large scale $V' \sim V$ and, at tree level, couple always to at least one exotic fermion [10]. This means that, for low energy processes involving ordinary fermions, the contribution of the new charged gauge bosons will be present only at loop level. Thus, we expect that the tree-level FCNC effects transmitted by the exotic neutral gauge bosons will dominate.

Our main interest, therefore, is in the neutral gauge boson sector which consists of four physical fields: the massless photon A_μ and three massive gauge bosons which come from the diagonalization of the mixing $Z_\mu - Z'_\mu - Z''_\mu$. In terms of the electroweak basis, they are given by

$$\begin{aligned} A^\mu &= S_W A_3^\mu + C_W \left[\frac{T_W}{\sqrt{3}} \left(A_8^\mu - 2 \frac{A_{15}^\mu}{\sqrt{2}} \right) + (1 - T_W^2)^{1/2} B^\mu \right], \\ Z^\mu &= C_W A_3^\mu - S_W \left[\frac{T_W}{\sqrt{3}} \left(A_8^\mu - 2 \frac{A_{15}^\mu}{\sqrt{2}} \right) + (1 - T_W^2)^{1/2} B^\mu \right], \\ Z'^\mu &= \frac{1}{\sqrt{3}} (1 - T_W^2)^{1/2} \left(A_8^\mu - 2 \frac{A_{15}^\mu}{\sqrt{2}} \right) - T_W B^\mu, \\ Z''^\mu &= 2A_8^\mu / \sqrt{6} + A_{15}^\mu / \sqrt{3}. \end{aligned} \quad (12)$$

Since we are interested in the low energy phenomenology, we can choose $V \simeq V'$. If we also take $v \simeq v'$, the current $Z''^\mu \equiv Z_3^\mu$ decouples from the other two and acquires a squared mass $M_{Z_3}^2 = (g_4^2/2)(V^2 + v^2)$ [7–10]. This fact produces an enormous simplification in the study of the low energy deviations of the Z couplings to the SM families which now come from the diagonalization of the mass matrix

$$M_{(Z,Z')} = \frac{g_4^2}{C_W^2} \begin{pmatrix} v^2 & \delta v^2 S_W \\ \delta v^2 S_W & \frac{\delta^2}{S_W^2} (V^2 C_W^4 + v^2 S_W^4) \end{pmatrix}, \quad (13)$$

where $\delta = g_X/g_4$, and $S_W = \delta/\sqrt{2\delta^2 + 1}$ and C_W are the sine and cosine of the electroweak mixing angle, respectively. The corresponding mass eigenstates are: $Z_1^\mu = Z^\mu \cos\theta + Z'^\mu \sin\theta$ and $Z_2^\mu = -Z^\mu \sin\theta + Z'^\mu \cos\theta$, where the mixing angle θ is given by

$$\tan(2\theta) = \frac{v^2 S_W^2 \sqrt{C_{2W}}}{v^2 (1 + S_W^2)^2 + V^2 C_W^4 - 2v^2}, \quad (14)$$

with $C_{2W} = C_W^2 - S_W^2$. Since $V \gg v$ this mixing angle is expected to be very small.

A. Neutral currents

The Lagrangian for the neutral currents $J_\mu(\text{EM})$, $J_\mu(Z)$, $J_\mu(Z')$, and $J_\mu(Z'')$, in the basis in which all the fields are gauge eigenstates, is given by

$$\begin{aligned} -\mathcal{L}^{\text{NC}} &= e A^\mu J_\mu(\text{EM}) + \frac{g_4}{C_W} Z^\mu J_\mu(Z) + g_X Z'^\mu J_\mu(Z') \\ &\quad + \frac{g_4}{2\sqrt{2}} Z''^\mu J_\mu(Z''), \end{aligned} \quad (15)$$

where $e = g S_W = g_X C_W \sqrt{1 - T_W^2} > 0$. Calling q_f the electric charge of the fermion f in units of e , the currents are given by

$$\begin{aligned}
 J_\mu(\text{EM}) = & \frac{2}{3} \left[\bar{u}_3 \gamma_\mu u_3 + \bar{U}_3 \gamma_\mu U_3 \right. \\
 & + \left. \sum_{i=1}^2 (\bar{u}_i \gamma_\mu u_i + \bar{U}_i \gamma_\mu U_i) \right] \\
 & - \frac{1}{3} \left[\bar{d}_3 \gamma_\mu d_3 + \bar{D}_3 \gamma_\mu D_3 \right. \\
 & + \left. \sum_{i=1}^2 (\bar{d}_i \gamma_\mu d_i + \bar{D}_i \gamma_\mu D_i) \right] \\
 & - \sum_{\alpha=1}^3 (\bar{e}_\alpha^- \gamma_\mu e_\alpha^- + \bar{E}_\alpha^- \gamma_\mu E_\alpha^-) = \sum_f q_f \bar{f} \gamma_\mu f, \quad (16)
 \end{aligned}$$

$$\begin{aligned}
 J_\mu(Z) = & J_{\mu,L}(Z) - S_W^2 J_\mu(\text{EM}) \\
 = & \frac{1}{2} [\bar{u}_{3L} \gamma_\mu u_{3L} - \bar{d}_{3L} \gamma_\mu d_{3L} \\
 & - \sum_{i=1}^2 (\bar{d}_{iL} \gamma_\mu d_{iL} - \bar{u}_{iL} \gamma_\mu u_{iL}) \\
 & + \sum_{\alpha=1}^3 (\bar{\nu}_{\alpha L} \gamma_\mu \nu_{\alpha L} - \bar{e}_{\alpha L}^- \gamma_\mu e_{\alpha L}^-)] - S_W^2 J_\mu(\text{EM}), \quad (17)
 \end{aligned}$$

$$\begin{aligned}
 J_\mu(Z') = & J_{\mu,L}(Z') - T_W J_\mu(\text{EM}) \\
 = & (2T_W)^{-1} [T_W^2 \bar{u}_{3L} \gamma_\mu u_{3L} - T_W^2 \bar{d}_{3L} \gamma_\mu d_{3L} \\
 & - \bar{D}_{3L} \gamma_\mu D_{3L} + \bar{U}_{3L} \gamma_\mu U_{3L} \\
 & - \sum_{i=1}^2 (T_W^2 \bar{d}_{iL} \gamma_\mu d_{iL} - T_W^2 \bar{u}_{iL} \gamma_\mu u_{iL} \\
 & - \bar{U}_{iL} \gamma_\mu U_{iL} + \bar{D}_{iL} \gamma_\mu D_{iL}) \\
 & + \sum_{\alpha=1}^3 (T_W^2 \bar{\nu}_{\alpha L} \gamma_\mu \nu_{\alpha L} - T_W^2 \bar{e}_{\alpha L}^- \gamma_\mu e_{\alpha L}^- \\
 & - \bar{E}_{\alpha L}^- \gamma_\mu E_{\alpha L}^- + \bar{N}_{\alpha L}^0 \gamma_\mu N_{\alpha L}^0)] - T_W J_\mu(\text{EM}), \quad (18)
 \end{aligned}$$

and

$$\begin{aligned}
 J_\mu(Z'') = & \sum_{i=1}^2 (-\bar{d}_{iL} \gamma_\mu d_{iL} - \bar{u}_{iL} \gamma_\mu u_{iL} \\
 & + \bar{U}_{iL} \gamma_\mu U_{iL} + \bar{D}_{iL} \gamma_\mu D_{iL}) + \bar{u}_{3L} \gamma_\mu u_{3L} \\
 & + \bar{d}_{3L} \gamma_\mu d_{3L} - \bar{D}_{3L} \gamma_\mu D_{3L} - \bar{U}_{3L} \gamma_\mu U_{3L} \\
 & + \sum_{\alpha=1}^3 (\bar{\nu}_{\alpha L} \gamma_\mu \nu_{\alpha L} + \bar{e}_{\alpha L}^- \gamma_\mu e_{\alpha L}^- \\
 & - \bar{E}_{\alpha L}^- \gamma_\mu E_{\alpha L}^- - \bar{N}_{\alpha L}^0 \gamma_\mu N_{\alpha L}^0). \quad (19)
 \end{aligned}$$

Equation (17) shows that we can identify Z_μ as the neutral gauge boson of the SM because $J_\mu(Z)$ is just the generalization of the SM neutral current. Moreover, from (18) it is

straightforward to see that the neutral gauge boson Z'_μ does not transmit FCNC at low energy since it couples diagonally and universally to ordinary fermions.

Notice, on the other hand, that $J_\mu(Z'')$ is a pure left-handed current and that, notwithstanding the neutral gauge boson $Z''_\mu \equiv Z''_3$ does not mix neither with Z_μ nor with Z'_μ (in the case $V' \simeq V$ and $v' \simeq v$), it still couples nonuniversally to ordinary fermions. As a matter of fact, even though the Z''_μ couplings are diagonal, its couplings to the third family of quarks are different from the ones to the first two families. Thus, at low energy, we will have tree-level FCNC transmitted by Z''_μ which are induced by fermion mixing. This means that in the corresponding low energy effective Lagrangian the chiral Z''_μ couplings will in general explicitly depend on the entries of the unitary matrices $V_{L,R}^\psi$ that diagonalize the quark Yukawa matrices. So, in the analysis of these FCNC effects a convenient parametrization of the $V_{L,R}^\psi$ matrices must be chosen.

At this point we must notice that if the mass splitting between Z_2 and Z_3 is so large as the constraints from neutral meson mixing suggest, we must be aware about the fact that the loop contributions coming from the exchange of Z_2 can compete with the tree-level effects transmitted by Z_3 . But, as already discussed, these constraints are rather unreliable, and since the mass matrix in (13) can be exactly diagonalized we can estimate this mass splitting in the case we are considering, that is $V \simeq V' \gg v' \simeq v$ which, in turn, implies a small mixing. So, neglecting the mixing we have: $m_{Z_2}^2 \approx g_4^2 \delta^2 V^2 / T_W^2 = g_4^2 V^2 / (1 - T_W^2)$ which, compared with $m_{Z_3}^2 = (g_4^2/2)(V^2 + v^2) \approx (g_4^2/2)V^2$, shows that in our approximation these masses are of the same order V and, consequently, tree-level FCNC mediated by Z_3 will dominate.

B. The effective Lagrangian

We will use the formalism developed in Ref. [12] where general expressions for calculating FCNC effects in models predicting the existence of one extra neutral gauge boson are presented. We start by generalizing this formalism to the case of $N - 1$ extra neutral gauge bosons and then we restrict ourselves to the case $N - 1 = 2$.

The Lagrangian for neutral currents in Eq. (15) can be rewritten and generalized as

$$\begin{aligned}
 \mathcal{L}_{\text{NC}} = & -e A_\mu J^\mu(\text{EM}) - g_1 Z_{1,\mu}^0 J^{(1)\mu} \\
 & - \sum_{\alpha=2}^{N-1} g_\alpha Z_{\alpha,\mu}^0 J^{(\alpha)\mu}, \quad (20)
 \end{aligned}$$

where $Z_1^0 \equiv Z$ denotes the neutral gauge boson of the SM and Z_α^0 are the new heavy Z bosons (which in general mix with Z_1^0).

Following the notation of Ref. [12], the currents can be written as

$$J_\mu^{(m)} = \sum_\psi \sum_{i,j} \bar{\psi}_i \gamma_\mu [\epsilon_{Lij}^{\psi(m)} P_L + \epsilon_{Rij}^{\psi(m)} P_R] \psi_j, \quad (21)$$

where the sum extends over all quarks and leptons $\psi_{i,j}$ and $P_{R,L} = (1 \pm \gamma_5)/2$. $\epsilon_{R,Lij}^{\psi(1)} = \epsilon_{R,L}(i) \delta_{ij}$ denotes the SM chiral couplings and $\epsilon_{R,Lij}^{\psi(m)} (m \neq 1)$ denotes the chiral couplings of the heavy gauge bosons.

After the spontaneous symmetry breaking the physical massive bosons Z_α are linear combinations of the gauge eigenstates Z_α^0 :

$$Z_\alpha = \sum_{\beta=1}^N U_{\alpha\beta} Z_\beta^0, \quad (22)$$

where U is an orthogonal $N \times N$ matrix.

The chiral Z_α^0 couplings to fermions in the fermion mass eigenstate basis read

$$E_{L,R}^{\psi(\alpha)} \equiv V_{L,R}^\psi \epsilon_{L,R}^{\psi(\alpha)} V_{L,R}^{\psi\dagger}, \quad (23)$$

where the CKM mixing matrix is given by the combination

$$V_{\text{CKM}} = V_L^\mu V_L^{d\dagger}. \quad (24)$$

The four-fermion effective Lagrangian, for the general case of N neutral gauge bosons, reads [13]

$$- \mathcal{L}_{\text{eff}} = \frac{4G_F}{\sqrt{2}} \sum_{\alpha=1}^N \rho_\alpha \left(\sum_{\beta=1}^N U_{\alpha\beta} \frac{g_\beta}{g_1} J_\beta^\mu \right)^2, \quad (25)$$

where $\rho_\alpha \equiv m_W^2/(m_\alpha^2 \cos^2 \theta_W)$ and m_α is the mass of Z_α .

Using Eqs. (21) and (23) into (25) a general expression for the effective Lagrangian, written in a way useful for further calculations, is obtained as

$$- \mathcal{L}_{\text{eff}} = \frac{4G_F}{\sqrt{2}} \sum_{\psi,\chi} \sum_{k,l} \sum_{i,j,m,n} \sum_{X,Y} W_{XY}^{kl,ijmn} (\bar{\psi}_i \gamma_\mu P_X \psi_j) \times (\bar{\chi}_m \gamma^\mu P_Y \chi_n), \quad (26)$$

where $k, l = 1, 2, 3$; X and Y run over the chiralities L, R ; ψ and χ represent classes of fermions with the same SM quantum numbers, i.e. u, d, e^-, ν , while i, j, m, n are family indexes. $W_{\text{PQ}}^{kl,ijmn}$ is given by

$$W_{XY}^{kl,ijmn} = \frac{g_k g_l}{g_1^2} \left(\sum_{r=1}^N \rho_r U_{rk} U_{rl} \right) E_{Xij}^{\psi(k)} E_{Ymn}^{\chi(l)}, \quad (27)$$

where, in order to have a compact expression for the Lagrangian in Eq. (26), the summation is written so that takes elements of the matrix U .

Another way of writing Eq. (26), using the notation of Ref. [12], is

$$- \mathcal{L}_{\text{eff}} = \frac{4G_F}{\sqrt{2}} \sum_{\psi,\chi} \sum_{i,j,m,n} [C_{mn}^{ij} Q_{mn}^{ij} + \tilde{C}_{mn}^{ij} \tilde{Q}_{mn}^{ij} + D_{mn}^{ij} O_{mn}^{ij} + \tilde{D}_{mn}^{ij} \tilde{O}_{mn}^{ij}], \quad (28)$$

with the local operators given by

$$\begin{aligned} Q_{mn}^{ij} &= (\psi_i \gamma^\mu P_L \psi_j) (\chi_m \gamma_\mu P_L \chi_n), \\ \tilde{Q}_{mn}^{ij} &= (\psi_i \gamma^\mu P_R \psi_j) (\chi_m \gamma_\mu P_R \chi_n), \\ O_{mn}^{ij} &= (\psi_i \gamma^\mu P_L \psi_j) (\chi_m \gamma_\mu P_R \chi_n), \\ \tilde{O}_{mn}^{ij} &= (\psi_i \gamma^\mu P_R \psi_j) (\chi_m \gamma_\mu P_L \chi_n), \end{aligned} \quad (29)$$

and the coefficients are

$$\begin{aligned} C_{mn}^{ij} &= \sum_{kl} W_{LL}^{kl,ijmn}, & \tilde{C}_{mn}^{ij} &= \sum_{kl} W_{RR}^{kl,ijmn}, \\ D_{mn}^{ij} &= \sum_{kl} W_{LR}^{kl,ijmn}, & \tilde{D}_{mn}^{ij} &= \sum_{kl} W_{RL}^{kl,ijmn}. \end{aligned} \quad (30)$$

The former expressions are perfectly general for any number of new Z bosons. In the case of only one extra Z boson, from these expressions and using an orthogonal 2×2 transformation matrix U parametrized by a mixing angle θ , it is straightforward to obtain the formulae for C_{kl}^{ij} , \tilde{C}_{kl}^{ij} , D_{kl}^{ij} and \tilde{D}_{kl}^{ij} in Ref. [12].

1. The $N = 3$ case

Let us now restrict ourselves to the case $N - 1 = 2$ corresponding to the 3-4-1 extension so that in Eq. (20) we have $Z_2^0 \equiv Z'$, $Z_3^0 \equiv Z''$, and a comparison with Eq. (15) gives: $g_1 = g_4/C_W$, $g_2 = g_X$ and $g_3 = g_4/(2\sqrt{2})$. These gauge couplings are expressed in terms of the $SU(2)_L \otimes U(1)_Y$ couplings of the SM by the gauge matching conditions in Eq. (4) and, as it is done in studies of low energy FCNC effects associated to extra neutral gauge bosons when only SM particles are present in the effective theory, we neglect their renormalization group evolution from the high scale $V' \simeq V$ down to the M_W scale (see, for example, Refs. [14,15]).

Since in our case the $Z - Z' - Z''$ mixing occurs between Z and Z' only, the matrix U takes the form

$$U = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (31)$$

Now, as already stated, the couplings of Z''_μ to the third family of quarks are different from the ones to the first two families, thus allowing FCNC at tree level induced by fermion mixing. These FCNC have consequences on the predictions of $\Delta S = 2$ and $\Delta B = 2$ observables in the down-quark sector as the well-measured ΔM_K , $\Delta M_{d/s}$, ε_K and $\sin\Phi_d$. In order to build the theoretical expressions both for these observables and for the related rare decays we will study let us extract, from Eq. (19), the couplings of Z''_μ to ordinary down-type quarks. In the gauge eigenstate basis $Q^d = (d_1 d_2 d_3)^T$ the interaction Lagrangian can be written as

$$- \mathcal{L}_{\text{NC}}^d = \frac{g_4}{2\sqrt{2}} \sum_{ij=1}^3 [\bar{Q}_i^d \gamma^\mu (\epsilon_{Lij}^{d(3)} P_L + \epsilon_{Rij}^{d(3)} P_R) Q_j^d] Z_\mu'' \quad (32)$$

where $\epsilon_L^{d(3)} = -I_{3 \times 3} + 2\text{diag}(0, 0, 1)$ and $\epsilon_R^{d(3)} = 0$. Then, the chiral Z_3 couplings, in the mass eigenstate basis $\mathcal{Q}^d = (dsb)^T$, are

$$E_L^{d(3)} = -I_{3 \times 3} + 2V_L^d \text{diag}(0, 0, 1)V_L^{d\dagger}; \quad E_R^{d(3)} = 0. \quad (33)$$

Clearly, FCNC arise in this extension from the non-diagonal elements in the 3×3 matrix $E_L^{d(3)}$. That is, the coefficients which determine the contribution of \mathcal{L}_{eff} to $\Delta S = 2$ and $\Delta B = 2$ processes are $E_{Lij}^{d(3)}$ with $i \neq j$. Therefore, using (26) and (33), the corresponding effective interaction Lagrangian is given by

$$- \mathcal{L}_{\text{eff}} = \frac{4G_F}{\sqrt{2}} \left(\frac{g_3}{g_1}\right)^2 \rho_3 (\bar{Q}_i^d \gamma_\mu E_{Lij}^{d(3)} P_L Q_j^d) \times (\bar{Q}_m^d \gamma^\mu E_{Lmn}^{d(3)} P_L Q_n^d). \quad (34)$$

The nondiagonal elements of $E_L^{d(3)}$ in Eq. (33) read

$$E_{Lij}^{d(3)} = 2V_{Li3}^d V_{Lj3}^{d*}, \quad (35)$$

whereas the ratio (g_3/g_1) can be written in terms of the Weinberg angle as

$$\left(\frac{g_3}{g_1}\right)^2 = \frac{\cos^2 \theta_W}{8}. \quad (36)$$

As it is well known, in the SM the matrix entries in V_L^d are not observables while the observable quantities only concern the entries in the V_{CKM} matrix. However, from Eq. (34) it is clear that the new physics contributions will be proportional to the entries V_{Li3}^d in the down-quark mixing matrix. We must, then, conveniently parametrize the V_L^d matrix so that we can estimate the size of its entries. This, in turn, requires to determine the number of independent parameters in V_L^d . A unitary 3×3 matrix has, in general, nine independent parameters: three real angles and six complex phases. In the CKM mixing matrix five phases can be absorbed in redefinitions of the quark fields, which leaves us with three real angles and one complex phase. In the case of the V_L^d matrix, by analogous redefinitions of the three ordinary down-type quarks, we can absorb three complex phases so that we remain with three real angles and three complex phases. The V_L^d matrix can consequently be conveniently parametrized as the product of three rotations each one of them containing a complex phase [16]. We thus get

$$V_L^d = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23}^d & s_{23}^d e^{-i\delta_{23}^d} \\ 0 & -s_{23}^d e^{i\delta_{23}^d} & c_{23}^d \end{pmatrix} \begin{pmatrix} c_{13}^d & 0 & s_{13}^d e^{-i\delta_{13}^d} \\ 0 & 1 & 0 \\ -s_{13}^d e^{i\delta_{13}^d} & 0 & c_{13}^d \end{pmatrix} \begin{pmatrix} c_{12}^d & s_{12}^d e^{-i\delta_{12}^d} & 0 \\ -s_{12}^d e^{i\delta_{12}^d} & c_{12}^d & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (37)$$

Performing the product we have

$$V_L^d = \begin{pmatrix} c_{12}^d c_{13}^d & s_{12}^d c_{13}^d e^{-i\delta_{12}^d} & s_{13}^d e^{-i\delta_{13}^d} \\ -s_{12}^d c_{23}^d e^{i\delta_{12}^d} - c_{12}^d s_{23}^d s_{13}^d e^{i(\delta_{13}^d - \delta_{23}^d)} & c_{12}^d c_{23}^d - s_{12}^d s_{23}^d s_{13}^d e^{i(\delta_{13}^d - \delta_{12}^d - \delta_{23}^d)} & s_{23}^d c_{13}^d e^{-i\delta_{23}^d} \\ s_{12}^d s_{23}^d e^{i(\delta_{12}^d + \delta_{23}^d)} - c_{12}^d c_{23}^d s_{13}^d e^{i\delta_{13}^d} & -c_{12}^d s_{23}^d e^{i\delta_{23}^d} - s_{12}^d c_{23}^d s_{13}^d e^{i(\delta_{13}^d - \delta_{12}^d)} & c_{23}^d c_{13}^d \end{pmatrix} \quad (38)$$

Another way V_L^d can be parametrized is [17]

$$V_L^d = P \tilde{V} K, \quad (39)$$

where $P = \text{diag}(e^{i\xi_1}, 1, e^{i\xi_3})$, $K = \text{diag}(e^{i\alpha_1}, e^{i\alpha_2}, e^{i\alpha_3})$ while, using the standard parametrization, the unitary matrix \tilde{V} can be written in terms of three mixing angles θ_{12} , θ_{23} and θ_{13} and one phase φ [11]. Note that this parametrization includes six complex phases but again, by redefinitions of the ordinary down-type quarks, we can absorb three of them.

It is easy to show that the theoretical expressions resulting from both parametrizations are equivalent via redefinition of phases [See Eqs. (62)–(64) below and Ref. [5]].

III. $\Delta F = 2$ OBSERVABLES

We now proceed to build the theoretical expressions for the $\Delta F = 2$ ($F = S, B$) neutral meson mixing observables ΔM_K , $\Delta M_{d/s}$, ε_K and $\sin\Phi_d$ in such a way that the Z_μ'' contributions enter into the expressions as corrections to the SM predictions, following a similar procedure as for the minimal 3-3-1 model in Ref. [4] and for the economical 3-3-1 model in Ref. [5]. These expressions are functions of the matrix element

$$M_{12}^P \equiv \frac{\langle P^0 | \mathcal{H}_{\text{eff}} | \bar{P}^0 \rangle}{2m_P}, \quad (40)$$

where P stands for K , B_s , or B_d . In our case M_{12}^P receives both SM contributions arising from standard one-loop

diagrams and contributions coming from tree-level Z_3 exchange, that is

$$M_{12}^P = M_{12}^{P(\text{SM})} + M_{12}^{P(3-4-1)}. \quad (41)$$

The expressions for the $\Delta F = 2$ observables are

$$\Delta m_K = 2 \text{Re}(M_{12}^K), \quad (42)$$

$$\Delta m_d = 2|M_{12}^{B_d}|, \quad (43)$$

$$\Delta m_s = 2|M_{12}^{B_s}|, \quad (44)$$

$$\varepsilon_K = e^{i\phi_\varepsilon} \sin\phi_\varepsilon \left(\frac{\text{Im}(M_{12}^K)}{\Delta m_K} + P_0 \right), \quad (45)$$

$$\Phi_d = \arg(M_{12}^{B_d}), \quad (46)$$

where the term P_0 is due to long-distance contributions to Kaon mixing. The experimental values for these observables are consigned in Table II. It is customary to set $\phi_\varepsilon = \pi/4$ and to neglect P_0 [23,24]. For this reason a multiplicative correction factor for ε_K , that accounts for $\phi_\varepsilon \neq \pi/4$ and $P_0 \neq 0$, is introduced [18,25]. Then, the expression for ε_K becomes

$$\varepsilon_K = \kappa_\varepsilon \exp\left(i\frac{\pi}{4}\right) \frac{\text{Im}(M_{12}^K)}{\sqrt{2}\Delta m_K}. \quad (47)$$

The well-known SM contributions to M_{12}^P are given by [23,24]

$$M_{12}^{K(\text{SM})} = \frac{G_F^2}{12\pi^2} f_K^2 B_K m_K m_W^2 [\lambda_c^{*2} \eta_1 S_0(x_c) + \lambda_t^{*2} \eta_2 S_0(x_t) + 2\lambda_c^* \lambda_t^* \eta_3 S_0(x_c, x_t)], \quad (48)$$

and

$$M_{12}^{B_q(\text{SM})} = \frac{G_F^2}{12\pi^2} f_{B_q}^2 B_{B_q} \eta_B m_{B_q} m_W^2 S_0(x_t) (V_{tq}^* V_{tb})^2, \quad (49)$$

where q stands for d or s . f_P is the P -meson decay constant, B_P is the corresponding renormalization scale and scheme invariant bag parameter and the basic electro-weak loop contributions, without QCD corrections, are expressed through the functions $S_0(x_i)$ ($x_i = m_i^2/m_W^2$), namely

TABLE II. Values of the experimental and theoretical quantities used as input parameters.

Input	Value	Reference
G_F [GeV]	$1.16637(1) \times 10^{-5}$	
α	$7.2973525376(50) \times 10^{-3}$	
$\alpha_s(m_Z)$	0.1184(7)	
$\sin^2\theta_W(m_Z)(\overline{MS})$	0.23116(13)	[11]
$ V_{ud} $	0.9728(30)	
$ V_{us} $	0.2250(27)	
$ V_{ub} $	0.00427(38)	
$ V_{cd} $	0.230(11)	
$ V_{cs} $	0.98(10)	
$ V_{cb} $	0.0415(7)	
$ V_{td} $	0.0084(6)	
$ V_{ts} $	0.0387(21)	
$ V_{tb} $	0.88(7)	[11]
m_W [GeV]	80.399(23)	
m_Z [GeV]	91.1876(21)	
m_c [GeV]	1.268(9)	
m_t [GeV]	172.4(1.2)	[11]
γ	78(12)	[18]
$f_K\sqrt{B_K}$ [GeV]	$133(55) \times 10^{-3}$	
$f_{B_d}\sqrt{B_{B_d}}$ [GeV]	$216(15) \times 10^{-3}$	
$f_{B_s}\sqrt{B_{B_s}}$ [GeV]	$275(13) \times 10^{-3}$	[18]
η_1	1.32(32)	[19,20]
η_2	0.5765(65)	[19,20]
η_3	0.47(4)	[20,21]
η_B	0.551(7)	[19,22]
κ_ε	0.92(1)	[18]
m_{K^0} [GeV]	$497.614(24) \times 10^{-3}$	
m_{B_d} [GeV]	$5279.50(30) \times 10^{-3}$	
m_{B_s} [GeV]	$5366.3(6) \times 10^{-3}$	
Δm_K [ps^{-1}]	$0.5292(9) \times 10^{-2}$	
Δm_{B_d} [ps^{-1}]	0.507(5)	
Δm_{B_s} [ps^{-1}]	17.77(12)	
$ \varepsilon_K $	$2.228(11) \times 10^{-3}$	
$\sin\Phi_d$	0.673(23)	[11]

$$S_0(x_c) \doteq x_c, \quad (50)$$

$$S_0(x_t) = \frac{4x_t - 11x_t^2 + x_t^3}{4(1-x_t)^2} - \frac{3x_t^3 \ln x_t}{2(1-x_t)^3}, \quad (51)$$

$$S_0(x_c, x_t) = x_c \left[\ln \frac{x_t}{x_c} - \frac{3x_t}{4(1-x_t)} - \frac{3x_t^2 \ln x_t}{4(1-x_t)^2} \right]. \quad (52)$$

Renormalization group short-distance QCD effects are described through the renormalization scheme independent factors η_1 , η_2 , η_3 and η_B [19–22], and the CKM factors are given by $\lambda_i = V_{is}^* V_{id}$.

Now, from the effective Lagrangian in Eq. (34) the expressions for the 3-4-1 contributions can be obtained.

Since only left-left (LL) operators appear in the effective Lagrangian, the QCD renormalization group evolution of the Wilson coefficients $C(\mu)$ after including the Z_3 contributions is exactly the same as in the SM. Then, the relevant elements of the evolution matrix $U(\mu, \mu_3)$ from the high scale $\mu_3 = m_{Z_3}$ down to the scale μ , in the notation of Ref. [26], can be calculated from

$$[\eta(\mu, \mu_3)]_{\text{VLL}} = [\eta(\mu, \mu_t)]_{\text{VLL}}[\eta(\mu_t, \mu_3)]_{\text{VLL}}, \quad (53)$$

where $\mu_t = \mathcal{O}(m_t)$.

In general, for $\mu < \mu'$, the correction factor $[\eta(\mu, \mu')]_{\text{VLL}}$ in an effective theory with f active quark flavors is given by

$$[\eta(\mu, \mu')]_{\text{VLL}} = [\eta^{(0)}(\mu)]_{\text{VLL}} + \frac{\alpha_s^{(f)}(\mu)}{4\pi} [\eta^{(1)}(\mu)]_{\text{VLL}}, \quad (54)$$

where $[\eta^{(0)}(\mu)]_{\text{VLL}}$ and $[\eta^{(1)}(\mu)]_{\text{VLL}}$ are the leading order (LO) and the next-to-leading order (NLO) factors, respectively, and in the right-hand side we have suppressed the high scale μ' .

For K and B meson mixing the factor $[\eta(\mu, \mu_t)]_{\text{VLL}}$ in Eq. (53) has been calculated in Ref. [26] using $\mu_t = m_t(m_t) = 166$ GeV. The results are: $[\eta^K(\mu_L, \mu_t)]_{\text{VLL}} = 0.788$, for $K^0 - \bar{K}^0$ mixing with $\mu_L = 2$ GeV the lattice scale where the matching with the lattice results of the associated hadronic matrix elements is done, and $[\eta^B(\mu_b, \mu_t)]_{\text{VLL}} = 0.842$, for $B_{d/s}^0 - \bar{B}_{d/s}^0$ mixing with $\mu_b = 4.4$ GeV the mass scale of the bottom quark. In order to calculate $[\eta(\mu, \mu_3)]_{\text{VLL}}$ we first evaluate $\alpha_s^{(f)}(\mu_3)$ to LO in the $\overline{\text{MS}}$ scheme in an effective theory with $f = 6$ quark flavors and we impose the continuity relation $\alpha_s^{(6)}(\mu_t) = \alpha_s^{(5)}(\mu_t)$. To this purpose we use

$$\begin{aligned} \frac{1}{\alpha_s^{(5)}(\mu_t)} &= \frac{1}{\alpha_s^{(5)}(m_Z)} + \frac{\beta_0^{(5)}}{4\pi} \ln \frac{\mu_t^2}{m_Z^2}, \\ \frac{1}{\alpha_s^{(6)}(\mu_3)} &= \frac{1}{\alpha_s^{(5)}(\mu_t)} + \frac{\beta_0^{(6)}}{4\pi} \ln \frac{\mu_3^2}{\mu_t^2}, \end{aligned} \quad (55)$$

where $\beta_0^{(f)} = 11 - (2/3)f$ and for $\alpha_s^{(5)}(m_Z)$ we use the central value $\alpha_s^{(5)}(m_Z) = 0.1184$. In this way $[\eta(\mu, \mu_3)]_{\text{VLL}}$ can be obtained from Eq. (54) with [26]

$$\begin{aligned} [\eta^{(0)}(\mu_t)]_{\text{VLL}} &= \eta_6^{6/21}, \\ [\eta^{(1)}(\mu_t)]_{\text{VLL}} &= 1.3707(1 - \eta_6)\eta_6^{6/21}, \end{aligned} \quad (56)$$

where $\eta_6 = \alpha_s^{(6)}(\mu_3)/\alpha_s^{(6)}(\mu_t)$. The numerical value of $[\eta(\mu, \mu_3)]_{\text{VLL}}$ depends of the value of the high energy scale $\mu_3 = m_{Z_3}$. Since in our approximation the Z_2 and Z_3 mass scales are of the same order (see Sec. II), we will select values for m_{Z_3} in the typical range 1–5 TeV.

With this in mind and defining

$$\begin{aligned} U_{\text{VLL}}^K &\equiv [\eta^K(\mu_L, \mu_t)]_{\text{VLL}}[\eta(\mu_t, \mu_3)]_{\text{VLL}}, \\ U_{\text{VLL}}^B &\equiv [\eta^B(\mu_b, \mu_t)]_{\text{VLL}}[\eta(\mu_t, \mu_3)]_{\text{VLL}}, \end{aligned} \quad (57)$$

the 3-4-1 contribution to M_{12}^P is given by

$$M_{12}^{P(3-4-1)} = \frac{8\sqrt{2}G_F}{3} U_{\text{VLL}}^P \rho_3 \left(\frac{g_3}{g_1}\right)^2 m_P f_P^2 \hat{B}_P \lambda_P^2, \quad (58)$$

where

$$\lambda_K = s_{13}^d s_{23}^d c_{13}^d e^{i\phi'}, \quad (59)$$

$$\lambda_{B_d} = s_{13}^d c_{23}^d c_{13}^d e^{i\phi'}, \quad (60)$$

$$\lambda_{B_s} = s_{23}^d c_{23}^d (c_{13}^d)^2 e^{-i\phi_3}, \quad (61)$$

and the phases are defined as

$$\phi' = -\delta_{13}^d, \quad (62)$$

$$\phi'' = -(\delta_{13}^d - \delta_{23}^d), \quad (63)$$

$$\phi_3 = \phi'' - \phi'. \quad (64)$$

As can be seen from Eq. (58), the 3-4-1 contributions $M_{12}^{P(3-4-1)}$ are given in terms of five unknown independent parameters, namely, the mass of the Z_3 gauge boson, the angles θ_{13}^d , θ_{23}^d and the two complex phases ϕ' and ϕ'' , where the last four parameters come from the V_L^d mixing matrix.

The $\Delta F = 2$ observables discussed in this Section are, in principle, sufficient to set bounds on the relevant parameters of V_L^d in Eq. (38).

IV. THEORETICAL EXPRESSIONS FOR RARE DECAY AMPLITUDES

In the next Section, using the observables related to meson mixing, the bounds on the parameter space associated to the relevant entries of the down-quark mixing matrix will be established. Our next task will be to study the implications of these bounds on several rare decay amplitudes following a similar approach as in Ref. [4]. The decays we will study are the theoretically very clean rare decays $K^+ \rightarrow \pi^+ \nu \bar{\nu}$, $K_L \rightarrow \pi^0 \nu \bar{\nu}$, and the clean rare decays $B_{d/s} \rightarrow \mu^+ \mu^-$, $K_L \rightarrow \pi^0 l^+ l^-$ ($l = e, \mu$).

As it is known, the rare decays we are interested in are governed both by electroweak- and photon-penguins and by leptonic box diagram contributions. In the SM these contributions are described, in the LO, by the corresponding Inami-Lim functions $C_0(x_t)$, $D_0(x_t)$ and $B_0(x_t)$ which, in the expressions for the decay amplitudes, always appear in the gauge invariant combinations: $X_0(x_t) = C_0(x_t) - 4B_0(x_t)$, $Y_0(x_t) = C_0(x_t) - B_0(x_t)$, $Z_0(x_t) = C_0(x_t) + (1/4)D_0(x_t)$ [27]. In the NLO these combinations will be written as $X(x_t)$, $Y(x_t)$ and $Z(x_t)$. Since the Z_3 contribution

to the effective Hamiltonians for the rare decays we are considering have the same operator structure than the ones in the SM, that is they have a $(V - A)(V - A)$ form, its effects on the various decays can be encoded by appropriate redefinitions of the $X(x_i)$, $Y(x_i)$ and $Z(x_i)$ functions in the form

$$\begin{aligned} X'(x_i) &= X^{(\text{SM})}(x_i) + \Delta X, \\ Y'(x_i) &= Y^{(\text{SM})}(x_i) + \Delta Y, \\ Z'(x_i) &= Z^{(\text{SM})}(x_i) + \Delta Z. \end{aligned} \quad (65)$$

As we will see, in the 3-4-1 extension the redefined functions are in general complex. We will write them as they appear in Ref. [23] so that the modifications lead to the correct results in the 3-4-1 extension including the corresponding imaginary part. For the decays $K_L \rightarrow \pi^0 l^+ l^-$, in which there is also a right-handed contribution, the Z_3 effects will be absorbed into the matching conditions of the Wilson coefficients.

A. $K \rightarrow \pi \nu \bar{\nu}$

For $K \rightarrow \pi \nu \bar{\nu}$ there exists a charged decay $K^+ \rightarrow \pi^+ \nu \bar{\nu}$, and a neutral one $K_L \rightarrow \pi^0 \nu \bar{\nu}$ [28]. For both of them the NLO effective Hamiltonian is given by

$$\begin{aligned} H_{\text{eff}}^{\text{SM}} &= \frac{G_F}{\sqrt{2}} \frac{\alpha}{2\pi \sin^2 \theta_W} \sum_{l=e,\mu,\tau} (V_{cs}^* V_{cd} X_{\text{NL}}^l + V_{ts}^* V_{td} X(x_i)) \\ &\quad \times (\bar{s}d)_{V-A} (\bar{\nu}_l \nu_l)_{V-A}, \end{aligned} \quad (66)$$

where the functions X_{NL} and $X(x_i)$ represent the charm and top-loop contributions, respectively. Summing over the three neutrino flavors and collecting the charm contributions in $P_c(X) = 0.41 \pm 0.05$ [29], the BR for $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ can be expressed as

$$\begin{aligned} \text{BR}(K^+ \rightarrow \pi^+ \nu \bar{\nu}) &= \kappa_+ \cdot \left[\left(\text{Im} \left(\frac{\lambda_t}{\lambda^5} X(x_i) \right) \right)^2 + \left(\text{Re} \left(\frac{\lambda_c}{\lambda} P_c(X) \right) \right. \right. \\ &\quad \left. \left. + \text{Re} \left(\frac{\lambda_t}{\lambda^5} X(x_i) \right) \right)^2 \right], \end{aligned} \quad (67)$$

where [30]

$$\begin{aligned} \kappa_+ &= r_{K^+} \frac{3\alpha^2 \text{BR}(K^+ \rightarrow \pi^0 e^+ \nu)}{2\pi^2 \sin^4 \theta_W} \lambda^8 \\ &= (5.26 \pm 0.06) \times 10^{-11} \left[\frac{\lambda}{0.225} \right]^8. \end{aligned} \quad (68)$$

For $K_L \rightarrow \pi^0 \nu \bar{\nu}$ it is found

$$\text{BR}(K_L \rightarrow \pi^0 \nu \bar{\nu}) = \kappa_L \cdot \left(\text{Im} \left(\frac{\lambda_t}{\lambda^5} X(x_i) \right) \right)^2, \quad (69)$$

with

$$\kappa_L = \kappa_+ \frac{r_{K_L}}{r_{K^+}} \frac{\tau(K_L)}{\tau(K^+)} = (2.29 \pm 0.03) \times 10^{-10} \left[\frac{\lambda}{0.225} \right]^8, \quad (70)$$

where λ is a parameter of the Wolfenstein parametrization of the CKM matrix [31] and is set equal to s_{12} of the standard parametrization [11].

For both decays the contribution from the 3-4-1 extension to the effective Hamiltonian comes from a tree diagram transmitted by the Z'' boson and has the form

$$H_{\text{eff}}^{Z''} = \sum_{l=e,\mu,\tau} \frac{2G_F}{\sqrt{2}} \left(\frac{g_3}{g_1} \right)^2 \rho_3 V_{L23}^d V_{L13}^{d*} (\bar{s}d)_{V-A} (\bar{\nu}_l \nu_l)_{V-A}. \quad (71)$$

This contribution can be included as a modification ΔX of the function $X(x_i)$ which reads

$$\Delta X_{K\pi\nu\nu} = \frac{4\pi \sin^2 \theta_W}{\alpha} \left(\frac{g_3}{g_1} \right)^2 \rho_3 \frac{V_{L23}^d V_{L13}^{d*}}{V_{ts}^* V_{td}}. \quad (72)$$

B. $B_{d/s} \rightarrow \mu^+ \mu^-$

These decays are entirely determined by the top contribution so that the SM effective Hamiltonian in the NLO is given by

$$H_{\text{eff}}^{B_{d/s}\mu\mu} = -\frac{G_F}{\sqrt{2}} \frac{\alpha}{2\pi s_W^2} (V_{tb}^* V_{td/s}) Y(x_t) (\bar{b}q)_{V-A} (\bar{\mu}\mu)_{V-A}. \quad (73)$$

From here, the expressions for the branching fractions are obtained as

$$\begin{aligned} \text{BR}(B_q \rightarrow \mu^+ \mu^-) &= \tau_{B_q} \frac{G_F^2}{\pi} m_{B_q} \left(\frac{\alpha f_{B_q} m_\mu}{4\pi \sin^2 \theta_W} \right)^2 \\ &\quad \times \sqrt{1 - 4 \frac{m_\mu^2}{m_{B_q}^2} |V_{tb}^* V_{tq} Y(x_t)|^2}. \end{aligned} \quad (74)$$

The 3-4-1 contribution to $B_{d/s} \rightarrow \mu^+ \mu^-$ is found to be

$$H_{\text{eff}}^{Z''} = \frac{2G_F}{\sqrt{2}} \left(\frac{g_3}{g_1} \right)^2 \rho_3 V_{L33}^d V_{L13/23}^{d*} (\bar{b}q)_{V-A} (\bar{\mu}\mu)_{V-A}, \quad (75)$$

which we now absorb into a modification of $Y(x_i)$ as

$$\Delta Y_{B\mu\mu} = -\frac{4\pi \sin^2 \theta_W}{\alpha} \left(\frac{g_3}{g_1} \right)^2 \rho_3 \frac{V_{L33}^d V_{L13/23}^{d*}}{V_{tb}^* V_{td/ts}}. \quad (76)$$

C. $K_L \rightarrow \pi^0 l^+ l^-$

In the SM the CP -violating part of the effective Hamiltonian, after neglect QCD penguin operators, is given by

$$H_{\text{eff}}^{K\pi ll} = -\frac{G_F}{\sqrt{2}} V_{ts}^* V_{td} (y_{7V} Q_{7V} + y_{7A} Q_{7A}), \quad (77)$$

where $Q_{7V} = (\bar{s}d)_{V-A}\bar{e}\gamma^\mu e$ and $Q_{7A} = (\bar{s}d)_{V-A}\bar{e}\gamma^\mu\gamma^5 e$ are the vector- and axial-vector operators which contribute and that originate from γ - and Z -penguin and box diagrams. The matching conditions of the Wilson coefficients y_{7V} and y_{7A} are given by

$$y_{7V} = \frac{\alpha}{2\pi} \left(\frac{Y_0(x_t)}{\sin^2\theta_W} - 4Z_0(x_t) + P_0 \right), \quad (78)$$

and

$$y_{7A} = -\frac{\alpha}{2\pi} \frac{Y_0(x_t)}{\sin^2\theta_W}. \quad (79)$$

In y_{7V} a small term P_E has been neglected and we use the normalization $P_0 = 2.89 \pm 0.06$ [32,33].

For these decays the contributions from new physics lead to the effective Hamiltonian

$$H_{\text{eff}}^{Z'} = \frac{2G_F}{\sqrt{2}} \left(\frac{g_3}{g_1} \right)^2 \rho_3 V_{L23}^d V_{L13}^{d*} (Q_{7V} - Q_{7A}). \quad (80)$$

Following [4], instead of absorbing these new contributions into modifications of the Inami-Lim functions, they will be absorbed into the matching conditions of the Wilson coefficients in the form

$$\Delta y_{7V} = -2 \left(\frac{g_3}{g_1} \right)^2 \rho_3 \frac{(V_{L23}^d V_{L13}^{d*})}{V_{ts}^* V_{td}}, \quad (81)$$

$$\Delta y_{7A} = 2 \left(\frac{g_3}{g_1} \right)^2 \rho_3 \frac{(V_{L23}^d V_{L13}^{d*})}{V_{ts}^* V_{td}}. \quad (82)$$

The expressions for the BR, including the long-distance indirectly CP -violating terms and their interference with the short-distance contributions, are [34]

$$Br(K_L \rightarrow \pi^0 \ell^+ \ell^-) = (C_{\text{dir}}^\ell \pm C_{\text{int}}^\ell |a_s| + C_{\text{mix}}^\ell |a_s|^2 + C_{\text{CPC}}^\ell) \times 10^{-12}, \quad (83)$$

where

$$C_{\text{dir}}^e = (4.62 \pm 0.24)(\omega_{7V}^2 + \omega_{7A}^2), \quad (84)$$

$$C_{\text{dir}}^\mu = (1.09 \pm 0.05)(\omega_{7V}^2 + 2.32\omega_{7A}^2),$$

$$C_{\text{int}}^e = (11.3 \pm 0.3)\omega_{7V}, \quad C_{\text{int}}^\mu = (2.63 \pm 0.06)\omega_{7V}, \quad (85)$$

$$C_{\text{mix}}^e = 14.5 \pm 0.05, \quad C_{\text{mix}}^\mu = 3.36 \pm 0.20, \quad (86)$$

$$C_{\text{CPC}}^e \simeq 0, \quad C_{\text{CPC}}^\mu = 5.2 \pm 1.6, \quad (87)$$

$$|a_s| = 1.2 \pm 0.2, \quad (88)$$

with $w_{7A,7V} = \text{Im}(\lambda_t y_{7A,7V}) / \text{Im}\lambda_t$.

V. CONSTRAINTS AND IMPLICATIONS FOR RARE DECAYS

In this Section we numerically evaluate the theoretical expressions obtained in the two previous sections. As noted above, the 3-4-1 contributions $M_{12}^{P(3-4-1)}$ to the matrix elements in Eqs. (42)–(46) are given in terms of the five unknown independent parameters m_{Z_3} , the angles θ_{13}^d , θ_{23}^d and two complex phases, ϕ' and ϕ'' . We perform two related numerical analysis: in the first one, we use the well-measured observables ΔM_K , $\Delta M_{d/s}$, ε_K and $\sin\Phi_d$ to constrain the angles and phases appearing in the V_L^d mixing matrix for selected values of the Z_3 mass. In the second analysis we study the implications of these bounds on the rare decays previously mentioned. In this context we are mainly interested in obtaining upper and lower bounds for these decays as a function of the Z_3 mass in order to confront them with the present experimental data and with the SM predictions. The updated experimental data for the input parameters are also given in Table II.

A. Constraints from $K^0 - \bar{K}^0$ and $B_q^0 - \bar{B}_q^0$ mixing

We start by focusing on the bounds on the parameter space of the 3-4-1 extension associated to the downlike quark mixing parameters. Two possible analyses can be carried out. The first one, followed in several works both in 3-3-1 models [2] and in 3-4-1 extensions [7,9,10], consist in to assume a texture for the mixing matrix in order to obtain bounds on the mass of the heavy gauge bosons. In the opposite approach, one can fix these masses and obtain some information on the size of the corresponding mixing matrix elements [3–5]. Here we follow this second approach with fixed values of m_{Z_3} in the range 1–5 TeV.

In previous studies on flavour physics observables in different models, the numerical analysis was simplified by setting all input parameters to their central values and allowing instead ΔM_K , ε_K , ΔM_d , ΔM_s , $|\varepsilon_K|$, and $\sin\Phi_d$ to differ from their experimental values by $\pm 50\%$, $\pm 40\%$, $\pm 40\%$, $\pm 40\%$, $\pm 20\%$ and $\pm 8\%$, respectively. This simplifying assumption was justified in order to determine the size of possible effects on observables that have not been well measured so far. In recent analysis, however, an improved error analysis has been done that enable us to draw more accurate conclusions in view of the recent significant improvements in the experimental constraints and in the lattice determinations of the nonperturbative parameter. Therefore, in what follows, we will take all input parameters to be flatly distributed within their 1σ ranges indicated in Table II. At the same time, we require the observables $|\varepsilon_K|$, ΔM_d , ΔM_s and $\sin\Phi_d$, resulting from the SM and the 3-4-1 contributions, to lie within their experimental 1σ ranges. In the case of ΔM_K , where the theoretical uncertainty is large due to unknown long-distance contributions, we allow the generated value to lie within $\pm 30\%$ of its experimental central value.

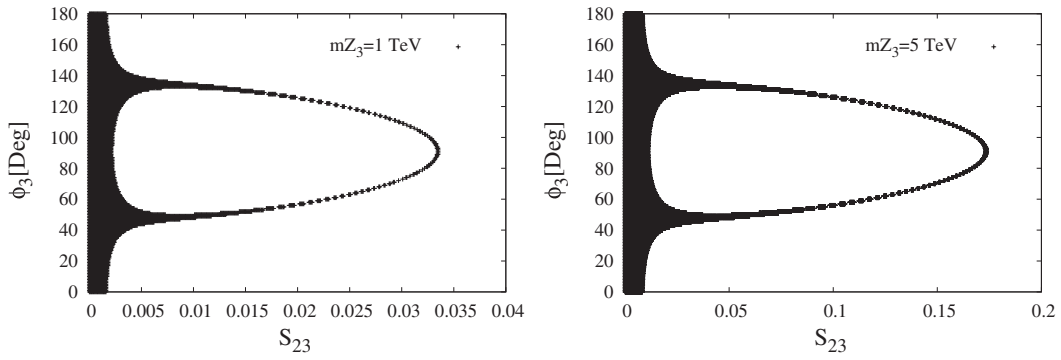


FIG. 1. Allowed $s_{23}^d - \phi_3$ region for $s_{13}^d = 0$, and $m_{Z_3} = 1$ TeV (Left), $m_{Z_3} = 5$ TeV (right).

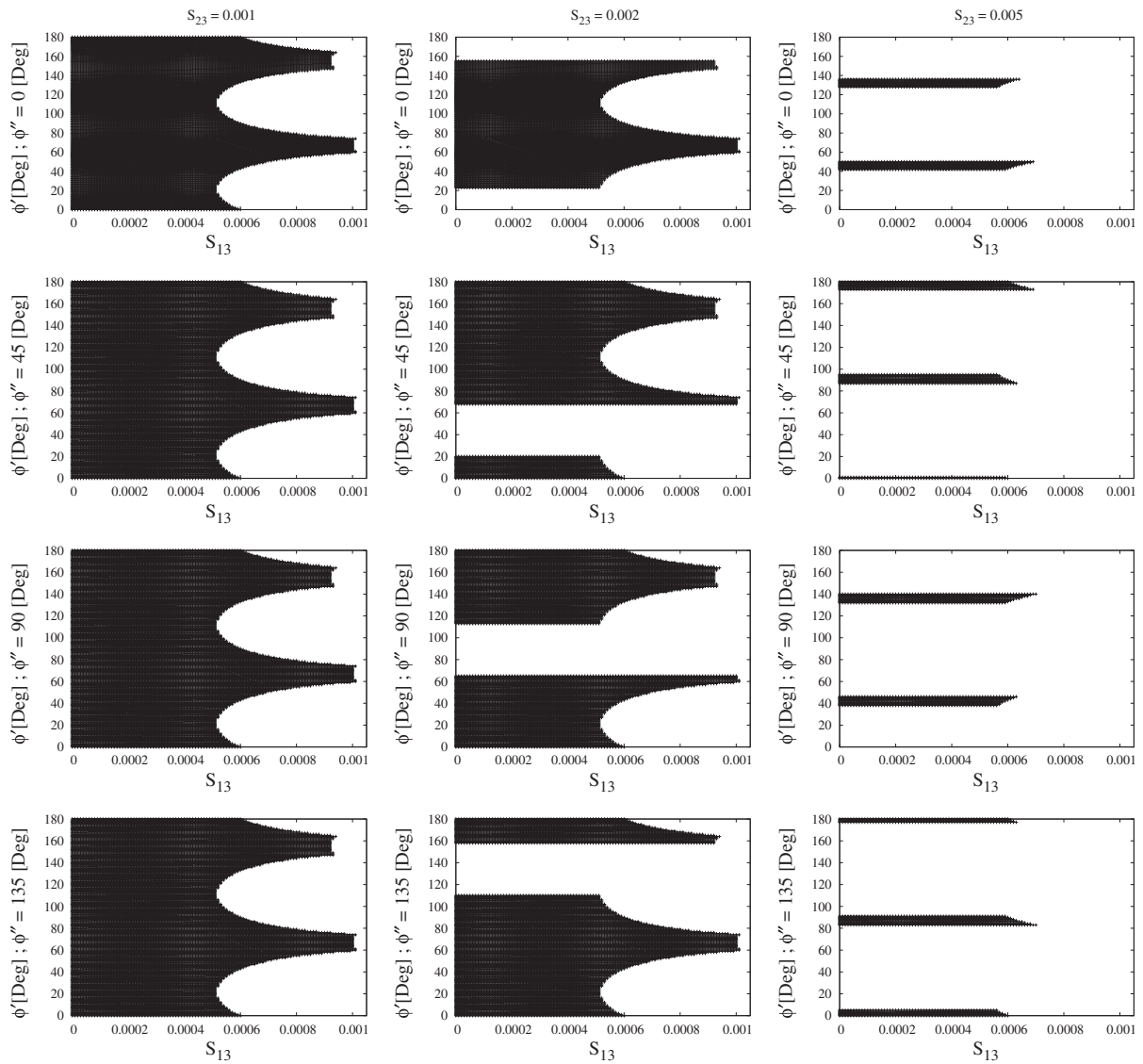


FIG. 2. Allowed $s_{13}^d - \phi'$ region for $m_{Z_3} = 1$ TeV and for different values of ϕ'' and increasing values of s_{23}^d : 0.001 (left), 0.002 (center) and 0.005 (right).

TABLE III. Upper limit for θ_{13} and θ_{23} for different values of m_{Z_3} .

m_{Z_3}	$\theta_{13,\max}$	$\theta_{23,\max}$
1 TeV	8.73×10^{-4}	3.14×10^{-2}
3 TeV	2.97×10^{-3}	9.42×10^{-2}
5 TeV	5.06×10^{-3}	1.75×10^{-1}

In dealing with the values of the CKM matrix parameters quoted in Table II, instead of using the full CKM angle fits, we take only the values related to direct experimental measures with the largest uncertainties taken into account. In this way we avoid possible modifications from the 3-4-1 contributions to the one-loop SM processes included in the global fits and leave the largest room available for possible new physics effects, respecting the well-measured values. For the CP -violating parameter γ we take its direct determination from the model-independent UTfit analysis of $B \rightarrow D^{(*)}K^{(*)}$ decays.

We then proceed to scan the parameter space fixing m_{Z_3} and generating a large number of points that we call an ‘‘effective’’ parameter space. For each fixed value of the Z_3 mass, this space consists of more than 1×10^6 points that fulfil the requirements listed above, where all angles are varied in the interval $[0, \pi/2]$, all phases between 0 and 2π , and all input parameters are varied in their 1σ ranges.

With these data we can plot contours setting bounds on some of the parameters. For example, let us take the mixing angle $\theta_{13}^d = 0$. In this case, Eq. (61) determines the allowed region in the $s_{23}^d - \phi_3$ plane, which is shown in Fig. 1 for $m_{Z_3} = 1$ TeV and $m_{Z_3} = 5$ TeV. These plots allows us to set upper bounds on $|s_{23}^d|$ which depend on the value of m_{Z_3} . It is seen that the upper bound for s_{23}^d increase for increasing values of m_{Z_3} . Figure 1 also shows that, at the 1σ confidence level, there is a large and typical region which is excluded and that prefers values for ϕ_3 around 48° and 136° independently of the Z_3 mass.

Now we explore, for several values of m_{Z_3} , the allowed regions in the plane $s_{13}^d - \phi'$ using nonzero values of s_{23}^d and selected values of ϕ'' . This is shown in Fig. 2 for $m_{Z_3} = 1$. We have taken values for s_{23}^d consistent with the ones in the allowed $s_{23}^d - \phi_3$ plane and we have fixed ϕ'' at the values 0° , 45° , 90° , and 135° . This figure shows that for increasing values of s_{23}^d and depending of the ϕ'' value,

the allowed region reduces considerably involving both the selection of very specific values of ϕ' and a decrease in the upper bound for s_{13}^d .

With the data from the effective parameter space we can estimate the order of magnitude of the upper limits for θ_{13}^d and θ_{23}^d for several values of the Z_3 mass. The results are collected in Table III which shows that these upper bounds increase for increasing values of m_{Z_3} . This allow us to elucidate the size of the allowed region in the parameter space.

B. Implications for rare decays

In order to study the implications of the obtained bounds for the modification in the rare decay amplitudes given in Section IV, we move over all the effective parameter space for each selected value of m_{Z_3} , and we calculate the amplitudes for the different decays. At difference with the analysis in Ref. [4], our procedure allows us to set not only upper bounds but also lower bounds on the corresponding BR as a function of the Z_3 mass. The present experimental data for these decays [11] and the SM predictions [29,34,35] are given in Table IV. Our goal will be to determine these upper and lower bounds and establish their compatibility with the data in this Table.

For the decays $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ and $K_L \rightarrow \pi^0 \nu \bar{\nu}$, using Eqs. (67), (69), and (72) and the effective parameter space, the bounds are given in Fig. 3. For the decay $K^+ \rightarrow \pi^+ \bar{\nu} \nu$ we see from Table IV that the experimental central value is greater than the SM prediction. Figure 3 shows that for low values of m_{Z_3} the upper limit reaches the experimental central value, but for larger m_{Z_3} both the upper and lower limits go closer to the SM prediction. For the decay $K_L \rightarrow \pi^0 \nu \bar{\nu}$ we can see that big enhancements are expected for low m_{Z_3} . The bigger enhancements are expected for m_{Z_3} values less that ~ 2.5 TeV. For the decay $K^+ \rightarrow \pi^+ \bar{\nu} \nu$ the greatest contributions come from the CP -conserving case and for $K_L \rightarrow \pi^0 \nu \bar{\nu}$ come from the CP -violating case in accordance with the setting in [4], but with the difference that we have used all the effective parameter space which includes the CP -conserving, the CP -violating and the mixed cases.

To appreciate more clearly the departures from the SM, the ratios $\text{BR}(K^+ \rightarrow \pi^+ \nu \bar{\nu})/\text{BR}(K^+ \rightarrow \pi^+ \nu \bar{\nu})_{\text{SM}}$ and $\text{BR}(K_L \rightarrow \pi^0 \nu \bar{\nu})/\text{BR}(K_L \rightarrow \pi^0 \nu \bar{\nu})_{\text{SM}}$ vs m_{Z_3} are shown

TABLE IV. Present experimental data and SM predictions for the rare decays considered in this work.

Label	Decay	Experimental	SM
$\text{BR}(K^+)$	$\text{BR}(K^+ \rightarrow \pi^+ \nu \bar{\nu})$	$(1.7 \pm 1.1) \times 10^{-10}$	$(7.81_{-0.71}^{+0.80} \pm 0.29) \times 10^{-11}$
$\text{BR}(K_L \nu)$	$\text{BR}(K_L \rightarrow \pi^0 \nu \bar{\nu})$	$< 6.7 \times 10^{-8}$ (90% C.L.)	$(2.43_{-0.37}^{+0.40} \pm 0.06) \times 10^{-11}$
$\text{BR}(B_s)$	$\text{BR}(B_s \rightarrow \mu^+ \mu^-)$	$< 1 \times 10^{-7}$ (90% C.L.)	$(3.35 \pm 0.32) \times 10^{-9}$
$\text{BR}(B_d)$	$\text{BR}(B_d \rightarrow \mu^+ \mu^-)$	$< 3 \times 10^{-8}$ (90% C.L.)	$(1.03 \pm 0.09) \times 10^{-10}$
$\text{BR}(K_L e)$	$\text{BR}(K_L \rightarrow \pi^0 e^+ e^-)$	$< 28 \times 10^{-11}$ (90% C.L.)	$(3.54_{-0.85}^{+0.98}) \times 10^{-11}$
$\text{BR}(K_L \mu)$	$\text{BR}(K_L \rightarrow \pi^0 \mu^+ \mu^-)$	$< 38 \times 10^{-11}$ (90% C.L.)	$(1.41_{-0.26}^{+0.28}) \times 10^{-11}$

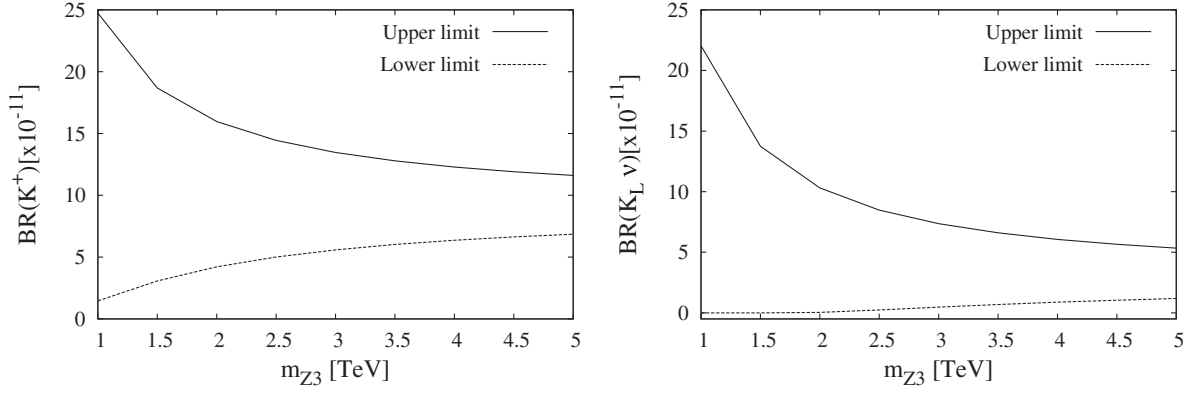


FIG. 3. Upper and lower bounds on the BR for the decay $K^+ \rightarrow \pi^+ \bar{\nu} \nu$ (left), and for the decay $K_L \rightarrow \pi^0 \nu \bar{\nu}$ (right).

in Fig. 4. For the decay $K^+ \rightarrow \pi^+ \bar{\nu} \nu$ we find enhancements around 2.8 times the SM prediction for low values of m_{Z_3} , but small departures from the SM for large Z_3 mass. For $K_L \rightarrow \pi^0 \nu \bar{\nu}$, enhancements of around 7.6 times the SM value are found for low values of m_{Z_3} .

Comparing the upper and lower bounds on both decays we see that the possibility of enhancement is always larger than the possibility of suppression, except for large m_{Z_3} where both of them are comparable.

The plane $\text{BR}(K_L \rightarrow \pi^0 \nu \bar{\nu}) - \text{BR}(K^+ \rightarrow \pi^+ \bar{\nu} \nu)$ is shown in Fig. 5. The upper and lower branches correspond mainly to the CP -violating case and the horizontal branch to the CP -conserving case. It is implicit in this figure that the mixed case is very restricted making the expressions in Eqs. (72), (76), (81), and (82) mainly real or imaginary. Notice that the pattern seen is similar to the one obtained in the Littlest Higgs model with T parity in Ref. [36] and in the minimal 3-3-1 model in Ref. [4].

For the decays $B_s \rightarrow \mu^+ \mu^-$ and $B_d \rightarrow \mu^+ \mu^-$, using Eqs. (74) and (76), the limits are shown in Fig. 6. From the data in Table IV we see that, for all the selected range of values of m_{Z_3} , the upper bounds are smaller than the

experimental upper limit. For the decay $B_d \rightarrow \mu^+ \mu^-$ the upper and lower bounds are lower than the SM predictions.

$\text{BR}(B_s \rightarrow \mu^+ \mu^-)/\text{BR}(B_s \rightarrow \mu^+ \mu^-)_{\text{SM}}$ and $\text{BR}(B_d \rightarrow \mu^+ \mu^-)/\text{BR}(B_d \rightarrow \mu^+ \mu^-)_{\text{SM}}$ vs m_{Z_3} are shown in Fig. 7. For $B_s \rightarrow \mu^+ \mu^-$ we see that both enhancements and suppressions are possible but both of them decrease with increasing values of m_{Z_3} and are very small with respect to the SM prediction for $m_{Z_3} = 5$ TeV. For $B_d \rightarrow \mu^+ \mu^-$ suppression of around 0.87 times the SM value is possible

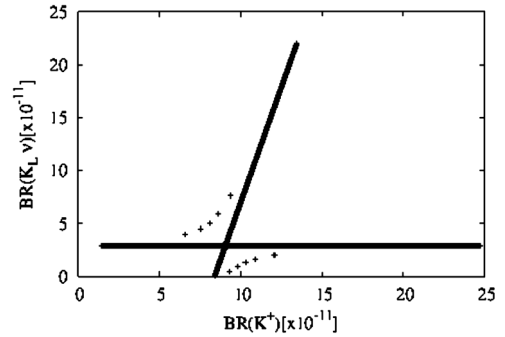


FIG. 5. $\text{BR}(K_L \rightarrow \pi^0 \nu \bar{\nu}) - \text{BR}(K^+ \rightarrow \pi^+ \bar{\nu} \nu)$ plane for $m_{Z_3} = 1$ TeV.

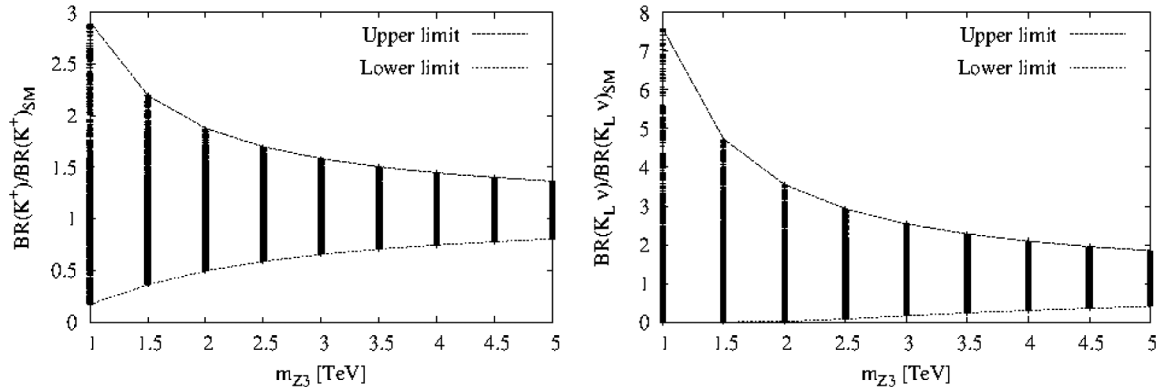


FIG. 4. $\text{BR}(K^+ \rightarrow \pi^+ \bar{\nu} \nu)/\text{BR}(K^+ \rightarrow \pi^+ \bar{\nu} \nu)_{\text{SM}}$ (left) and $\text{BR}(K_L \rightarrow \pi^0 \nu \bar{\nu})/\text{BR}(K_L \rightarrow \pi^0 \nu \bar{\nu})_{\text{SM}}$ (right). Upper and lower bounds for different values of m_{Z_3} are shown.

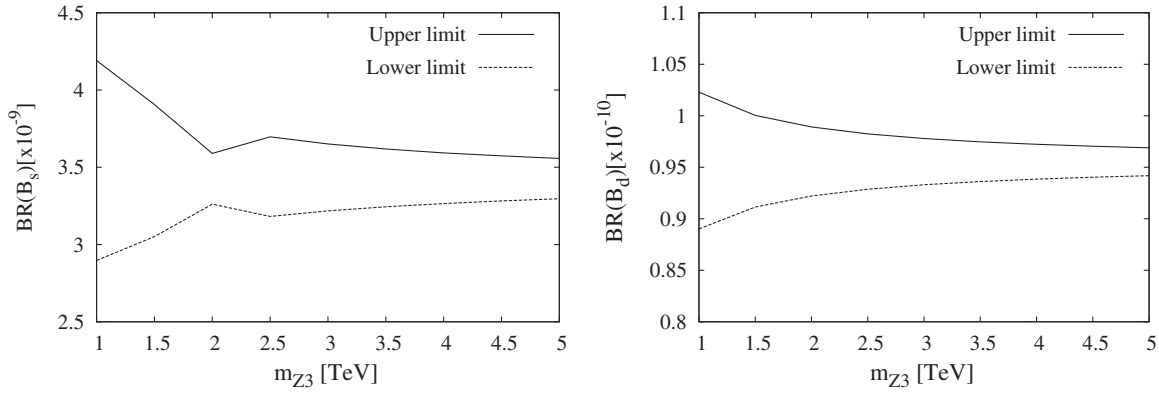


FIG. 6. Upper and lower bound on the BR for the decay $B_s \rightarrow \mu^+ \mu^-$ (left), and for the decay $B_d \rightarrow \mu^+ \mu^-$ (right) as a function of m_{Z_3} .

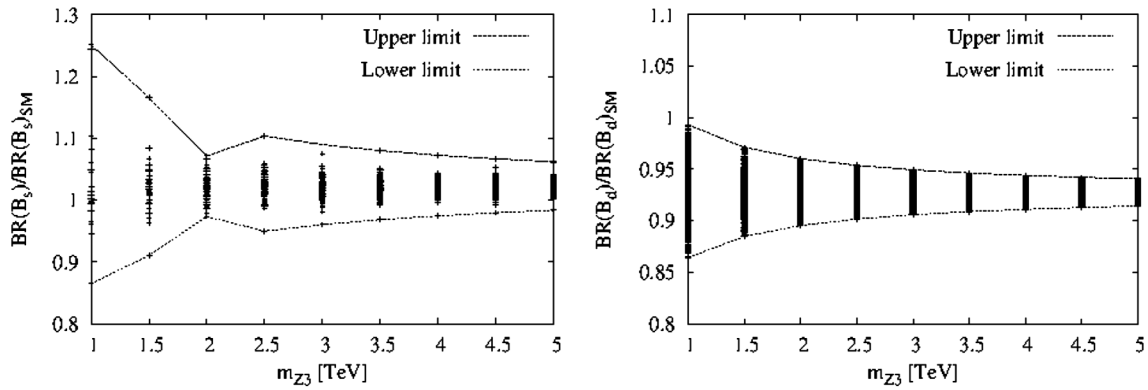


FIG. 7. $BR(B_s \rightarrow \mu^+ \mu^-) / BR(B_s \rightarrow \mu^+ \mu^-)_{SM}$ (left) and $BR(B_d \rightarrow \mu^+ \mu^-) / BR(B_d \rightarrow \mu^+ \mu^-)_{SM}$ (right) for different values of m_{Z_3} , including upper and lower bounds.

for $m_{Z_3} \approx 1$ TeV, but for higher m_{Z_3} the suppression can reach 0.92 times the SM prediction.

Going back to Fig. 6 it is interesting to note that, due to the small difference between the upper and lower bounds for $BR(B_s \rightarrow \mu^+ \mu^-)$ and $BR(B_d \rightarrow \mu^+ \mu^-)$, they are suppressed in this 3-4-1 extension even with the inclusion of new CP -violating phases, as compared with the case for

the decays in Fig. 3. A similar behavior is reported in [4] for the minimal 3-3-1 model. It is important to remark that, as in the 3-3-1 model, a strong enhancement of this BR would rule out the extension we are studying.

We conclude analyzing the results for the decays $K_L \rightarrow \pi^0 e^+ e^-$ and $K_L \rightarrow \pi^0 \mu^+ \mu^-$. Using Eqs. (81)–(83), the resulting limits are shown in Fig. 8. At this point it is

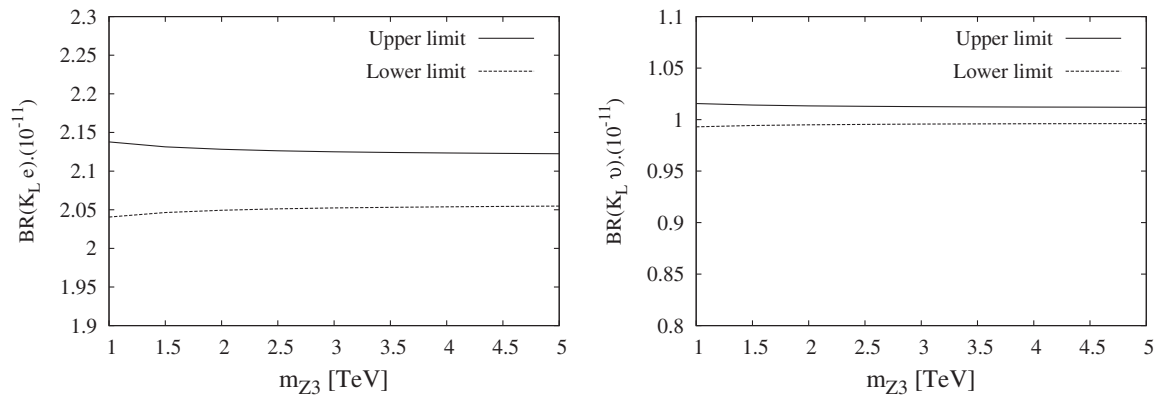


FIG. 8. Upper and lower bound on the BR for the decay $K_L \rightarrow \pi^0 e^+ e^-$ (left), and for the decay $K_L \rightarrow \pi^0 \mu^+ \mu^-$ (right).

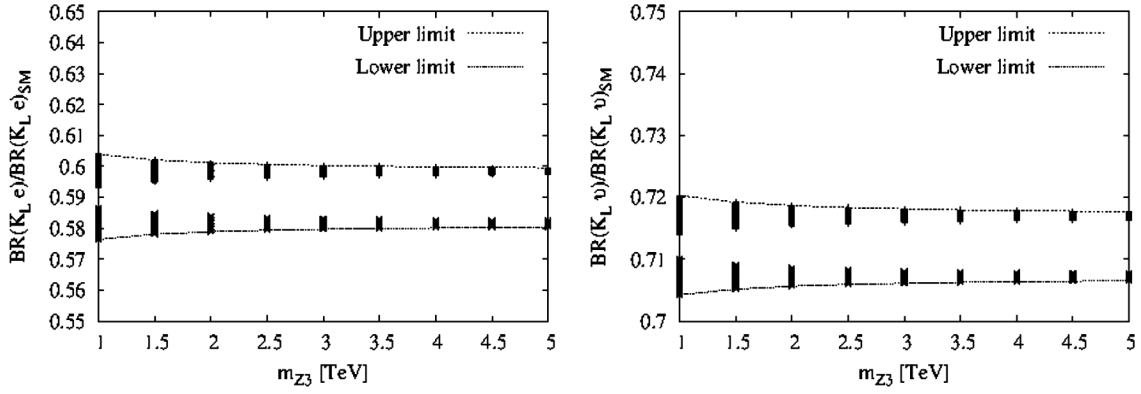


FIG. 9. Upper and lower bounds for the ratios $\text{BR}(K_L \rightarrow \pi^0 e^+ e^-)/\text{BR}(K_L \rightarrow \pi^0 e^+ e^-)_{\text{SM}}$ (left) and $\text{BR}(K_L \rightarrow \pi^0 \mu^+ \mu^-)/\text{BR}(K_L \rightarrow \pi^0 \mu^+ \mu^-)_{\text{SM}}$ (right), for different values of m_{Z_3} .

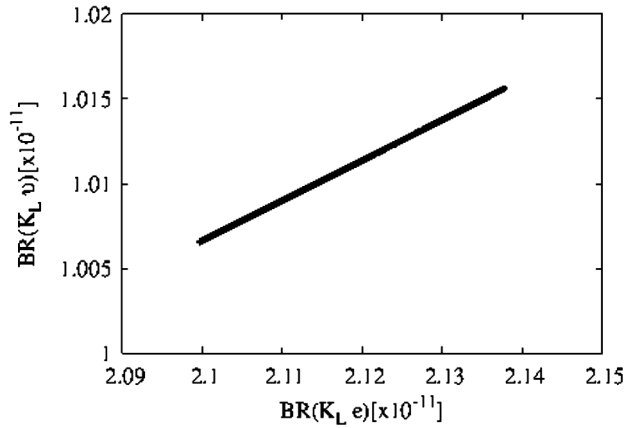


FIG. 10. $\text{BR}(K_L \rightarrow \pi^0 \mu^+ \mu^-)$ - $\text{BR}(K_L \rightarrow \pi^0 e^+ e^-)$ plane for $m_{Z_3} = 1$ TeV.

important to clarify that the upper bound is built using the + sign in Eqs. (83) and the lower bound is obtained using the - sign. From the data in Table IV we can see that these limits are lower than the SM predictions, but consistent with the upper experimental bounds. Notice also that the

upper and lower limits are both almost independent of the Z_3 mass.

$\text{BR}(K_L \rightarrow \pi^0 e^+ e^-)/\text{BR}(K_L \rightarrow \pi^0 e^+ e^-)_{\text{SM}}$ and $\text{BR}(K_L \rightarrow \pi^0 \mu^+ \mu^-)/\text{BR}(K_L \rightarrow \pi^0 \mu^+ \mu^-)_{\text{SM}}$ vs m_{Z_3} are shown in Fig. 9 from which we can see that these decays are suppressed compared to the SM prediction. Moreover, a comparison with Fig. 7 shows that, in the 3-4-1 extension, these BR are more suppressed than the ones for the $B_s \rightarrow \mu^+ \mu^-$ and $B_d \rightarrow \mu^+ \mu^-$ decays.

In the spirit of [32], we plot the plane $\text{BR}(K_L \rightarrow \pi^0 \mu^+ \mu^-)$ - $\text{BR}(K_L \rightarrow \pi^0 e^+ e^-)$ for $m_{Z_3} = 1$ TeV in Fig. 10. This shows the correlation between these decays. It can be seen that they are very restricted. Comparing Fig. 10 with the corresponding figures in Refs. [32,34], it is noted that they are very close to the plots for the SM in these works. This fact can also be observed from a comparison between the two plots in Fig. 8.

In Fig. 11, the planes $\text{BR}(K_L \rightarrow \pi^0 e^+ e^-)$ - $\text{BR}(K_L \rightarrow \pi^0 \nu \bar{\nu})$ and $\text{BR}(K_L \rightarrow \pi^0 \mu^+ \mu^-)$ - $\text{BR}(K_L \rightarrow \pi^0 \nu \bar{\nu})$ are shown. We find that, as it happens in many other models, the decays $K_L \rightarrow \pi^0 \mu^+ \mu^-$ and $K_L \rightarrow \pi^0 e^+ e^-$ are subject to weaker modifications than

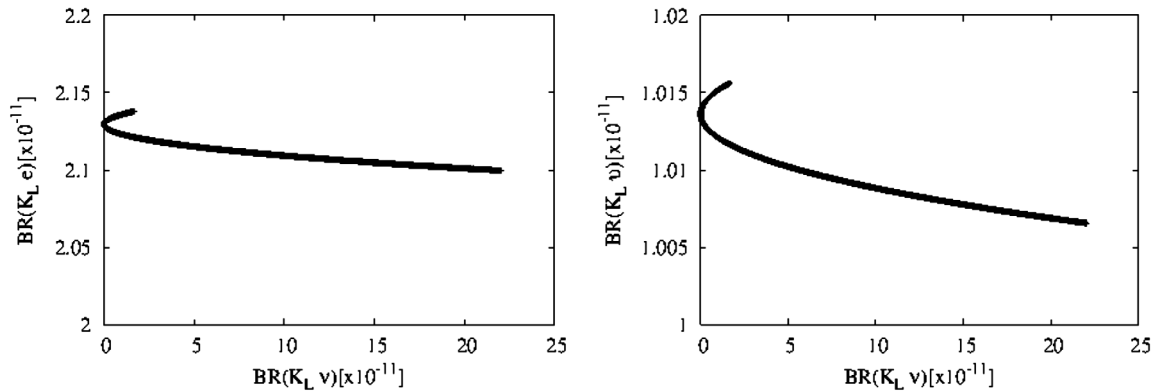


FIG. 11. $\text{BR}(K_L \rightarrow \pi^0 e^+ e^-)$ - $\text{BR}(K_L \rightarrow \pi^0 \nu \bar{\nu})$ plane (left) and $\text{BR}(K_L \rightarrow \pi^0 \mu^+ \mu^-)$ - $\text{BR}(K_L \rightarrow \pi^0 \nu \bar{\nu})$ plane (right) for $m_{Z_3} = 1$ TeV.

the decay $K_L \rightarrow \pi^0 \nu \bar{\nu}$. We remark that this is contrary to the result found in [4] for the minimal 3-3-1 model. This result, then, can allow to test the 3-4-1 extension studied here when more accurate measurements of both decays become available.

VI. SUMMARY

In this work, in the framework of a 3-4-1 extension of the SM characterized by the values $b = 1$, $c = -2$ in the most general expression for the electric charge generator and with chiral anomalies cancelling between the families, we have analyzed the effects of FCNC transmitted by one of the two extra neutral gauge bosons predicted by this construction in the case in which this new gauge boson Z_3 does not mix with the other two neutral gauge bosons. In this context we have studied in detail the constraints that well-measured $\Delta F = 2$ observables ($F = S, B$) impose on the parameter space associated to the V_L^d matrix describing mixing in the down-quark sector. This parameter space consists of five variables, namely, the angles θ_{13}^d , θ_{23}^d and the CP -violating phases ϕ^l , ϕ'' , related to the V_L^d mixing matrix, and the Z_3 mass, and we have used the well-measured quantities ΔM_K , $\Delta M_{d/s}$, ε_K and $\sin\Phi_d$. For fixed values of some of the relevant V_L^d parameters, allowed regions for the remaining parameters were plotted in Figs. 1 and 2 when m_{Z_3} is allowed to vary in the typical range 1–5 TeV. A calculation of the upper limits for θ_{13} and θ_{23} , for different values of m_{Z_3} , is presented in Table III which shows that these bounds increase when m_{Z_3} increases. This allowed us to appreciate the behavior of the parameter space for selected values of the Z_3 mass.

We have then taken these results to obtain upper and lower bounds for the BR of several clean rare decay processes, namely $K^+ \rightarrow \pi^+ \bar{\nu} \nu$, $K_L \rightarrow \pi^0 \nu \bar{\nu}$, $K_L \rightarrow \pi^0 l^+ l^-$ ($l = e, \mu$) and $B_{d/s} \rightarrow \mu^+ \mu^-$. We find sizeable enhancements and/or suppressions with respect to the SM prediction mainly for m_{Z_3} values in the range 1–2 TeV. In this range large enhancements, as compared with the suppressions, are predicted for the BR of the decays $K^+ \rightarrow \pi^+ \bar{\nu} \nu$ and $K_L \rightarrow \pi^0 \nu \bar{\nu}$. For $BR(B_s \rightarrow \mu^+ \mu^-)$ we find

enhancements slightly larger than suppressions, while for $BR(B_d \rightarrow \mu^+ \mu^-)$ we find suppression for all the considered range of values of m_{Z_3} . Finally, for $BR(K_L \rightarrow \pi^0 l^+ l^-)$ we find upper and lower bounds smaller than the SM prediction but consistent with the upper experimental limit. These departures from the SM, presented in Figs. 4, 7, and 9, can be considered as signals of the 3-4-1 extension under consideration when looking at the data. From these plots is clear that the decays $K^+ \rightarrow \pi^+ \bar{\nu} \nu$ and $K_L \rightarrow \pi^0 \nu \bar{\nu}$ show the greatest differences with the SM predictions, but the decays $K_L \rightarrow \pi^0 l^+ l^-$ ($l = e, \mu$) and $B_{d/s} \rightarrow \mu^+ \mu^-$ are very restricted in the 3-4-1 extension. It is important to note that, as in the minimal 3-3-1 model, a strong enhancement of the BR for the decays $B_{d/s} \rightarrow \mu^+ \mu^-$ would rule out the 3-4-1 extension studied here.

The plane $BR(K_L \rightarrow \pi^0 \nu \bar{\nu}) - BR(K^+ \rightarrow \pi^+ \bar{\nu} \nu)$ in Fig. 5 allowed us to compare our results with the ones in the littlest Higgs model with T parity [36] and with the ones in the minimal 3-3-1 model [4], and it also helped us to better understand the structure of the expressions in Eqs. (72), (76), (81), and (82). It is important to remark that, contrary to what it is done in [4], our results are obtained without the need of forcing a zero value neither for the real part nor for the imaginary part of these expressions.

In Figs. 10 and 11 we also plotted the planes $BR(K_L \rightarrow \pi^0 \mu^+ \mu^-) - BR(K_L \rightarrow \pi^0 e^+ e^-)$, $BR(K_L \rightarrow \pi^0 e^+ e^-) - BR(K_L \rightarrow \pi^0 \nu \bar{\nu})$ and $BR(K_L \rightarrow \pi^0 \mu^+ \mu^-) - BR(K_L \rightarrow \pi^0 \nu \bar{\nu})$. It can be seen that the decays $K_L \rightarrow \pi^0 \mu^+ \mu^-$ and $K_L \rightarrow \pi^0 e^+ e^-$ are subject to weaker modifications than the decay $K_L \rightarrow \pi^0 \nu \bar{\nu}$. When these plots are compared with similar ones obtained in other models, we find clear differences. In this sense, these results can allow to test the 3-4-1 extension under study here when more accurate measurements of the corresponding decays become available.

ACKNOWLEDGMENTS

We acknowledge financial support from DIME at Universidad Nacional de Colombia-Sede Medellín.

-
- [1] M. Singer, J. W. F. Valle, and J. Schechter, *Phys. Rev. D* **22**, 738 (1980); F. Pisano and V. Pleitez, *Phys. Rev. D* **46**, 410 (1992); V. Pleitez and M. D. Tonasse, *Phys. Rev. D* **48**, 2353 (1993); **48**, 5274 (1993); D. Ng, *Phys. Rev. D* **49**, 4805 (1994); H. N. Long, *Phys. Rev. D* **53**, 437 (1996); **54**, 4691 (1996); V. Pleitez, *Phys. Rev. D* **53**, 514 (1996); W. A. Ponce, J. B. Florez, and L. A. Sánchez, *Int. J. Mod. Phys. A* **17**, 643 (2002); W. A. Ponce, Y. Giraldo, and L. A. Sánchez, *Phys. Rev. D* **67**, 075001 (2003); D. A. Gutiérrez, W. A. Ponce, and L. A. Sánchez, *Int. J. Mod. Phys. A* **21**, 2217 (2006); *Eur. Phys. J. C* **46**, 497 (2006); A. E. Carcamo, R. Martínez, and F. Ochoa, *Phys. Rev. D* **73**, 035007 (2006); J. C. Salazar, W. A. Ponce, and D. A. Gutiérrez, *Phys. Rev. D* **75**, 075016 (2007).
- [2] D. Gómez Dumm, F. Pisano, and V. Pleitez, *Mod. Phys. Lett. A* **9**, 1609 (1994); J. A. Rodríguez and M. Sher, *Phys. Rev. D* **70**, 117702 (2004); R. A. Diaz, R. Martínez, and F. Ochoa, *Phys. Rev. D* **72**, 035018 (2005).
- [3] J. T. Liu and D. Ng, *Phys. Rev. D* **50**, 548 (1994).

- [4] C. Promberger, S. Schatt, and F. Schwab, *Phys. Rev. D* **75**, 115007 (2007).
- [5] J. M. Cabarcas, D. Gómez Dumm, and R. Martínez, *Phys. Rev. D* **77**, 036002 (2008).
- [6] R. Foot, H. N. Long, and T. A. Tran, *Phys. Rev. D* **50**, R34 (1994); F. Pisano and V. Pleitez, *Phys. Rev. D* **51**, 3865 (1995); Fayyazuddin and Riazuddin, *Phys. Rev. D* **30**, 1041 (1984); *J. High Energy Phys.* **12** (2004) 013; A. Palcu, *Mod. Phys. Lett. A* **24**, 1247 (2009); **24**, 1731 (2009).
- [7] L. A. Sánchez, F. A. Pérez, and W. A. Ponce, *Eur. Phys. J. C* **35**, 259 (2004); W. A. Ponce, D. A. Gutiérrez, and L. A. Sánchez, *Phys. Rev. D* **69**, 055007 (2004); L. A. Sánchez, L. A. Wills-Toro, and J. I. Zuluaga, *Phys. Rev. D* **77**, 035008 (2008).
- [8] W. A. Ponce and L. A. Sánchez, *Mod. Phys. Lett. A* **22**, 435 (2007).
- [9] J. L. Nisperuza and L. A. Sánchez, *Phys. Rev. D* **80**, 035003 (2009).
- [10] S. Villada and L. A. Sánchez, *J. Phys. G* **36**, 115002 (2009).
- [11] K. Nakamura (Particle Data Group), *J. Phys. G* **37**, 075021 (2010).
- [12] P. Langacker and M. Plumacher, *Phys. Rev. D* **62**, 013006 (2000).
- [13] L. S. Durkin and P. Langacker, *Phys. Lett.* **166B**, 436 (1986).
- [14] V. Barger, C. W. Chiang, P. Langacker, and H. S. Lee, *Phys. Lett. B* **580**, 186 (2004).
- [15] J. Hua, C. S. Kim, and Y. Li, *Eur. Phys. J. C* **69**, 139 (2010).
- [16] M. Blanke, A. J. Buras, A. Poschenrieder, S. Recksiegel, C. Tarantino, S. Uhlig, and A. Weiler, *Phys. Lett. B* **646**, 253 (2007).
- [17] G. C. Branco, M. N. Rebelo, and J. I. Silva-Marcos, *Phys. Lett. B* **597**, 155 (2004).
- [18] J. Laiho, E. Lunghi, and R. S. Van de Water, *Phys. Rev. D* **81**, 034503 (2010).
- [19] A. J. Buras, M. Jamin, and P. H. Weisz, *Nucl. Phys.* **B347**, 491 (1990).
- [20] J. Brod and M. Gorbahn, *Phys. Rev. D* **82**, 094026 (2010).
- [21] S. Herrlich and U. Nierste, *Nucl. Phys.* **B476**, 27 (1996).
- [22] J. Urban, F. Krauss, U. Jentschura, and G. Soff, *Nucl. Phys.* **B523**, 40 (1998).
- [23] G. Buchalla, A. J. Buras, and M. E. Lautenbacher, *Rev. Mod. Phys.* **68**, 1125 (1996).
- [24] A. J. Buras, arXiv:hep-ph/0505175.
- [25] A. J. Buras and D. Guadagnoli, *Phys. Rev. D* **79**, 053010 (2009); **78**, 033005 (2008).
- [26] A. J. Buras, S. Jager, and J. Urban, *Nucl. Phys.* **B605**, 600 (2001).
- [27] G. Buchalla, A. J. Buras, and M. K. Harlander, *Nucl. Phys.* **B349**, 1 (1991).
- [28] A. J. Buras, F. Schwab, and S. Uhlig, *Rev. Mod. Phys.* **80**, 965 (2008).
- [29] A. J. Buras, M. Gorbahn, U. Haisch, and U. Nierste, *J. High Energy Phys.* **11** (2006) 002; J. Brod and M. Gorbahn, *Phys. Rev. D* **78**, 034006 (2008).
- [30] G. Isidori, F. Mescia, and C. Smith, *Nucl. Phys.* **B718**, 319 (2005).
- [31] L. Wolfenstein, *Phys. Rev. Lett.* **51**, 1945 (1983).
- [32] G. Isidori, C. Smith, and R. Unterdorfer, *Eur. Phys. J. C* **36**, 57 (2004).
- [33] G. Buchalla, G. D'Ambrosio, and G. Isidori, *Nucl. Phys.* **B672**, 387 (2003).
- [34] F. Mescia, C. Smith, and S. Trine, *J. High Energy Phys.* **08** (2006) 088.
- [35] J. Brod, M. Gorbahn, and E. Stamou, *Phys. Rev. D* **83**, 034030 (2011).
- [36] M. Blanke, A. J. Buras, A. Poschenrieder, S. Recksiegel, C. Tarantino, S. Uhlig, and A. Weiler, *J. High Energy Phys.* **01** (2007) 066.