Padé-Borel approximation of the continuum limit of strong coupling lattice fields: Two-dimensional nonlinear $O(N)$ sigma model at $N \geq 3$

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Based on the strong coupling expansion, we reinvestigate the two-dimensional $O(N)$ sigma model by the use of Padé-Borel approximants. The conventional strong coupling expansion of the mass square M in momentum space in $\beta = 1/g^2$ is inverted to give β expanded in $1/M$. Borel transform of β with respect to M is carried out and the result is improved as the rational function by the Padé method. We find the behavior of Padé-Borel transformed bare coupling at 18th order is consistent for $N \geq 3$ with that of continuum scaling to the four-loop perturbation theory. We estimate the nonperturbative mass gap at $N \geq 3$ and find agreement with the exact result by Hasenfratz *et. al.*

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I. INTRODUCTION

Nearly four decades ago, the quark confinement was shown by Wilson at the strong bare coupling region [1]. For weak coupling, perturbation theory clarified for the Yang-Mills system that bare coupling g tends to vanish as the lattice spacing $a \to 0$ [2]. The motivation of the present work is to attempt to extrapolate the large a behavior of bare coupling to the asymptotically free behavior at weak coupling. For the purpose, we like to reformulate the strong coupling expansion by changing the primary variable from bare coupling to the lattice spacing itself.

Lattice serves us a suitable regularization, since in lattice field theories the lattice spacing a explicitly appears in the action and enters into the physical quantities. For instance, the dimensionless correlation length ξ represents a physical length scale divided by a. It is given at strong coupling as a series $\mathcal{R}$ in $(g^2)^{-1}$ and it determines, in an implicit manner, the a dependence of the bare coupling. Mutual roles of g and ξ are exchanged by inverting the relation $\mathcal{R}$. Thus we address the question whether the small ξ series of g allows us to confirm directly the weak coupling behavior predicted by perturbation theory.

In the present paper, in a nonlinear $O(N)$ model at two dimensions, we make an attempt to approximate the asymptotic behavior of bare coupling in the continuum limit via its large a expansion. The model is of interest as a testing ground of our approach, since it enjoys asymptotic freedom and dynamical mass generation for $N \geq 3$ [3]. In addition to the large N limit, we also consider the case of finite N.

As the basic variable, rather than the correlation length in lattice space, we adopt mass M in momentum space defined by the zero momentum limit of the two-point field correlation. The lattice spacing a is included in the mass which is rescaled to be dimensionless and then M must vanish in the continuum limit $a \to 0$ [see Eq. (2.7)]. Now the strong coupling expansion gives a series of M in $\beta = (g^2)^{-1}$. By inverting the series, we express β as a power series in $1/M$, which is equivalent to large a expansion. As it would be, the nave series fails to confirm the continuum behavior of β. However, it is nontrivial and interesting to examine, when both Padé and Borel techniques are applied on the series, whether the continuum scaling emerges at finite N or not.

Before proceeding to following sections, we remark on the role of the Borel transform in our approach. We use the Borel transform as a device of dilation operation around the continuum limit. The response of scale transformation on $f(M)$ is probed by rescaling M to λM in f and taking the $\lambda \to 0$ limit. Then, it is said f scales with the exponent Δ if

$$f(\lambda M) \to \lambda^{-\Delta} f(M), \qquad \lambda \to 0. \tag{1.1}$$

The above criterion of scaling is implemented by introducing δ defined by

$$\lambda = 1 - \delta, \qquad 0 \leq \delta \leq 1 \tag{1.2}$$

and performing expansion to some finite orders in δ [4,5]. Suppose the function approaches $M^{-\Delta}$. Then, expanding it to δ^L and setting $\delta = 1$, one has $(M(1-\delta))^{-\Delta} \to \frac{L!}{\Gamma(1+\Delta)\Gamma(L-\Delta+1)} M^{-\Delta}$. Further, if we take the limit $M \to \infty$, $L \to \infty$ with $M/L = \bar{M}$ fixed, we obtain

$$M^{-\Delta} \to \frac{1}{\Gamma(1+\Delta)} \left(\frac{M}{L}\right)^{-\Delta} = \frac{1}{\Gamma(1+\Delta)} \bar{M}^{-\Delta}. \tag{1.3}$$

That is, the limit $\lambda \to 0$ has transitioned to the limit $\delta \to 1$ with the cut off L. Then the scaling behavior with exponent Δ manifests itself in the power of $\bar{M}(= M/L)$. Note that the universal quantity Δ is left unchanged. On the other hand, when same operation is acted on f in the series form $\sum a_k M^{-k}$ valid at large M, we have $\bar{f} = \sum (a_k/k!) \bar{M}^{-k}$ with a larger convergence radius, which is just the Borel transform of the original series. We thus interpret the Borel transform as a realization of scale transformation. We do

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not need integrating $\bar{f}$ back to f. Though the information of $f(M)$ over the whole range of M is not obtained, what we need in lattice field theories is the behavior of $f(M)$ in the neighborhood of $M = 0$.

II. DESCRIPTION OF THE MODEL

On the two-dimensional square lattice, the continuous spin fields $\vec{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_N)$ are set on every site. The action of the system is given by

$$S = -\beta \sum_{\mathbf{n}} \sum_{\mu=1,2} \vec{\sigma}_{\mathbf{n}} \cdot \vec{\sigma}_{\mathbf{n}+\mathbf{e}_\mu}, \tag{2.1}$$

where $\mathbf{e}_1 = (1, 0)$, $\mathbf{e}_2 = (0, 1)$, and

$$\beta = \frac{1}{g^2}. \tag{2.2}$$

The fields are constrained to satisfy at every site, $\vec{\sigma}^2 = N$.

The mass variable M defined via the zero momentum limit of the propagator $(\sum_{\mathbf{n}} \exp(i\mathbf{p}\cdot\mathbf{n})\langle\vec{\sigma}(\mathbf{0})\cdot\vec{\sigma}(\mathbf{n})\rangle)^{-1}$ is given by

$$M = \frac{2D\chi}{\mu}, \tag{2.3}$$

where susceptibility χ and second moment μ are, respectively, given by $\chi = \frac{1}{N}\sum_{\mathbf{n}}\langle\vec{\sigma}(\mathbf{0})\cdot\vec{\sigma}(\mathbf{n})\rangle$ and $\mu = \frac{1}{N}\sum_{\mathbf{n}}\mathbf{n}^2\langle\vec{\sigma}(\mathbf{0})\cdot\vec{\sigma}(\mathbf{n})\rangle$. D denotes the dimension of lattice space and $D = 2$ in the present work. Let us summarize the continuum limit of the model and large a expansion of β.

The perturbative renormalization group predicts that, for $N \geq 3$, the correlation length behaves at weak coupling as

$$\xi = C_\xi \exp\left[\frac{2\pi N\beta}{N-2}\right]\left(\frac{2\pi N\beta}{N-2}\right)^{-1/(N-2)}\left(1 + \sum_{k=1}^{\infty}\frac{a_k}{\beta^k}\right), \tag{2.4}$$

where the multiplied constant C_ξ is specified only nonperturbatively. Hasenfratz *et. al.* has computed it via a thermodynamic Bethe ansatz [6], giving

$$C_\xi = 32^{-1/2}\left(\frac{e^{1-\pi/2}}{8}\right)^{1/(N-2)}\Gamma\left(1 + \frac{1}{N-2}\right). \tag{2.5}$$

The terms $a_k\beta^{-k}$ ($k = 1, 2, 3, \ldots$) in (2.4) are contributions of $k+2$-loop levels and three- [7] and four-loop [8] results were computed in the literature. They are given as

$$\begin{aligned} a_1 &= \frac{1}{N(N-2)}(-0.0490 - 0.0141N), \\ a_2 &= \frac{1}{N^2(N-2)^2} \times (0.0444 + 0.0216N \\ &\quad + 0.0045N^2 - 0.0129N^3). \end{aligned} \tag{2.6}$$

Though three and higher loop contributions disappear in the continuum limit for the bare coupling, we cannot take out the limit because only the series to finite order is at hand. Hence, we include known three- and four-loop contributions in our analysis.

It is known that $M^{-1/2}$ has functional form of β, the same as Eq. (2.4) but with another multiplicative constant, say C_M. However, Monte Carlo data [9] showed that the difference is less than a percent at $N = 4$. Since the two constants agree with each other in the large N limit, the difference between C_ξ and C_M may actually be negligible for all $N \geq 3$. Thus the estimation of the mass gap via strong coupling expansion becomes the estimation of C_ξ and this is one of the aims of our work.

Since the mass M approaches ξ^{-2} in the continuum limit, the physical mass of dimension 2 is given by

$$m_{\text{phys}}^2 = \lim_{a\to 0} M a^{-2} = C_\xi^{-2}\Lambda_L^2, \tag{2.7}$$

where Λ_L is the finite mass scale given by

$$\Lambda_L = a^{-2}\exp\left[\frac{2\pi N\beta}{N-2}\right]\left(\frac{2\pi N\beta}{N-2}\right)^{-1/(N-2)}\left(1 + \sum_{k=1}^{\infty}\frac{a_k}{\beta^k}\right). \tag{2.8}$$

From (2.4) and $\xi \sim M^{-1/2}$, we have continuum β to four-loop order as a function of M,

$$\begin{aligned} \beta \sim{}& \frac{N-2}{4\pi N}\log\frac{x}{C_\xi^2} + \frac{1}{2\pi N}\log\left(\frac{1}{2}\log\frac{x}{C_\xi^2}\right) \\ &+ \frac{-2\pi N(N-2)a_1 + \log(\frac{1}{2}\log\frac{x}{C_\xi^2})}{\pi N(N-2)\log\frac{x}{C_\xi^2}} \\ &+ \frac{1}{\pi N(N-2)^2(\log\frac{x}{C_\xi^2})^2}\Big[4\pi N(N-2)(-a_1 + N\pi a_1^2 \\ &- 2\pi N a_2) + 2(1 + 2\pi N(N-2)a_1)\log\left(\frac{1}{2}\log\frac{x}{C_\xi^2}\right) \\ &- \log\left(\frac{1}{2}\log\frac{x}{C_\xi^2}\right)^2\Big], \end{aligned} \tag{2.9}$$

where

$$x = \frac{1}{M}. \tag{2.10}$$

On the series expansion at large M, we borrow the result in the work of Butera and Comi [10] who computed a strong coupling series of χ and μ to β^{21}. Using the result, we have expansion of M in powers of β,

$$\begin{aligned} M ={}& \frac{1}{\beta} - 4 + \frac{2(3+2N)}{2+N}\beta \\ &+ \frac{2(16 + 32N + 17N^2 + 2N^3)}{(2+N)^2(4+N)}\beta^3 \\ &- \frac{16(-1+N)}{(2+N)^2}\beta^4 + O(\beta^5). \end{aligned} \tag{2.11}$$

By inverting the above relation, we have

$$\beta = x - 4x^2 + \frac{2(10N + 19)x^3}{N + 2} - \frac{8(14N + 25)x^4}{N + 2} + \frac{2(338N^3 + 2593N^2 + 6084N + 4512)x^5}{(N + 2)^2(N + 4)} + \cdots. \tag{2.12}$$

Based on the series (2.12), we discuss the approximation of the continuum limit by the use of the Padé-Borel approximation scheme. We attempt to recover the asymptotically free behavior (2.9) from (2.12) and then estimate C_ξ.

III. LARGE N LIMIT

The large N limit serves as a good benchmark of our approach. So we consider that case first and then turn to finite N in the next section.

In the large N limit, only the one-loop contribution to β survives to give

$$\beta \sim \frac{1}{4\pi}\log(x/C_\xi^2), \quad C_\xi = (32)^{-1/2} = 0.176\,776\,69\ldots. \tag{3.1}$$

As briefly presented in the introduction, the Borel transform is given by a certain limit of delta expansion [5]. Explicitly, the logarithm is expanded and gives at $\delta = 1$ that $\log(x/(1-\delta)) \to \log x + \sum_{l=1}^{L}\frac{1}{l}$ to the order L. Then using the asymptotic expansion $\sum_{l=1}^{L} 1/l = \log L + \gamma_E + 0(L^{-1})$ (γ_E denotes Euler's constant), we have $\log x \to \log(xL) + \gamma_E$ in the $L \to \infty$ limit. Let x be small enough with $\bar{x} = xL$ kept finite, then the result represents the Borel transform of $\log x$. Denoting the operation of the Borel transform by $\mathcal{B}$ we thus find $\mathcal{B}[\log x] = \log\bar{x} + \gamma_E$. Using an abbreviated symbol $\bar{\beta} = \mathcal{B}[\beta]$, we then obtain

$$\bar{\beta} \sim \frac{1}{4\pi}(\log(\bar{x}/C_\xi^2) + \gamma_E) = \bar{\beta}_{\rm cont}. \tag{3.2}$$

The large M expansion of β reads

$$\beta = x - 4x^2 + 20x^3 - 112x^4 + 676x^5 - 4304x^6 + \cdots. \tag{3.3}$$

The Borel transform of the above series results to divide the nth order coefficient by the factorial of n,

$$\bar{\beta} = \bar{x} - \frac{4}{2!}\bar{x}^2 + \frac{20}{3!}\bar{x}^3 - \frac{112}{4!}\bar{x}^4 + \frac{676}{5!}\bar{x}^5 - \frac{4304}{6!}\bar{x}^6 + \cdots. \tag{3.4}$$

Then as a crucial step, we use the Padé method to extrapolate the above series to larger $\bar{x}$. The resultant Padé-Borel approximants enable us to capture the scaling behavior to be seen in the scaling region as we can see below.

As a preliminary study, we have examined the behaviors of $[m/n]$ approximants of $\bar{\beta}$ over almost all possible pairs of m, n at orders $m + n = 4, 5, \ldots, 20$. On the contrary to the condensed matter models undergoing second-order phase transition, critical behavior of the present model is known from perturbation theory as logarithmic and slowly varying. Hence it is conceivable that good behaviors come from the cases where the difference between m and n is small. The numerical experiment confirmed this is indeed the case. We have also compared the approximants of three types, Padé-Borel, Borel-only, and Padé-only improvements. The result at sixth order is shown in Fig. 1. As already reported in [5], Borel-only improvement is not sufficient for observing the asymptotic freedom. The Padé-only case ($\beta_{[3/3]}$ in Fig. 1) is also insufficient as is clear from Fig. 1. However, the Padé-Borel approximant shows enough improvement for quantitative approximation. Though the Padé-only approximation is found to be improved at higher orders, the best performance is achieved by the Padé-Borel approximant at every order we analyzed. We therefore focus on the Padé-Borel approximant hereafter.

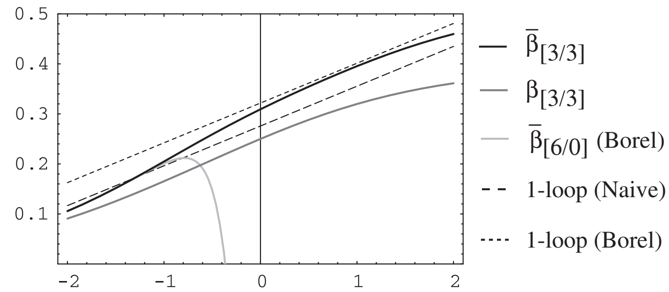

FIG. 1. Plot of improved β and $\bar{\beta}$ at sixth order. Two dashed lines (one for β and the other for $\bar{\beta}$) represent behaviors at continuum. The horizontal axis corresponds to $\log\bar{x} = \log(1/\bar{M})$ and $\log x = \log(1/M)$ (for the Padé-only case).

Now, let us turn to the evaluation of the mass gap by estimating C_ξ. Since we know information at weak coupling, the estimation is carried out by fitting $\bar{\beta}_{\rm cont}$ to $[m/n]$ order approximants of $\bar{\beta}$, $\bar{\beta}_{[m/n]}$, by adjusting the value of C_ξ. In practice, we consider the difference between $\bar{\beta}_{[m/n]}$ and $\bar{\beta}_{\rm cont}$ and plot the difference by changing the value of C_ξ. At just proper value of C_ξ, the two functions touch each other at a point $\bar{x}_0$ and the difference is tiny over an interval including $\bar{x}_0$. A typical case is shown in Fig. 2 and the result of estimation of C_ξ is shown in Table I. For the reason previously written, we list only the results around the diagonal Padé. Though the reason is not known to us, the orders 6, 10, 14, and 18 give the best approximation among nearby orders.

IV. FINITE N DOWN TO $N = 3$

In this section we study the weak coupling behavior from the Padé-Borel approximants for a finite number of spin components. First we discuss the Borel transform of (2.9) to compare it with Padé-Borel approximants of large the M series (2.12).

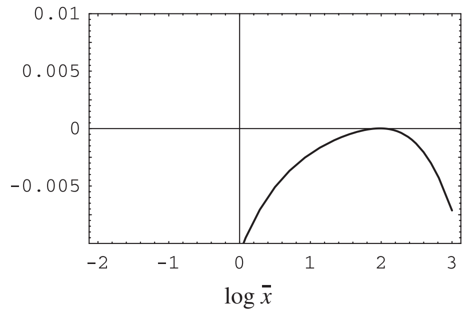

FIG. 2. Subtracted function $\bar{\beta}_{[9/9]} - \bar{\beta}_{\rm cont} = \bar{\beta}_{[9/9]} - \{\frac{1}{4\pi} \times (\log\bar{x}/C_\xi^2 + \gamma_E)\}$. Plotted curve is for $C_\xi = 0.178\,68$.

TABLE I. Evaluation result denoted as $C_{\rm app}$ of the nonperturbative constant C_ξ in $\frac{1}{4\pi}(\log(\bar{x}/C_\xi^2) + \gamma_E)$. Rigorous value of C_ξ is $(32)^{-1/2} = 0.176\,776\,69\ldots$.

$[m/n]$	$C_{\rm app}$	$[m/n]$	$C_{\rm app}$
[3/3]	0.183 27	[7/7]	0.179 11
[4/3]	0.187 34	[8/7]	0.180 41
[3/4]	0.187 22	[7/8]	0.180 38
[4/4]	0.184 63	[8/8]	0.179 72
[5/5]	0.181 38	[9/9]	0.178 68
[6/5]	0.182 66	[10/9]	0.179 00
[5/6]	0.182 64	[9/10]	0.179 01
[6/6]	0.181 78	[10/10]	0.178 75

Let us consider the Borel transform of the two-loop contribution. We find

$$\mathcal{B}[\log\log x] = \log\log\bar{x} + \frac{\gamma_E}{\log\bar{x}} + \frac{\zeta(2) - \gamma_E^2}{2(\log\bar{x})^2} + O((\log\bar{x})^{-3}). \tag{4.1}$$

The result is simplified by absorbing γ_E into the log. Then we obtain

$$\mathcal{B}[\log\log x] \sim \log(\log\bar{x} + \gamma_E) + \frac{\zeta(2)}{2(\log\bar{x} + \gamma_E)^2} + O((\log\bar{x})^{-3}). \tag{4.2}$$

Note that the second term should be included when the four-loop contribution is taken into account. At two- and three-loop orders, we need only the first term. In a similar manner, for contributions at three- and four-loop orders, we find as follows:

$$\mathcal{B}\left[\frac{1}{\log x}\right] = \frac{1}{\log\bar{x} + \gamma_E} + O((\log\bar{x})^{-3}), \tag{4.3}$$

$$\mathcal{B}\left[\frac{\log\log x}{\log x}\right] = \frac{\log(\log\bar{x} + \gamma_E)}{\log\bar{x} + \gamma_E} + O((\log\bar{x})^{-3}), \tag{4.4}$$

$$\mathcal{B}\left[\frac{1}{(\log x)^2}\right] = \frac{1}{(\log\bar{x} + \gamma_E)^2} + O((\log\bar{x})^{-3}), \tag{4.5}$$

$$\mathcal{B}\left[\frac{\log\log x}{(\log x)^2}\right] = \frac{\log(\log\bar{x} + \gamma_E)}{(\log\bar{x} + \gamma_E)^2} + O((\log\bar{x})^{-3}), \tag{4.6}$$

$$\mathcal{B}\left[\frac{(\log\log x)^2}{(\log x)^2}\right] = \frac{(\log(\log\bar{x} + \gamma_E))^2}{(\log\bar{x} + \gamma_E)^2} + O((\log\bar{x})^{-3}). \tag{4.7}$$

Thus the result of the Borel transform to the four-loop level reads

$$\begin{aligned}\bar{\beta} = {}& \frac{N-2}{4\pi N}\left(\log\frac{\bar{x}}{C_\xi^2} + \gamma_E\right) + \frac{1}{2\pi N}\log\left[\frac{1}{2}\left(\log\frac{\bar{x}}{C_\xi^2} + \gamma_E\right)\right] + \frac{-2\pi N(N-2)a_1 + \log[\frac{1}{2}(\log\frac{\bar{x}}{C_\xi^2} + \gamma_E)]}{\pi N(N-2)(\log\frac{\bar{x}}{C_\xi^2} + \gamma_E)} \\ & + \frac{1}{\pi N(N-2)^2(\log\frac{\bar{x}}{C_\xi^2} + \gamma_E)^2}\left[4\pi N(N-2)(-a_1 + N\pi a_1^2 - 2\pi N a_2) + 2(1 + 2\pi N(N-2)a_1)\log\left[\frac{1}{2}\left(\log\frac{\bar{x}}{C_\xi^2} + \gamma_E\right)\right]\right. \\ & \left. - \log\left[\frac{1}{2}\left(\log\frac{\bar{x}}{C_\xi^2} + \gamma_E\right)\right]^2\right] + \frac{\zeta(2)}{4\pi N(\log\frac{\bar{x}}{C_\xi^2} + \gamma_E)^2} \\ = {}& \bar{\beta}_{\rm cont}.\end{aligned} \tag{4.8}$$

At large $\bar{M}$, we have from (2.12),

$$\bar{\beta} = \bar{x} - \frac{4}{2!}\bar{x}^2 + \frac{2(10N+19)\bar{x}^3}{3!(N+2)} - \frac{8(14N+25)\bar{x}^4}{4!(N+2)} + \cdots. \tag{4.9}$$

As in the previous section, we further improve the large $\bar{M}$ series by the Padé method. We have checked that also at finite N, the diagonal Padé method provides the best behaviors. Skipping low-order results, we explicitly present only the results at 18th order for various N. Figure 3 shows the plots of $\bar{\beta}_{[9/9]}$ and $\bar{\beta}_{\rm cont}$ at one- and two-loop levels (at $N = 3, 4, 5$, $\bar{\beta}_{\rm cont}$ at three- and four-loop levels are also plotted) as functions of $\log\bar{x}$. At $N \geq 6$ three- and four-loop $\bar{\beta}_{\rm cont}$ are very close to that at two-loop at $\bar{x} > 0$ and we

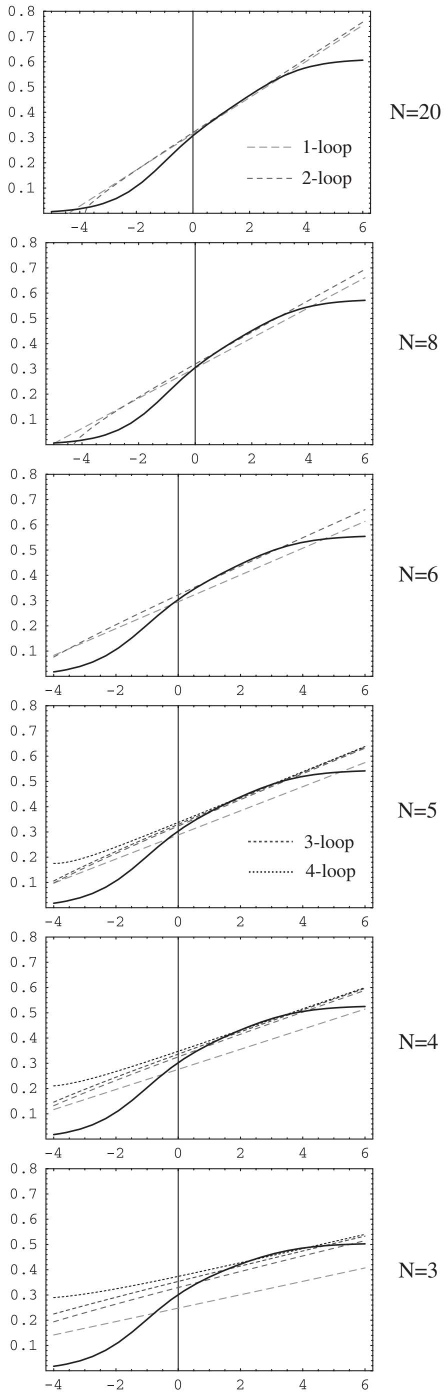

FIG. 3. Plots of $\bar{\beta}_{[9/9]}$ at $N = 20, 8, 6, 5, 4, 3$, and $\bar{\beta}_{\rm cont}$ at one- and two-loop results (plus three- and four-loop results for $N = 3$, 4, and 5) as functions of $\log\bar{x}$.

have omitted them. At $N = 3, 4$, though the scaling to the four-loop level is not so clear, the behavior of $\bar{\beta}_{[9/9]}$ is roughly consistent with the continuum one for $\log\bar{x} \in [2, 4]$. At $N = 5$, linearlike behavior with correct slope is observed around $\log\bar{x} \sim 2$, which signals scaling behavior. From $N \sim 8$, we observe continuum scaling at the two-loop level.

TABLE II. Result of estimation of the constant C_ξ (implied by $C_{\rm app}$). The last column shows the result of Botera and Comi [10].

N	$C_{\rm app}^{2\text{-loop}}$	$C_{\rm app}^{3\text{-loop}}$	$C_{\rm app}^{4\text{-loop}}$	C_ξ	$C_{\rm BC}$
3	0.0068	0.0094	0.0112	0.0125	
4	0.0336	0.0373	0.0398	0.0416	0.039
5	0.0584	0.0615	0.0639	0.0652	0.065
6	0.0771	0.0797	0.0818	0.0826	0.084
7	0.0913	0.0934	0.0953	0.0955	
8	0.1021	0.1038	0.1055	0.1054	0.106
9	0.1106	0.1121	0.1136	0.1132	
10	0.1175	0.1187	0.1201	0.1195	0.121
11	0.1231	0.1242	0.1255	0.1247	
12	0.1278	0.1288	0.1299	0.1290	0.130
13	0.1318	0.1326	0.1337	0.1327	
14	0.1352	0.1359	0.1369	0.1358	0.137
15	0.1381	0.1388	0.1397	0.1386	

Now, having examined continuum scaling, we evaluate constant C_ξ as in the same manner at $N = \infty$. Namely, we consider $\bar{\beta}_{[9/9]} - \bar{\beta}_{\rm cont}$ and search for the value of C_ξ by fitting $\bar{\beta}_{\rm cont}$ to $\bar{\beta}_{[9/9]}$ by changing values of C_ξ. The result is summarized in Table II and Fig. 4. We may say that, for all $N \geq 3$, especially for $N = 3$ and 4, $\bar{\beta}_{[9/9]}$ yields a good four-loop estimation of the nonperturbative constant C_ξ. However we see that, at $N \geq 8$, the estimated value $C_{\rm app}$ is slightly larger than the exact one. Note that an excess of the estimation is observed also in the large N limit. On the contrary, for $N \leq 7$, $C_{\rm app}$ is slightly smaller than C_ξ. We would like to discuss the issue in the next section. To summarize, we conclude that the approximation level is satisfactory.

V. DISCUSSION

From previous two sections, at larger N, we found that the four-loop estimation of C_ξ gives excess to the exact value. For example, at $[9/9]$ approximants, the excess reads ~ 0.0001, ~ 0.0011, and ~ 0.0019 at $N = 8$, 15, and ∞, respectively. As long as $\bar{x}$ is not so small, every multiple-loop (above the one-loop level) contribution decreases as N becomes large and vanishes at $N = \infty$. Hence,

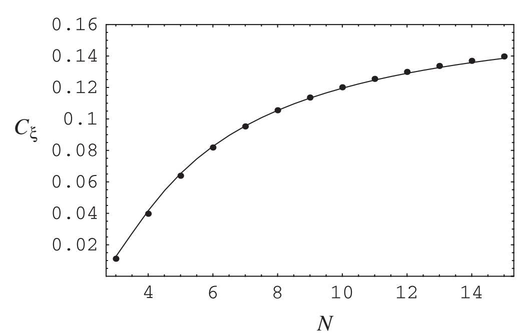

FIG. 4. Plot of true value of C_ξ (solid curve) and its estimation at $N = 3, 4, 5, 6, \ldots, 15$ (black points) carried out via Padé-Borel approximants $\bar{\beta}_{[9/9]}$ compared with four-loop $\bar{\beta}_{\rm cont}$.

at large enough N, the five, sixth, $\cdots$-loop contributions may be safely neglected in our study. Then the main factor of the discrepancy would come from the lattice artifact. To find evidence, let us discuss the large N limit because that case provides us a quantitative example as we can see below.

At $N = \infty$, β behaves for small M as

$$\beta = -\frac{1}{4\pi}\log\frac{M}{32} + \frac{M}{32\pi}\left(\log\frac{M}{32} + 1\right) + O(M^2 \log M). \tag{5.1}$$

Note that this is just the one-loop result. The second and higher order terms represent lattice artifacts which disappear in the continuum limit. They involve the logarithm and delay the approach of β to the continuum limit. In fact, use of the Borel transform has the notable advantage that it reduces the correction to

$$\bar{\beta} = -\frac{1}{4\pi}\left(\log\frac{\bar{M}}{32} - \gamma_E\right) - \frac{\bar{M}}{32\pi} + O(\bar{M}^2). \tag{5.2}$$

Here we have used

$$\mathcal{B}[M \log M] = -\bar{M}, \tag{5.3}$$

$$\mathcal{B}[M] = 0. \tag{5.4}$$

For original β [see Eq. (5.1)], the second term is of order $M \log M$, but for $\bar{\beta}$, $\bar{M}$ and the deviation from the asymptotic scaling is much reduced when $\bar{M}$ is small enough. The correction, however, still affects the small $\bar{M}$ behavior of the transformed bare coupling. We have examined scaling and evaluated C_ξ by keeping the first-order correction $-\frac{\bar{M}}{32\pi}$. From Table III, it is apparent that incorporation of the $O(M \log M) = O(a^2 \log a^2)$ term improves the approximation. At [9/9] order the excess is only ~ 0.0001. This means that Padé-Borel approximants actually recover the small M behavior very well but, at the same time, the residual effect of the correction is still non-negligible for higher accuracy. We thus find that the main factor of the discrepancy comes from the lattice artifact as long as N is large enough.

TABLE III. Result of estimation of the constant C_ξ (implied by $C_{\rm app}$) in the large N limit when the correction $-\frac{\bar{M}}{32\pi}$ to the asymptotic scaling is taken into account. Only the results of diagonal approximants are shown. Rigorous value of C_ξ is $(32)^{-1/2} = 0.176\,776\,69\ldots$.

$[m/n]$	$C_{\rm app}$
[2/2]	0.187 126
[3/3]	0.177 873
[4/4]	0.178 509
[5/5]	0.177 355
[6/6]	0.177 484
[7/7]	0.176 892
[8/8]	0.177 036
[9/9]	0.176 891
[10/10]	0.176 900

Next, consider the case of lower $N = 3 \sim 7$, where the estimated value of C_ξ is slightly lower than the exact one. As a typical case, consider the $N = 3$ case. Three- and four-loop effects contribute to $\bar{\beta}_{\rm cont} \sim 0.4694$ at $x = \bar{x}_0$ ($\log \bar{x}_0 \sim 3.2$) by amounts ~ 0.02 and ~ 0.01, respectively. Though they carry with small fractions of total $\bar{\beta}$, they are not negligible at all, since $C_{\rm app}$ increases by 0.0026 and 0.0018 when three- and four-loop effects are taken into account, and the magnitude of C_ξ itself is small. Therefore, loop contributions above four would be still active for estimating C_ξ and even have the possibility to push $C_{\rm app}$ to be larger than C_ξ. For small N, in addition to the lattice artifact, a discrepancy may come also from lack of higher loops.

On the lattice artifact, it is crucial to reduce the effect for obtaining a precise result for all $N \geq 3$. It has been reported in [11] that the standard action gives $a^2(\log a^2)^{N/(N-2)}$ as the leading lattice artifact near the continuum limit. It has the maximum value at $N = 3$ giving the contribution $\sim a^2(\log a^2)^3$. The Borel transform would reduce the effect of such a logarithmic term but the effect would remain to obscure the asymptotic scaling at finite $\bar{x}$.

In general, the leading lattice artifact may not be known completely. Then, one way to resolve the issue is to construct or to use the lattice action in which such artifacts are reduced from the outset. As an example, we report the result of Symanzik's modification of lattice action [12] in the large N limit. In the Symanzik improvement program, one generalizes the action element from $1 - \vec{\sigma}_{\mathbf{n}} \cdot \vec{\sigma}_{\mathbf{n}+\mathbf{e}_\mu}$ to $\sum_{k=0}^{K} A_k \vec{\sigma}_{\mathbf{n}} \cdot \vec{\sigma}_{\mathbf{n}+k\mathbf{e}_\mu}$. By expanding the action in a and minimizing the lattice artifact at the level of action, one can obtain the optimized set of coefficients A_k ($k = 0, 1, 2, 3, \ldots, K$). Then, the direct effect is the modification of the unperturbed propagator from $[2\sum_{\mu=1,2}(1 - \cos a p_\mu)]^{-1} = [\sum_\mu a^2 p_\mu^2 - \sum_\mu a^4 p_\mu^4/12 + O(a^6 p^6)]^{-1}$ to the one closer to the continuum limit $[a^2 \sum_\mu p^2]^{-1}$. For instance, to the first order ($K = 2$) we have

$$\left[\sum_\mu\left(\tfrac{5}{2} - \tfrac{8}{3}\cos a p_\mu + \tfrac{1}{6}\cos 2a p_\mu\right)\right]^{-1} = \left[a^2 \sum_\mu p^2 - a^6 \sum_\mu p_\mu^6/90 + O(a^8 p^8)\right]^{-1}. \tag{5.5}$$

At the infinite order ($K = \infty$) the action becomes an infinite series composed of field couplings between two sites along μ of all distances. The result in momentum space is simple modification of the propagator to the continuum limit $[a^2 \sum_\mu p^2]^{-1}$. Since in the large N limit, β is given by the gap equation written only with the propagator with mass square M, we can easily obtain the large M series

TABLE IV. Ratio of $C_{\rm app}$ (approximant of C_ξ) to the exact value of C_ξ in standard, first-order, and infinite-order improved actions. The blanks represent absence of extremum zero of $\bar{\beta}_{[n/n]} - \bar{\beta}$. However, even in those cases, the difference (the subtracted function) exhibits almost stationary behavior around the point, say also $\bar{x}_0$, at which $\bar{\beta}_{[n/n]} - \bar{\beta}$ vanishes and the first derivative takes minimum value. The estimation of C_ξ at such $\bar{x}_0$ yields accurate values.

$[m/n]$	Standard	First order	Infinite order
[3/3]	1.036 73	1.011 02	
[4/4]	1.044 43	1.011 37	0.999 244
[5/5]	1.026 04	1.005 05	
[6/6]	1.028 30	1.005 16	0.999 914
[7/7]	1.013 20	1.001 68	
[8/8]	1.016 65	1.001 99	0.999 990
[9/9]	1.010 77	1.001 03	
[10/10]	1.011 45	1.001 03	0.999 999

both at first- and infinite-order improved actions (for a detailed presentation, see the first citation in Ref. [4]). For example, at the first order it follows that

$$\beta = \int_{-\pi}^{\pi} \frac{d^2p}{(2\pi)^2} \frac{1}{M + \sum_{\mu=1,2} (\frac{5}{2} - \frac{8}{3}\cos p_\mu + \frac{1}{6}\cos 2p_\mu)}$$
$$= \frac{1}{M} - \frac{5}{M^2} + \frac{1157}{36M^3} - \frac{8419}{36M^4} + O(M^{-5}). \quad (5.6)$$

At infinite-order improvement, the right-hand side becomes just the integral of $(M + \sum_{\mu=1,2} p_\mu^2)^{-1}$ and expansion of β in $1/M$ is straightforward. It now suffices for us to repeat the same procedure for the approximation of (3.1) and the constant C_ξ at the first and inifinite orders of improved actions. Here note that the change of action induces the change of the value of nonuniversal C_ξ. $C_\xi = 0.237\,760\,7\ldots$ and $0.285\,145\,6\ldots$ at first and infinite orders, respectively. Table IV summarizes the result of our approximation. The improved action improves the approximation accuracy both at the first and at infinite orders. Though the improved lattice action is conventionally used in the Monte Carlo analysis and perturbation theory, it is also useful in our approach.

In the present work, we have analyzed Padé-Borel approximants of strong coupling expansion in nonlinear sigma model and find good behaviors approximating the continuum limit. We close the paper by pointing out that, even working with the standard action, further higher order computation would improve the result for all N including the limit $N \to \infty$. Padé-Borel approximants may become effective at larger $\bar{x}$ (smaller $\bar{M}$) and the two unwanted effects, lattice artifacts and omitted loop contributions, would be weaker there. Then, continuum scaling at smaller $\bar{M}$ with a clearer sign of asymptotic freedom near $N = 3$ would be seen, which allows us accurate evaluation of the mass gap for all $N \geq 3$.

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