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Role of electric charge and cosmological constant in structure scalars

L. Herrera,[*,†] A. Di Prisco,[*,‡] and J. Ibáñez[§]

Departamento de Física Teórica e Historia de la Ciencia, Universidad del País Vasco, Bilbao, Spain

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The physical meaning of structure scalars is analyzed for charged dissipative spherical fluids and for neutral dust in the presence of cosmological constant. The role played by such factors in the structure scalars is clearly brought out and physical consequences are discussed. Particular attention needs to be paid to the changes introduced by the above mentioned factors in the inhomogeneity factor and the evolution of the expansion scalar and the shear tensor.

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I. INTRODUCTION

In a recent work [1], the full set of equations governing the structure and the evolution of self—gravitating spherically symmetric dissipative fluids with anisotropic stresses—was written down in terms of five scalar quantities obtained from the orthogonal splitting of the Riemann tensor in the context of general relativity. It was shown that these scalars (denoted by X_T, X_{TF}, Y_{TF}, Y_T, and Z) are directly related to fundamental properties of the fluid distribution, such as energy density, energy density inhomogeneity, local anisotropy of pressure, dissipative flux, and the active gravitational mass. In particular, the following properties of such quantities were established:

(i) X_T is the energy density of the fluid, whereas Z describes all possible dissipative fluxes [1].

(ii) In the absence of dissipation, X_{TF} controls inhomogeneities in the energy density [1].

(iii) Y_{TF} describes the influence of the local anisotropy of pressure and density inhomogeneity on the Tolman mass [1].

(iv) Y_T turns out to be proportional to the Tolman mass "density" for systems in equilibrium or quasi-equilibrium [1].

(v) The evolution of the expansion scalar and the shear tensor is fully controlled by Y_{TF} and Y_T [1–3].

Motivated by the deep physical meaning of structure scalars, we shall in this work calculate them for two situations of evident physical interest (see, for example, Refs. [4–11] and references therein), namely:

(i) charged fluids.

(ii) neutral dust with cosmological constant.

As we shall see here, both factors (electric charge and cosmological constant) affect the evolution of the system exclusively through their presence in some of the structure scalars, stressing further their relevance in the study of self-gravitating systems.

[*]Also at U. C. V, Caracas

[†]laherrera@cantv.net.ve

[‡]adiprisc@fisica.ciens.ucv.ve

[§]j.ibanez@ehu.es

II. GENERAL EQUATIONS AND DEFINITIONS

Full details of some intermediate calculations, notation, and basic equations can be found in Refs. [1–4], however, for self-consistency, we shall here provide a summary of the more essential equations and definitions.

We shall consider a general spherically symmetric line element of the form

$$ds^2 = -A^2dt^2 + B^2dr^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2) \qquad (1)$$

and a general fluid distribution whose energy-momentum tensor may be written as

$$T_{\alpha\beta} = (\mu + P_\perp)V_\alpha V_\beta + P_\perp g_{\alpha\beta} + (P_r - P_\perp)\chi_\alpha\chi_\beta + q_\alpha V_\beta + V_\alpha q_\beta + \epsilon l_\alpha l_\beta - 2\eta\sigma_{\alpha\beta}, \qquad (2)$$

where μ is the energy density, P_r the radial pressure, $P_\perp$ the tangential pressure, q^α the heat flux, ϵ the radiation density, η the coefficient of shear viscosity, $\sigma_{\alpha\beta}$ the shear tensor, V^α the four velocity of the fluid, χ^α a unit four-vector along the radial direction, and l^α a radial null four-vector. The four-vectors above for Eq. (1) are

$$V^\alpha = A^{-1}\delta_0^\alpha, \qquad q^\alpha = qB^{-1}\delta_1^\alpha,$$
$$l^\alpha = A^{-1}\delta_0^\alpha + B^{-1}\delta_1^\alpha, \qquad \chi^\alpha = B^{-1}\delta_1^\alpha, \qquad (3)$$

where q is a function of t and r, and $q^\alpha = q\chi^\alpha$.

If the fluid is charged, we shall need to add the electromagnetic contribution to the fluid distribution.

The electromagnetic energy tensor $S_{\alpha\beta}$ is given by (see Ref. [4] for details)

$$S_{\alpha\beta} = \frac{1}{4\pi}\left(F_\alpha{}^\gamma F_{\beta\gamma} - \frac{1}{4}F^{\gamma\delta}F_{\gamma\delta}g_{\alpha\beta}\right), \qquad (4)$$

where $F_{\alpha\beta}$ is the electromagnetic field tensor. The electric charge interior to radius r is time independent and given by

$$s(r) = 4\pi\int_0^r \varsigma BR^2dr, \qquad (5)$$

where the charge density ς is a function of t and r.

Next, for the four-acceleration, the expansion scalar, and the shear tensor, we have

$$a_1 = \frac{A'}{A}, \qquad a^2 = a^\alpha a_\alpha = \left(\frac{A'}{AB}\right)^2, \tag{6}$$

with $a^\alpha = a\chi^\alpha$,

$$\Theta = V^\alpha{}_{;\alpha} = \frac{1}{A}\left(\frac{\dot{B}}{B} + 2\frac{\dot{R}}{R}\right), \tag{7}$$

and

$$\sigma_{11} = \frac{2}{3}B^2\sigma, \qquad \sigma_{22} = \frac{\sigma_{33}}{\sin^2\theta} = -\frac{1}{3}R^2\sigma, \tag{8}$$

with

$$\sigma^2 = \frac{3}{2}\sigma^{\alpha\beta}\sigma_{\alpha\beta} = \frac{1}{A^2}\left(\frac{\dot{B}}{B} - \frac{\dot{R}}{R}\right)^2, \tag{9}$$

where dots and primes denote differentiation with respect to t and r, respectively.

The mass function $m(t, r)$ is given by

$$m = \frac{(R)^3}{2}R_{23}{}^{23} + \frac{s^2}{2R} = \frac{R}{2}\left[\left(\frac{\dot{R}}{A}\right)^2 - \left(\frac{R'}{B}\right)^2 + 1\right], \tag{10}$$

which can be rewritten as

$$E \equiv \frac{R'}{B} = \left(1 + U^2 - \frac{2m(t, r)}{R} + \frac{s^2}{R^2}\right)^{1/2}, \tag{11}$$

where U is the areal velocity of the collapsing fluid, i.e. $U = \frac{1}{A}\dot{R}$.

From Eq. (10), we may obtain (see Eq. (38) in Ref. [4])

$$m = \int_0^r 4\pi R^2\left(\tilde{\mu} + \tilde{q}\frac{U}{E}\right)R'dr + \frac{s^2}{2R} + \frac{1}{2}\int_0^r \frac{s^2}{R^2}R'dr \tag{12}$$

(assuming a regular center to the distribution, so $m(0)=0$), or

$$\frac{3m}{R^3} = 4\pi\tilde{\mu} - \frac{4\pi}{R^3}\int_0^r R^3\tilde{\mu}'dr + \frac{4\pi}{R^3}\int_0^r 3\tilde{q}UBR^2dr + \frac{3}{R^3}\left(\frac{s^2}{2R} + \frac{1}{2}\int_0^r \frac{s^2}{R^2}R'dr\right), \tag{13}$$

where $\tilde{\mu} = \mu + \epsilon$ and $\tilde{q} = q + \epsilon$.

The Weyl tensor ($C_{\alpha\mu\beta\nu}$), as usual, may be decomposed in its electric and magnetic parts; however, due to the spherical symmetry, the magnetic part vanishes, and so the Weyl tensor is expressed in terms of its electric part alone.

The electric part of Weyl tensor is defined by

$$E_{\alpha\beta} = C_{\alpha\mu\beta\nu}V^\mu V^\nu, \tag{14}$$

which may also be written as

$$E_{\alpha\beta} = \mathcal{E}\left(\chi_\alpha\chi_\beta - \frac{1}{3}h_{\alpha\beta}\right), \tag{15}$$

where

$$h_{\alpha\beta} = g_{\alpha\beta} + V_\alpha V_\beta, \tag{16}$$

and

$$\mathcal{E} = \frac{1}{2A^2}\left[\frac{\ddot{R}}{R} - \frac{\ddot{B}}{B} - \left(\frac{\dot{R}}{R} - \frac{\dot{B}}{B}\right)\left(\frac{\dot{A}}{A} + \frac{\dot{R}}{R}\right)\right] + \frac{1}{2B^2}\left[\frac{A''}{A} - \frac{R''}{R} + \left(\frac{B'}{B} + \frac{R'}{R}\right)\left(\frac{R'}{R} - \frac{A'}{A}\right)\right] - \frac{1}{2R^2}. \tag{17}$$

Using Einstein equations (10), (13), and (17), we can write

$$\mathcal{E} = 4\pi(2\eta\sigma - \Pi) + \frac{3s^2}{2R^4} + \frac{4\pi}{R^3}\int_0^r R^3\tilde{\mu}'dr - \frac{12\pi}{R^3}\int_0^r \tilde{q}UBR^2dr - \frac{3}{2R^3}\int_0^r \frac{s^2R'}{R^2}dr, \tag{18}$$

where $\Pi = \tilde{P}_r - P_\perp$ and $\tilde{P}_r = P_r + \epsilon$.

III. STRUCTURE SCALARS FOR THE CHARGED FLUID

We can now calculate the structure scalars for our charged fluid. To do so, let us define tensors $Y_{\alpha\beta}$ and $X_{\alpha\beta}$ by:

$$Y_{\alpha\beta} = R_{\alpha\gamma\beta\delta}V^\gamma V^\delta, \tag{19}$$

$$X_{\alpha\beta} = {}^*R^*_{\alpha\gamma\beta\delta}V^\gamma V^\delta = \frac{1}{2}\eta_{\alpha\gamma}{}^{\epsilon\rho}R^*_{\epsilon\rho\beta\delta}V^\gamma V^\delta, \tag{20}$$

where $R^*_{\alpha\beta\gamma\delta} = \frac{1}{2}\eta_{\epsilon\rho\gamma\delta}R_{\alpha\beta}{}^{\epsilon\rho}$.

Tensors $Y_{\alpha\beta}$ and $X_{\alpha\beta}$ may be expressed as

$$Y_{\alpha\beta} = \frac{1}{3}Y_T h_{\alpha\beta} + Y_{TF}\left(\chi_\alpha\chi_\beta - \frac{1}{3}h_{\alpha\beta}\right), \tag{21}$$

$$X_{\alpha\beta} = \frac{1}{3}X_T h_{\alpha\beta} + X_{TF}\left(\chi_\alpha\chi_\beta - \frac{1}{3}h_{\alpha\beta}\right). \tag{22}$$

Then, after lengthy but simple calculations, using field equations (see Eqs. (20, 21, 22, 23) in Ref. [4]) and Eq. (17), we obtain

$$Y_T = 4\pi(\tilde{\mu} + 3\tilde{P}_r - 2\Pi) + \frac{s^2}{R^4}, \qquad Y_{TF} = \mathcal{E} - 4\pi(\Pi - 2\eta\sigma) + \frac{s^2}{R^4}, \tag{23}$$

$$X_T = 8\pi\tilde{\mu} + \frac{s^2}{R^4}, \qquad X_{TF} = -\mathcal{E} - 4\pi(\Pi - 2\eta\sigma) + \frac{s^2}{R^4}. \tag{24}$$

Using Eqs. (18) and (23), we may write Y_{TF} as

$$Y_{TF} = -8\pi\Pi + 16\pi\eta\sigma + \frac{5s^2}{2R^4} - \frac{3}{2R^3}\int_0^r \frac{s^2}{R^2}R'dr + \frac{4\pi}{R^3}\int_0^r R^3\left(\tilde{\mu}' - \frac{3\tilde{q}BU}{R}\right)dr. \tag{25}$$

At this point, it would be useful to introduce the following "effective" variables:

$$-(T_0^0 + S_0^0) \equiv \mu_{\rm eff} = \tilde{\mu} + \frac{s^2}{8\pi R^4}, \qquad (26)$$

$$T_1^1 + S_1^1 \equiv P_r^{\rm eff} = \left(\tilde{P}_r - \frac{4}{3}\eta\sigma\right) - \frac{s^2}{8\pi R^4}, \qquad (27)$$

$$T_2^2 + S_2^2 \equiv P_\perp^{\rm eff} = \left(P_\perp + \frac{2}{3}\eta\sigma\right) + \frac{s^2}{8\pi R^4}, \qquad (28)$$

and

$$P_r^{\rm eff} - P_\perp^{\rm eff} \equiv \Pi^{\rm eff} = \Pi - 2\eta\sigma - \frac{s^2}{4\pi R^4}. \qquad (29)$$

As it is evident from above, the effective variables are just the corresponding ordinary variables with all contributions (from viscosity and electric charge) included. In terms of the former, the structure scalars read

$$Y_{TF} = -8\pi\Pi^{\rm eff} + \frac{4\pi}{R^3}\int_0^r R^3\left(\mu'_{\rm eff} - \frac{3\tilde{q}BU}{R}\right)dr, \qquad (30)$$

$$X_{TF} = -\frac{4\pi}{R^3}\int_0^r R^3\left(\mu'_{\rm eff} - \frac{3\tilde{q}UB}{R}\right)dr, \qquad (31)$$

$$Y_T = 4\pi(\tilde{\mu}_{\rm eff} + 3\tilde{P}_r^{\rm eff} - 2\Pi^{\rm eff}), \qquad (32)$$

$$X_T = 8\pi\tilde{\mu}_{\rm eff}. \qquad (33)$$

The remarkable fact emerging from these expressions is that the charge contribution is always absorbed into the effective variables. In the absence of electrical charge, the structure scalars are obtained from Eqs. (30)–(33), just replacing the effective variables by the corresponding ordinary ones.

In order to delve deeper into the question about the role of electric charge in the structure and evolution of compact objects—and how this reflects in the structure scalar—we shall consider three very important equations in general relativity. These are the evolution equation for the expansion scalar (Raychaudhuri), the evolution equation for the shear [1,3,12,13], and a differential equation relating the energy density inhomogeneity with the Weyl tensor and other physical variables [1,12–14]. The Raychaudhuri equation reads, in our case,

$$V^\alpha\Theta_{;\alpha} + \frac{1}{3}\Theta^2 + \frac{2}{3}\sigma^2 - a^\alpha_{;\alpha} = -Y_T, \qquad (34)$$

which has exactly the same form as in the noncharged case (see Eq. (32) in Ref. [3]). For the shear evolution equation, we find

$$Y_{TF} = \chi^\alpha a_{;\alpha} + a^2 - \frac{aR'}{BR} - V^\alpha\sigma_{;\alpha} - \frac{2}{3}\Theta\sigma - \frac{\sigma^2}{3}, \qquad (35)$$

which, again, has exactly the same form as in the noncharged case (see Eq. (45) in Ref. [3]).

Finally, the differential equation for the Weyl tensor and the energy density inhomogeneity can be written as

$$(X_{TF} + 4\pi\mu_{\rm eff})' = -X_{TF}\frac{3R'}{R} + 4\pi\tilde{q}B(\Theta - \sigma), \qquad (36)$$

which is exactly the same expression for the noncharged fluid, replacing the effective energy density by the energy density (see Eq. (37) in Ref. [14]).

We shall next consider the case of dust with cosmological constant.

IV. STRUCTURE SCALARS FOR DUST WITH COSMOLOGICAL CONSTANT

Let us consider a spherically symmetric distribution of dust with nonvanishing cosmological constant. Then the energy-momentum tensor takes the simple form

$$T_{\alpha\beta} = 8\pi\mu V_\alpha V_\beta, \qquad (37)$$

and Einstein equations read

$$G_{\alpha\beta} = T_{\alpha\beta} - \Lambda g_{\alpha\beta}, \qquad (38)$$

where Λ is the cosmological constant.

Since the fluid is obviously geodesic for our comoving observers, we have $A' = 0$ and, rescaling the time coordinate t, we can put $A = 1$.

The mass function now can be casted into the form

$$m = 4\pi\int_0^r \mu R^2 R' dr + \frac{\Lambda}{6}R^3. \qquad (39)$$

From the above, the following equations may be obtained, which are the equivalent to Eqs. (13) and (18) in the case of dust with cosmological constant,

$$\frac{3m}{R^3} = 4\pi\mu + \frac{\Lambda}{2} - \frac{4\pi}{R^3}\int_0^r \mu' dr, \qquad (40)$$

$$\mathcal{E} = \frac{4\pi}{R^3}\int_0^r R^3\mu' dr. \qquad (41)$$

From Eqs. (17) and (19)–(22), with the help of Einstein equations, we obtain for the structure scalars

$$Y_T = 4\pi\mu - \Lambda, \quad Y_{TF} = -X_{TF} = \mathcal{E}, \quad X_T = 8\pi\mu - \Lambda. \qquad (42)$$

Then, the evolution equations for the shear and expansion become

$$\mathcal{E} = Y_{TF} = -V^\alpha\sigma_{;\alpha} - \frac{2}{3}\Theta\sigma - \frac{\sigma^2}{3}, \qquad (43)$$

and

$$V^\alpha\Theta_{;\alpha} + \frac{1}{3}\Theta^2 + \frac{2}{3}\sigma^2 - a^\alpha_{;\alpha} = -4\pi\mu + \Lambda = -Y_T, \qquad (44)$$

whereas the differential equation for the inhomogeneity factor can be written as

$$(X_{TF} + 4\pi\mu)' = -X_{TF}\frac{3R'}{R}, \qquad (45)$$

from which it follows at once $\mu' = 0 \leftrightarrow X_{TF} = 0$, allowing us to identify X_{TF} as the inhomogeneity factor.

V. SUMMARY

In the case of the charged fluid, we have seen that the role of electrical charge in the structure and evolution of self-gravitating systems is completeley determined by structure scalars. Thus, the influence of charge, in the evolution of the expansion and the shear, reveals itself exclusively through its contribution to Y_T and Y_{TF}, respectively. The same can be said about the inhomogeneity factor, as it follows from Eq. (36). It is also worth stressing the fact that the charge contribution is always absorbed into the effective variables in a rather, intuitively, obvious way.

In the case of dust with cosmological constant, we see that the latter does not affect at all either the evolution of the shear or the inhomogeneity factor. Instead, it affects the evolution of the expansion scalar through the Λ term in Y_T. The fact that the cosmological constant does not affect the stability of the shear-free condition deserves to be emphasized.

It should be observed that, besides local anisotropy of pressure, dissipation, and shear viscosity, the inclusion of electric charge and cosmological constant exhausts all possible physical phenomena that we expect in a spherically symmetric relativistic fluid distribution. The fact that all of them act exclusively through their presence in structure scalars exhibits the universality of the latter.

The comments above reinforce our belief that structure scalars are called upon to play a major role in the study of self-gravitating systems.

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