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Lie-algebraic solutions of the type IIB matrix model

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A systematic search for Lie-algebra solutions of the type IIB matrix model is performed. Our survey is based on the classification of all Lie algebras for dimensions up to five and of all nilpotent Lie algebras of dimension six. It is shown that Lie-type solutions of the equations of motion of the type IIB matrix model exist and they correspond to certain nilpotent and solvable Lie algebras. Their representation in terms of Hermitian matrices is discussed in detail. These algebras give rise to certain noncommutative spaces for which the corresponding star products are provided. Finally the issue of constructing quantized compact nilmanifolds and solvmanifolds based on the above algebras is addressed.

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I. INTRODUCTION

String-inspired matrix models (MM) were proposed as nonperturbative definitions of M theory [1] and type IIB superstring theory [2]. Such MM provide an interesting and simple framework to study the dynamics of branes, both analytically and numerically.

Several solutions of the above MM have been identified. In particular, as far as the model of Ishibashi-Kawai-Kitazawa-Tsuchiya (IKKT) [2] is concerned, the first solutions appear in the original publication and they correspond to one or more *D*-strings. Odd-dimensional *Dp*-brane solutions, in accord with type IIB superstring theory, were described and studied in [3–7]. On the other hand, compact noncommutative (NC) spaces, such as fuzzy tori, fuzzy spheres and other fuzzy homogeneous spaces, were shown to provide solutions upon adding extra terms (deformations) in the original action [8–13]. Compact solutions of the undeformed MM were described only recently [14]. The relation of the IKKT model to toroidal compactifications was already discussed from a different point of view in the pioneering paper [15].

The fluctuations around solutions of the MM carry gauge degrees of freedom and provide a fruitful arena to study non-Abelian gauge theories on NC space-time [7]. Such backgrounds may therefore provide a natural setup for model building, much like vacua of the type IIB string theory with *D*-branes. Such attempts were made in [16–18]. More recently, configurations of intersecting NC branes in the IKKT model were studied and a realization of the gauge group and the chiral spectrum of the standard model was provided in [19]. Similar considerations in a field-theoretical context were discussed in [20–23].

In the present work we perform a survey on the possible Lie-type solutions to the (undeformed) IKKT matrix model. In other words, we examine which of the known and classified (semisimple, nilpotent and solvable) Lie algebras provide solutions to its equations of motion. Such an examination does not come without its restrictions. Indeed, the classification of Lie algebras beyond six dimensions becomes complicated. Therefore, our first restriction is to focus on all the Lie algebras of dimension up to five and all nilpotent Lie algebras of dimension six.[1] For these low numbers of dimensions full classification tables exist [25–27] and therefore our task becomes tractable. Moreover, we shall be interested in the following only in non-Abelian algebras. Abelian ones and algebraic sums of them are always solutions of the MM but they do not lead to interesting dynamics; therefore they will not be further discussed. Finally, let us mention that we work here with real Lie algebras; Lie algebras over different fields will not be considered.

Under the above requirements, scanning the classification tables of Lie algebras in Sec. III, we come up with the following result. There are only nine Lie-algebraic solutions to the IKKT MM, one of which is three-dimensional, one four-dimensional, three five-dimensional and four six-dimensional. Out of the above nine cases, seven correspond to nilpotent Lie algebras and two of them correspond to solvable ones. There are no semisimple Lie algebras providing tree-level solutions to the undeformed IKKT MM.

Having identified the Lie algebras which constitute solutions of the equations of motion of the model, the next step is to study whether they can be represented by Hermitian matrices. Evidently this is a necessary requirement in order for these Lie algebras to correspond to NC spaces which are indeed solutions of the MM. Utilizing the powerful results of Kirillov on the unitary representations of nilpotent Lie groups [28], such representations are indeed determined for most of the relevant Lie algebras.

Finally, the issue of constructing compact noncommutative spaces based on these algebras is addressed. The main possibility which arises in this context is to consider spaces obtained as the quotient of a nilpotent Lie group by a compact discrete subgroup of it. Such spaces are known

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[1]The same classification was used in [24] in the construction of Wess-Zumino-Witten models.

as nilmanifolds and they can be compact even when the nilpotent group is not [29]. Some of them are known to provide string flux vacua based on the ideas of the seminal paper [30] (see, e.g., [31–35] and references therein). In an appendix we collect some useful definitions on Lie algebras.

II. THE TYPE IIB MATRIX MODEL

A. Action and symmetries

Let us briefly describe the IKKT or IIB matrix model, which was originally proposed in [2] as a nonperturbative definition of the type IIB superstring theory. It is a zero-dimensional reduced matrix model defined by the action

$$S = -\frac{\Lambda^4}{g^2}\,\mathrm{Tr}\bigg(\frac{1}{4}[X_a, X_b][X^a, X^b] + \frac{1}{2}\,\bar{\psi}\Gamma^a[X_a, \psi]\bigg), \quad (1)$$

where X_a, $a = 0, \ldots, 9$ are ten Hermitian matrices, and ψ are 16-component Majorana-Weyl spinors of $SO(9,1)$. Indices are raised and lowered with the invariant tensor η_{ab}, or possibly δ_{ab} in the Euclidean version of the model where $SO(9,1)$ is replaced by $SO(10)$. The Γ^a are generators of the corresponding Clifford algebra. Λ is an energy scale, which we will set equal to one, $\Lambda = 1$, and work with dimensionless quantities. Finally, g is a parameter which can be related to the gauge coupling constant.

The symmetry group of the above model contains the $U(N)$ gauge group (where the limit $N \to \infty$ is understood) as well as the $SO(10)$ or $SO(9,1)$ global symmetry. Moreover, the model is $\mathcal{N} = 2$ supersymmetric, with supersymmetries realized by the following transformations,

$$\begin{aligned} \delta_\epsilon \psi &= \frac{i}{2}[X_a, X_b]\Gamma^{ab}\epsilon, \\ \delta_\epsilon X_a &= i\bar{\epsilon}\Gamma_a\psi, \\ \delta_\xi \psi &= \xi, \\ \delta_\xi X_a &= 0, \end{aligned} \quad (2)$$

where Γ^{ab} denotes the antisymmetrized product of gamma matrices as usual. Therefore, the amount of supersymmetry indeed matches that of the type IIB superstring. Let us also note that the homogeneous ϵ supersymmetry is inherited by the maximal $\mathcal{N} = 1$ supersymmetry of super-Yang-Mills theory in ten dimensions.

It is important to stress that due to its zero-dimensional nature, the IKKT model is not defined on any predetermined space-time background. Instead, space-time emerges as a particular solution of the model, as we discuss in the following. This picture provides a dynamical origin for geometry and space-time.

B. Equations of motion and basic solutions

Varying the action (1) with respect to the matrices X_a and setting $\psi = 0$, the following equations of motion are obtained,

$$[X_b, [X^a, X^b]] = 0. \quad (3)$$

Simple as they may appear, these equations admit diverse interesting and nontrivial solutions. Clearly, the simplest solution is given by a set of commuting matrices, $[X^a, X^b] = 0$. In that case, the matrices X^a can be simultaneously diagonalized and therefore they may be expressed as

$$X^a = \mathrm{diag}(X_1^a, X_2^a, \ldots, X_N^a). \quad (4)$$

However, such solutions are in a sense degenerate and do not lead to interesting dynamics. In the following, commutative solutions will not be considered any further.

For notational convenience let us now split the ten matrices X^a in two sets; we shall use the following notation,

$$X^a = \begin{pmatrix} X^\mu \\ X^i \end{pmatrix}, \quad (5)$$

where the X^μ, $\mu = 0, \ldots, 3$ correspond to the first four X^a matrices and the X^i, $i = 1, \ldots, 6$ to the six rest of the X^a matrices, respectively.[2] In this notation, another solution of the equations (3) is given by

$$X^a = \begin{pmatrix} \bar{X}^\mu \\ 0 \end{pmatrix}, \quad (6)$$

where $\bar{X}^\mu$ are the generators of the Moyal-Weyl quantum plane $\mathbb{R}^4_\theta$, which satisfy the commutation relation

$$[\bar{X}^\mu, \bar{X}^\nu] = i\theta^{\mu\nu}, \quad (7)$$

where $\theta^{\mu\nu}$ is a constant antisymmetric tensor. This solution corresponds to a single NC flat 3-brane, which corresponds to space-time emerging as a solution of the matrix model. Being a single brane, this solution is associated to an Abelian gauge theory. An obvious generalization of the above solution is given by

$$X^a = \begin{pmatrix} \bar{X}^\mu \\ 0 \end{pmatrix} \otimes \mathbb{1}_n, \quad (8)$$

which is interpreted as n coincident branes carrying a non-Abelian $U(n)$ gauge theory.

III. LIE-ALGEBRAIC SOLUTIONS

In the present section we search for solutions of the IKKT model which have the structure of a Lie algebra. This task may be split into two steps. First the Lie algebra should solve the equations of motion (3). Second, the algebras which pass the first test should possess representations in terms of Hermitian matrices, since only then they

[2] Let us stress that although a splitting of the type $10 = 4 + 6$ is considered here, this is not *a priori* favored by the dynamics of the model. For studies related to the four-dimensionality of space-time in the IKKT model, see, e.g., [6,36,37].

may be considered solutions of the IKKT model. We shall address these two issues separately below.

The Lie algebras we study here are algebras over the field of real numbers. The classification we follow appears in the tables of [27], which is a complete classification of real Lie algebras of dimension up to five and real nilpotent Lie algebras[3] of dimension six.[4] Let us mention that a certain classification of solvable Lie algebras of dimension six was partially given in [38] and later completed in [39] but here we shall restrict our discussion to only the cases mentioned above, appearing in [27], and leave a more complete analysis for a future work.

A note on notation is in order here. The Lie algebras under study will be denoted as $\mathcal{A}_{d,i}$, where d is the dimension of the algebra (the number of its generators) and i is just an enumerative index according to the tables of [27]. Moreover, when there is some parameter on which the algebra depends, it will appear as superscript, e.g., $\mathcal{A}^{\alpha}_{d,i}$ if there is one parameter α. Let us also note that the generators of an algebra will be denoted as X_a, $a = 1, \ldots, d$.

For the solutions that we find, the corresponding quadratic Casimir operators are presented, as well as the Killing form g_{ab} and the invariant metric Ω_{ab} (whenever it exists). The related definitions appear in the appendix. It is important to note that unlike semisimple Lie groups, where the invariant metric is proportional to the Killing form, for a general Lie group this is not true as will become obvious in some of the following examples.

As a final remark, let us explain that when we refer to a solution of the equations (3) it is implied that the rest of the matrices (i.e., ten minus the number of algebra generators) are taken to be zero. For example, in the case of a three-dimensional algebra with generators X_1, X_2 and X_3, a solution would be a set of commutation relations which solve Eq. (3) accompanied by $X_i = 0$, $i = 0, 4, \ldots, 9$. Of course such solutions may be subsequently combined with the basic solution of Sec. II.

A. Solutions of the equations of motion

One- and two-dimensional Lie algebras. There is one real one-dimensional Lie algebra, $\mathcal{A}_{1,1}$. Evidently, this is an Abelian algebra and it constitutes a solution of the IKKT model, albeit not an interesting one as we have already argued. Therefore it will not be considered further.

As far as two-dimensional Lie algebras are concerned, there exists the algebraic sum of two copies of $\mathcal{A}_{1,1}$, namely, $\mathcal{A}_{1,1} \oplus \mathcal{A}_{1,1}$, which we shall not consider for the above reasons. This will be true in all dimensions to follow from now on and algebraic sums of lower-dimensional Abelian algebras will not be considered further. Moreover, there exists one non-Abelian solvable Lie algebra in two dimensions, based on the following commutation relation,

$$[X_1, X_2] = iX_2. \tag{9}$$

However, it is clearly not a solution to the matrix model, since one may easily verify that

$$[X_1, [X_1, X_2]] = -X_2 \neq 0 \tag{10}$$

and therefore the corresponding equation of motion is not satisfied.

Three-dimensional Lie algebras. There exist nine real Lie algebras $\mathcal{A}_{3,i}$ in three dimensions.[5] One of them is nilpotent, six are solvable and two are the well-known semisimple ones, which are isomorphic to $su(2)$ and $su(1,1) \sim sl(2;\mathbb{R})$. Using their commutation relations appearing in [27] it is straightforward to verify that only one of them, the $\mathcal{A}_{3,1}$ one, provides a solution of the equations of motion (3). The only nontrivial commutation relation of this algebra is

$$[X_2, X_3] = iX_1. \tag{11}$$

This algebra is nilpotent and its only quadratic Casimir operator is $C^{(2)}(\mathcal{A}_{3,1}) = X_1^2$. Its Killing form vanishes identically, $g_{ab} = 0$, while

$$\Omega^{ab} = \mathrm{diag}(1, 0, 0), \tag{12}$$

which is degenerate.

Four-dimensional Lie algebras. There are 12 four-dimensional real Lie algebras $\mathcal{A}_{4,i}$ and, in particular, one nilpotent and 11 solvable ones. Out of these algebras we find only one that provides a solution to the equations (3). It is the $\mathcal{A}_{4,12}$ one, which is solvable and its commutation relations are

$$\begin{aligned} &[X_1, X_3] = iX_1, \qquad [X_2, X_3] = iX_2, \\ &[X_1, X_4] = -iX_2, \qquad [X_2, X_4] = iX_1. \end{aligned} \tag{13}$$

However, this algebra does not possess any quadratic Casimir operators (in fact it does not possess any invariants at all) and therefore it is of no further interest.

Five-dimensional Lie algebras. In five dimensions the number of real Lie algebras sums up to 40. Six of them are nilpotent and the rest are solvable. Scanning the commutation relations of the corresponding table in [27] we find three solutions to the equations (3), corresponding to the algebras $\mathcal{A}_{5,1}$, $\mathcal{A}_{5,4}$ and $\mathcal{A}_{5,39}$.

$\mathcal{A}_{5,1}$ is a nilpotent algebra with the following commutation relations,

$$[X_3, X_5] = iX_1, \qquad [X_4, X_5] = iX_2. \tag{14}$$

[3]For the definition of nilpotent and solvable Lie algebras the reader may consult the appendix.

[4]Nilpotent algebras of dimension seven are also classified but they are not finitely many.

[5]We always refer to algebras which are not algebraic sums of lower-dimensional ones.

Its invariants are X_1, X_2 and $X_2X_3 - X_1X_4$ and therefore its quadratic Casimir operator may be written as

$$C^{(2)}(\mathcal{A}_{5,1}) = pX_1^2 + qX_2^2 + r(X_2X_3 - X_1X_4), \tag{15}$$

where p, q, r are arbitrary real parameters. Since the algebra is nilpotent its Killing form is identically zero, while it holds that

$$\Omega^{ab} = \begin{pmatrix} p & 0 & 0 & -r & 0 \\ 0 & q & r & 0 & 0 \\ 0 & -r & 0 & 0 & 0 \\ r & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{16}$$

which is again degenerate and does not possess an inverse.

For $\mathcal{A}_{5,4}$, which is also nilpotent, the commutation relations are

$$[X_2, X_4] = iX_1, \qquad [X_3, X_5] = iX_1. \tag{17}$$

Its only invariant is X_1 and therefore it holds that

$$C^{(2)}(\mathcal{A}_{5,4}) = X_1^2, \tag{18}$$

which leads to $\Omega^{ab} = \mathrm{diag}(1, 0, 0, 0, 0)$.

Finally, $\mathcal{A}_{5,39}$ is solvable with commutation relations

$$[X_1, X_4] = iX_1, \quad [X_2, X_4] = iX_2, \quad [X_1, X_5] = -iX_2,$$
$$[X_2, X_5] = iX_1, \quad [X_4, X_5] = iX_3. \tag{19}$$

Its only invariant is X_3 and therefore we find $C^{(2)}(\mathcal{A}_{5,39}) = X_3^2$, $g_{ab} = \mathrm{diag}(0, 0, 0, 2, -2)$ and $\Omega^{ab} = \mathrm{diag}(0, 0, 1, 0, 0)$. The last two cases are obviously degenerate too.

Six-dimensional nilpotent Lie algebras. There are 22 real nilpotent Lie algebras of dimension six which are not algebraic sums of lower-dimensional ones. Four of them provide solutions to the equations (3) and, in particular, the $\mathcal{A}_{6,3}$, $\mathcal{A}_{6,4}$, $\mathcal{A}_{6,5}^{\alpha}$ and $\mathcal{A}_{6,14}^{-1}$. In the two latter cases the algebras have a continuous parameter α which in the last case is fixed to -1 in order to provide the desired solution. Let us mention again that the Killing form for all the nilpotent Lie algebras is identically zero.

The algebra $\mathcal{A}_{6,4}$ has the following commutation relations,

$$[X_1, X_2] = iX_5, \quad [X_1, X_3] = iX_6, \quad [X_2, X_4] = iX_6. \tag{20}$$

Its invariants are X_5 and X_6 and therefore

$$C^{(2)}(\mathcal{A}_{6,4}) = pX_5^2 + qX_6^2, \tag{21}$$

which gives the degenerate metric

$$\Omega^{ab} = \mathrm{diag}(0, 0, 0, 0, p, q). \tag{22}$$

Similarly, for $\mathcal{A}_{6,5}^{\alpha}$ the commutation relations read as

$$\begin{aligned} &[X_1, X_3] = iX_5, \qquad [X_1, X_4] = iX_6, \\ &[X_2, X_3] = i\alpha X_6, \qquad [X_2, X_4] = iX_5, \qquad \alpha \neq 0. \end{aligned} \tag{23}$$

The invariants are again X_5 and X_6 and therefore the results of the previous case apply in the present one as well.

The algebra $\mathcal{A}_{6,14}^{-1}$ has the following commutation relations,

$$\begin{aligned} &[X_1, X_3] = iX_4, \qquad [X_1, X_4] = iX_6, \\ &[X_2, X_3] = iX_5, \qquad [X_2, X_5] = -iX_6, \end{aligned} \tag{24}$$

with invariants X_6 and $X_5^2 - X_4^2 + 2X_3X_6$. Therefore the quadratic Casimir operator is

$$C^{(2)}(\mathcal{A}_{5,14}^{-1}) = pX_6^2 + q(X_5^2 - X_4^2 + 2X_3X_6), \tag{25}$$

which gives

$$\Omega^{ab} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q \\ 0 & 0 & 0 & -q & 0 & 0 \\ 0 & 0 & 0 & 0 & q & 0 \\ 0 & 0 & -q & 0 & 0 & p \end{pmatrix}, \tag{26}$$

which is again degenerate.

The most interesting solution corresponds to the algebra $\mathcal{A}_{6,3}$. This one has the following commutation relations,

$$[X_1, X_2] = iX_6, \quad [X_1, X_3] = iX_4, \quad [X_2, X_3] = iX_5. \tag{27}$$

Its invariants are X_4, X_5, X_6 and $X_1X_5 + X_3X_6 - X_2X_4$ and therefore the general form of its quadratic Casimir operator reads as

$$C^{(2)}(\mathcal{A}_{6,3}) = pX_4^2 + qX_5^2 + rX_6^2 + s(X_1X_5 + X_3X_6 - X_2X_4). \tag{28}$$

Now the corresponding metric is given by

$$\Omega^{ab} = \begin{pmatrix} 0 & 0 & 0 & 0 & s & 0 \\ 0 & 0 & 0 & -s & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & s \\ 0 & -s & 0 & p & 0 & 0 \\ s & 0 & 0 & 0 & q & 0 \\ 0 & 0 & s & 0 & 0 & r \end{pmatrix}, \tag{29}$$

which is the first nondegenerate case that we encounter in our analysis. The determinant of the metric is

$$|\Omega^{ab}| = -s^6 \neq 0 \tag{30}$$

and therefore it is invertible with inverse

$$\Omega_{ab} = \begin{pmatrix} -q/s^2 & 0 & 0 & 0 & 1/s & 0 \\ 0 & -p/s^2 & 0 & -1/s & 0 & 0 \\ 0 & 0 & -r/s^2 & 0 & 0 & 1/s \\ 0 & -1/s & 0 & 0 & 0 & 0 \\ 1/s & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/s & 0 & 0 & 0 \end{pmatrix}. \tag{31}$$

The six eigenvalues of the latter are

$$\frac{1}{s^2}(-x_i \pm \sqrt{x_i^2 + 4s^2}), \qquad i = 1, 2, 3, \tag{32}$$

where $x_1 = p$, $x_2 = q$ and $x_3 = r$. We observe that there are three positive and three negative eigenvalues; therefore the algebra is noncompact.

Finally, let us close this subsection by mentioning that there is one further nontrivial case in six dimensions, which is the algebraic sum of two three-dimensional nilpotent algebras $\mathcal{A}_{3,1}$, namely, $\mathcal{A}_{3,1} \oplus \mathcal{A}_{3,1}$. The properties of this case are directly derived from the properties of $\mathcal{A}_{3,1}$, which were presented before.

B. Representations in terms of Hermitian matrices

Let us now discuss the representation of the above algebras in terms of Hermitian matrices. First let us note that a complete study of the unitary representations of nilpotent Lie groups was performed in [28], which facilitates our task.

The method that will be followed consists of the following steps. First the central elements, i.e., the elements which commute with all the algebra generators, are identified. These elements are mapped to operators which are multiples of the identity. Then, in order to completely define the representation one has to map the remaining elements to Hermitian matrices. It turns out that this last step amounts in mapping these elements to the usual operators for coordinates and momenta in quantum mechanics. Let us now present a case by case analysis following the above steps for the algebras which were identified in the previous subsection.

$\mathcal{A}_{3,1}$ *case*. This algebra has one central element, the X_1. Therefore we map this element to a multiple of the identity,

$$X_1 = \theta \mathbb{1}, \qquad \theta \in \mathbb{R}. \tag{33}$$

The remaining elements now satisfy the commutation relation,

$$[X_2, X_3] = i\theta \mathbb{1}, \tag{34}$$

which reduces to the Moyal-Weyl case. Clearly, X_2 and X_3 may then be represented by the usual Hermitian matrices corresponding to the coordinate and momentum operators of quantum mechanics. These matrices are of course infinite-dimensional and they have the well-known form

$$P = \sqrt{\frac{1}{2}}\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \dots \\ 1 & 0 & \sqrt{2} & 0 & 0 & \dots \\ 0 & \sqrt{2} & 0 & \sqrt{3} & 0 & \dots \\ 0 & 0 & \sqrt{3} & 0 & 0 & \dots \\ \vdots & & & & & \end{pmatrix},$$
$$Q = \sqrt{\frac{1}{2}}\begin{pmatrix} 0 & i & 0 & 0 & 0 & \dots \\ -i & 0 & i\sqrt{2} & 0 & 0 & \dots \\ 0 & -i\sqrt{2} & 0 & i\sqrt{3} & 0 & \dots \\ 0 & 0 & -i\sqrt{3} & 0 & 0 & \dots \\ \vdots & & & & & \end{pmatrix}. \tag{35}$$

Then the solution we have obtained is

$$\{X_1 = \theta \mathbb{1}, X_2 = \sqrt{\theta} Q, X_3 = \sqrt{\theta} P\}. \tag{36}$$

$\mathcal{A}_{4,12}$ *case*. As we already mentioned in the previous subsection, this algebra does not possess any invariants and therefore the method we follow here cannot be applied.

$\mathcal{A}_{5,1}$ *case*. The central elements of this algebra are X_1 and X_2 and the combination $X_2X_3 - X_1X_4$. Therefore we set

$$X_1 = \theta_1 \mathbb{1} \quad \text{and} \quad X_2 = \theta_2 \mathbb{1}. \tag{37}$$

Then the commutation relations read as

$$[X_3, X_5] = i\theta_1 \mathbb{1}, \qquad [X_4, X_5] = i\theta_2 \mathbb{1}. \tag{38}$$

Moreover the last quadratic invariant has to be fixed,

$$X_2X_3 - X_1X_4 = \theta \mathbb{1}. \tag{39}$$

The resulting solution is

$$\Big\{X_1 = \theta_1 \mathbb{1}, X_2 = \theta_2 \mathbb{1}, X_3 = \sqrt{\theta_1} Q, X_4 = \frac{\theta_2}{\sqrt{\theta_1}} Q - \frac{\theta}{\theta_1}\mathbb{1}, X_5 = \sqrt{\theta_1} P\Big\}. \tag{40}$$

$\mathcal{A}_{5,4}$ *case*. The unique central element in the present case is X_1. Therefore we set

$$X_1 = \theta \mathbb{1}, \tag{41}$$

which leads to the commutation relations

$$[X_2, X_4] = i\theta \mathbb{1}, \qquad [X_3, X_5] = i\theta \mathbb{1}. \tag{42}$$

These relations may be interpreted as two quantum planes in the directions (24) and (35), respectively, with the same quantization parameter θ. The solution is

$$\{X_1 = \theta \mathbb{1}, X_2 = \sqrt{\theta} Q, X_3 = \sqrt{\theta} Q', X_4 = \sqrt{\theta} P, X_5 = \sqrt{\theta} P'\}, \tag{43}$$

where (Q, P) and (Q', P') are two sets of Hermitian matrices, representing the two different quantum planes and therefore mutually commuting.

$\mathcal{A}_{5,39}$ *case*. This is a solvable Lie algebra and the method we follow here does not directly apply. Therefore this case is less clear and it will not be treated any further.

$\mathcal{A}_{6,3}$ *case*. The central elements in the present case are X_4, X_5, X_6 and $X_1X_5 + X_3X_6 - X_2X_4$. Therefore we set

$$X_4 = \theta_4 \mathbb{1}, \qquad X_5 = \theta_5 \mathbb{1} \quad \text{and} \quad X_6 = \theta_6 \mathbb{1}. \tag{44}$$

The commutation relations take the form

$$[X_1, X_2] = i\theta_6 \mathbb{1}, \qquad [X_1, X_3] = i\theta_4 \mathbb{1}, \qquad [X_2, X_3] = i\theta_5 \mathbb{1}, \tag{45}$$

while the quadratic invariant is fixed according to

$$X_1X_5 + X_3X_6 - X_2X_4 = \theta \mathbb{1}. \tag{46}$$

The resulting solution in this case is

$$\Big\{X_1 = \sqrt{\theta_6}Q, X_2 = \sqrt{\theta_6}P, \\ X_3 = -\frac{\theta_5}{\sqrt{\theta_6}}Q + \frac{\theta_4}{\sqrt{\theta_6}}P + \frac{\theta}{\theta_6}\mathbb{1}, X_4 = \theta_4\mathbb{1}, \\ X_5 = \theta_5\mathbb{1}, X_6 = \theta_6\mathbb{1}\Big\}. \tag{47}$$

$\mathcal{A}_{6,4}$ *and* $\mathcal{A}^{\alpha}_{6,5}$ *cases*. The central elements for these algebras are X_5 and X_6. Therefore in both cases we set

$$X_5 = \theta_5\mathbb{1}, \qquad X_6 = \theta_6\mathbb{1}. \tag{48}$$

Then, for the first case we obtain the commutation relations

$$[X_1, X_2] = i\theta_5\mathbb{1}, \qquad [X_1, X_3] = i\theta_6\mathbb{1}, \qquad [X_2, X_4] = i\theta_6\mathbb{1}, \tag{49}$$

while for the second case

$$[X_1, X_3] = i\theta_5\mathbb{1}, \qquad [X_1, X_4] = i\theta_6\mathbb{1}, \\ [X_2, X_3] = i\alpha\theta_6\mathbb{1}, \qquad [X_2, X_4] = i\theta_5\mathbb{1}. \tag{50}$$

The resulting solutions take the following form,

$$\Big\{X_1 = \sqrt{\theta_6}Q - \frac{\theta_5}{2\sqrt{\theta_6}}P', X_2 = \sqrt{\theta_6}Q' + \frac{\theta_5}{2\sqrt{\theta_6}}P, \\ X_3 = \sqrt{\theta_6}P, X_4 = \sqrt{\theta_6}P', X_5 = \theta_5\mathbb{1}, X_6 = \theta_6\mathbb{1}\Big\}, \tag{51}$$

and

$$\Big\{X_1 = \sqrt{\theta_5}Q + \frac{\theta_6}{\sqrt{\theta_5}}Q', X_2 = \sqrt{\theta_5}Q' + \frac{\alpha\theta_6}{\sqrt{\theta_5}}Q, \\ X_3 = \sqrt{\theta_5}P, X_4 = \sqrt{\theta_5}P', X_5 = \theta_5\mathbb{1}, X_6 = \theta_6\mathbb{1}\Big\}, \tag{52}$$

respectively, where (Q, P) and (Q', P') are again two sets of mutually commuting representations.

$\mathcal{A}^{-1}_{6,14}$ *case*. The central elements in this case are X_6 and $X_5^2 - X_4^2 + 2X_3X_6$ and we set

$$X_6 = \theta_6\mathbb{1}. \tag{53}$$

Then the commutation relations read as

$$[X_1, X_3] = iX_4, \qquad [X_1, X_4] = i\theta_6\mathbb{1}, \\ [X_2, X_3] = iX_5, \qquad [X_2, X_5] = -i\theta_6\mathbb{1}, \tag{54}$$

while there is also a relation of the form

$$X_5^2 - X_4^2 + 2X_3X_6 = \theta\mathbb{1}. \tag{55}$$

The solution in this case is

$$\Big\{X_1 = \sqrt{\theta_6}Q, X_2 = \sqrt{\theta_6}P', X_3 = \frac{1}{2}\Big(P^2 - Q'^2 + \frac{\theta}{\theta_6}\mathbb{1}\Big), \\ X_4 = \sqrt{\theta_6}P, X_5 = \sqrt{\theta_6}Q', X_6 = \theta_6\mathbb{1}\Big\}. \tag{56}$$

Having identified the above Lie-algebraic solutions of the IKKT model, in the following subsection we discuss their relation to NC geometry in a more general context.

C. Noncommutative spaces and * products

NC spaces. The construction of "NC spaces" is based on a shift from the space itself to the algebra of functions defined on it [40–44]. Therefore, strictly speaking a NC space is not a space in the classical sense but instead it corresponds to an associative but not necessarily commutative algebra $\mathcal{A}$, accompanied with a set of relations.

In order to be more specific, let us consider an associative algebra $\mathcal{A}$ with generators X_a, $a = 1, \ldots, N$. These generators satisfy certain commutation relations of the general form

$$[X_a, X_b] = i\theta_{ab}(X), \tag{57}$$

where $\theta_{ab}(X)$ is an arbitrary function of the generators X_a. Then the above algebraic structure defines a NC space and the generators of the algebra are commonly referred to as "coordinates" of the NC space [45]. The case of constant θ_{ab} corresponds to the Moyal-Weyl quantum plane, which was encountered in Sec. II as the basic solution of the IKKT MM.

The cases we already studied in the present section correspond to a Lie-algebra structure. In other words, the function θ_{ab} is linear in the generators X_a and the commutation relations read as

$$[X_a, X_b] = if_{ab}{}^{c}X_c. \tag{58}$$

The most prominent representatives of such a structure are the fuzzy two-sphere [46] and its higher-dimensional generalizations [47], the fuzzy complex projective spaces [48,49] and others [50–52]. All these NC spaces are compact, since they are based on compact semisimple Lie algebras. As we saw, these compact NC spaces do not directly provide solutions of the undeformed IKKT MM.[6] However, we proved that there exist solutions of the IKKT model with the structure (58), albeit based on noncompact algebras. Indeed, it is obvious from our previous analysis that in each case there is a set of generators and relations, along with the prescribed Casimir operators which fix the representation of the algebra. Thus all the cases that were discussed correspond to well-defined NC spaces. Moreover, in all the cases the number of generators minus the number of invariants (whose fixing specifies the representation) is always even. This means that the resulting solutions describe noncompact, noncommutative even-dimensional branes, similarly to the Dp-brane solutions of the IIB string theory (with p odd), appropriately embedded in 10-dimensional $\mathbb{R}^{10}$. We shall return to this and present a more detailed description at the end of the present section.

[6] See however [14].

Weyl quantization and $*$ products. As we discussed above, the shift from spaces to algebras paves the road to NC geometry and provides a natural setup to construct NC/quantized spaces which correspond to certain algebraic structures. Indeed, a natural way to quantize a manifold is to consider an appropriate algebra of functions on it and instead quantize the algebra, either by truncating it or deforming its product structure. The latter possibility belongs in the broad context of deformation quantization [53], whose most prominent physical example is phase-space (Weyl) quantization [54].

Let us briefly discuss Weyl quantization in the case of a Lie-algebra structure following [45] and apply it in the specific cases studied here. A more formal and rigorous discussion based on the pioneering work of Kontsevich [55] may be found in [56]. Let us denote classical (commutative) coordinates by x^a, $a = 1, \ldots, N$ and elements of $\mathcal{A}$ (NC coordinates) by X^a, as before. An operator $W(f)$ may be associated to every classical function $f(x)$, given by

$$W(f) = \frac{1}{\sqrt{(2\pi)^n}} \int d^n k e^{ik_a X^a} \tilde{f}(k), \tag{59}$$

where $\tilde{f}$ is the Fourier transform of f,

$$\tilde{f}(k) = \frac{1}{\sqrt{(2\pi)^n}} \int d^n x e^{ik_a x^a} f(x). \tag{60}$$

Multiplying operators of the kind appearing in (59) results in new operators, which might or might not be associated to classical functions as well. In the case that this is possible, i.e., when

$$W(f)W(g) = W(h), \tag{61}$$

the corresponding function h will be identified with a deformed product of f and g, which is denoted by $*$, i.e.,

$$h = f * g. \tag{62}$$

More explicitly, we can write the product of the operators as

$$W(f)W(g) = \frac{1}{(2\pi)^n} \int d^n k d^n p e^{ik_a X^a} e^{ip_b X^b} \tilde{f}(k) \tilde{g}(p). \tag{63}$$

Then the function $f * g$ exists if the product of the two exponentials in the integrand can be calculated by the Baker-Campbell-Hausdorff formula. In the case of a Lie structure one can write

$$e^{ik_a X^a} e^{ip_b X^b} = e^{iP_a(k,p)X^a}, \tag{64}$$

where

$$P_a = k_a + p_a + \tfrac{1}{2} g_a(k, p), \tag{65}$$

where the g_a contain the information about the NC structure. Having determined the functions g_a it is straightforward to write down the explicit formula for the $*$ product,

$$f * g = e^{(i/2)x^a g_a(i(\partial/\partial y), i(\partial/\partial z))} f(y) g(z)|_{y,z\to x}. \tag{66}$$

Let us now determine the functions g_a for the cases that are studied here. By direct calculation we obtain the following results for each of the seven nilpotent cases, where the brackets appearing in the subscripts denote antisymmetrization with weight one:

$$\mathcal{A}_{3,1}\colon\ g_1 = ik_{[2}p_{3]}, \qquad g_{2,3} = 0.$$

$$\mathcal{A}_{5,1}\colon\ g_1 = ik_{[3}p_{5]}, \qquad g_2 = ik_{[4}p_{5]}, \qquad g_{3,4,5} = 0.$$

$$\mathcal{A}_{5,4}\colon\ g_1 = i(k_{[2}p_{4]} + k_{[3}p_{5]}), \qquad g_{2,3,4,5} = 0.$$

$$\mathcal{A}_{6,3}\colon\ g_{1,2,3} = 0, \qquad g_4 = ik_{[1}p_{3]}, \\ g_5 = ik_{[2}p_{3]}, \qquad g_6 = ik_{[1}p_{2]}.$$

$$\mathcal{A}_{6,4}\colon\ g_{1,2,3,4} = 0, \qquad g_5 = ik_{[1}p_{2]}, \\ g_6 = i(k_{[1}p_{3]} + k_{[2}p_{4]}).$$

$$\mathcal{A}_{6,5}\colon\ g_{1,2,3,4} = 0, \qquad g_5 = i(k_{[1}p_{3]} + k_{[2}p_{4]}), \\ g_6 = i(k_{[1}p_{4]} + k_{[2}p_{3]}).$$

$$\mathcal{A}_{6,14}\colon\ g_{1,2,3} = 0, \qquad g_4 = ik_{[1}p_{3]}, \qquad g_5 = ik_{[2}p_{3]}, \\ g_6 = i\Big(k_{[1}p_{4]} - k_{[2}p_{5]} + \frac{i}{6}(k_1 k_{[1}p_{3]} - k_2 k_{[2}p_{3]} \\ - p_1 k_{[1}p_{3]} + p_2 k_{[2}p_{3]})\Big).$$

Plugging these expressions in the Eq. (66) gives directly the corresponding $*$ product in each case. It is worth noting that due to the nature of nilpotent algebras the Baker-Campbell-Hausdorff formula terminates and the above functions determine exactly the exponent of the $*$ product.

D. Geometric interpretation and supersymmetry

In order to conclude this section let us discuss some important issues related to the solutions which were described above.

The first issue we would like to discuss concerns the geometric interpretation of the solutions. The cases based on the algebras $\mathcal{A}_{3,1}$, $\mathcal{A}_{5,1}$ and $\mathcal{A}_{5,4}$ can be easily interpreted. Indeed, the $\mathcal{A}_{3,1}$ solution (36) represents a static D-string extending in the X_3 direction and shifted by θ in the X_1 direction (here X_2 may be thought of as the timelike matrix of the model in its Lorentzian version). Such a solution was obtained in [2]. However, it is worth noting that in the present case the Lie-algebra structure partially determines the embedding of the D-string in $\mathbb{R}^{10}$. The D-string *has to be* shifted by θ along X_1; setting θ to zero is not allowed here.

The solution (40) based on $\mathcal{A}_{5,1}$ represents a configuration of two static D-strings, one extending in the X_3

direction and one extending along the X_4 direction. Both D-strings are also shifted by θ_1 in the X_1 direction and by θ_2 in the X_2 direction. Obviously, these D-strings are not parallel but instead they intersect perpendicularly. It is interesting that such a configuration of intersecting D-strings is naturally obtained in our context.

The solution (43) based on $\mathcal{A}_{5,4}$ represents a static $D3$-brane shifted by θ in the X_1 direction. Such a solution was previously described in [3–7].

As for the solution (47) based on $\mathcal{A}_{6,3}$, we observe that it involves one set of (Q, P) matrices and therefore it should correspond again to a pair of D-strings, one extending along X_2 and the other along X_3. Both are shifted in the directions X_4, X_5, X_6 by θ_4, θ_5, θ_6, respectively. The difference here in comparison to previous cases, especially to the similar case of $\mathcal{A}_{5,1}$, is that although the first D-string is again static, the second one is not. Indeed, it is obvious from the solution (47) that the D-string extending along the X_3 direction is time-dependent (the timelike matrix here is X_1) and therefore it is moving and it carries nonzero momentum.

Similar considerations hold for the cases $\mathcal{A}_{6,4}$, $\mathcal{A}_{6,5}^{\alpha}$ and $\mathcal{A}_{6,14}^{-1}$. In these cases the corresponding solutions involve two sets of (Q, P) matrices and therefore they should all be associated to $D3$-branes. Their difference from the similar $D3$-brane solution based on $\mathcal{A}_{5,4}$ lies in their different embedding in $\mathbb{R}^{10}$, specified by the solutions (51), (52), and (56) respectively. Moreover, certain of these solutions may describe moving branes. For example, as far as the solution (51) is concerned, if the timelike direction is identified, e.g., with the matrix X_3, then the $D3$-brane is time-dependent along the direction X_2; i.e., it carries momentum in this direction.

A second important issue regards the supersymmetry of the above configurations. As we briefly discussed in Sec. II, the IKKT model has $\mathcal{N} = 2$ supersymmetry given by the transformations (2). The single Dp-brane solutions of the model (such as the single D-string and $D3$-brane solutions obtained above) are in fact $\frac{1}{2}$ Bogomol'nyi-Prasad-Sommerfield (BPS) states; i.e. they preserve only one supersymmetry [2]. This may be easily seen, since in all but one of the nilpotent cases it turns out that the commutator $[X_a, X_b]$ is a c number, say, θ_{ab}. Then setting $\xi = \frac{1}{2}\theta_{ab}\Gamma^{ab}\epsilon$ one obtains

$$(\delta_\epsilon - \delta_\xi)\psi = 0, \qquad (\delta_\epsilon - \delta_\xi)X_a = 0, \tag{67}$$

which shows that half of the supersymmetry is preserved [2]. Let us note that the case of $\mathcal{A}_{6,14}^{-1}$ does not lie within the above argument since the corresponding commutators are not c numbers as can be seen by (54).

Moreover, the cases of intersecting D-strings call for special treatment. Let us consider the case based on the algebra $\mathcal{A}_{5,1}$. The nontrivial commutation relations appear in (38). Then, in line with the previous argument, a supersymmetric configuration corresponds to $\xi = \frac{1}{2}\theta_1\Gamma^{35}\epsilon + \frac{1}{2}\theta_2\Gamma^{45}\epsilon$. Therefore, upon this choice one supersymmetry is again preserved for this configuration.

As a final issue, let us briefly discuss the relevance of these solutions for the description of space-time and the construction of gauge theories on it. In Sec. II we presented the four-dimensional Moyal-Weyl quantum plane solution $\mathbb{R}^4_\theta$, which may be considered as the space-time brane. In such a description, space-time ceases to be a smooth manifold below some scale set by the noncommutativity parameter θ, where one may talk about fuzzy or quantum space-time. Such a picture arises naturally when gravity and quantum mechanics are considered (see [57]). This role may be played by the four-dimensional solutions which were presented above, e.g., by the one based on the algebra $\mathcal{A}_{5,4}$.

However, let us note that more ingredients are necessary if one aims at the construction of particle physics models based on the above solutions. As we argued before, the single brane solutions are generically $\frac{1}{2}$ BPS states and therefore in four dimensions they would lead to a $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. This is interesting on its own but it is clearly not realistic, mainly due to the absence of chiral fermions. In order to obtain a more realistic structure it is necessary to break the large amount of supersymmetries to $\mathcal{N} = 1$, where chiral fermions may in principle be accommodated. Although we are not going to delve into details on mechanisms which address the above issue, let us mention that the most promising possibility would be to consider backgrounds based on multiple intersecting higher-dimensional branes. Such backgrounds were recently studied in [19]. There it was shown that under certain conditions one can accommodate chiral fermions at brane intersections and construct semirealistic models. One necessary condition in that context turns out to be that the branes should be $D7$ ones. Although the solutions presented in the present paper correspond only to $D1$ and $D3$ branes, it is not difficult to combine them with the basic solution (7) in order to construct higher-dimensional ones. As a concrete example, let us consider the combined solution

$$X^a = \begin{pmatrix} \bar{X}^\mu \\ \tilde{X}^i \end{pmatrix}, \tag{68}$$

with $\bar{X}^\mu$ satisfying the commutation relations (7) of the $\mathbb{R}^4_\theta$, $\tilde{X}^i$ for $i = 1, \ldots, 5$ satisfying the commutation relations of the algebra $\mathcal{A}_{5,1}$ [i.e., corresponding to the solution (40)] and $\tilde{X}^6 = 0$. This is then clearly a solution of the IKKT model corresponding to a flat $D7$ brane. The study of intersections of such solutions and their implications for the construction of models for particle physics is interesting and should be addressed elsewhere. Here we just state that the welcome results of [19] are expected to hold in the present cases as well.

IV. TOWARDS QUANTIZED COMPACT NILMANIFOLDS

In the present section we deviate from the search for solutions of the IKKT model and we pose the following question: are there compact manifolds based on the algebras discussed in Sec. III? This question is of interest because once it is answered in the affirmative it will open up the possibility to consider new compactifications of the IKKT model, based on the Lie algebras presented in the previous section.

The simplest way to construct a manifold out of a nilpotent or a solvable Lie algebra $\mathcal{A}$ is to consider the action of a dicrete (sub)group Γ on its Lie group. Then the quotient[7] $M = A/\Gamma$ is called a nilmanifold or a solvmanifold, respectively. A very important result for our purposes states that in the nilpotent case such a construction is possible if and only if the corresponding Lie algebra has rational structure constants in some basis [29]. This guarantees that in the nilpotent cases that we study here a Γ as above always exists.[8] For the solvable cases the situation is more complicated but it will soon become evident that the ones we met in Sec. III are not of further interest for our purposes.

An important issue which we would like to mention regards the compactness of a nilmanifold. It is true that even starting with a noncompact group A it is possible to construct a compact manifold by considering its quotient by a compact discrete subgroup of it. A necessary condition for compactness is that the group is unimodular; i.e., its structure constants satisfy $f^a{}_{ab} = 0$ (this was already discussed in [30]). This condition is not sufficient but for nilpotent groups the requirement of rational structure constants is enough.

As a first check on whether we can construct compact manifolds based on the algebras that were singled out in Sec. III, let us try to verify the condition of unimodularity. It is straightforward to check (by mere inspection of the commutation relations) that $\mathcal{A}_{3,1}$, $\mathcal{A}_{5,1}$, $\mathcal{A}_{5,4}$, $\mathcal{A}_{6,3}$, $\mathcal{A}_{6,4}$, $\mathcal{A}^{\alpha}_{6,5}$ and $\mathcal{A}^{-1}_{6,14}$ indeed pass the test, while on the other hand $\mathcal{A}_{4,12}$ and $\mathcal{A}_{5,39}$ fail to do so. Therefore the two latter cases do not give rise to compact manifolds. It is worth noting that these two cases are exactly the only solvable ones that we found in Sec. III and therefore our present analysis shows that there are no compact solvmanifolds[9] corresponding to algebras which solve the equations (3). Therefore in the following only nilmanifolds will be discussed. For tables of nilmanifolds and solvmanifolds in six dimensions the reader may consult [34,58]. Moreover, six-dimensional solvmanifolds were used in flux compactifications in [59], where a rather detailed review on solvmanifolds may be found.

Let us proceed by giving two explicit examples of the construction of a nilmanifold. The first one is well-known and it corresponds to the simplest case of the algebra $\mathcal{A}_{3,1}$ with nontrivial commutation relation $[X_2, X_3] = X_1$ [60]. A basis for the algebra is given by the following 3×3 upper triangular matrices,[10]

$$X_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \tag{69}$$

Then, any element of the corresponding group $A_{3,1}$ may be parametrized as

$$g = \begin{pmatrix} 1 & x_2 & x_1 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{pmatrix}. \tag{70}$$

This is clearly a noncompact group. According to the above discussion, in order to produce a compact manifold out of it, a compact discrete subgroup Γ has to be considered. Such a subgroup is given by those elements $g \in A_{3,1}$ which have integer values of x_2, x_3 and cx_1, where c is a positive integer. Then the quotient $A_{3,1}/\Gamma$ is indeed a compact nilmanifold. In the physics literature this manifold is known as a twisted torus and it corresponds to a (twisted) fibration of a torus over another torus (see, e.g., [34]).

Let us explain the above construction in more detail. Consider a representative element $g \in A_{3,1}$, as in (70). Then the Maurer-Cartan 1-form e is given by

$$e = g^{-1} dg, \tag{71}$$

which gives

$$e = \begin{pmatrix} 0 & dx_2 & dx_1 - x_2 dx_3 \\ 0 & 0 & dx_3 \\ 0 & 0 & 0 \end{pmatrix}. \tag{72}$$

The 1-form e is Lie-algebra valued, $e = e^a X_a$, and its components are

$$e^1 = dx^1 - x_2 dx_3, \qquad e^2 = dx_2, \qquad e^3 = dx_3, \tag{73}$$

which evidently satisfy the Maurer-Cartan equations

$$de^a = -\frac{1}{2} f^a_{bc} e^b \wedge e^c, \tag{74}$$

[7] Here A denotes the group associated to the algebra $\mathcal{A}$.
[8] This does not mean though that it has to be unique.
[9] This argument holds of course only for up to five-dimensional solvable algebras which we consider here.

[10] Of course these matrices are not Hermitian and therefore they cannot be directly related to solutions of the IKKT model. However, their use facilitates significantly the geometric description of the corresponding manifolds. We shall comment on the relevance of such manifolds for the IKKT model at the end of this section.

since $de^2 = de^3 = 0$ and $de^1 = -e^2 \wedge e^3$. The important observation here is that in order to compactify the group one has to introduce a twist. Indeed, while for the directions x_1 and x_3 the compactification is achieved by the identifications

$$(x_1, x_2, x_3) \sim (x_1 + a, x_2, x_3) \sim (x_1, x_2, x_3 + b), \qquad a, b \in \mathbb{Z}, \tag{75}$$

one cannot do the same for x_2; i.e., the identification $(x_1, x_2, x_3) \sim (x_1, x_2 + c, x_3)$ obviously does not work. Instead, the correct identification is

$$(x_1, x_2, x_3) \sim (x_1 + cx_3, x_2 + c, x_3). \tag{76}$$

Under (75) and (76) the desired (twisted) compactification is achieved.

The above example serves as a prototype for any other. One can always write down a basis for the algebra in terms of upper triangular matrices and compactify the corresponding group by modding out a discrete subgroup corresponding to elements with integer entries. Let us work out in some detail a less trivial, six-dimensional example, based on the algebra $\mathcal{A}_{6,3}$. A basis for this algebra is given by the following 6×6 upper triangular matrices:

$$X_1 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \\ & & 0 & 1 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 1 & 0 \\ & & 0 & 0 & 0 & 1 \\ & & & 0 & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix},$$
$$X_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ & 0 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix}, \quad X_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ & 0 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix}, \quad X_6 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix}. \tag{77}$$

The corresponding general group element is given by

$$g = \begin{pmatrix} 1 & x_2 & x_1 & x_6 & x_5 & x_4 \\ & 1 & 0 & 0 & x_3 & 0 \\ & & 1 & x_2 & 0 & x_3 \\ & & & 1 & 0 & 0 \\ & & & & 1 & 0 \\ & & & & & 1 \end{pmatrix}. \tag{78}$$

The Maurer-Cartan 1-form may be computed and it has the following form,

$$e = \begin{pmatrix} 0 & dx_2 & dx_1 & dx_6 - x_1 dx_2 & dx_5 - x_2 dx_3 & dx_4 - x_1 dx_3 \\ & 0 & 0 & 0 & dx_3 & 0 \\ & & 0 & dx_2 & 0 & dx_3 \\ & & & 0 & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix}, \tag{79}$$

with components

$$e^1 = dx_1, \qquad e^2 = dx_2, \qquad e^3 = dx_3, \qquad e^4 = dx_4 - x_1 dx_3, \qquad e^5 = dx_5 - x_2 dx_3, \qquad e^6 = dx_6 - x_1 dx_2. \tag{80}$$

The necessity for twists is again evident. Indeed, it is straightforward to see that for the directions x_3, x_4, x_5 and x_6 we can consider the identifications

$$(x_1, x_2, x_3, x_4, x_5, x_6) \sim (x_1, x_2, x_3 + c, x_4, x_5, x_6) \\ \sim (x_1, x_2, x_3, x_4 + d, x_5, x_6) \\ \sim (x_1, x_2, x_3, x_4, x_5 + e, x_6) \\ \sim (x_1, x_2, x_3, x_4, x_5, x_6 + f), \\ c, d, e, f \in \mathbb{Z}, \tag{81}$$

while for the x_1 and x_2 ones the correct identifications are

$$(x_1, x_2, x_3, x_4, x_5, x_6) \sim (x_1 + a, x_2, x_3, x_4 + ax_3, x_5, x_6 + ax_2) \\ \sim (x_1, x_2 + b, x_3, x_4, x_5 + bx_3, x_6), \\ a, b \in \mathbb{Z}. \tag{82}$$

Under (81) and (82) we obtain the desired twisted compactification.

Following the above procedure, nilmanifolds corresponding to the seven nilpotent Lie algebras which were singled out in Sec. III may be constructed. The deformation quantization of the first one, based on $\mathcal{A}_{3,1}$, was performed in [60]. A detailed discussion on the quantization of the rest of the cases is beyond the scope of the present paper. However, the pursuit of this task would be of interest, e.g., for compactifications of the IKKT model. In [1,15] toroidal compactifications of the Banks-Fischler-Shenker-Susskind and IKKT matrix models were considered. They correspond to a restriction of the matrix action if we consider matrix configurations which satisfy certain periodicity conditions related to the compact directions on a multidimensional torus. These conditions cannot be satisfied by finite-dimensional matrices; however, solutions can be found for operators in an infinite-dimensional Hilbert space. In the above point of view, it would be natural to consider twisted toroidal compactifications based on the manifolds presented earlier in this section. In that case, one should impose twisted periodicity conditions, dictated by the identifications (75), (76), (81), and (82), and determine (infinite-dimensional) matrix configurations which provide solutions to the compactified matrix model. This task deserves separate treatment and we leave a detailed study for future work.

Moreover, in [15] compactifications on NC tori were studied and they were shown to correspond to supergravity backgrounds with constant 3-form flux. Then one could expect that similar compactifications on NC twisted tori could account for nongeometric flux vacua [61,62], incorporating the nongeometric fluxes in the geometry of the NC space. We hope to report on this in a future publication.

V. DISCUSSION AND CONCLUSIONS

In the present paper we performed a survey of Lie-algebraic solutions to the IKKT matrix model. Up to now it was known that manifolds with Lie-type noncommutativity are either solutions of deformed MM [9–13] or else some split noncommutativity has to be introduced [14]. Moreover, the above compact solutions are all based on compact semisimple Lie algebras. Our investigation revealed the possibility of obtaining (noncompact) solutions to the undeformed IKKT model without deformations or additional requirements, which are based on nilpotent and solvable Lie algebras.

More specifically, scanning the classification tables of [27] we found seven nilpotent Lie algebras (one three-dimensional, two five-dimensional and four 6-dimensional ones) and two solvable ones (one four-dimensional and one five-dimensional) which solve the equations of motion of the IKKT model. Subsequently, we discussed the representation of these algebras by Hermitian matrices. This is always possible for the nilpotent cases, thus proving that they indeed constitute solutions to the IKKT model. It is straightforward to combine these solutions with the basic four-dimensional solution of the IKKT model, which was presented in Sec. II and corresponds to NC space-time as a Moyal-Weyl quantum plane $\mathbb{R}^4_\theta$.

In addition, we addressed the problem of constructing compact NC spaces associated to these algebras. The simplest constructions of compact spaces based on non-semisimple Lie algebras are the so-called nilmanifolds and solvmanifolds, also known as twisted tori in the physics literature. These correspond to quotients of the group of a nilpotent or solvable Lie algebra, respectively, by a compact discrete subgroup of it. We argued that for the two solvable cases we found, there are no associated compact spaces. However, all the cases of nilpotent Lie algebras give rise to certain compact nilmanifolds. These nilmanifolds can be formally quantized via Weyl quantization. Although it cannot be argued at this stage that these compact manifolds are solutions of the IKKT model as well, it would be interesting to investigate the compactification of the model on them along the lines of [15].

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APPENDIX: DEFINITIONS ON LIE ALGEBRAS

Lie algebras are classified according to their properties in simple, semisimple, Abelian, nilpotent and solvable. In this brief appendix let us collect some useful definitions, which appear often in the main text. A standard reference is [63].

Let us denote a Lie algebra by $\mathcal{A}$ and its generators by X_a. These generators satisfy the commutation relations

$$[X_a, X_b] = f^c_{ab} X_c, \tag{A1}$$

where f^c_{ab} are the structure constants. Knowledge of the structure constants is enough to determine the Killing form g_{ab} according to the formula

$$g_{ab} = f^d_{ac} f^c_{bd}. \tag{A2}$$

According to Cartan's criterion a Lie algebra is semisimple if and only if its Killing form is nondegenerate. Accordingly, the Killing form of a nilpotent Lie algebra vanishes identically.

Given a Lie algebra one may search for its invariants, i.e., functions of its generators which commute with all the generators. The most important of these invariants is the quadratic Casimir operator, which we denote as

$$C^{(2)}(\mathcal{A}) = \Omega^{ab} X_a X_b, \tag{A3}$$

with Ω^{ab} the elements of a symmetric matrix. If the matrix Ω^{ab} is invertible, then one may form its inverse Ω_{ab}, which is symmetric, nondegenerate and invariant under the adjoint action of the corresponding group. In other words, Ω_{ab} is a metric on the corresponding group manifold. In fact, for semisimple algebras Ω_{ab} is proportional to the Killing form. This is no longer true for nonsemisimple ones.

Let us define the derived algebra $\mathcal{A}'$ of a Lie algebra $\mathcal{A}$ as

$$\mathcal{A}' = [\mathcal{A}, \mathcal{A}]. \tag{A4}$$

Moreover, let us introduce two generalizations of the derived algebra and, in particular, the so-called upper central series or derived series of $\mathcal{A}$, defined as

$$\mathcal{A}^{\{i\}} = [\mathcal{A}^{\{i-1\}}, \mathcal{A}^{\{i-1\}}], \qquad i \geq 2, \tag{A5}$$

and the lower central series, defined as

$$\mathcal{A}_{\{i\}} = [\mathcal{A}, \mathcal{A}_{\{i-1\}}], \qquad i \geq 2. \tag{A6}$$

In the above iterative definitions it holds that $\mathcal{A}_{\{1\}} = \mathcal{A}^{\{1\}} = \mathcal{A}'$. Then we have the following two definitions:

(i) A Lie algebra is called solvable if its derived series becomes zero after a finite number of steps, i.e., $\exists i_0$, such that $\mathcal{A}^{\{i_0\}} = 0$.

(ii) A Lie algebra is called nilpotent of step i_0 if its lower central series becomes zero after a finite number of steps, i.e., $\exists i_0$, such that $\mathcal{A}_{\{i_0\}} = 0$.

Clearly nilpotency implies solvability.

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