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In this paper, we propose a first-order action functional for a large class of systems that generalize the relativistic perfect fluids in the Kähler parametrization to noncommutative spacetimes. The noncommutative action is parametrized by two arbitrary functions $K(z, \bar{z})$ and $f(\sqrt{-}j^2)$ that depend on the fluid potentials and represent the generalization of the Kähler potential of the complex surface parametrized by z and \bar{z} , respectively, and the characteristic function of each model. We calculate the equations of motion for the fluid potentials and the energy-momentum tensor in the first order in the noncommutative parameter. The density current does not receive any noncommutative corrections and it is conserved under the action of the commutative generators P_μ but the energy-momentum tensor is not. Therefore, we determine the set of constraints under which the energy-momentum tensor is divergenceless. Another set of constraints on the fluid potentials is obtained from the requirement of the invariance of the action under the generalization of the volume preserving transformations of the noncommutative spacetime. We show that the proposed action describes noncommutative fluid models by casting the energy-momentum tensor in the familiar fluid form and identifying the corresponding energy and momentum densities. In the commutative limit, they are identical to the corresponding quantities of the relativistic perfect fluids. The energy-momentum tensor contains a dissipative term that is due to the noncommutative spacetime and vanishes in the commutative limit. Finally, we particularize the theory to the case when the complex fluid potentials are characterized by a function $K(z, \bar{z})$ that is a deformation of the complex plane and show that this model has important common features with the commutative fluid such as infinitely many conserved currents and a conserved axial current that in the commutative case is associated to the topologically conserved linking number.

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I. INTRODUCTION

The formulation of a theory of the noncommutative fluids was motivated initially by the observation that the Abelian noncommutative Chern-Simons theory at level n is equivalent to the Laughlin theory at level $1/n$ [1,2] thus establishing a connection among the theories of noncommutative fields, fluid dynamics, quantum Hall effect and the matrix theory. The connection between the fractional quantum Hall effect and the noncommutative field theory has been subsequently studied for the Haldane model in Ref. [3] while the noncommutative fluid model from Ref. [1] was used to determine the density fluctuations in Ref. [4] and the topological order of the fractional Hall effect in Ref. [5] (see for a review Ref. [6]). A different motivation for the study of the noncommutative fluids is given by the fact that the volume-preserving transformations leave invariant the structure of noncommutative configuration spaces as well as the equations of motion of the non-Abelian Lagrangian fluids [7–10]. More recently, different fluid models have

appeared in the context of $U(1)$ gauge fields in curved noncommutative spaces [11] and in the study of the cosmological perturbations of the perfect fluid [12]. Reference [13] proposed a generalized symplectic structure of two models of irrotational and rotational noncommutative nonrelativistic fluids, respectively.

When studying the noncommutative fluids, it is certainly important to investigate models that reduce to relativistic fluids in the limit of commutative spacetime. This task is facilitated by the existence of a formulation in terms of the action functional of a large class of relativistic (perfect) fluids. In this formulation the fluid degrees of freedom that enter a first-order Lagrangian are given by the fluid potentials in either the (real) Clebsch parametrization [14] or the (complex) Kähler parametrization [15]. Although a proof of the equivalence of the two parameterizations is missing, it is known that both of them remove the obstruction to define a consistent Lagrangian which is due to the Chern-Simons term that is necessary in order to describe the nonzero vorticity and can be generalized to include the supersymmetry [15,16]. The complex parametrization of the fluid potentials has two interesting properties. First, there are infinitely many conserved charges for the non-singular Kähler potentials that characterize a geodesically

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complete complex manifold. Second, the Hamiltonian dynamics is governed by a set of simple second-class constraints among the fluid degrees of freedom. In particular, that Hamiltonian structure of the constraints has permitted a detailed analysis of the metafluid dynamics in Ref. [17], the formulation of the conformal fluids in Ref. [18] and the quantization of a large class of non-supersymmetric fluids in Ref. [19]. Also, the Kähler parametrization has been used to formulate the supersymmetric hydrodynamics in Ref. [20] and to construct the Navier-Stokes equations from the AdS/CFT and fluid correspondence in Ref. [21].

In this paper, we propose an action for the noncommutative fluid that generalizes the action of the relativistic fluid in the Kähler parametrization to the noncommutative space M_λ defined by the relations

$$[x^\mu, x^\nu] = i\lambda^{\mu\nu}, \quad (1)$$

where $\mu, \nu = 0, 1, 2, 3$ and $\lambda^{\mu\nu}$ is a constant antisymmetric matrix. Our action reduces to the previous action from Ref. [15] in the commutative limit $\lambda^{\mu\nu} \rightarrow 0$. The noncommutative action is not Poincaré invariant since the relevant group in the general noncommutative space given by the relation (1) is the volume-preserving group rather than a deformation of the Poincaré group. We determine a set of constraints on the fluid potentials such that the Lagrangian is invariant under the volume-preserving group. By choosing the commuting conjugate operators P_μ to x^μ , we show that, contrary to the commutative case, the energy-momentum tensor of the noncommutative fluid is not divergenceless under the action of P_μ 's. However, we are able to determine a set of constraints for the fluid potentials under which the energy-momentum tensor is conserved under the commutative translations.

This paper is organized as follows. In the next section, we propose the action of a large class of noncommutative fluids parametrized by the generalizations of the Kähler potential and an arbitrary function on the fluid currents that characterizes particular models from this class. Also, we derive the equations of motion at first order in $\lambda_{\mu\nu}$. In Sec. III, we derive the energy-momentum tensor and the equation of state. In the commutative limit, they take the form of the corresponding equations of the relativistic perfect fluid. In Sec. IV, we determine the constraints on the fluid potentials under which the noncommutative Lagrangian is invariant at zeroth and first order in the noncommutative parameter. In Sec. V, we present a simpler model which generalizes the fluid potentials on the complex plane. We show that in this model there are infinitely many conserved currents as in the commutative case, which makes the model particularly interesting because, in general, the generalizations of the fluid do not have this property. The last section is devoted to discussions.

II. NONCOMMUTATIVE FLUID ACTION

The class of relativistic perfect fluids on the four-dimensional Minkowski space M can be described in terms of the scalar potentials $\{\theta(x), z(x), \bar{z}(x)\}$ which are smooth functions from $C^\infty(M) = \{f: M \rightarrow \mathbb{C}\}$. The potential $\theta(x)$ is purely real while the fields $z(x)$ and $\bar{z}(x)$ are complex conjugate to each other, respectively. The class is parametrized by two arbitrary functions: $K(z, \bar{z})$ that is the Kähler potential associated to the two dimensional manifold of coordinates z and \bar{z} and $f(\rho)$ which depends on the local fluid density ρ . The relativistic fluid is characterized by the equations of state that involve the local pressure p and the energy density ε , respectively. The dynamics conserves the energy-momentum tensor $T_{\mu\nu}$ and the fluid density current j^μ and can be derived from a first-order Lagrangian functional in the potentials.¹ The Lagrangian has two more symmetries: the parametrization of the fluid potentials which leads to the conservation of infinitely many two-dimensional currents J_μ and the axial symmetry which leads to the conservation of the topological charge ω that describes the linking number of the vortices formed in the fluid [15,19].

Consider the noncommutative space M_λ with the algebra of complex function $\mathcal{F}(M_\lambda)$. A well-known property [22] is that this structure is isomorphic to the algebra $(C^\infty(M), *)$ where $*$: $C^\infty(M) \times C^\infty(M) \rightarrow C^\infty(M)$ is the Moyal product defined as

$$f * g = f e^{(i/2)\lambda_{\mu\nu}\vec{\partial}_\mu\vec{\partial}_\nu} g. \quad (2)$$

We take for the tangent space mapping

$$[\partial_\mu, \partial_\nu] = 0. \quad (3)$$

Since the algebra of functions contains the same objects with the usual dot product replaced by the star product, the perfect fluid is still characterized by its potentials $\{\theta(x), z(x), \bar{z}(x)\}$ with the interaction given by the star multiplication which could possibly affect the physical properties of the system. The action functional of the noncommutative fluid that generalizes the commutative action from Ref. [15] is given by the following relation:

$$S[j^\mu, \theta, z, \bar{z}] = \int d^4x [-j^\mu * (\partial_\mu \theta + i\partial_z K * \partial_\mu z - i\partial_{\bar{z}} K * \partial_\mu \bar{z})] - f(\sqrt{-j^\mu * j_\mu}). \quad (4)$$

The Lagrangian from Eq. (4) describes a large class of noncommutative fluids parametrized by the arbitrary functions $K(z, \bar{z})$ and $f(\sqrt{-j^\mu * j_\mu})$. In what follows, we are going to study the action (4) for a general noncommutative field j^μ until Sec. IV where we will investigate the

¹The metric on the Minkowski space has the signature $(-, +, +, +)$. The current four-vector is defined as $j^\mu = \rho u^\mu$ where $u^\mu = dx^\mu/d\tau$ is the velocity four-vector and $u^\mu u_\mu = -1$.

consequences of the generalization of the relation ρu^μ to the noncommutative theory. In general, $K(z, \bar{z})$ is not associated with a noncommutative Kähler manifold which can be viewed as a deformation quantization of a Kähler manifold (see, e.g. Refs. [23–25]). However, the commutative sector of $K(z, \bar{z})$ is the Kähler potential on the commutative sector of the (z, \bar{z}) manifold. In what follows, we make the simplifying truncation of the partial derivatives of the generalized Kähler potential at zero order in $\lambda_{\mu\nu}$ which allows one to apply the Leibniz rule. If higher orders in the noncommutative parameter are considered, the Leibniz rule does not generally hold. The function $f(\sqrt{-j^\mu * j_\mu})$ should coincide with $f(\sqrt{-j^\mu j_\mu})$ in the commutative limit $\lambda_{\mu\nu} \rightarrow 0$. In this way, one can establish the correspondence principle between the noncommutative perfect fluids given by the action (4) and the commutative perfect fluids studied in Refs. [15,19]. For small values of $\lambda_{\mu\nu}$, the linearized Lagrangian from the equation (4) takes the form

$$\begin{aligned} \mathcal{L}[j^\mu, \theta, z, \bar{z}] = & -j^\mu (\partial_\mu \theta + i \partial_z K \cdot \partial_\mu z - i \partial_{\bar{z}} K \cdot \partial_\mu \bar{z}) \\ & + \frac{1}{2} \lambda^{\alpha\beta} j^\mu (\partial_\alpha \partial_z K \cdot \partial_\beta \partial_\mu z - \partial_\alpha \partial_{\bar{z}} K \cdot \partial_\beta \partial_\mu \bar{z}) \\ & - \frac{i}{2} \lambda^{\alpha\beta} \partial_\alpha j^\mu \cdot \partial_\beta (\partial_\mu \theta + i \partial_z K \cdot \partial_\mu z \\ & - i \partial_{\bar{z}} K \cdot \partial_\mu \bar{z}) - f \left(\sqrt{-j^2 - \frac{i}{2} \lambda^{\alpha\beta} \partial_\alpha j^\mu \partial_\beta j_\mu} \right). \end{aligned} \quad (5)$$

A first difference to be noted between the commutative and the noncommutative fluids is that the current j^μ is propagating in the noncommutative case. Also, even in the lowest order in the noncommutative parameter, the Lagrangian contains higher-order derivatives in the fields. The Euler-Lagrange equations of motion can be obtained in the usual way by imposing the invariance of the action (4) under infinitesimal variations of the fields with vanishing boundary conditions for the fields and the derivatives. As can be seen from Eq. (4), the equations of motion have the general form

$$\frac{\delta \mathcal{L}}{\delta \phi} = \frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial x^\alpha} \left(\frac{\partial \mathcal{L}}{\partial (\partial_\alpha \phi)} \right) + \frac{\partial^2}{\partial x^\alpha \partial x^\beta} \left(\frac{\partial \mathcal{L}}{\partial (\partial_{\alpha\beta}^2 \phi)} \right) = 0. \quad (6)$$

By calculating Eq. (6) for the scalar potential $\theta(x)$, one can easily show that

$$\partial_\mu j^\mu = 0. \quad (7)$$

The equation of motion of the current j^μ takes the following form

$$\begin{aligned} f' \frac{j_\mu}{\sqrt{-j^2 - \frac{i}{2} \lambda^{\alpha\beta} \partial_\alpha j^\nu \partial_\beta j_\nu}} \\ = (\partial_\mu \theta + i \partial_z K \cdot \partial_\mu z - i \partial_{\bar{z}} K \cdot \partial_\mu \bar{z}) \\ - \frac{1}{2} \lambda^{\alpha\beta} (\partial_\alpha \partial_z K \cdot \partial_\beta \partial_\mu z - \partial_\alpha \partial_{\bar{z}} K \cdot \partial_\beta \partial_\mu \bar{z}). \end{aligned} \quad (8)$$

Here, f' denotes the derivative of f with respect to its variable. The equation of motion of the potential $z(x)$ can be obtained in the same way from the Eq. (6). After some algebra, one can show that it has the following form:

$$\begin{aligned} 2i j^\mu \partial_{z\bar{z}}^2 K \cdot \partial_\mu \bar{z} \\ + \frac{1}{2} \lambda^{\alpha\beta} j^\mu (\partial_z \partial_\alpha \partial_z K \cdot \partial_\beta \partial_\mu z - \partial_z \partial_\alpha \partial_{\bar{z}} K \cdot \partial_\beta \partial_\mu \bar{z}) \\ + \frac{1}{2} \lambda^{\alpha\beta} \partial_\alpha j^\mu \cdot (\partial_z \partial_\beta \partial_z K \cdot \partial_\mu z - \partial_z \partial_\beta \partial_{\bar{z}} K \cdot \partial_\mu \bar{z}) \\ + \frac{1}{2} \lambda^{\alpha\beta} \partial_\alpha [j^\mu (\partial_{z\bar{z}}^2 K \cdot \partial_\beta \partial_\mu z - \partial_{z\bar{z}}^2 K \cdot \partial_\beta \partial_\mu \bar{z})] \\ + \frac{1}{2} \lambda^{\alpha\beta} \partial_\beta [\partial_\alpha j^\mu (\partial_{z\bar{z}}^2 K \cdot \partial_\mu z - \partial_{z\bar{z}}^2 K \cdot \partial_\mu \bar{z})] \\ + \frac{1}{2} \lambda^{\alpha\beta} \partial_{\beta\mu}^2 (j^\mu \partial_\alpha \partial_z K) + \frac{1}{2} \lambda^{\alpha\beta} \partial_\alpha j^\mu \cdot \partial_{\beta\mu}^2 \partial_z K = 0. \end{aligned} \quad (9)$$

The equation of motion of $\bar{z}(x)$ can be obtained from Eq. (9) by replacing the appropriate derivative with respect to z by the derivative with respect to \bar{z} or by using (6). By either way, the result is

$$\begin{aligned} -2i j^\mu \partial_{z\bar{z}}^2 K \cdot \partial_\mu z \\ + \frac{1}{2} \lambda^{\alpha\beta} j^\mu (\partial_{\bar{z}} \partial_\alpha \partial_z K \cdot \partial_\beta \partial_\mu z - \partial_{\bar{z}} \partial_\alpha \partial_{\bar{z}} K \cdot \partial_\beta \partial_\mu \bar{z}) \\ + \frac{1}{2} \lambda^{\alpha\beta} \partial_\alpha j^\mu \cdot (\partial_{\bar{z}} \partial_\beta \partial_z K \cdot \partial_\mu z - \partial_{\bar{z}} \partial_\beta \partial_{\bar{z}} K \cdot \partial_\mu \bar{z}) \\ - \frac{1}{2} \lambda^{\alpha\beta} \partial_\alpha [j^\mu (\partial_{z\bar{z}}^2 K \cdot \partial_\beta \partial_\mu z - \partial_{z\bar{z}}^2 K \cdot \partial_\beta \partial_\mu \bar{z})] \\ - \frac{1}{2} \lambda^{\alpha\beta} \partial_\beta [\partial_\alpha j^\mu (\partial_{z\bar{z}}^2 K \cdot \partial_\mu z - \partial_{z\bar{z}}^2 K \cdot \partial_\mu \bar{z})] \\ - \frac{1}{2} \lambda^{\alpha\beta} \partial_{\beta\mu}^2 (j^\mu \cdot \partial_\alpha \partial_{\bar{z}} K) - \frac{1}{2} \lambda^{\alpha\beta} \partial_\alpha j^\mu \cdot \partial_{\beta\mu}^2 \partial_{\bar{z}} K = 0. \end{aligned} \quad (10)$$

Note that the derivatives with respect to the spacetime coordinates do not commute with the derivatives with respect to the complex fields z and \bar{z} , respectively. The first of the equations of motion (7) has a simple interpretation. It shows that the current j^μ is invariant under the transformations generated by the operators $P_\mu = \partial_\mu$. This equation does not receive any noncommutative corrections and it is in agreement with the generalization of the translation group defined by Eq. (3). The remaining equations of motion do not have such simple interpretation but more algebra shows that they reduce to the corresponding equations in the commutative limit. In particular, Eqs. (9) and (10) do not imply any longer that there are infinitely many conserved currents associated with the reparametrization invariance of any Kähler surface. We will return to this point in Sec. V.

III. ENERGY-MOMENTUM TENSOR

The class of perfect relativistic fluids in the Minkowski spacetime which are generalized to the noncommutative

spacetime by the action (4) is characterized by the divergenceless density current and the divergenceless energy-momentum tensor. These properties are related to the equations of motion of the fluid and with the translation invariance of the Lagrangian. As we have seen in the previous section, the density current of the noncommutative fluid is divergenceless and, by identifying the generators of the

translations with the derivatives ∂_μ , it is related to the translation invariance, too.

The energy-momentum tensor of the noncommutative fluid can be defined by coupling it with a c -number metric tensor $g_{\mu\nu}(x)$ and by taking the functional derivative of the action with respect to the metric. In this way, we obtain the relation

$$\begin{aligned}
 T_{\mu\nu} = \eta_{\mu\nu} & \left[-j^\gamma(\partial_\gamma\theta + i\partial_z K \cdot \partial_\gamma z - i\partial_{\bar{z}} K \cdot \partial_\gamma \bar{z}) + \frac{1}{2} \lambda^{\alpha\beta} j^\gamma (\partial_\alpha \partial_z K \cdot \partial_\beta \partial_\gamma z - \partial_\alpha \partial_{\bar{z}} K \cdot \partial_\beta \partial_\gamma \bar{z}) \right. \\
 & \left. - \frac{i}{2} \lambda^{\alpha\beta} \partial_\alpha j^\gamma \cdot \partial_\beta (\partial_\gamma \theta + i\partial_z K \cdot \partial_\gamma z - i\partial_{\bar{z}} K \cdot \partial_\gamma \bar{z}) - f \left(\sqrt{-j^\mu j_\mu + \frac{i}{2} \lambda^{\alpha\beta} \partial_\alpha j^\mu \cdot \partial_\beta j_\mu} \right) \right] \\
 & + 2j_\mu (\partial_\nu \theta + i\partial_z K \cdot \partial_\nu z - i\partial_{\bar{z}} K \cdot \partial_\nu \bar{z}) - \lambda^{\alpha\beta} j_\mu (\partial_\alpha \partial_z K \cdot \partial_\beta \partial_\nu z - \partial_\alpha \partial_{\bar{z}} K \cdot \partial_\beta \partial_\nu \bar{z}) \\
 & + i\lambda^{\alpha\beta} \partial_\alpha j_\mu \cdot \partial_\beta (\partial_\nu \theta + i\partial_z K \cdot \partial_\nu z - i\partial_{\bar{z}} K \cdot \partial_\nu \bar{z}) - f' \cdot \left(j_\mu j_\nu + \frac{i}{2} \lambda^{\alpha\beta} \partial_\alpha j_\mu \cdot \partial_\beta j_\nu \right). \quad (11)
 \end{aligned}$$

In general, the divergence of the energy-momentum tensor (11) will not vanish. In order for this to happen, one has to impose constraints on the fields. It can be shown that by using the equations of motion (23) and (24) the energy-momentum tensor is divergenceless for the solutions of the following constraints:

$$\partial_\nu \left(f' \frac{j^\mu j_\mu}{\sqrt{-j^2 - \frac{i}{2} \lambda^{\beta\gamma} \partial_\beta j^\nu \partial_\gamma j_\nu}} - f \right) - j_\mu \partial^\mu \left(f' \frac{j_\nu}{\sqrt{-j^2 - \frac{i}{2} \lambda^{\beta\gamma} \partial_\beta j^\nu \partial_\gamma j_\nu}} \right) = 0, \quad (12)$$

$$\partial_\nu \partial_\alpha j^\mu \cdot \partial_\beta (\partial_\mu \theta + i\partial_z K \cdot \partial_\mu z - i\partial_{\bar{z}} K \cdot \partial_\mu \bar{z}) - \partial_\alpha j_\mu \cdot \partial^\mu \partial_\beta (\partial_\nu \theta + i\partial_z K \cdot \partial_\nu z - i\partial_{\bar{z}} K \cdot \partial_\nu \bar{z}) + \partial_\alpha j_\mu \cdot \partial^\mu \partial_\beta j_\nu = 0. \quad (13)$$

In the form given by Eq. (11), it is unclear how the commutative perfect fluid is generalized to the noncommutative space. In order to make the relationship between the two more transparent, we take for the noncommutative j^μ the following natural generalization of the current:

$$j^\mu = \rho * u^\mu, \quad (14)$$

where $u^\mu = dx^\mu/d\tau$ does depend on τ only. Then it is easy to verify that

$$f \left(\sqrt{-j^\mu j_\mu + \frac{i}{2} \lambda^{\alpha\beta} \partial_\alpha j^\mu \cdot \partial_\beta j_\mu} \right) = f \left(\sqrt{-j^\mu j_\mu} \right). \quad (15)$$

By performing the corresponding simplification and by using the equation of motion of j^μ (23), one can show that the energy-momentum tensor has the following form

$$T_{\mu\nu} = \eta_{\mu\nu} p(\lambda) + [\varepsilon(\lambda) + p(\lambda)] u_\mu u_\nu + i\lambda^{\alpha\beta} \partial_\alpha \rho \cdot u_\mu \partial_\beta (\partial_\nu \theta + i\partial_z K \cdot \partial_\nu z - i\partial_{\bar{z}} K \cdot \partial_\nu \bar{z}), \quad (16)$$

where

$$\begin{aligned}
 p(\lambda) = \rho f' - f - j^\gamma (\partial_\gamma \theta + i\partial_z K \cdot \partial_\gamma z - i\partial_{\bar{z}} K \cdot \partial_\gamma \bar{z}) + \frac{1}{2} \lambda^{\alpha\beta} j^\gamma (\partial_\alpha \partial_z K \cdot \partial_\beta \partial_\gamma z - \partial_\alpha \partial_{\bar{z}} K \cdot \partial_\beta \partial_\gamma \bar{z}) \\
 - \frac{i}{2} \lambda^{\alpha\beta} \partial_\alpha j^\gamma \cdot \partial_\beta (\partial_\gamma \theta + i\partial_z K \cdot \partial_\gamma z - i\partial_{\bar{z}} K \cdot \partial_\gamma \bar{z}), \quad (17)
 \end{aligned}$$

$$\begin{aligned}
 \varepsilon(\lambda) = f + j^\gamma (\partial_\gamma \theta + i\partial_z K \cdot \partial_\gamma z - i\partial_{\bar{z}} K \cdot \partial_\gamma \bar{z}) - \frac{1}{2} \lambda^{\alpha\beta} j^\gamma (\partial_\alpha \partial_z K \cdot \partial_\beta \partial_\gamma z - \partial_\alpha \partial_{\bar{z}} K \cdot \partial_\beta \partial_\gamma \bar{z}) \\
 + \frac{i}{2} \lambda^{\alpha\beta} \partial_\alpha j^\gamma \cdot \partial_\beta (\partial_\gamma \theta + i\partial_z K \cdot \partial_\gamma z - i\partial_{\bar{z}} K \cdot \partial_\gamma \bar{z}). \quad (18)
 \end{aligned}$$

The above relations show that the action (4) is the generalization of the perfect fluid to the noncommutative case because Eqs. (16)–(18) reduce in the limit $\lambda^{\alpha\beta} \rightarrow 0$ to the known relations for the energy-momentum tensor, the pressure and the energy density [15]. The pressure is the generalization of the Legendre transformation of the specific energy to the noncommutative fluid. The divergencelessness of the energy-momentum tensor is apparent in Eq. (16) from which we note the last term that involves the product between the velocity and the combination of potentials that include the nonzero vorticity. This resembles a dissipative term that is a consequence of the noncommutative structure of the spacetime. If we require that all momentum density be generated by the flow of the energy density, it follows that

$$\lambda^{\alpha\beta} j^\mu \partial_\alpha j_\mu \cdot \partial_\beta (\partial_\nu \theta + i \partial_z K \cdot \partial_\nu z - i \partial_{\bar{z}} K \cdot \partial_\nu \bar{z}) = 0. \quad (19)$$

If the fluid is generalized to include more conserving charges, one could use Eq. (19) to define u^μ which is the analogue of choosing the frame for the commutative fluid.

IV. VOLUME PRESERVING SYMMETRY

The noncommutative structure of spacetime given by Eq. (1) is invariant under the following generalization of the volume preserving transformations [14]:

$$\delta x_\mu = [x_\mu, h], \quad (20)$$

where the parameter $h(x)$ is an arbitrary continuous function on x^μ 's. The brackets from the above equation involve the Moyal product and at the first order in $\lambda^{\mu\nu}$ take the form

$$[f, g] = i \lambda^{\mu\nu} \partial_\mu f \cdot \partial_\nu g. \quad (21)$$

In general, the Lagrangian given in relation (5) is not invariant under the transformations (20) due to the arbitrariness of the functions $\theta(x)$, $z(x)$, $\bar{z}(x)$, $K(z, \bar{z})$ and $f(x)$. Thus, by requiring that the Lagrangian be invariant under the volume-preserving transformations, constraints need to be imposed on these functions. It can be easily verified that the fields of the theory transform under Eq. (20) as follows:

$$\begin{aligned} \delta \phi &= [\phi, h], & \delta \psi^\mu &= [\psi^\mu, h], \\ \delta(\partial_\mu \phi) &= [\partial_\mu \phi, h] + [\phi, \partial_\mu h], \end{aligned} \quad (22)$$

where ϕ and ψ^μ are scalar and vector fields, respectively. The transformation of the derivative holds for vector fields, too. By varying the Lagrangian (5) with respect to Eq. (20), one obtains a bi-polynomial in the powers m of the antisymmetric matrix $\lambda^{\mu\nu}$ and the degree n of the derivatives of the arbitrary parameter $h(x)$. Consequently, the invariance of the Lagrangian is guaranteed if the terms of its variation vanish at each order in m and n , respectively. By keeping in mind this organization, we obtain from the terms linear in $\lambda^{\mu\nu}$ the following equations:

$$\begin{aligned} f' \frac{j^\mu \partial_\alpha j_\mu}{\sqrt{-j^2 - \frac{i}{2} \lambda^{\beta\gamma} \partial_\beta j^\nu \partial_\gamma j_\nu}} \\ = -\partial_\alpha [j^\mu (\partial_\mu \theta + i \partial_z K \cdot \partial_\mu z - i \partial_{\bar{z}} K \cdot \partial_\mu \bar{z})], \end{aligned} \quad (23)$$

$$j^\mu (\partial_\alpha \theta + i \partial_z K \cdot \partial_\alpha z - i \partial_{\bar{z}} K \cdot \partial_\alpha \bar{z}) = 0. \quad (24)$$

The quadratic terms in $\lambda^{\mu\nu}$ involve second- and third-order partial derivatives of h . The second-order derivatives couple with $\lambda^{\mu\nu}$ as well as j^μ and different couplings generate independent constraints. The result is the following set of equations

$$\begin{aligned} \partial_\gamma j^\mu \partial_\alpha (\partial_\mu \theta + i \partial_z K \cdot \partial_\mu z - i \partial_{\bar{z}} K \cdot \partial_\mu \bar{z}) \\ + \partial_\alpha j^\mu \partial_\gamma (\partial_\mu \theta + i \partial_z K \cdot \partial_\mu z - i \partial_{\bar{z}} K \cdot \partial_\mu \bar{z}) \\ + 2j^\mu [\partial_\alpha \partial_z K \cdot \partial_\gamma] \partial_\mu z - (z \leftrightarrow \bar{z}) = 0, \end{aligned} \quad (25)$$

$$\begin{aligned} \partial_\alpha j^\mu [\partial_\gamma \partial_\beta \theta + i \partial_\beta \partial_z K \cdot \partial_\gamma z + i \partial_z K \cdot \partial_\gamma \partial_\beta z \\ - (z \leftrightarrow \bar{z})] + j^\mu [\partial_\alpha \partial_z K \cdot \partial_\gamma \partial_\mu z - (z \leftrightarrow \bar{z})] = 0, \end{aligned} \quad (26)$$

$$\begin{aligned} \partial_\alpha j^\mu (\partial_\gamma \theta + i \partial_z K \cdot \partial_\gamma z - i \partial_{\bar{z}} K \cdot \partial_\gamma \bar{z}) \\ + j^\mu (\partial_\alpha \partial_z K \cdot \partial_\gamma z - \partial_\alpha \partial_{\bar{z}} K \cdot \partial_\gamma \bar{z}) = 0, \end{aligned} \quad (27)$$

where we have used the standard antisymmetrization convention with respect to the spacetime indices $a_{[\mu} b_{\nu]} = \frac{1}{2}(a_\mu b_\nu - a_\nu b_\mu)$. Constraints with higher powers of $\lambda^{\mu\nu}$ arise from higher-order corrections to the Lagrangian. If the spacetime noncommutativity is assumed to hold at high energy, only the linear terms in the antisymmetric matrix are relevant to the theory and the invariance of the Lagrangian under the generalized volume transformations is determined by the constraints (23) and (24) alone. Also, if the theory is studied on shell, some simplification of the above set of constraints is obtained.

V. A SIMPLER MODEL

The noncommutative perfect fluids discussed in the previous sections form a general class since the functions $K(z, \bar{z})$ and $f(\sqrt{-j^\mu * j_\mu})$ are not required to satisfy any property other than differentiability to an arbitrary order. This makes the dynamics quite complicated, even at first order in the noncommutative parameter. A slightly simpler model can be obtained by taking

$$K(z, \bar{z}) = z * \bar{z}, \quad f(\sqrt{-j^\mu * j_\mu}) = \frac{c}{2} \rho^2 = -\frac{c}{2} j^2, \quad (28)$$

where c is a c -number constant. In this model, the function $K(z, \bar{z})$ represents the generalization of the Kähler potential of the complex plane and, at the first order in the noncommutative parameter, it is a noncommutative deformation of the complex plane. The particular form of the function f is typical to the perfect fluid. The Lagrangian (5) of this particular model can be casted into the following form at first-order $\lambda^{\alpha\beta}$:

$$\begin{aligned}
 \mathcal{L} = & -j^\mu (\partial_\mu \theta + i\bar{z} \cdot \partial_\mu z - iz \cdot \partial_\mu \bar{z}) \\
 & + \frac{1}{2} \lambda^{\alpha\beta} j^\mu (\partial_\alpha \bar{z} \cdot \partial_\beta \partial_\mu z - \partial_\alpha z \cdot \partial_\beta \partial_\mu \bar{z}) \\
 & - \frac{i}{2} \lambda^{\alpha\beta} \partial_\alpha j^\mu \cdot \partial_\beta (\partial_\mu \theta + i\bar{z} \cdot \partial_\mu z - iz \cdot \partial_\mu \bar{z}) \\
 & + \frac{c}{2} j^2. \tag{29}
 \end{aligned}$$

The equations of motion can be obtained by using the relations (28) into the general equations (7)–(10) or by recalculating them from scratch:

$$\partial_\mu j^\mu = 0, \tag{30}$$

$$\begin{aligned}
 c j_\mu - (\partial_\mu \theta + i\bar{z} \cdot \partial_\mu z - iz \cdot \partial_\mu \bar{z}) \\
 + \frac{1}{2} \lambda^{\alpha\beta} (\partial_\alpha \bar{z} \cdot \partial_\beta \partial_\mu z - \partial_\alpha z \cdot \partial_\beta \partial_\mu \bar{z}) = 0, \tag{31}
 \end{aligned}$$

$$j^\mu \partial_\mu \bar{z} = j^\mu \partial_\mu z = 0. \tag{32}$$

The first remark that one can make about the dynamics of this particular model is that the equations of motion of θ , z and \bar{z} potentials do not receive any noncommutative correction. Next, we note that the relations (32) imply the existence of an infinite set of currents,

$$J_\mu[G] = -2G(z, \bar{z}) \cdot j_\mu, \tag{33}$$

where the generators $G(z, \bar{z})$ are arbitrary commutative functions on their arguments. The currents $J_\mu[G]$ are divergenceless at zero order in the noncommutative parameter because at this order the Leibniz rule holds. To the currents (33) correspond the conserved charges

$$Q[G] = \int d^3x J^0[G]. \tag{34}$$

These properties show that the particular model described by the functions (28) shares similar properties with the whole class of the commutative relativistic perfect fluids and with a special regime of the supersymmetric fluids [15,19].

Next, we can particularize the constraints (23)–(27) on the field potentials under which the Lagrangian (29) becomes invariant under the volume-preserving symmetry. If we consider the on-shell invariance, then the constraints take the simpler form:

$$c j^\mu \partial_\alpha j_\mu + \partial_\alpha (j^\mu \partial_\mu \theta) = 0, \tag{35}$$

$$j^\mu (\partial_\alpha \theta + i\bar{z} \partial_\alpha z - iz \partial_\alpha \bar{z}) = 0, \tag{36}$$

$$\begin{aligned}
 \partial_\gamma j^\mu \partial_\alpha (\partial_\mu \theta + i\bar{z} \partial_\mu z - iz \partial_\mu \bar{z}) \\
 + \partial_\alpha j^\mu \partial_\gamma (\partial_\mu \theta + i\bar{z} \partial_\mu z - iz \partial_\mu \bar{z}) + 2j^\mu [\partial_{[\alpha} \bar{z} \partial_\gamma] \partial_\mu z \\
 - (z \leftrightarrow \bar{z})] = 0, \tag{37}
 \end{aligned}$$

$$\begin{aligned}
 \partial_\alpha j^\mu [\partial_\gamma \partial_\beta \theta + i \partial_\beta \bar{z} \cdot \partial_\gamma z + i\bar{z} \cdot \partial_\gamma \partial_\beta \bar{z} \\
 - (z \leftrightarrow \bar{z})] + j^\mu [\partial_\alpha \bar{z} \cdot \partial_\gamma \partial_\mu z - (z \leftrightarrow \bar{z})] = 0, \tag{38}
 \end{aligned}$$

$$\begin{aligned}
 \partial_\alpha j^\mu (\partial_\gamma \theta + i\bar{z} \partial_\gamma z - iz \partial_\gamma \bar{z}) + j^\mu (\partial_\alpha \bar{z} \cdot \partial_\gamma z - \partial_\alpha z \cdot \partial_\gamma \bar{z}) = 0. \tag{39}
 \end{aligned}$$

The fluid properties of the model are described by the energy-momentum tensor and the equation of state which can be easily obtained from Eqs. (16)–(18) and put into the following form:

$$\begin{aligned}
 T_{\mu\nu} = & \eta_{\mu\nu} p(\lambda) + [\varepsilon(\lambda) + p(\lambda)] u_\mu u_\nu \\
 & + i \lambda^{\alpha\beta} \partial_\alpha \rho \cdot u_\mu \partial_\beta (\partial_\nu \theta + i\bar{z} \partial_\nu z - iz \partial_\nu \bar{z}), \tag{40}
 \end{aligned}$$

where

$$\begin{aligned}
 p(\lambda) = & \rho f' - f - j^\mu \partial_\mu \theta \\
 & + \frac{1}{2} \lambda^{\alpha\beta} j^\mu (\partial_\alpha \bar{z} \cdot \partial_\beta \partial_\mu z - \partial_\alpha z \cdot \partial_\beta \partial_\mu \bar{z}) \\
 & - \frac{i}{2} \lambda^{\alpha\beta} \partial_\alpha j^\mu \cdot \partial_\beta (\partial_\mu \theta + i\bar{z} \cdot \partial_\mu z - iz \cdot \partial_\mu \bar{z}), \tag{41}
 \end{aligned}$$

$$\begin{aligned}
 \varepsilon(\lambda) = & f + j^\mu \partial_\mu \theta \\
 & - \frac{1}{2} \lambda^{\alpha\beta} j^\mu (\partial_\alpha \bar{z} \cdot \partial_\beta \partial_\mu z - \partial_\alpha z \cdot \partial_\beta \partial_\mu \bar{z}) \\
 & + \frac{i}{2} \lambda^{\alpha\beta} \partial_\alpha j^\mu \cdot \partial_\beta (\partial_\mu \theta + i\bar{z} \partial_\mu z - iz \partial_\mu \bar{z}). \tag{42}
 \end{aligned}$$

From these equations, we see that the present model represents a generalization of the relativistic perfect fluid which preserves the infinite conserved currents associated with the reparameterization invariance of the complex manifold which is described by the complex potentials z and \bar{z} at zeroth order in the noncommutative parameter.

VI. CONCLUSIONS AND DISCUSSIONS

In this paper, we have proposed the functional action (4) for a large class of noncommutative fluids that generalizes the relativistic perfect fluids formulated in the Kähler parametrization to the noncommutative spacetime. The noncommutative fluids are characterized by $K(z, \bar{z})$ and $f(\sqrt{-j^\mu * j_\mu})$ which generalize the corresponding arbitrary functions from the commutative case with the restriction of the partial derivatives to the zeroth order in the noncommutative parameter that makes the Leibniz property hold. Without this technical restriction, there are more contributions at first order in $\lambda^{\mu\nu}$. We have derived the equations of motion of the fluid potentials to the first order in the noncommutative parameter. Also, we have calculated the energy-momentum tensor. The equation of motion for the θ field (7) does not receive any noncommutative corrections and it represents the divergencelessness of the density current j^μ like in the commutative case. However, the energy-momentum tensor is not divergenceless. That implies that $T_{\mu\nu}$ is not invariant under translations if the dual operators $P_\mu = \partial_\mu$ commute with each other. If one requires that the energy-momentum tensor of

the noncommutative theory be invariant, the constraints (12) and (13) should be imposed on the fields. Note that Eq. (7) holds in the κ -Minkowski spacetime, too. Actually, the current conservation suggests that the action (4) be valid in all noncommutative spaces where the translations are generated by commuting $P_\mu = \partial_\mu$. The equation of motion of the current j^μ contains commutative terms that are the same as the ones obtained for commutative fluids and noncommutative corrections. Also, one can show that the equations (9) and (10) for the fields z and \bar{z} can be reduced to the corresponding equations in the commutative case if the current conservation (7) is used in those terms that are independent of $\lambda^{\mu\nu}$. By particularizing the functions $K(z, \bar{z})$ and f to the relations (28), we have shown that other properties of the commutative fluids can be generalized to the noncommutative ones. In particular, the models specified by (28) have infinitely many conserved currents $J_\mu[G]$ in the Leibniz approximation for the partial

derivatives. This feature alone makes the model quite interesting, since in general the currents are not conserved for generalizations of the perfect fluid.

Another important quantity that is conserved in the commutative case is the axial current which is related to the topologically conserved linking number of vortices [15]. Therefore, it is desirable to see if the noncommutative fluids have divergenceless axial currents. We can generalize the axial current k_μ to the noncommutative case by applying the correspondence principle adopted in this paper

$$K^\mu = \epsilon^{\mu\nu\xi\lambda}(\partial_\nu\theta + i\partial_z K * \partial_\nu z - i\partial_{\bar{z}} K * \partial_\nu \bar{z}) * \partial_\xi(\partial_\lambda\theta + i\partial_z K * \partial_\lambda z - i\partial_{\bar{z}} K * \partial_\lambda \bar{z}), \quad (43)$$

where $\epsilon^{\mu\nu\xi\lambda}$ is the four-dimensional antisymmetric tensor with $\epsilon^{0123} = 1$. If we calculate the divergence of K^μ at first order in $\lambda^{\mu\nu}$, we see, after lengthy calculations, that it fails to be zero by a term of the form

$$\begin{aligned} & -2i\epsilon^{\mu\nu\xi\lambda}\lambda^{\alpha\beta}(\partial_{\bar{z}\bar{z}}^2 K \partial_\mu \bar{z} \partial_\alpha \partial_\nu z + \partial_z \partial_{\bar{z}}^2 K \partial_\mu \bar{z} \partial_\alpha z \partial_\nu z + \partial_{\bar{z}}^2 \partial_z K \partial_\mu \bar{z} \partial_\alpha \bar{z} \partial_\nu z + \partial_{\bar{z}\bar{z}}^2 K \partial_\mu \partial_\alpha \bar{z} \partial_\nu z) \\ & \times (\partial_{\bar{z}\bar{z}}^2 K \partial_\xi \bar{z} \partial_\beta \partial_\lambda z + \partial_z \partial_{\bar{z}}^2 K \partial_\xi \bar{z} \partial_\beta z \partial_\lambda z + \partial_{\bar{z}}^2 \partial_z K \partial_\xi \bar{z} \partial_\beta \bar{z} \partial_\lambda z + \partial_{\bar{z}\bar{z}}^2 K \partial_\xi \partial_\beta \bar{z} \partial_\lambda z) \\ & - i\epsilon^{\mu\nu\xi\lambda}\lambda^{\alpha\beta} \partial_{z\bar{z}} K \partial_\mu \bar{z} \partial_\nu z (\partial_z^2 K \partial_\xi \partial_\alpha z \partial_\beta \partial_\lambda z - \partial_{\bar{z}}^2 K \partial_\xi \partial_\alpha \bar{z} \partial_\beta \partial_\lambda \bar{z}). \end{aligned} \quad (44)$$

This relation shows that K^μ would not be conserved unless further constraints were imposed on the potentials. However, we can show that for the particular model presented in the section V,

$$\partial_\mu K^\mu = 0. \quad (45)$$

Thus, the generalization of the Kähler potential for the complex plane and the perfect fluid shares most of the properties with the commutative fluids.

It is interesting to investigate further the noncommutative fluids of the type presented in this paper along several lines. One of the most important problems is to describe concrete models that preserve the noncommutative Poincaré symmetry. This can be achieved by taking for M_λ the κ -Minkowski spacetime. As mentioned above, the noncommutative generalization of the translation operators satisfy Eq. (1) so all the conclusions derived for it concerning the invariance of the density current and the energy-momentum tensor are expected to continue true. Another interesting issue is to analyze the fluids obtained

by relaxing the Leibniz rule for the partial derivative and work within the full noncommutative structure. This would modify all the equations of motion and the constraints by adding extra terms that contain $\lambda^{\mu\nu}$. Therefore, one should be able to recover the relativistic fluid in the commutative limit as we have done in the present paper. However, the conservation of the generalized parametrization currents might not hold without other constraints. And finally, it would be interesting to study the symplectic structure on the phase space of the fluid induced by the underlying noncommutative structure of spacetime.

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- [1] L. Susskind, [arXiv:hep-th/0101029](https://arxiv.org/abs/hep-th/0101029).
- [2] S. Bahcall and L. Susskind, *Int. J. Mod. Phys. B* **5**, 2735 (1991).
- [3] A. El Rhalami, E. M. Sahraoui, and E. H. Saidi, *J. High Energy Phys.* **05** (2002) 004.

- [4] J. L. F. Barbon and A. Paredes, *Int. J. Mod. Phys. A* **17**, 3589 (2002).
- [5] J. L. F. Barbon and D. Gerber, *Int. J. Mod. Phys. A* **22**, 5287 (2007).
- [6] A. P. Polychronakos, [arXiv:0706.1095](https://arxiv.org/abs/0706.1095).

- [7] R. Jackiw, S. Y. Pi, and A. P. Polychronakos, *Ann. Phys. (N.Y.)* **301**, 157 (2002).
- [8] R. Jackiw, [arXiv:physics/0209108](https://arxiv.org/abs/physics/0209108).
- [9] R. Jackiw, *Nucl. Phys. B, Proc. Suppl.* **127**, 53 (2004).
- [10] R. Jackiw, V. P. Nair, S. Y. Pi, and A. P. Polychronakos, *J. Phys. A* **37**, R327 (2004).
- [11] S. A. Alavi, *Chinese Physics Letters* **23**, 2637 (2006).
- [12] A. De Felice, J. M. Gerard, and T. Suyama, *Phys. Rev. D* **81**, 063527 (2010).
- [13] M. V. Marcial, A. C. R. Mendes, C. Neves, W. Oliveira, and F. I. Takakura, *Phys. Lett. A* **374**, 3608 (2010).
- [14] R. Jackiw, [arXiv:physics/0010042](https://arxiv.org/abs/physics/0010042).
- [15] T. S. Nyawelo, J. W. van Holten, and S. GrootNibbelink, *Phys. Rev. D* **68**, 125006 (2003).
- [16] R. Jackiw and A. P. Polychronakos, *Phys. Rev. D* **62**, 085019 (2000).
- [17] D. Baleanu, *Czech. J. Phys.* **55**, 473 (2005).
- [18] P. D. Jarvis and J. W. van Holten, *Nucl. Phys.* **B734**, 272 (2006).
- [19] L. Holender, M. A. Santos, and I. V. Vancea, *Phys. Rev. D* **77**, 045024 (2008).
- [20] T. S. Nyawelo, *Nucl. Phys.* **B672**, 87 (2003).
- [21] P. A. Grassi, A. Mezzalana, and L. Sommovigo, [arXiv:1107.2780](https://arxiv.org/abs/1107.2780).
- [22] A. Connes, *Noncommutative Geometry* (Academic Press, San Diego, California, 1994).
- [23] H. Omori, Y. Maeda, N. Miyazaki, and A. Yoshioka, Report No. KSTS/RR-97/003, 1997.
- [24] N. Reshetikhin and L. A. Takhtajan, in *L. D. Faddeev's Seminar on Mathematical Physics*, American Mathematical Society Translations Series 2 Vol. 201 (American Mathematical Society, Providence, Rhode Island, 2000), p. 257-276.
- [25] M. Schlichenmaier, *Adv. Theor. Math. Phys.* **2010**, 927280 (2010).