Proximity force approximation for the Casimir energy as a derivative expansion

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The proximity force approximation (PFA) has been widely used as a tool to evaluate the Casimir force between smooth objects at small distances. In spite of being intuitively easy to grasp, it is generally believed to be an uncontrolled approximation. Indeed, its validity has only been tested in particular examples, by confronting its predictions with the next-to-leading-order (NTLO) correction extracted from numerical or analytical solutions obtained without using the PFA. In this article we show that the PFA and its NTLO correction may be derived within a single framework, as the first two terms in a derivative expansion. To that effect, we consider the Casimir energy for a vacuum scalar field with Dirichlet conditions on a smooth curved surface described by a function ψ in front of a plane. By regarding the Casimir energy as a functional of ψ, we show that the PFA is the leading term in a derivative expansion of this functional. We also obtain the general form of the corresponding NTLO correction, which involves two derivatives of ψ. We show, by evaluating this correction term for particular geometries, that it properly reproduces the known corrections to PFA obtained from exact evaluations of the energy.

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I. INTRODUCTION

In the last years, there have been important theoretical and experimental advances in the analysis of the Casimir effect [1].

Until the recent development of theoretical methods that allowed for the exact evaluation of the Casimir energy for several geometries, the interaction between different bodies has been mostly computed using the so-called proximity force approximation (PFA) [2]. This approximation, expected to be reliable as long as the interacting surfaces are smooth, almost parallel, and very close, makes use of Casimir's expression for the energy per unit area for two parallel plates at a distance a apart. For the case of a single massless scalar field and Dirichlet conditions (the case we deal with in this paper) it is given by

$$E_{\rm pp}(a) = -\frac{\pi^2}{1440a^3}. \tag{1}$$

The PFA then approximates the interaction between two Dirichlet surfaces separated by a gap of spatially varying width z, as follows:

$$E_{\rm PFA} = \int_\Sigma d\sigma E_{\rm pp}(z), \tag{2}$$

where Σ is one of the two surfaces. Quite obviously, this formula does not take into account the nonparallelism of the surfaces. Moreover, the result may depend on the particular surface Σ chosen to perform the integral.

As the PFA was believed to be an uncontrolled approximation, its accuracy has been assessed only in some of the particular geometries where it was possible to compute the Casimir energy numerically or analytically. On general grounds, denoting by $\mathcal{L}$ a typical length associated with the curvature of one of the surfaces (assumed much smaller than the curvature of the second one) and by a the minimum distance between surfaces, one expects that

$$E_C = E_{\rm PFA}\left\{1 + \gamma\frac{a}{\mathcal{L}} + \mathcal{O}\left[\left(\frac{a}{\mathcal{L}}\right)^2\right]\right\}, \tag{3}$$

where γ is a constant, whose numerical value fixes the accuracy of the PFA in each particular geometry [the situation could be more complex, since the corrections to PFA may contain nonanalytic corrections as $(\frac{a}{\mathcal{L}})^n \log(\frac{a}{\mathcal{L}})$]. One can write similar expressions for geometries that involve two surfaces of similar curvature.

In this paper we explore the following simple idea. The Casimir energy can be thought of as a functional of the shape of the surfaces of the interacting bodies. As the PFA should be adequate for almost plane surfaces, a derivative expansion [3] of this functional should reproduce, to lowest order, the PFA. Moreover, the terms involving derivatives of the functions that describe the shape of the surfaces should contain the corrections to the PFA. We will show that this is indeed the case, and that it is possible to find a general formula to compute the first corrections to PFA for rather arbitrary surfaces.

Just to avoid some technical complications, we consider a massless scalar field in the presence of a curved surface in front of a plane. We will assume that the quantum field satisfies Dirichlet boundary conditions on both surfaces. Generalizations to other boundary conditions and to the electromagnetic field will be analyzed in a forthcoming work.

This paper is organized as follows. In Sec. II, we describe the model and derive a formal expression for the Casimir energy in the geometry described above. Then, in Sec. III, we perform a derivative expansion in the expression for the Casimir energy to obtain the main result of the paper: a general formula for the interaction energy between an arbitrary curved surface and a plane, containing up to two derivatives of the function ψ that describes the curved surface. The leading term of the expansion corresponds to the PFA, while the term with derivatives is the first nontrivial correction.

In Sec. IV we present some examples: a sphere, a cylinder, or a corrugated surface in front of a plane. We show, by comparing with existing analytical results, that the derivative expansion of the Casimir energy describes correctly both the PFA and its first correction for all of these geometries. We also compute the derivative expansion of the Casimir energy for geometries involving parabolic mirrors. Section V contains the conclusions of our work.

II. FORMAL EXPRESSION FOR THE VACUUM ENERGY

We shall consider a model consisting of a massless real scalar field φ in 3 + 1 dimensions, coupled to two mirrors that impose Dirichlet boundary conditions. In our Euclidean conventions, we use x_0, x_1, x_2, x_3 to denote the spacetime coordinates, x_0 being the imaginary time.

The mirrors occupy two surfaces, denoted by L and R, defined by the equations $x_3 = 0$ and $x_3 = \psi(x_1, x_2)$, respectively.

Following the functional approach to the Casimir effect, we introduce $\mathcal{Z}$, which may be interpreted as the zero temperature limit of a partition function, for the scalar field in the presence of the two mirrors. It may be written as follows:

$$\mathcal{Z} = \int \mathcal{D}\varphi \delta_L(\varphi)\delta_R(\varphi)e^{-S_0(\varphi)}, \tag{4}$$

where S is the free real scalar field Euclidean action

$$S_0(\varphi) = \frac{1}{2}\int d^4x(\partial\varphi)^2, \tag{5}$$

while δ_L (δ_R) imposes Dirichlet boundary conditions on the L (R) surface.

Exponentiating the two delta functions by introducing two auxiliary fields, λ_L and λ_R, we obtain for $\mathcal{Z}$ an equivalent expression:

$$\mathcal{Z} = \int \mathcal{D}\varphi\mathcal{D}\lambda_L\mathcal{D}\lambda_R e^{-S(\varphi;\lambda_L,\lambda_R)}, \tag{6}$$

with

$$S(\varphi;\lambda_L,\lambda_R) = S_0(\varphi) - i\int d^4x\varphi(x)[\lambda_L(x_\parallel)\delta(x_3) + \lambda_R(x_\parallel)\delta(x_3 - \psi(\mathbf{x}_\parallel))] \tag{7}$$

where we have introduced the notations $x_\parallel \equiv (x_0, x_1, x_2)$ and $\mathbf{x}_\parallel \equiv (x_1, x_2)$.

Integrating out φ, we see that $\mathcal{Z}_0$, corresponding to the field φ, in the absence of boundary conditions factors out, while the rest becomes an integral over the auxiliary fields:

$$\mathcal{Z} = \mathcal{Z}_0\int \mathcal{D}\lambda_L\mathcal{D}\lambda_R e^{-\frac{1}{2}\int d^3x_\parallel\int d^3y_\parallel\sum_{\alpha,\beta}\lambda_\alpha(x_\parallel)\mathbb{T}_{\alpha\beta}\lambda_\beta(y_\parallel)}, \tag{8}$$

where α, $\beta = L$, R and we have introduced the objects

$$\mathbb{T}_{LL}(x_\parallel, y_\parallel) = \langle x_\parallel, 0|(-\partial^2)^{-1}|y_\parallel, 0\rangle, \tag{9}$$

$$\mathbb{T}_{LR}(x_\parallel, y_\parallel) = \langle x_\parallel, 0|(-\partial^2)^{-1}|y_\parallel, \psi(\mathbf{y}_\parallel)\rangle, \tag{10}$$

$$\mathbb{T}_{RL}(x_\parallel, y_\parallel) = \langle x_\parallel, \psi(\mathbf{x}_\parallel)|(-\partial^2)^{-1}|y_\parallel, 0\rangle, \tag{11}$$

$$\mathbb{T}_{RR}(x_\parallel, y_\parallel) = \langle x_\parallel, \psi(\mathbf{x}_\parallel)|(-\partial^2)^{-1}|y_\parallel, \psi(\mathbf{y}_\parallel)\rangle, \tag{12}$$

where we use a "bra-ket" notation to denote matrix elements of operators, and ∂^2 is the four-dimensional Laplacian. Thus, for example,

$$\langle x|(-\partial^2)^{-1}|y\rangle = \int \frac{d^4k}{(2\pi)^4}\frac{e^{ik\cdot(x-y)}}{k^2}. \tag{13}$$

The vacuum energy of the system, $E_{\rm vac}$, subtracting the zero-point energy of the free field (contained in $\mathcal{Z}_0$), is

$$E_{\rm vac} = \lim_{T\to\infty}\left(\frac{\Gamma}{T}\right) = \frac{1}{2T}\,\mathrm{Tr}\,\log\mathbb{T}, \tag{14}$$

where T is the extent of the time dimension (or β^{-1}, in the thermal partition function setting), $\Gamma \equiv -\log\frac{\mathcal{Z}}{\mathcal{Z}_0}$ and the trace is meant to act on both discrete and continuous indices.

Note that $E_{\rm vac}$ still contains "self-energy" contributions, due to the vacuum distortion produced by each mirror, even when the other is infinitely far apart. This piece (irrelevant to the force between mirrors) shall be subtracted, in order to obtain a finite Casimir energy, in the calculations below.

III. DERIVATIVE EXPANSION

We present here a derivation of the first two terms in a derivative expansion of the Casimir energy for the system defined in the previous section.

To that end, and for calculational purposes, it is convenient to consider first a simplified situation: we split ψ into two components,

$$\psi(\mathbf{x}_\parallel) = a + \eta(\mathbf{x}_\parallel), \tag{15}$$

where a (assumed to be greater than zero) is the spatial average of ψ, and therefore a constant, whereas η contains the varying piece of ψ. The simplified case amounts to expanding up to the second order in η. Since the derivatives of ψ equal the derivatives of η, to find the terms with up to two derivatives of ψ, it is sufficient to expand Γ up to the second order in η, keeping up to the second order term in an expansion in derivatives:

$$\Gamma(a, \eta) = \Gamma^{(0)}(a) + \Gamma^{(1)}(a, \eta) + \Gamma^{(2)}(a, \eta) + \cdots \quad (16)$$

where the index denotes the order in derivatives. Each term will be a certain coefficient times the spatial integral over $\mathbf{x}_\parallel$ of a local term, depending on a and derivatives of η.

So far this is a perturbative expansion in η and its derivatives. However, to the same order in derivatives, it is quite straightforward to include the corrections which are of the same order in derivatives but of arbitrary order in η. Indeed, to do this, in the terms obtained in (16), one just has to replace a by ψ and also η by ψ, before performing the spatial integrals. This procedure accounts for all the terms of higher order in η, and the same order in derivatives, that contribute to the respective order the derivative expansion. Formally, this procedure may be represented as follows:

$$\Gamma^{(l)}(\psi) = \Gamma^{(l)}(a, \eta)|_{a\to\psi, \eta\to\psi} \quad (17)$$

for each term in (16).

Let us calculate the different terms in the derivative expansion for Γ, following this procedure.

Expanding first the matrix $\mathbb{T}$ in powers of η

$$\mathbb{T} = \mathbb{T}^{(0)} + \mathbb{T}^{(1)} + \mathbb{T}^{(2)} + \cdots, \quad (18)$$

we obtain $\Gamma = \Gamma^{(0)} + \Gamma^{(1)} + \Gamma^{(2)} + \cdots$, where

$$\begin{aligned} \Gamma^{(0)} &= \tfrac{1}{2}\mathrm{Tr}\log\mathbb{T}^{(0)}, \\ \Gamma^{(1)} &= \tfrac{1}{2}\mathrm{Tr}\log[(\mathbb{T}^{(0)})^{-1}\mathbb{T}^{(1)}], \\ \Gamma^{(2)} &= \tfrac{1}{2}\mathrm{Tr}\log[(\mathbb{T}^{(0)})^{-1}\mathbb{T}^{(2)}] \\ &\quad - \tfrac{1}{4}\mathrm{Tr}\log[(\mathbb{T}^{(0)})^{-1}\mathbb{T}^{(1)}(\mathbb{T}^{(0)})^{-1}\mathbb{T}^{(1)}], \end{aligned} \quad (19)$$

where, in $\Gamma^{(l)}$, we need to keep up to l derivatives of η.

The zeroth-order term is thus simply obtained by replacing first ψ by a constant, a, and then subtracting the contribution corresponding to $a \to \infty$, to get rid of the divergent self-energies. This yields,

$$\Gamma^{(0)}(a) = \tfrac{1}{2}\mathrm{Tr}\log[1 - (T^{(0)}_{LL})^{-1}T^{(0)}_{LR}(T^{(0)}_{RR})^{-1}T^{(0)}_{RL}] \quad (20)$$

where the $T^{(0)}_{\alpha\beta}$ elements are identical to the ones one would have for the two flat parallel mirrors at a distance a apart. As mentioned above, we have then to replace a by ψ at the end of the calculation. After evaluating the trace, we obtain

$$\Gamma^{(0)} = \frac{T}{2}\int d^2\mathbf{x}_\parallel \int \frac{d^3k_\parallel}{(2\pi)^3}\log[1 - e^{-2k_\parallel a}]. \quad (21)$$

We then replace $a \to \psi$ to extract the zeroth-order Casimir energy,

$$\begin{aligned} E^{(0)}_{\rm vac} &= \frac{1}{2}\int d^2\mathbf{x}_\parallel \int \frac{d^3k_\parallel}{(2\pi)^3}\log[1 - e^{-2k_\parallel \psi(\mathbf{x}_\parallel)}] \\ &= -\frac{\pi^2}{1440}\int d^2\mathbf{x}_\parallel \frac{1}{\psi(\mathbf{x}_\parallel)^3}, \end{aligned} \quad (22)$$

which equals the PFA approximation to the vacuum energy.

The first order term in the derivative expansion Γ vanishes identically, while for the second order one we have two contributions:

$$\Gamma^{(2)} = \Gamma^{(2,1)} + \Gamma^{(2,2)} \quad (23)$$

where,

$$\Gamma^{(2,1)} = \tfrac{1}{2}\mathrm{Tr}\log[(\mathbb{T}^{(0)})^{-1}\mathbb{T}^{(2)}] \quad (24)$$

and

$$\Gamma^{(2,2)} = -\frac{1}{4}\mathrm{Tr}\log[(\mathbb{T}^{(0)})^{-1}\mathbb{T}^{(1)}(\mathbb{T}^{(0)})^{-1}\mathbb{T}^{(1)}], \quad (25)$$

where we have to keep up to two derivatives of η.

The form of those terms can be obtained in a quite straightforward fashion; indeed, we first note that, in Fourier space, and before expanding to second order in momentum (derivatives), they have the structure

$$\Gamma^{(2,j)} = \frac{T}{2}\int \frac{d^2k}{(2\pi)^2} f^{(2,j)}(\mathbf{k})|\tilde{\eta}(\mathbf{k})|^2 \quad (26)$$

($j = 1, 2$), where $\mathbf{k} = (k_1, k_2)$, $\tilde{\eta}$ is the Fourier transform of η, and the $f^{(2,j)}$ kernels are the $k_0 \to 0$ (i.e., static) limits of

$$\begin{aligned} f^{(2,1)}(k) &= -\int \frac{d^3p}{(2\pi)^3}\frac{|p||p+k|}{1 - e^{-2|p+k|a}}, \\ f^{(2,2)}(k) &= -\int \frac{d^3p}{(2\pi)^3}\frac{|p||p+k|e^{-2|p+k|a}(1 + e^{-2|p|a})}{(1 - e^{-2|p|a})(1 - e^{-2|p+k|a})}. \end{aligned}$$

Besides, we need to subtract an a-independent self-energy contribution, obtained by taking $a \to \infty$ in the expressions above. Putting together the two terms above, and subtracting the $a \to \infty$ limit, the total contribution to $\Gamma^{(2)}$ adopts the form

$$\Gamma^{(2)} = \frac{T}{2}\int \frac{d^2k}{(2\pi)^2} f^{(2)}(\mathbf{k})|\tilde{\eta}(\mathbf{k})|^2 \quad (27)$$

with

$$f^{(2)}(k) = -2\int \frac{d^3p}{(2\pi)^3}\frac{|p||p+k|}{(1 - e^{-2|p|a})(e^{2|p+k|a} - 1)}, \quad (28)$$

where we just need to extract its $\mathbf{k}^2$ term in a Taylor expansion at zero momentum. Namely, $f^{(2)}(\mathbf{k}) \simeq \chi\mathbf{k}^2$, where

$$\chi = \frac{1}{2}\left[\frac{\partial^2 f^{(2)}(k)}{\partial k^2}\right]_{k\to 0} = -\int \frac{d^3p}{(2\pi)^3}\frac{|p|}{(1-e^{-2|p|a})}\lim_{k\to 0}\frac{\partial^2}{\partial k^2}\left[\frac{|p+k|}{(e^{2|p+k|a}-1)}\right].$$

The resulting integral may be exactly calculated,

$$\chi = -\frac{\pi^2}{1080a^3}. \tag{29}$$

Thus,

$$\Gamma^{(2)}(a,\eta) = -\frac{T}{2}\frac{\pi^2}{1080}\int\frac{d^2k}{(2\pi)^2}\frac{\mathbf{k}^2}{a^3}|\tilde{\eta}(\mathbf{k})|^2 = -\frac{T}{2}\frac{\pi^2}{1080}\int d^2\mathbf{x}_\parallel \frac{1}{a^3}(\partial_\alpha\eta)^2, \tag{30}$$

where, to obtain the second order contribution in derivatives to the vacuum energy, we need to replace $a \to \psi$, $\eta \to \psi$, and cancel the T factor, obtaining

$$E^{(2)}_{\rm vac} = \frac{\Gamma^{(2)}(\psi)}{T} = -\frac{1}{2}\frac{\pi^2}{1080}\int d^2\mathbf{x}_\parallel \frac{(\partial_\alpha\psi)^2}{\psi^3}, \tag{31}$$

where the index α runs from 1 to 2.

Putting together the terms up to second order, the expression for the energy becomes

$$E_{\rm DE} \equiv E^{(0)}_{\rm vac} + E^{(2)}_{\rm vac} = -\frac{\pi^2}{1440}\int d^2\mathbf{x}_\parallel \frac{1}{\psi^3}\left[1 + \frac{2}{3}(\partial_\alpha\psi)^2\right]. \tag{32}$$

This is the main result of this paper. The first term is the PFA for the Casimir energy. The second term contains the first nontrivial correction to PFA for an arbitrary surface. We could have guessed the form of both terms in the final formula by using dimensional and symmetry arguments. The global factor could also be determined by considering the particular case of parallel plates. Therefore, the calculation presented above, besides confirming the general arguments, provides the relative weight between both terms, which turns out to be $2/3$, regardless of the form of the surface.

IV. EXAMPLES

We provide here some applications of the general formula for the Casimir interaction energy.

A. A corrugated surface in front of a plane

Let us first consider a corrugated surface in front of a plane. For simplicity we assume sinusoidal corrugations in the direction of x_1

$$\psi(x_1) = a + \epsilon\sin\left(\frac{2\pi x_1}{\lambda}\right), \tag{33}$$

where a is the mean distance to the flat surface, ϵ is the amplitude, and λ the wavelength of the corrugation. We assume a square plane of side L, which is much larger than any other length in the problem.

The derivative expansion for the Casimir energy is given by

$$E_{\rm DE} = -\frac{\pi^2}{1440}\left[\int d^2\mathbf{x}_\parallel \frac{1}{(a+\epsilon\sin\frac{2\pi x_1}{\lambda})^3} \times\left(1 + \frac{2}{3}\left(\frac{2\pi}{\lambda}\right)^2\epsilon^2\cos^2\frac{2\pi x_1}{\lambda}\right)\right]. \tag{34}$$

In this case, the derivative expansion is an expansion in powers of a/λ and ϵ/λ, i.e. λ is the largest relevant distance in the problem. In order to compare with previous results in the literature [4], we will further assume that $\epsilon \ll a$. In this limit we obtain

$$E_{\rm DE} \simeq -\frac{\pi^2L^2}{1440a^3}\left[1 + 3\left(\frac{\epsilon}{a}\right)^2 + \frac{4\pi^2}{3}\left(\frac{\epsilon}{\lambda}\right)^2\right]. \tag{35}$$

This expression coincides with the small a/λ expansion of the result obtained in Ref. [4]. Indeed, in that work the interaction energy was written as

$$\frac{E_{\rm vac}}{L^2} = -\frac{\pi^2}{1440a^3} - \frac{\epsilon^2}{a^5}G_{\rm TM}\left(\frac{a}{\lambda}\right), \tag{36}$$

where $G_{\rm TM}(x)$ can be written in terms of Polylogarithm functions [5]. One can readily compute the small argument expansion of $G_{\rm TM}$ to obtain

$$G_{\rm TM}(x) \simeq \frac{\pi^2}{480} + \frac{\pi^4x^2}{1080}. \tag{37}$$

After inserting this expansion into Eq. (36), the result coincides with the derivative expansion Eq. (35).

B. A sphere in front of a plane

We now consider a sphere of radius R at a distance a from a plane. The evaluation of the Casimir energy in the electromagnetic case for this configuration has been performed in Refs. [6,7], while the evaluation for scalar fields has been previously reported in Ref. [8]. See also [9,10] for asymptotic expansions in the scalar and electromagnetic cases near the proximity limit.

For this geometry, we expect the derivative expansion to be adequate in the limit $a \ll R$. It is worth noting that the surface of the sphere cannot be described by a single valued function $x_3 = \psi(x_1, x_2)$. Note that even if we consider an hemisphere, the derivatives of ψ will be divergent on the equator. For these reasons, the derivative expansion will not converge. In spite of this, we will see that it still gives quantitative adequate results even beyond the lowest order approximation.

In order to avoid these problems, we will consider only the region of the sphere that is closer to the plane. This is the usual approach when computing the Casimir energy using the PFA. The final result will not depend on the part of the sphere considered. Denoting by (ρ, φ) the polar coordinates in the (x_1, x_2) plane the function ψ reads

$$\psi(\rho) = a + R\left(1 - \sqrt{1 - \frac{\rho^2}{R^2}}\right). \tag{38}$$

This function describes a hemisphere when $0 \le \rho \le R$. As mentioned above, the derivative expansion will be well defined if we restrict the integrations to the region $0 \le \rho \le \rho_M < R$.

Inserting this expression for ψ into the derivative expansion for the Casimir energy, one can perform explicitly the integrations and obtain an analytic expression $E_{\rm DE}(\rho_M, a, R)$. We do not present this rather long expression here, but only the leading terms in an expansion in powers of a/R, which is given by

$$E_{\rm vac}^{(0)} \simeq -\frac{\pi^3}{1440}\frac{R}{a^2}\left[1 - \frac{a}{R}\right], \tag{39}$$

$$E_{\rm vac}^{(2)} \simeq -\frac{\pi^3}{1080a}, \tag{40}$$

and therefore

$$E_{\rm DE} \simeq -\frac{\pi^3}{1440}\frac{R}{a^2}\left(1 + \frac{1}{3}\frac{a}{R}\right). \tag{41}$$

It is noteworthy that, up to this order, the result does not depend on ρ_M. Moreover, the result is in agreement with the asymptotic expansion obtained from the exact formula for this configuration [9], and with the former numerical evaluation in [11].

It is interesting to remark that $E_{\rm vac}^{(0)}$ includes part of the next-to-leading-order corrections. It is correct to keep the second term in Eq. (39) only when the contribution coming from $E_{\rm vac}^{(2)}$ is also taken into account.

C. A cylinder in front of a plane

Let us now consider a cylinder of radius R and length $L \gg R$ at a distance a from a plane. The Casimir energy for this configuration was first evaluated in the PFA in Ref. [12]. The exact result was first derived in Ref. [13]. The caveats mentioned in the above subsection also apply for this geometry. We will consider the function ψ given by

$$\psi(x_1) = a + R\left(1 - \sqrt{1 - \frac{x_1^2}{R^2}}\right), \tag{42}$$

with $-x_M < x_1 < x_M < R$ in order to cover the part of the cylinder that is closer to the plane. The calculation is similar to the previous case, and the final result is

$$E_{\rm DE} \simeq -\frac{\pi^3 L}{1920\sqrt{2}}\frac{R^{1/2}}{a^{5/2}}\left(1 + \frac{7}{36}\frac{a}{R}\right). \tag{43}$$

Once more, up to this order, the result does not depend on x_M. Moreover, it is in agreement with the asymptotic expansion obtained from the exact formula for the cylinder-plane geometry and numerical findings [14–17].

D. A parabolic cylinder in front of a plane

We compute here the Casimir interaction energy between a parabolic cylinder of length L in front of a plane. The surface is defined by the function

$$\psi(x_1) = a + \frac{x_1^2}{2R}, \tag{44}$$

with $-x_M < x_1 < x_M < R$. Once more, we only consider the portion of the curved surface that is closer to the plane [note that the functions defining the cylinder Eq. (42) and the parabolic cylinder Eq. (44) coincide up to first order in x_1/R]. The integrations needed to compute the derivative expansion of the Casimir energy are very simple. Expanding the result in powers of a/R we obtain

$$E_{\rm DE} \simeq -\frac{\pi^3 L}{1920\sqrt{2}}\frac{R^{1/2}}{a^{5/2}}\left(1 + \frac{4}{9}\frac{a}{R}\right). \tag{45}$$

The final answer is independent of x_M and the leading order coincides with that of the cylinder in front of a plane.

E. A paraboloid in front of a plane

As a final example we consider a paraboloid, defined by

$$\psi(\rho) = a + \frac{\rho^2}{2R}, \tag{46}$$

with $0 < \rho < \rho_M < R$, in front of a plane.

The approximation for the vacuum energy reads

$$E_{\rm DE} \simeq -\frac{\pi^3}{1440}\frac{R}{a^2}\left(1 + \frac{4}{3}\frac{a}{R}\right). \tag{47}$$

As in all the previous examples, the result does not depend on the region of integration defined by ρ_M. Moreover, the leading order is equal to that of the sphere in front of a plane, as expected from the fact that the functions describing both surfaces Eqs. (38) and (46) coincide in the region closer to the plane.

V. CONCLUSIONS

We have shown that the PFA can be thought of as akin to a derivative expansion of the Casimir energy with respect to the shape of the surfaces. Our main result, given in Eq. (32), shows that the lowest order (the "effective potential") reproduces the PFA. Moreover, when the first nontrivial correction containing two derivatives of ψ is also included, the general formula gives the next-to-leading-order correction to PFA for a general surface.

Several remarks are in order here. To begin with, at least for the surfaces considered in this paper, the PFA becomes a well-defined and controlled approximation scheme: the leading corrections are small when $|\partial_\alpha \psi| \ll 1$ or, in other words, when the curved surface is almost parallel to the plane. Higher order corrections will be negligible when, in addition to this condition, the scale of variation of the shape of the surface is much larger than the local distance between surfaces. It is also clear that the corrections to PFA only contain local information about the geometry of the surface, and do not include correlations between different points of the surface.

Although we applied our general result to the case of a cylinder and a sphere in front of a plane, these geometries present additional complications, because they cannot be described by a single function ψ. Moreover, the derivatives of ψ diverge when the surface becomes perpendicular to the plane, and therefore it is clear that the derivative expansion will not converge. In spite of this, it is remarkable that Eq. (32) describes the interaction energy for these configurations including the first nontrivial correction to PFA. Strictly speaking, for these geometries we are computing the interaction energy between a plane and a large curved surface which, in the region closest to the plane, has a cylindrical or spherical shape.

We expect the main idea presented in this paper to be generalizable in several directions, as for instance for a scalar field satisfying Neumann or Robin boundary conditions, and also to the electromagnetic field satisfying perfect conductor boundary conditions on the surfaces. In all these cases, we expect the derivative expansion to be of the form

$$E_{\rm DE} = -\frac{\pi^2}{1440}\int d^2\mathbf{x}_\parallel \frac{1}{\psi^3}[\beta_1 + \beta_2(\partial_\alpha \psi)^2], \tag{48}$$

where the constants β_i will depend on the kind of fields and boundary conditions considered.

Other interesting generalizations would be to consider two curved surfaces, and the case of imperfect boundary conditions. Moreover, as the applications of the PFA are not restricted to the Casimir energy, the derivative expansion could also be useful to compute gravitational [18], electrostatic [19], or even nuclear forces [20].

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