

Gribov pendulum in the Coulomb gauge on curved spacesFabrizio Canfora,^{1,*} Alex Giacomini,^{2,†} and Julio Oliva^{2,‡}¹*Centro de Estudios Científicos (CECS), Casilla 1469, Valdivia, Chile*²*Instituto de Ciencias Físicas y Matemáticas, Facultad de Ciencias, Universidad Austral de Chile, Valdivia, Chile*

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In this paper the generalization of the Gribov pendulum equation in the Coulomb gauge for curved space-times is analyzed on static spherically symmetric backgrounds. A rigorous argument for the existence and uniqueness of solution is provided in the asymptotically AdS case. The analysis of the strong and weak boundary conditions is equivalent to analyzing an effective one-dimensional Schrödinger equation. Necessary conditions in order for spherically symmetric backgrounds to admit solutions of the Gribov pendulum equation representing copies of the vacuum satisfying the strong boundary conditions are given. It is shown that asymptotically flat backgrounds do not support solutions of the Gribov pendulum equation of this type, while on asymptotically AdS backgrounds such ambiguities can appear. Some physical consequences are discussed.

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I. INTRODUCTION

The Yang-Mills Lagrangian L is one of the basic blocks of the standard model:

$$L = \text{tr} F_{\mu\nu} F^{\mu\nu}, \quad (F_{\mu\nu})^a = (\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu])^a. \quad (1)$$

The degrees of freedom of the theory are encoded in the connection $(A_\mu)^a$, which is a Lie algebra valued one form. The action functional is invariant under finite gauge transformations, which act on the gauge potential as

$$A_\mu \rightarrow U^\dagger A_\mu U + U^\dagger \partial_\mu U \quad (2)$$

whereas the physical observables are invariant under proper gauge transformations. The latter has to be everywhere smooth, and it has to decrease fast enough at infinity such that a suitable norm, to be specified later, converges.¹ This invariance is related with the existence of first-class constraints, which in turn imply that the degrees of freedom of the theory are less than the number of algebraically independent components of the gauge potential.

Up to now, the program of using from the very beginning gauge-invariant variables has been completed only in the cases of topological field theories in $2 + 1$ dimensions [2], while it is still far from clear how to perform practical computations in a completely gauge-invariant way for Yang-Mills theories in $2 + 1$ and $3 + 1$ dimensions. Furthermore, the gauge-fixing problem is also relevant in the classical theory since, when using the Dirac bracket formalism, the Faddeev-Popov determinant appears in the

denominators of the Dirac-Poisson brackets (see, for instance, the detailed analysis in [3]).

A gauge-fixing condition is the common practical solution, the most convenient choices being the Coulomb gauge and the Lorenz gauge²:

$$\partial^i A_i = 0, \quad \partial^\mu A_\mu = 0; \quad (3)$$

where $i = 1, \dots, D$ are the spacelike indices and $\mu = 0, 1, \dots, D$ are space-time indices.

This procedure has enormous value, allowing perturbative computations around the trivial vacuum $A_\mu = 0$. However, the existence of a proper gauge transformation (2) preserving one of the conditions (3) would spoil the whole quantization procedure. In [5], Gribov showed that³ a *proper gauge fixing* is not possible.

In the path integral formalism, an ambiguity in the gauge fixing corresponds to smooth zero modes of the Faddeev-Popov (FP) operator satisfying suitable boundary conditions. In order to define the path integral in the presence of Gribov copies, it has been suggested to exclude classical A_μ backgrounds which generate zero modes of the FP operator (see, in particular, [5,7–11]; two nice reviews are [12,13]). This possibility is consistent with the usual perturbative point of view since, in the case of $SU(N)$ Yang-Mills theories, for a “small enough” potential A_μ (with respect to a suitable functional norm [11]), there are no zero-modes of the FP operator in the Landau or Coulomb gauge.

It is also worth emphasizing that the issue of gauge-fixing ambiguities cannot be ignored in any case. In particular,

²Other gauge fixings are possible such as the axial gauge, the temporal gauge, etc.; nevertheless these choices have their own problems (see, for instance, [4]).

³Furthermore, it has been shown by Singer [6], that if Gribov ambiguities occur in Coulomb gauge, they occur in all the gauge-fixing conditions involving derivatives of the gauge field.

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¹A key reference on the problem of defining a proper gauge transformation is [1].

even if gauge-fixing choices free of Gribov ambiguities can be found, still the presence of Gribov ambiguities in other gauges gives rise to a breaking of the BRS symmetry at a nonperturbative level (see, for instance, [14–17]).

Abelian gauge theories on flat space-time are devoid of this problem, since the Gribov copy equation for the smooth gauge parameter ϕ is

$$\partial_i \partial^i \phi = 0 \quad \text{or} \quad \partial_\mu \partial^\mu \phi = 0 \quad (4)$$

which on flat space-time (once the time coordinate has been Wick-rotated: $t \rightarrow i\tau$) has no smooth nontrivial solutions fulfilling the physical boundary conditions. In fact, the situation changes dramatically when we consider an Abelian gauge field propagating on a curved background: it was shown in [18] that, quite generically, a proper gauge fixing in the Abelian case cannot be achieved. Furthermore, it has been recently pointed out [19] that, at least in the case of gravitational theories in $2 + 1$ dimensions, gauge-fixing ambiguities may provide one with a valuable tool to achieve SUSY breaking.

For these reasons, the issue of the Gribov copies in the case of non-Abelian gauge theories on curved spaces as well as on spaces with nontrivial topologies is of interest. In many physically relevant situations (such as close to a black hole, in neutron stars and even more in quarks and hybrid star [20] and in cosmological setups) the curved nature of space-time cannot be ignored. Thus, in those situations it is important to consider the dynamics of QCD on a curved background. In the present paper we will analyze the issue of the appearance of Gribov copies by analyzing the curved generalization of the Gribov pendulum equation in the Coulomb gauge.⁴ Here we will consider the class of static curved space-times with spherical symmetry as backgrounds. We will construct necessary conditions in order for spherically symmetric backgrounds to admit solutions of the Gribov pendulum equation representing copies of the vacuum and satisfying the strong boundary conditions. We will show with explicit examples that the curvature of the space-time can generate quite nontrivial deformations of the Gribov horizon.

The paper is organized as follows. In Sec. II, the curved generalization of the Gribov pendulum in the Coulomb gauge will be constructed, and the strong and weak boundary conditions will be given. In Sec. III we analyze the existence of copies in the background corresponding to AdS space-time. In Sec. IV it will be shown that smooth solutions of the Gribov pendulum equation exist and the analysis of the boundary conditions in terms of an effective Schrödinger equation will be also discussed. In Sec. V, background metrics admitting copies of the vacuum satisfying the strong boundary condition will be

constructed. Some conclusions will be drawn in the last section.

II. CURVED GRIBOV PENDULUM

The main goal of the present paper is to analyze the new features of Gribov ambiguities in the Coulomb gauge on a curved spherically symmetric background. The metric of the curved backgrounds which will be considered here is

$$ds^2 = -g^2(r)dt^2 + f^2(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (5)$$

The Coulomb gauge condition on the non-Abelian gauge potential A_μ^a of the $SU(2)$ gauge group reads

$$A_0^a = 0; \quad \nabla^i A_i^a = 0, \quad (6)$$

where the spatial indices correspond to $i = 1, 2, 3$ and the ∇^i stands for the Levi-Civita connection of the metric (5), with spatial indices. It is easy to see that due to the form of the metric (5), the Coulomb gauge condition transforms covariantly with respect to the three-dimensional spatial metric ds_Σ^2 of $t = \text{const}$ surfaces

$$ds_\Sigma^2 = (g_\Sigma)_{ij} dx^i dx^j := f^2(r)dr^2 + r^2 d\Omega^2, \quad (7)$$

where $d\Omega$ stands for the line element of the two sphere. The gauge fixing (6) then can be written as

$$\nabla^i A_i^a = \frac{1}{\sqrt{\det g_\Sigma}} \partial_j (\sqrt{\det g_\Sigma} g_\Sigma^{jj} A_i^a) = 0. \quad (8)$$

Let us consider an element of the gauge group of the following form:

$$U(x^\mu) = \exp\left(i \frac{\alpha(r)}{2} x^i \sigma_i\right) \quad (9)$$

where σ_i are the flat Pauli matrices and x^i is a normalized radial contravariant vector on g_Σ , which in the above coordinate system reads

$$\vec{x} = (\sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta),$$

so that

$$\mathbf{I} = (x^i \sigma_i)(x^j \sigma_j), \quad (10)$$

\mathbf{I} being the 2×2 identity matrix. It can be seen that $U^\dagger = U^{-1}$. Let us consider a background gauge potential A_i^* of the following form:

$$A_i^* = i \varepsilon_{ijk} \frac{x^j \sigma^k}{r^2} \varphi(r), \quad (11)$$

where ε_{ijk} is the three-dimensional Levi-Civita tensor. Note that A_i^* is divergence free for any radial function $\varphi(r)$:

$$\nabla^i A_i^* = 0, \quad \forall \varphi(r). \quad (12)$$

We will choose the above gauge potential in Eq. (11). Even if this is not the most general transverse potential, we choose it because it discloses very clearly the differences

⁴We will consider the Coulomb instead of the Landau gauge in order to avoid the subtleties related to the Wick rotation on curved space-times.

between the solutions of the Gribov pendulum equation on flat and curved space-times.⁵ We are now in the position to derive the curved generalization of the Gribov pendulum equation. One has to ask then for a gauge transformation of the non-Abelian gauge potential A_i^* in Eq. (11) generated by the group element U in Eq. (9) satisfying

$$\nabla^i(U^{-1}A_i^*U + U^{-1}\partial_i U) = 0. \quad (13)$$

The existence of solutions for the equation above is a necessary condition for the appearance of Gribov copies of A_i^* . Explicitly on the spherically symmetric space-times we are considering the gauge-fixing equations imply that

$$\left(\frac{r^2\alpha'}{f}\right)' = 2f(1-2\varphi)\sin\alpha, \quad (14)$$

where primes denote derivation with respect to the radial coordinate. Note that this equation is invariant under the transformation $f \rightarrow Cf$ and $(1-2\varphi) \rightarrow C^{-2}(1-2\varphi)$, provided C is a constant.

Strong and weak boundary conditions on curved spaces

Here, we will discuss the weak and strong boundary conditions for the function α in Schwarzschild-like coordinates as in Eq. (5). The importance of carefully distinguishing copies satisfying strong and weak boundary conditions comes from the following fact (well known in the flat case). When the solution of the Gribov pendulum equation satisfies the weak boundary conditions

$$\alpha \xrightarrow{r \rightarrow \infty} (2n+1)\pi + O(1/r^\eta), \quad \eta > 0, \quad (15)$$

the corresponding copy

$$\begin{aligned} U(x^\mu) &= \exp\left(i\frac{\alpha(r)}{2}x^j\sigma_j\right) \\ &= \mathbf{1}\cos\left(\frac{\alpha(r)}{2}\right) + ix^j\sigma_j\sin\left(\frac{\alpha(r)}{2}\right), \end{aligned} \quad (16)$$

does not approach to an element of the center of the gauge group at spatial infinity. A copy of this type is not problematic since it can be discarded with the argument that it changes the definition of (non-Abelian) charge at infinity and so it does not give rise to a proper gauge transformation (see [1,5]).

On the other hand, when a solution of the Gribov pendulum equation satisfies the strong boundary conditions, the corresponding copy does approach to an element of the center of the gauge group at spatial infinity. A copy of this type is particularly problematic since it belongs to the class of proper gauge transformations and would represent a failure of the whole gauge-fixing procedure. Of course,

⁵The more general ansatz (see, for instance, [12]) for the background gauge potential which gives rise to a spherically symmetric Gribov pendulum equation does not add new qualitative features

this was one of the main arguments behind the Gribov-Zwanziger idea of ‘‘cutting’’ the path integral when the first copies satisfying the strong boundary conditions appear. Indeed, the worst case would be to have a copy of the vacuum $A_\mu = 0$ fulfilling the strong boundary conditions since, in this case, not even usual perturbation theory leading to the standard Feynman rules in the Landau or Coulomb gauge would be well defined. In QCD on flat space this does not happen but we will show here that whenever the theory is considered in a curved background, the situation becomes much more delicate.

Weak boundary conditions: The weak boundary condition for a copy on the metric (5), corresponds to look for a solution of the curved Gribov pendulum equation Eq. (14) which behaves as

$$\begin{aligned} \alpha \xrightarrow{r \rightarrow \infty} (2n+1)\pi + O(1/r^\eta), \quad \eta > 0, \quad \varphi \xrightarrow{r \rightarrow \infty} \text{const} + o(1/r), \\ \alpha \xrightarrow{r \rightarrow 0} 2m\pi + O(r^\gamma), \quad \gamma > 0, \quad m, n \in \mathbb{Z}. \end{aligned} \quad (17)$$

As it occurs on flat space-time [12], as far as the behavior of the solution α close to the origin is concerned, both in the case of weak and in the case of strong boundary conditions one has to require that the condition in Eq. (17) holds otherwise the copy generated by the solution α would not be regular at the origin. As it will be discussed in the next sections, in the case in which a star is considered as a gravitational background, the situation is quite different.

Since the Christoffel symbols do not enter directly in the expression $U^{-1}A_\mu U + U^{-1}\partial_\mu U$ (since U behaves as a scalar under diffeomorphisms), in terms of α both the strong and the weak boundary conditions keep forms similar to the corresponding flat cases. In particular, this implies that also on a spherically symmetric curved space as in Eq. (5) the gauge transformation generated by an element of the group of the form $U(x^\mu) = \exp(i\frac{\alpha(r)}{2}x^i\sigma_i)$ will change the definition of non-Abelian charge as a surface integral at spatial infinity unless $\alpha(r)$ approaches to $2n\pi$ (strong boundary condition).

On flat space-time, a vector potential A_μ^C which generates a Coulomb-like electric field decays as

$$A_\mu^C \underset{r \rightarrow \infty}{\approx} \frac{1}{r} + O\left(\frac{1}{r^p}\right) \quad \text{with } p > 1, \quad (18)$$

in order for the corresponding electric field to decay as $1/r^2$. On AdS space-time, the metric function f is given in Eq. (24). Therefore, the electric field has to decay also as $1/r^2$ in order to generate a finite charge. One can see this as follows: the electric (as well as the non-Abelian) charge can be written in this way

$$Q^{(a)} = - \int_{\partial\Sigma} d^2x \sqrt{\gamma^{\partial\Sigma}} n_\mu s_\nu F^{(a)\mu\nu}, \quad (19)$$

where a is in the adjoint representation of $su(2)$, $\partial\Sigma$ is the boundary of the spacelike section Σ , with induced metric

$\gamma^{\partial\Sigma}$, n_μ is a normalized future pointing timelike vector ($n_\mu n^\mu = -1$) and s_μ is normal to $\partial\Sigma$ and normalized as $s_\mu s^\mu = 1$. Thus, in the AdS case, in order to have a finite charge the electric field has to decay as $1/r^2$ and correspondingly the vector potential generating an electric field has to decay as $1/r$.

Therefore, as it happens on flat space-times (see, for instance, [5]), in the case of the weak boundary conditions, one has to require the function $\varphi(r)$ appearing in the ansatz for the transverse vector potential in Eq. (11), to decay as

$$\varphi(r) \underset{r \rightarrow \infty}{\approx} \text{const} + O(1/r). \quad (20)$$

Strong boundary conditions: The strong boundary condition on the metric (5) given in Schwarzschild-like coordinates corresponds to ask that when $r \rightarrow \infty$, the solution of the curved Gribov pendulum equation Eq. (14) behaves as

$$\alpha \underset{r \rightarrow \infty}{\rightarrow} 2n\pi + O(1/r^\eta), \quad \varphi(r) \underset{r \rightarrow \infty}{\approx} \frac{1}{r^\varepsilon}, \quad \eta, \quad \varepsilon > 0, \quad (21)$$

while close to the origin the condition in Eq. (17) must hold in order the copy generated by α to be regular.

At a first glance, the curved Gribov pendulum in vacuum (which corresponds to Eq. (14) with $\varphi = 0$) could look like a flat Gribov pendulum (in which case $f = 1$) in a non-trivial background gauge field. If this would be the case, then it would also be easy to construct examples of curved background supporting copies of the vacuum satisfying the strong boundary conditions. Obviously, a non-Abelian gauge theory on a curved background supporting copies of the vacuum satisfying the strong boundary conditions would be pathological. However in many important cases (such as constant curvature backgrounds and spherically symmetric black hole space-times), such a resemblance is misleading. As it will be shown in the next sections, in these cases solutions of the curved Gribov pendulum equation representing copies of the vacuum satisfying the strong boundary conditions cannot be constructed.

Defining $\tau = \tau(r)$ by

$$\tau' = \frac{\partial\tau}{\partial r} = \frac{f}{r^2}, \quad (22)$$

the curved Gribov pendulum Eq. (14) can be transformed in the following useful form

$$\frac{\partial^2\alpha}{\partial\tau^2} = 2r^2(1 - 2\varphi) \sin\alpha, \quad (23)$$

where the variable r has to be expressed in terms of τ using Eq. (22). As it will be discussed in the next section, this form of the equation allows to deal in a very effective way with the problem of imposing strong and weak boundary conditions.

We will first focus on the case of AdS as a background metric. According to the AdS/CFT correspondence, it is

possible to explore the nonperturbative regime of supersymmetric Yang-Mills theories by performing semiclassical computations in the bulk of asymptotically AdS background [21,22]. Recently, this correspondence has also been extended to the context of condensed matter physics (see for two recent reviews [23]).

The metric (5) reduces to the metric on AdS space-time, provided

$$f(r) = \frac{1}{\sqrt{1 + \frac{r^2}{l^2}}}; \quad (24)$$

where l is the AdS curvature. Equation (22) implies that

$$\tau = -\frac{\sqrt{1 + \frac{r^2}{l^2}}}{r} \Rightarrow r^2 = \frac{1}{(l\tau)^2 - 1}, \quad \tau < 0. \quad (25)$$

In particular at spatial infinity we have

$$r \rightarrow +\infty \Leftrightarrow l\tau \rightarrow -1^-.$$

Therefore, in the AdS case the Gribov pendulum Eq. (23) can be rewritten as

$$\frac{\partial^2\alpha}{\partial\tau^2} = V_{\text{AdS}}(\tau) \sin\alpha, \quad (26)$$

$$V_{\text{AdS}}(\tau) := \frac{2(1 - 2\varphi)}{(l\tau)^2 - 1}. \quad (27)$$

On the other hand, on flat space-time $f(r) = 1$ so that $\tau = -\frac{1}{r}$ and

$$r \rightarrow +\infty \Leftrightarrow \tau \rightarrow 0^-. \quad (28)$$

Equation (14), then reduces to the flat Gribov pendulum equation which reads

$$\frac{\partial^2\alpha}{\partial\tau^2} = V_{\text{flat}}(\tau) \sin\alpha, \quad (29)$$

$$V_{\text{flat}}(\tau) := \frac{2(1 - 2\varphi)}{\tau^2}. \quad (30)$$

It is worth emphasizing here that, when one writes the curved Gribov pendulum equation in terms of the coordinate τ , the main difference between the AdS and the flat cases occurs close to the singularities of the effective potentials⁶ $V_{\text{AdS}}(\tau)$ and $V_{\text{flat}}(\tau)$ appearing in Eqs. (27) and (30). Close to the singularity (when $l\tau \rightarrow -1^-$), the effective AdS potential $V_{\text{AdS}}(\tau)$ diverges as $1/\tau$ while the effective potential $V_{\text{flat}}(\tau)$ corresponding to the flat metric diverges when $\tau \rightarrow 0$ as $1/\tau^2$. On the other hand, as soon as one moves away from the corresponding singularities (namely, when $|\tau| > 1$), the Gribov pendulum equations in the AdS and flat cases look the same.

⁶The reason to call $V_{\text{AdS}}(\tau)$ and $V_{\text{flat}}(\tau)$ effective potentials will be manifest in the next sections.

The curved Gribov pendulum equation corresponding to the Coulomb gauge on the spherically symmetric background in Eq. (5) can be derived as the Euler-Lagrange equation of the following functional:

$$N[\alpha] = \int \sqrt{\det g_\Sigma} d^3x \text{Tr}[(U^{-1}A_i^*U + U^{-1}\partial_i U)^2]. \quad (31)$$

When one inserts into the above expression Eqs. (9) and (11) one gets the following useful expression (which reduces to the known flat case [5] when $f = 1$):

$$N[\alpha] = \int_a^\infty \frac{dr}{f} \left\{ (r\alpha')^2 + 8f^2(1 - 2\varphi) \left[1 - \cos^2\left(\frac{\alpha}{2}\right) \right] \right\}. \quad (32)$$

The main goal of this paper is to show that the solutions of the curved Gribov pendulum equation on curved space-time can behave in a totally different manner with respect to the flat case. In particular, we will show that there are many physically interesting curved backgrounds that may admit copies of the vacuum satisfying the strong boundary condition (to be defined in the next section). If one accepts the interpretation of [5,7–10,15,16], our results would imply that the infrared structure of QCD on curved space-times could be quite different from the infrared structure on flat space-times.

III. ON THE EXISTENCE AND UNIQUENESS OF SOLUTIONS ON ADS

In this section we will describe the mathematical technique, based on the contraction theorem, which allows to prove existence and uniqueness of nonlinear Gribov pendulum equations for r larger than a suitable critical radius (defined below). When r is small, provided

$$f^2(r) = 1 + O(r^2), \quad (33)$$

the metric approaches to flat metric and, for globally flat background metrics, the issue of existence and uniqueness of solutions of the Gribov pendulum equation is well understood. Moreover, the most interesting technical differences in the procedure with respect to the flat case when AdS or asymptotically AdS space-time are considered as backgrounds appear for r larger than a critical radius (see also the comments after Eq. (30)). Thus, we will focus on the analysis of the problem for r larger than a critical radius defined below. The goal of this section is to provide one with a rigorous justification of the effective Schrödinger approach to the analysis of the weak and strong boundary conditions, which is useful on curved backgrounds approaching AdS in the asymptotic region.

The statement of the theorem ([24,25]) is the following:

Let S a complete metric (Banach) space. A metric space is a space in which a distance $d(X, Y)$ between any pair of elements of the space is defined

$$d(X, Y) \in \mathbb{R}, \quad X, Y \in S. \quad (34)$$

Complete metric space means that, with respect to the metric, from every Cauchy sequence one can extract a convergent subsequence (see, for instance, [24]). Let T be a map from the metric space S into itself:

$$T[\cdot]: S \rightarrow S. \quad (35)$$

If the map $T[\cdot]$ is a contraction, namely, for all $X \in S$ and $Y \in S$

$$d(T[X], T[Y]) \leq M d(X, Y), \quad \text{with } M < 1, \quad (36)$$

then the map $T[\cdot]$ has only one fixed point. In other words, if the map $T[\cdot]$ is a contraction of a complete metric space then there exist a unique solution to the equation

$$T[X] = X. \quad (37)$$

Hereafter we will focus on the asymptotic region, defined by $r \rightarrow \infty$ (in a precise sense as it will be explained in a moment).

The idea is to write the nonlinear equation one is interested in (Eq. (14) in our case) in the form of a fixed point equation for a suitable map and then, try to prove that the map is a contraction for some complete metric space. Let us define the following operator T_φ :

$$T_\varphi[\alpha](r) \equiv A + B \int_{r^*}^r \frac{ds}{s^2 \sqrt{1+s^2}} + \int_{r^*}^r \frac{1}{s^2 \sqrt{1+s^2}} \times \left[\int_{r^*}^s \frac{2(1-2\varphi(\rho)) \sin\alpha(\rho)}{\sqrt{1+\rho^2}} d\rho \right] ds, \quad (38)$$

where A and B are arbitrary constants, and r^* defines a critical radius. It is easy to see that the Gribov pendulum equation in Eq. (14) in the AdS case can be written as a fixed point equation for the operator defined in Eq. (38); i.e., if $\tilde{\alpha}$ is a fixed point of T_φ

$$T_\varphi[\tilde{\alpha}](r) \equiv \tilde{\alpha}(r) \quad (39)$$

then the same $\tilde{\alpha}$ is a solution of Eq. (14) with AdS as a background geometry. This can be seen directly by applying consecutively two derivatives at the right-hand side of Eq. (38). Note that we have fixed the AdS radius l to 1. Thus, we will prove that the above operator has a fixed point by using the above-mentioned theorem. Let us define S as the space of functions which are continuous and bounded on $[r^*, \infty[$, i.e.

$$S \equiv \{\alpha \mid \alpha \in C[r^*, \infty[, \quad |\alpha(r)| < M_\alpha \quad \forall r \in [r^*, \infty[. \quad (40)$$

The radius r^* will be determined in a moment and the constants A and B correspond to the value of $\tilde{\alpha}$ and its derivative at r^* , respectively:

$$\tilde{\alpha}(r^*) = A, \quad \tilde{\alpha}'(r^*) = \frac{B}{(r^*)^2 \sqrt{1+(r^*)^2}}. \quad (41)$$

This functional space is a Banach space (see, for instance, [24]) with respect to the following distance $d(\alpha, \beta)$:

$$d(\alpha, \beta) = \sup_{r \in [r^*, \infty[} |\alpha(r) - \beta(r)|. \quad (42)$$

$$I_1(r) = \int_{r^*}^r \frac{ds}{s^2 \sqrt{1 + s^2}}, \quad (44)$$

It is easy to see that the operator T_φ defined on (38), maps $S_{A,B}$ into itself since

$$|\alpha(r)| < M_\alpha \Rightarrow |T_\varphi[\alpha](r)| < \tilde{M}_\alpha, \quad (43) \quad |I_1(r)| \leq \frac{1}{r^*} \quad \forall r. \quad (45)$$

where \tilde{M}_α may be different from M_α . Indeed, let us consider the following function $I_1(r)$:

Hence one has

$$|T_\varphi[\alpha](r)| \leq |A| + \frac{|B|}{r^*} + \int_{r^*}^r \frac{ds}{s^2 \sqrt{1 + s^2}} \left[\int_{r^*}^s \frac{2|(1 - 2\varphi(\rho))| |\sin\alpha(\rho)|}{\sqrt{1 + \rho^2}} d\rho \right] \leq \quad (46)$$

$$\leq |A| + \frac{|B|}{r^*} + |1 + 2M_\varphi| \int_{r^*}^r \frac{ds}{s^2 \sqrt{1 + s^2}} \left[\int_{r^*}^s \frac{2}{\sqrt{1 + \rho^2}} d\rho \right] < \quad (47)$$

$$< |A| + \frac{|B|}{r^*} + |1 + 2M_\varphi| \int_{r^*}^r \frac{2(s - r^*) ds}{s^2 \sqrt{1 + s^2}} < \quad (48)$$

$$< |A| + \frac{|B|}{r^*} + 2|1 + 2M_\varphi| \int_{r^*}^r \frac{ds}{s \sqrt{1 + s^2}} < +\infty, \quad (49)$$

where we used the fact that φ in Eq. (11) is bounded and smooth everywhere so that

$$|(1 - 2\varphi(r))| < |1 + 2M_\varphi| \quad \forall r, \quad (50)$$

$$M_\varphi = \sup_{r \in [0, \infty[} |\varphi(r)| < \infty. \quad (51)$$

We will show that it is possible to choose the radius r^* such that the operator T_φ is a contraction of the Banach space in Eq. (40) with the distance in Eq. (42). To see this, one has to compute $|T_\varphi[\alpha] - T_\varphi[\beta]|$ where $\alpha, \beta \in S$:

$$|T_\varphi[\alpha] - T_\varphi[\beta]| \leq \int_{r^*}^r \frac{ds}{s^2 \sqrt{1 + s^2}} \left[\int_{r^*}^s \frac{2|(1 - 2\varphi(\rho))| |\sin\alpha(\rho) - \sin\beta(\rho)|}{\sqrt{1 + \rho^2}} d\rho \right] \leq \quad (52)$$

$$\leq c(\sup|\alpha(r) - \beta(r)|) |1 + 2M_\varphi| \int_{r^*}^r \frac{ds}{s^2 \sqrt{1 + s^2}} \left[\int_{r^*}^s \frac{2}{\sqrt{1 + \rho^2}} d\rho \right] < \quad (53)$$

$$< c(|1 + 2M_\varphi|) d(\alpha, \beta) \int_{r^*}^r \frac{2(s - r^*) ds}{s^2 \sqrt{1 + s^2}} < \quad (54)$$

$$< 2c(|1 + 2M_\varphi|) d(\alpha, \beta) \int_{r^*}^r \frac{ds}{s^2} < \frac{2c(|1 + 2M_\varphi|)}{r^*} d(\alpha, \beta) \Rightarrow \quad (55)$$

$$\sup|T_\varphi[\alpha] - T_\varphi[\beta]| = d(T_\varphi[\alpha], T_\varphi[\beta]) < \frac{2c(|1 + 2M_\varphi|)}{r^*} d(\alpha, \beta) \quad (56)$$

where we have used the trigonometric identity

$$\sin\alpha - \sin\beta = 2 \cos\left(\frac{\alpha + \beta}{2}\right) \sin\left(\frac{\alpha - \beta}{2}\right), \quad (57)$$

as well as the inequalities

$$|\sin x| \leq |x|, \quad |\cos x| \leq 1, \quad \forall x. \quad (58)$$

Equations (55) and (56) show that a sufficient condition in order for T_φ to be a contraction is to choose r^* such that:

$$\frac{2|1 + 2M_\varphi|}{r^*} < 1. \quad (59)$$

Thus, if one chooses r^* satisfying the inequality in Eq. (59) then Eq. (14) has a unique solution in the AdS case. It is

worth noting that in all the previous steps the presence of the curved metric (through the AdS factor $1/\sqrt{1+r^2}$) helped in obtaining the required bounds. Of course, if the constant A is chosen to be a multiple of π and B vanishes then, because of the above result, the unique solution is the constant, i.e.

$$A = n\pi \vee B = 0 \Rightarrow \alpha(r) = n\pi \quad \forall r \geq r^*. \quad (60)$$

One can observe that the solution is at least $C^2[r^*, \infty[$ since, as the integral form of the equation shows, one can take at least two derivatives. Furthermore, not only the solution but also the first and the second derivative of the solution are bounded as one can deduce from the equation in the ‘‘fixed point’’ form in Eqs. (38) and (39). This implies that, necessarily, one has

$$\alpha(r) \xrightarrow{r \rightarrow \infty} n\pi,$$

otherwise the second derivative would not be bounded.

Schrödinger equation approach

In order to analyze the issue of existence of copies satisfying strong boundary conditions, one can use an effective one-dimensional Schrödinger equation. In the AdS case, it is useful to consider the equation with the change of coordinate in Eqs. (22) and (25), and the corresponding Gribov pendulum equation in Eqs. (26) and (27). Because of the theorem discussed in the previous section, we know that bounded smooth solutions exist when $r \rightarrow \infty (l\tau \rightarrow -1^-)$. Therefore, when

$$V(\tau) \xrightarrow{l\tau \rightarrow -1^-} \infty, \quad (61)$$

in order for the solution to be bounded (taking into account that both the first and the second derivatives of the solution must be bounded as well as):

$$\alpha \xrightarrow{r \rightarrow \infty} n\pi + O(1/r) \Leftrightarrow \quad (62)$$

$$\alpha \xrightarrow{l\tau+1 \rightarrow 0^-} n\pi + O(l\tau + 1). \quad (63)$$

Consequently the following, leading order approximation is justified

$$\sin \alpha \xrightarrow{l\tau+1 \rightarrow 0^-} (-1)^n \alpha, \quad (64)$$

where n odd (even) corresponds to the weak (strong) boundary conditions. For these reasons, one is allowed to approximate for $r \gg r^*$ Eq. (26) as follows

$$\frac{\partial^2 \alpha}{\partial \tau^2} = (-1)^n V(\tau) \alpha = W(\tau) \alpha \quad (65)$$

which can be analyzed as a Schrödinger-like equation:

$$-u'' + W(\tau)u = Eu, \quad \tau \in]-\infty, -\frac{1}{l}[\quad (66)$$

$$W(\tau) = (-1)^n V(\tau), \quad E = 0. \quad (67)$$

Thus, the question of existence of normalizable copies reduces to the question of existence of nontrivial normalizable eigenvectors (bound states) of the above Schrödinger-like problem with zero eigenvalue such that

$$u \xrightarrow{l\tau+1 \rightarrow 0^-} 0. \quad (68)$$

As far as the vacuum copies in AdS are concerned, $V(\tau)$ in Eqs. (65) and (27) is always a positive and monotone function in $]-\infty, -1/l[$ and diverges to $+\infty$ when $l\tau + 1 \rightarrow 0^-$ so that, in order for the effective potential W in Eq. (67) to have bound states, the only possibility is that n in Eq. (62) is an odd number. Namely, on AdS, there are no vacuum copies satisfying the strong boundary conditions.⁷

On the other hand, in order to have a copy satisfying the strong boundary condition, it is enough as it happens on flat space-times, to consider φ in Eq. (65) which makes $V(\tau)$ negative enough in order to produce a ‘‘valley’’ in the effective potential which supports a bound state, even when n in Eq. (62) is an even number.

It is worth emphasizing that this effective Schrödinger approach when applied to the flat case, in which the effective potential is (see Eqs. (29) and (30))

$$W(\tau) = (-1)^n \frac{2(1-2\varphi)}{\tau^2}, \quad (69)$$

reproduces the well-known results such as the absence of vacuum copies satisfying the strong boundary conditions and also the need to have a factor $1-2\varphi$ ‘‘negative enough’’ as to produce a valley supporting a nontrivial bound state in a very intuitive manner. Furthermore, in this framework it is quite apparent the difference between the asymptotic behaviors in the flat and the AdS cases. The absolute value of the effective potential in the asymptotic region in the AdS case⁸ diverges as $1/\tau$ while in the flat case it diverges as $1/\tau^2$.

⁷Note that it is always possible to choose the integration constant such that the bound state vanishes in $l\tau + 1$ as required by the physics of the problem.

⁸As mentioned before, in the AdS case the asymptotic region $r \rightarrow \infty$ in the τ coordinate is defined as $l\tau + 1 \rightarrow 0^-$ while in the flat case the asymptotic region $r \rightarrow \infty$ in the τ coordinate is defined as $\tau \rightarrow 0^-$.

IV. METRICS WITH COPIES OF THE VACUUM SATISFYING THE STRONG BOUNDARY CONDITION

In this section, we will describe sufficient conditions in order for the spherically symmetric background metric in Eq. (5) to support copies of the vacuum. A background of this type would be quite pathological and one may wonder whether if, at least in a semiclassical approach to quantum gravity, backgrounds admitting copies of the vacuum should be discarded. This is a reasonable consistency criterion since one would like QCD perturbation theory to be well defined. Indeed, according to the point of view in [5,7–10,15,16], on such background space-times allowing copies of the vacuum satisfying the strong boundary conditions not even perturbation theory around a vanishing gauge field would be well defined.

A simple method inspired by the well-known work of Heney [26] to deduce necessary conditions for the appearance of copies of the vacuum, is to interpret Eq. (14) in the case in which the background gauge field in Eq. (11) vanishes, as an equation for the metric function f appearing in (5), assuming that α is everywhere regular and satisfies the strong boundary conditions when $r \rightarrow \infty$. In the case in which the space-time is everywhere regular (the case of black hole and stars will be considered in the next subsections) one can express f in terms of the copy α as follows:

$$f(r)^2 = \frac{(r^2 \partial_r \alpha)^2}{C + 4 \int_0^r x^2 (\partial_x \alpha) (\sin \alpha(x)) dx} \quad (70)$$

$$\int_0^\infty x^2 (\partial_x \alpha) (\sin \alpha(x)) dx = \int_0^{2\pi} (x(\alpha))^2 \sin \alpha d\alpha \quad (76)$$

$$= \int_0^\pi (x(\alpha))^2 \sin \alpha d\alpha + \int_\pi^{2\pi} (x(\alpha))^2 \sin \alpha d\alpha, \quad (77)$$

where

$$\int_0^\pi (x(\alpha))^2 \sin \alpha d\alpha > 0, \quad \int_\pi^{2\pi} (x(\alpha))^2 \sin \alpha d\alpha < 0.$$

Since $\partial_r \alpha > 0$ then $x(\alpha)$ is an increasing function of α , therefore the absolute value of the second integral on the right-hand side of Eq. (77) is larger than the first¹⁰ and consequently the integral in Eq. (76) is negative. Thus, with the choice in Eqs. (72) and (73) for r large enough f^2 is negative and solutions α satisfying the strong boundary conditions cannot appear. By repeating basically the same

⁹Replacing $f^2 \rightarrow f^{-1}$ this expression reduced to Eq. (22) of [18] for $\sin(\alpha) \approx \alpha$.

¹⁰Note that the two integrals in Eq. (77) would be equal and opposite without the factor $(x(\alpha))^2$.

where C is an integration constant.⁹ As mentioned before, in order for the metric to be regular close to the origin one has to require that

$$f(r)^2 \underset{r \rightarrow 0}{\approx} 1 + kr^2 + O(r^4), \quad (71)$$

where k is a real constant. By a direct expansion, one can see that this implies that $C = 0$ in Eq. (70). To fix the ideas, one can take a function α increasing monotonically from 0 to 2π at infinity:

$$\alpha(0) = 0, \quad \alpha(r) \underset{r \rightarrow \infty}{\rightarrow} 2\pi, \quad (72)$$

$$\partial_r \alpha > 0 \quad \forall r > 0, \quad \partial_r \alpha \underset{r \rightarrow \infty}{\rightarrow} 0. \quad (73)$$

With this choice the integrand in the denominator in Eq. (70) does not change sign for small r but one would have $f(r)^2 < 0$ for r large enough. To see this it is convenient to consider the following change of variable in the integral in the denominator in Eq. (70)

$$\partial_x \alpha dx = d\alpha \Rightarrow \quad (74)$$

$$\int_0^r x^2 (\partial_x \alpha) (\sin \alpha(x)) dx = \int_{\alpha(0)}^{\alpha(r)} (x(\alpha))^2 \sin \alpha d\alpha, \quad (75)$$

where $x(\alpha)$ is the inverse function of $\alpha(x)$ (which exist because of our hypothesis). The integral in the denominator in Eq. (70) up to infinity reads

argument, one can see that the same conclusion would hold for any choice in which α is monotone $\forall r > 0$. It is worth pointing out that the situation does not change qualitatively if one chooses α as a monotone decreasing function with $\alpha(0) = 2\pi$ and $\alpha(r) \rightarrow_{r \rightarrow \infty} 0$.

If one chooses a function α which is not monotone, then $\partial_r \alpha$ vanishes at least once for $r^* > 0$. Let us assume first that $\partial_r \alpha$ vanishes just once at r^* , and that this is a simple zero

$$\alpha(0) = 0, \quad \alpha(r) \underset{r \rightarrow \infty}{\rightarrow} 0, \quad \partial_r \alpha|_{r=r^*} = 0, \quad \pi < \alpha(r^*) < 2\pi, \quad (78)$$

$$\partial_r \alpha > 0 \quad \forall r < r^*, \quad \partial_r \alpha < 0 \quad \forall r > r^*, \quad \partial_r \alpha \underset{r \rightarrow \infty}{\rightarrow} 0. \quad (79)$$

The condition that $\alpha(r^*) > \pi$ is necessary in order for f^2 to be regular at r^* , as also the denominator in Eq. (70) has to vanish in order to compensate for the zero in the numerator:

$$\int_0^{r^*} x^2 (\partial_x \alpha) (\sin \alpha(x)) dx = 0.$$

$$\int_{\alpha(r^*)}^{\infty} x^2 (\partial_x \alpha) (\sin \alpha(x)) dx = \int_{\alpha(r^*)}^0 (x(\alpha))^2 \sin \alpha d\alpha \quad (80)$$

$$= \int_{\alpha(r^*)}^{\pi} (x(\alpha))^2 \sin \alpha d\alpha + \int_{\pi}^0 (x(\alpha))^2 \sin \alpha d\alpha = I_1 + I_2, \quad (81)$$

where

$$\begin{aligned} I_1 &= \int_{\alpha(r^*)}^{\pi} (x(\alpha))^2 \sin \alpha d\alpha \\ &= - \int_{\pi}^{\alpha(r^*)} (x(\alpha))^2 \sin \alpha d\alpha > 0, \end{aligned} \quad (82)$$

$$I_2 = \int_{\pi}^0 (x(\alpha))^2 \sin \alpha d\alpha = - \int_0^{\pi} (x(\alpha))^2 \sin \alpha d\alpha < 0. \quad (83)$$

Because of our hypothesis (see Eq. (79)) in the interval $]r^*, \infty[$ the function $\alpha(r)$ is a decreasing function of r and, consequently, in the same interval the inverse function $x(\alpha)$ is a decreasing function of α , so that the absolute value of the second integral on the right-hand side of Eq. (81)

In this case, it can be shown that for r large enough $f(r)^2$ becomes negative as well. Indeed, using the change of variable in Eqs. (74) and (75), one can evaluate the integral in the denominator in Eq. (70) from r^* up to infinity:

(which is negative, see Eq. (83)), is larger than the absolute value of the first integral on the right-hand side of Eq. (81) (which is positive, see Eq. (82)). Therefore, the integral on the left-hand side of Eq. (80) is negative and this implies that also under the hypothesis in Eqs. (78) and (79) f^2 becomes negative for r large enough and so solutions of the Gribov pendulum equations satisfying the strong boundary conditions cannot be constructed. Following the same reasoning, it is easy to show that also if one admits that α has more than one point where the first derivative vanishes it is impossible to have f^2 everywhere positive from 0 to ∞ , smooth and well defined with, at the same time, α fulfilling the strong boundary conditions. Indeed, nothing would change by replacing the hypothesis in Eqs. (78) and (79) with

$$\begin{aligned} \alpha(0) &= 2m\pi, & \alpha(r) &\xrightarrow{r \rightarrow \infty} 2n\pi, & \partial_r \alpha|_{r=r_i} &= 0, & i &= 1, \dots, p, \\ \partial_r \alpha &> 0 \quad \forall 0 < r < r_1, & \partial_r \alpha &< 0 \quad \forall r_1 < r < r_2, \dots, & \partial_r \alpha &\xrightarrow{r \rightarrow \infty} 0: \end{aligned}$$

it is enough to repeat the previous argument starting with the last point in which $\partial_r \alpha$ vanishes.

A. Space-time outside a black hole

Let us now consider the cases of spherically symmetric space-times which describe the exterior of a black hole. Since when one considers the Euclidean version of a black hole space-time, if there is a curvature singularity the origin $r = 0$ does not belong anymore to the space-time itself, and the condition in Eq. (17) (which ensures regularity at the origin in the standard case) does not apply anymore.

In the case of black hole space-times Eq. (70) which expresses the metric function f in terms of the copy α changes as follows

$$f(r)^2 = \frac{(r^2 \partial_r \alpha)^2}{C + 4 \int_{r_H}^r x^2 (\partial_x \alpha) (\sin \alpha(x)) dx} \quad (84)$$

where r_H is the radius of the event horizon. In the coordinate system given in Eq. (5), the event horizon can be characterized as a pole of $f(r)^2$. Since we are considering regular copies, we have to require that the derivative of α is bounded. Thus, one has to take $C = 0$ in Eq. (84) and, at the same time, the derivative of α at r_H does not vanish in such a way to get the desired pole. In this case, it is easy to convince oneself that there is no choice of α such that $f(r)^2$ is positive definite for $r > r_H$. To fix the ideas, one can take a function α increasing monotonically from the value at the horizon to 2π at infinity:

$$\pi < \alpha(r_H) < 2\pi, \quad (85)$$

$$\alpha(r) \xrightarrow{r \rightarrow \infty} 2\pi, \quad \partial_r \alpha > 0 \quad \forall r > r_H, \quad \partial_r \alpha \xrightarrow{r \rightarrow \infty} 0. \quad (86)$$

With this choice the integrand in the denominator in Eq. (84) does not change sign but one would have $f(r)^2 < 0$. If, instead, one assumes that

$$0 \leq \alpha(r_H) < \pi, \tag{87}$$

$$\alpha(r) \xrightarrow{r \rightarrow \infty} 2\pi, \quad \partial_r \alpha > 0 \quad \forall r > r_H, \quad \partial_r \alpha \xrightarrow{r \rightarrow \infty} 0, \tag{88}$$

it can be shown that for r large enough $f(r)^2$ becomes negative anyway. Using the change of variable in Eqs. (74) and (75), the integral in the denominator in Eq. (84) up to infinity reads

$$\int_{r_H}^{\infty} x^2 (\partial_x \alpha) (\sin \alpha(x)) dx = \int_{\alpha(r_H)}^{2\pi} (x(\alpha))^2 \sin \alpha d\alpha \tag{89}$$

$$= \int_{\alpha(r_H)}^{\pi} (x(\alpha))^2 \sin \alpha d\alpha + \int_{\pi}^{2\pi} (x(\alpha))^2 \sin \alpha d\alpha, \tag{90}$$

$$\int_{\alpha(r_H)}^{\pi} (x(\alpha))^2 \sin \alpha d\alpha > 0, \quad \int_{\pi}^{2\pi} (x(\alpha))^2 \sin \alpha d\alpha < 0.$$

Since $\partial_r \alpha > 0$, $x(\alpha)$ is an increasing function of α so that the absolute value of the second integral on the right-hand side of Eq. (90) (which is negative) is larger than the first and consequently the integral in Eq. (89) is negative. This implies that for r large enough $f(r)^2$ becomes negative. It is easy to see that the same would happen with any choice in which α is a monotone function $\forall r > r_H$ satisfying the strong boundary conditions. If one chooses a function α which is not monotone, then $\partial_r \alpha$ would vanish at least once for $r^* > r_H$. Thus, let us assume that $\partial_r \alpha$ vanishes just once at r^* . Then, one has to require that correspondingly also the denominator in Eq. (84) should vanish at r^* in such a way to have a finite and positive f^2 . One can assume that $\alpha(r_H) < \pi$ in such a way to ensure, at least

close to r_H , the positiveness of the integral:

$$0 < \alpha(r_H) < \pi, \quad \pi < \alpha(r^*) < 2\pi, \tag{91}$$

$$\partial_r \alpha > 0 \quad \forall r_H < r < r^*, \tag{92}$$

$$\partial_r \alpha < 0 \quad \forall r > r^*, \quad \alpha(r) \xrightarrow{r \rightarrow \infty} 0, \quad \partial_r \alpha \xrightarrow{r \rightarrow \infty} 0. \tag{93}$$

The above choice in Eqs. (91) and (93) for $\alpha(r^*)$ ensures that at least close to r^* (the denominator of) f^2 is positive. However, if one considers the integral from r^* to infinity in the denominator in Eq. (84) using the change of variable in Eqs. (74) and (75):

$$\int_{r^*}^{\infty} x^2 (\partial_x \alpha) (\sin \alpha(x)) dx = \int_{\alpha(r^*)}^0 (x(\alpha))^2 \sin \alpha d\alpha \tag{94}$$

$$= \int_{\alpha(r^*)}^{\pi} (x(\alpha))^2 \sin \alpha d\alpha + \int_{\pi}^0 (x(\alpha))^2 \sin \alpha d\alpha, \tag{95}$$

where

$$\int_{\alpha(r^*)}^{\pi} (x(\alpha))^2 \sin \alpha d\alpha = - \int_{\pi}^{\alpha(r^*)} (x(\alpha))^2 \sin \alpha d\alpha > 0,$$

$$\int_{\pi}^0 (x(\alpha))^2 \sin \alpha d\alpha = - \int_0^{\pi} (x(\alpha))^2 \sin \alpha d\alpha < 0,$$

once again one reaches the conclusion that the integral in Eq. (94) (and, consequently f^2) is negative because, for $r > r^*$, $x(\alpha)$ is a decreasing function of α so that the absolute value of the second integral on the right-hand side of Eq. (95) is larger than the absolute value of the first integral. It is easy to see that the same conclusion would hold in the case in which the derivative of α would vanish at more than one point. Hence, also if one assumes that α is not monotone, for r large enough f^2 becomes negative. Hence, on spherically symmetric black hole space-times as in Eq. (5) solutions of the Gribov pendulum equation

representing copies of the vacuum satisfying the strong boundary conditions cannot appear.

B. The space-time outside a star

In the previous subsections it has been shown that both, on spherically symmetric regular space-times and on spherically symmetric black hole space-times one cannot construct solutions satisfying the strong boundary conditions. The main technical reason is that both in Eq. (70) and in Eq. (84) one has to take $C = 0$. In the first case, this is necessary in order to achieve a space-time which is regular at the origin, while in the second case $C = 0$ ensures the appearance of the black hole horizon at r_H . Indeed, when $C = 0$ all the previous arguments on the change of sign of the integral in the denominator of the expression for f^2 work. However, the situation is radically different in the cases in which the background metric in Eq. (5) represents,

for instance, the exterior of a spherically symmetric star. In the case of a space-time representing the exterior of a star Eq. (70) which expresses the metric function f in terms of the copy α changes as follows

$$f(r)^2 = \frac{(r^2 \partial_r \alpha)^2}{C + 4 \int_{r_S}^r x^2 (\partial_x \alpha) (\sin \alpha(x)) dx}, \quad (96)$$

where r_S is the coordinate radius of the star. Unlike the black hole case in which one has to require that f^2 in Eq. (5) has a pole at r_H , in the case of a space-time representing the exterior of a star one has to require that f^2 evaluated at r_S should be finite and nonvanishing. This fact has the highly nontrivial consequence that, in this case, C can be chosen to be nonvanishing and this allows one to construct infinite examples of curved backgrounds supporting copies of the vacuum satisfying the strong boundary conditions. To see this, one can consider, for instance, a monotone function α varying from the value at the horizon to 2π at infinity:

$$\pi < \alpha(r_S) < 2\pi, \quad (97)$$

$$\alpha(r) \xrightarrow{r \rightarrow \infty} 2\pi, \quad \partial_r \alpha > 0 \quad \forall r > r_S, \quad \partial_r \alpha \xrightarrow{r \rightarrow \infty} 0. \quad (98)$$

Since in the denominator of Eq. (96) C can be chosen at will, one can take a positive value of C large enough to prevent any change of sign in the denominator:

$$C > 4 \int_{r_S}^{\infty} |x^2 (\partial_x \alpha) (\sin \alpha(x)) dx|;$$

the above condition also implies that the integral in the denominator of Eq. (96) has to converge. If the above constraint is satisfied, then f^2 is everywhere positive and the corresponding background metric supports, by construction, a copy of the vacuum satisfying the strong boundary conditions. This argument shows that there is a huge freedom in constructing background supporting such copies of the vacuum since the function α , besides the conditions in Eqs. (97) and (98), can be chosen arbitrarily.

A consequence of the present analysis is that a space-time supporting copies of the vacuum of the form in Eq. (9) with strong boundary condition cannot be asymptotically Minkowski (as it can be verified directly by expanding, for large r , Eq. (96)) whereas it can be asymptotically AdS provided

$$\alpha(r) \xrightarrow{r \rightarrow \infty} 2\pi + \frac{k}{r^2} + O(1/r^3),$$

$$\partial_r \alpha \xrightarrow{r \rightarrow \infty} -\frac{2k}{r^3} + O(1/r^4),$$

k being a real constant.

It is worth pointing out that in this construction the value of α at r_S must be different from its value at infinity, otherwise its derivative would be somewhere zero and the metric would be singular there. This implies that these vacuum Gribov copies cannot have trivial winding [12].

Of course, as it has been already emphasized, even when the copy has a nontrivial winding the corresponding gauge transformation represents a proper gauge transformation which cannot be discarded provided the strong boundary conditions are satisfied as it does not change the value of the observables.

Moreover, in the cases in which it is possible to find a copy also for $0 \leq r < r_S$ (the region which may represent the interior of the star), then one could match in a smooth (C^1) way the interior and the exterior copies to get a globally defined copy without any winding.¹¹ The C^1 matching of the copy appears to be possible due to the freedom given by the integration constants A and B appearing in (38).

V. CONCLUSIONS AND FURTHER COMMENTS

In this paper we analyzed the curved generalization of the Gribov pendulum in the Coulomb gauge on static spherically symmetric space-times. Using tools of non-linear functional analysis, we explored the issue of existence and uniqueness of solution of the Gribov pendulum on asymptotically AdS space-times in terms of an effective Schrödinger equation. Furthermore, we constructed necessary conditions in order for a curved static spherically symmetric background to admit copies of the vacuum satisfying the strong boundary conditions. An interesting consequence of the present analysis is that asymptotically Minkowski space-times do not admit vacuum copies of the Gribov form in Eq. (9) fulfilling the strong boundary conditions. This strongly suggests that as it happens in flat space-time, QCD at perturbative level is not affected by Gribov ambiguities in such cases.

The situation changes dramatically when one considers asymptotically AdS space-times. In these cases vacuum copies can appear depending on the structure of the interior bulk space-time. In particular black holes do not admit vacuum copies of the Gribov form (9) whereas an asymptotically AdS space-time containing, for instance, a star does admit vacuum copies satisfying the strong boundary conditions. An interesting issue arises if one considers the gravitational collapse of a star to a black hole with AdS asymptotics. This would imply a sudden change in the size of the Gribov horizon. According to the Gribov-Zwanziger approach this would imply a sudden change in the infrared behavior of QCD in these space-times.

Assuming the validity of the Gribov-Zwanziger procedure (which is supported by lattice data), the strong dependence of QCD on the structure of the interior bulk space-time may have also interesting consequences for the AdS/CFT correspondence. Indeed, the Gribov problem affects directly the gluon propagator and the AdS/CFT

¹¹It is worth remembering that in the expression for the winding number of a gauge transformation U only first derivatives of U appear so that the winding number is well defined whenever U is $C^1(M)$ (M being the space-time of interest).

correspondence is a statement about gauge-invariant operators. Notwithstanding, in the present framework the presence of Gribov copies is also relevant as far as confinement is concerned, which implies that one should also expect that gauge-invariant operators will be modified.

Our results suggest that the Gribov-Zwanziger confinement picture is stable under perturbations of the flat background metric that do not change the asymptotic structure. The reason is that for the family of copies considered here, the asymptotically flat case behaves in a very similar manner as the flat case. Our results also suggest that the pattern of appearance of Gribov copies inside a star (as well as outside a star with AdS asymptotic) should be very different from the flat case due to the presence of an intrinsic length scale of the problem (the radius of the star). In particular, in a curved background in which also the vacuum possesses copies satisfying the strong boundary conditions the very notion of asymptotic freedom could change dramatically. In fact, the usual scenario on flat spaces is that the deep ultraviolet region corresponds to the trivial vacuum $A_\mu = 0$, which is free of Gribov copies satisfying the strong boundary conditions. Thus, the absence of a region of the functional space of the gauge potential A_μ free of strong copies could be interpreted as the absence of a perturbative-deconfined region. This problem could be of great interest also because of its astrophysical implications and it is currently under investigation.

It is also worth pointing out a further interesting possibility related with the present scenario. Our results suggest that when a star living in an asymptotically AdS space-time undergoes a gravitational collapse to a black hole, the strong Gribov copies of the vacuum may disappear in very much the same way as it happens in the flat case. This suggests that a gravitational collapse in an asymptotically AdS space-time could induce a sort of phase transition for the QCD degrees of freedom outside the star collapsing to the black hole corresponding to the appearance, in the black hole phase, of a Gribov horizon around the trivial vacuum $A_\mu = 0$. This issue and also how such a phase transition may be visible in the dual boundary theory is not at all clear up to now and will be an issue of further investigation.

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