

Summing planar diagrams by an integrable bootstrap

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Correlation functions of matrix-valued fields are not generally known for massive renormalized field theories. We find the large- N limit of form factors of the $(1+1)$ -dimensional sigma model with $SU(N) \times SU(N)$ symmetry. These form factors give a correction to the free-field approximation for the $N = \infty$ Wightman function. The method is a combination of the $1/N$ expansion of the S matrix and Smirnov's form-factor axioms. We expand the renormalized field in terms of a free massive bosonic field as $N \rightarrow \infty$.

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I. INTRODUCTION

The planarity of Feynman diagrams in the large- N limit of matrix theories [1] has convinced many people that this limit is solvable. Unfortunately, little is known with precision about the $1/N$ expansion of $(N \times N)$ -matrix-valued field theories with *propagating degrees of freedom* (i.e., particles). Aside from maximally supersymmetric, conformal-invariant theories, the only exceptions are $(1+1)$ -dimensional quantum chromodynamics [2] and string models with Chan-Paton factors [3]. Massive matrix-field theories are not solvable by straightforward saddle-point approaches. The saddle-point method works only for field theories whose $N = \infty$ diagrams are not just planar, but linear. In this paper, we make some progress by melding the large- N expansion with the form-factor bootstrap. Perhaps our results will point to the solution of the planar limit in situations where this bootstrap does not work.

The S matrix of the $(1+1)$ -dimensional nonlinear sigma model with $SU(N) \times SU(N)$ symmetry is known. Unfortunately, its form factors are not, with the notable exception of $SU(2) \times SU(2) \simeq O(4)$ [4]. We study here the leading $1/N$ expansion of the form factors of this sigma model, also known as the principal chiral model. The bare field is a matrix $U(x)$, lying in the fundamental representation of $SU(N)$, where x^0 and x^1 are the time and space coordinates, respectively, of $(1+1)$ -dimensional Minkowski space-time. The action is

$$S = \frac{N}{2g_0^2} \int d^2x \eta^{\mu\nu} \text{Tr} \partial_\mu U(x)^\dagger \partial_\nu U(x), \quad (1.1)$$

where $\mu, \nu = 0, 1$, $U(x) \in SU(N)$ [that is, $U(x)$ is an $N \times N$ unitary matrix of determinant one], and the metric is that of flat Minkowski space, $\eta^{00} = 1$, $\eta^{11} = -1$, $\eta^{01} = \eta^{10} = 0$. The action does not change under the global transformation $U(x) \rightarrow V_L U(x) V_R$, for two constant matrices $V_L, V_R \in SU(N)$. We do not consider the addition of a

Wess-Zumino-Witten term to this action. The sigma model is asymptotically free. All the evidence indicates that the Hamiltonian spectrum has a mass gap m_1 , though no rigorous proof exists.

We study here the one-particle and three-particle form factors of the renormalized field operator $\Phi(x)$ (there are no two-particle form factors for $N > 3$). This field may be expressed in a theory with ultraviolet cutoff Λ as

$$\Phi(x) = Z(g_0, \Lambda)^{-1/2} U(x), \quad (1.2)$$

where g_0 is the coupling. The renormalization factor $Z(g_0(\Lambda), \Lambda)$ vanishes in the limit $\Lambda \rightarrow \infty$, where the running coupling $g_0(\Lambda)$ is defined so that the mass gap $m_1(g_0(\Lambda), \Lambda)$ is independent of Λ .

The S matrix of the principal chiral model has been found using the integrable bootstrap [5,6] and a subtle Bethe ansatz argument [7]. The essential ideas of the former approach begin from a general classification of $U(N)$ -symmetric S matrices for vector particles [8]. One such S matrix has no backward scattering [9]; hence the effective symmetry is $SU(N)$. The tensor product of two of these vector-particle S matrices yields the general S matrix with $SU(N) \times SU(N)$ symmetry, up to a Castillejo, Dalitz, and Dyson (CDD) factor. The requirement of a sine formula for bound-state masses (which follows from relativistic kinematics [10]) restricts the form of the CDD factor.

In this paper, we combine the $1/N$ expansion of the S matrix [6] with Smirnov's axioms [11], to obtain the three-particle form factors of the renormalized field operator $\Phi(x)$. The Lehmann, Symanzik, and Zimmermann (LSZ) reduction formula is used to fix the overall normalization [12].

There is an obvious advantage using the $1/N$ expansion to study correlation functions. Field theories with unitary symmetry have both fundamental or *elementary* particles and bound states. Particle masses are given by the sine formula mentioned above:

$$m_r = m_1 \frac{\sin \frac{\pi r}{N}}{\sin \frac{\pi}{N}}, \quad r = 1, \dots, N-1, \quad (1.3)$$

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where each choice of $r > 1$ corresponds to a bound state of r elementary particles. These bound states reveal themselves as poles in S -matrix elements. Particles with $r > 1$ make the determination of form factors difficult, though progress has been made [13]. The picture simplifies dramatically as $N \rightarrow \infty$, because the binding energy per particle number vanishes. The asymptotic states of the S matrix, with r or $N - r$ finite, consist only of $r = 1$ particles and $r = N - 1$ antiparticles, to any finite order of $1/N$. There are, however, bound states of infinite numbers of elementary particles, which correspond to keeping $r/N = \rho$ fixed, as $N \rightarrow \infty$ [14]. These bound states of infinitely many particles have mass $\approx Nm_1(\sin\rho)/\pi$, which becomes infinite in the 't Hooft limit, with m_1 fixed. There are continuously many such bound states, so their measure of integration must also be considered. We believe, however, that such bound states do not contribute to the $N \rightarrow \infty$ Wightman correlation function; they would produce unphysical cuts in momentum space. In an alternative large- N limit (not the 't Hooft limit, which we examine here), with $m_1/\sin\frac{\pi}{N} \approx Nm_1/\pi$ fixed, the parameter r/N becomes continuous, playing the role of a third space-time dimension [15].

The main drawback of our approach is that bound-state corrections are not analytic in powers of $1/N$. In our view, this is outweighed by the simplicity of the form-factor bootstrap in the planar limit.

Our interest in this problem began with applications of exact S matrices and form factors of the $SU(N)$ sigma model to $(2 + 1)$ -dimensional $SU(N)$ gauge theories [16]. The quark-antiquark potential [17] and the gluon mass spectrum [18] can be found at arbitrarily small but anisotropic gauge coupling. There is, unfortunately, a crossover from $(1 + 1)$ -dimensional to $(2 + 1)$ -dimensional behavior. A similar crossover is an obstacle to using the form factors of the two-dimensional Ising spin field to calculate critical exponents of the three-dimensional Ising model. Konik and Adamov were able to overcome this dimensional crossover for the Ising case with a density-matrix real-space renormalization group [19]. The triviality of the S matrix as $N \rightarrow \infty$ may help defeat the crossover for $SU(N)$ gauge theories. The reason is that the energy eigenstates of the $SU(\infty)_L \times SU(\infty)_R$ sigma model are simply Fock states of bosons, in the appropriate basis. Our hope is that this will make a real-space-renormalization-group approach feasible for the non-Abelian gauge theory.

We assume no previous knowledge of exact form factors. The reader unfamiliar with integrable-bootstrap

methods could simply take the $1/N$ -expanded form of the S matrix [in Eq. (2.6) below] on faith. Otherwise, we recommend starting with the summary by Zamolodchikov and Zamolodchikov [20]. The task of working through Ref. [20] may be simplified by consulting Ref. [21] (especially for infinite-product formulas for the S matrix) and the appendix of the first of Ref. [17] (in which some results are derived from scratch). We also recommend Ref. [10], in which the sine law is explained. From there, the papers on $U(N)$ - and $SU(N)$ -invariant theories of Berg *et al.* [8] and Kurak and Swieca [9] should be accessible. With this preparation, the reader should be ready to follow the derivation of the S matrix of the principal chiral model [5,6].

In the next section we discuss integrability and the $1/N$ expansion of the principal chiral model. We find the matrix element of the field operator between the vacuum and three-particle (more precisely one-antiparticle, two-antiparticle) state in Sec. III. We write the leading terms of the two-point Wightman function in Sec. IV. The form factors may be thought of as an expansion of the field operator in terms of a free field, which we briefly discuss in Sec. V. We present some conclusions and open questions in Sec. VI.

II. THE $1/N$ EXPANSION OF THE S MATRIX AND THE FIELD ALGEBRA

The basic Wightman correlation function is

$$\mathcal{W}(x) = \frac{1}{N} \langle 0 | \text{Tr} \Phi(0) \Phi(x)^\dagger | 0 \rangle, \quad (2.1)$$

where the scaling field Φ is defined by (1.2) and the normalization condition

$$\langle 0 | \Phi(0)_{b_0 a_0} | P, \theta, a_1, b_1 \rangle = N^{-1/2} \delta_{a_0 a_1} \delta_{b_0 b_1}, \quad (2.2)$$

where the ket on the right is a one-particle ($r = 1$) state, with rapidity θ (that is, with momentum components $p_0 = m \cosh\theta$, $p_1 = m \sinh\theta$) and we implicitly sum over left and right colors a_1 and b_1 , respectively.

The expression (2.2) is the most elementary form factor. It is similar to the definition of the scaling field in the Ising model [12]. We will determine the normalization of the other form factors using (2.2) and the LSZ reduction formula. The leading contribution to the Wightman function comes from the one-particle-intermediate-state approximation (or free-field approximation)

$$\mathcal{W}(x) \approx \frac{1}{N} \int \frac{d\theta}{4\pi} e^{im(x^0 \cosh\theta - x^1 \sinh\theta)} \langle 0 | \Phi(0)_{b_0 a_0} | P, \theta, a_1, b_1 \rangle_{\text{in}} \langle P, \theta, a_1, b_1 | \Phi(0)_{b_0 a_0}^* | 0 \rangle, \quad (2.3)$$

where m denotes m_1 and the sum over all repeated color indices is implicit. For $x^0 = 0$, $x^1 = \pm|x|$, this is

$$\mathcal{W}(x) \approx \frac{1}{4\pi} K_0(m|x|).$$

Note that this expression is of order $(1/N)^0$. We are assuming that there is no contribution from the one-particle state (with $r = N - 1$), i.e.,

$$\langle 0 | \Phi(0)_{b_0 a_0} | A, \theta, b_1, a_1 \rangle_{\text{in}} = 0.$$

The S matrix can be determined, assuming unitarity, factorization (the Yang-Baxter relation), and maximal analyticity. The basic $r = 1$ excitations have two color indices from 1 to N . One can view these excitations as a bound pair of two quarks of different color sectors (or alternatively as a quark in one color sector and an antiquark in the other). Such quarks can be regarded as the elementary physical excitations of the chiral Gross-Neveu model [8,9,22].

Next we show the S matrix of two elementary particles of the sigma model, with incoming rapidities θ_1 and θ_2 [we use the definition $(p_j)_0 = m \cosh \theta_j$, $(p_j)_1 = m \sinh \theta_j$, relating the momentum vector p_j and rapidity θ_j], outgoing rapidities θ'_1 and θ'_2 , and rapidity difference $\theta = \theta_{12} = \theta_1 - \theta_2$. This is

$$S_{PP} = S_{PP}(|\theta|) 4\pi \delta(\theta'_1 - \theta_1) 4\pi \delta(\theta'_2 - \theta_2),$$

where $S_{PP}(|\theta|)$ is a function which acts on the quantum numbers of the particles [in some papers, $\delta(p_j - p'_j)$ is written, incorrectly, in place of $4\pi \delta(\theta_j - \theta'_j)$]. The quantity $S_{PP}(|\theta|)$ is nearly always referred to as the S matrix in the literature. It is explicitly given by

$$S_{PP}(\theta) = \frac{\sin(\theta/2 - \pi i/N)}{\sin(\theta/2 + \pi i/N)} S_{\text{CGN}}(\theta)_L \otimes S_{\text{CGN}}(\theta)_R, \quad (2.4)$$

where $S_{\text{CGN}}(\theta)_{L,R}$, for either the subscript L (left) or R (right), is the S matrix of two elementary excitations of the chiral Gross-Neveu model:

$$S_{\text{CGN}}(\theta) = \frac{\Gamma(i\theta/2\pi + 1)\Gamma(-i\theta/2\pi - 1/N)}{\Gamma(i\theta/2\pi + 1 - 1/N)\Gamma(-i\theta/2\pi)} \left(1 - \frac{2\pi i}{N\theta} P\right), \quad (2.5)$$

where P switches the colors of the elementary Gross-Neveu particles. S -matrix elements for which one or both particles have $r > 1$ can be found by fusion.

We shall define the generalized S matrix to be (2.4) with $|\theta|$ replaced by $\theta = \theta_{12} = \theta_1 - \theta_2$. This is consistent with the definition given in Ref. [23] (where it is called the auxiliary S matrix).

The first few terms of the $1/N$ expansion of (2.4) are [6]

$$S_{PP}(\theta) = [1 + O(1/N^2)] \times \left[1 - \frac{2\pi i}{N\theta} (P \otimes 1 + 1 \otimes P) - \frac{4\pi^2}{N^2 \theta^2} P \otimes P \right]. \quad (2.6)$$

We can find the scattering matrix of one-particle and one-antiparticle $S_{AP}(\theta)$ from (2.6), using crossing.

There is an exceptional value of θ where the particle-particle S matrix does not become unity as $N \rightarrow \infty$. One of these is at $\theta = 0$. For vanishing relative rapidity, Eq. (2.4) yields $S_{PP}(0) = -P \otimes P$, independently of N ; thus the expansion (2.6) is not valid at $\theta = 0$. A similar breakdown of the $1/N$ expansion at $\theta = 0$ occurs for models with $O(N)$ symmetry [20,24]. This point corresponds to the threshold $s = 4m^2$, where s is the Mandelstam variable, related to the relative rapidity by $s = 2m^2 + 2m^2 \cosh \theta$. At this threshold, both particles have vanishing momenta in the center-of-mass frame, and exchange their left and right colors with probability one. In relativistic scattering theory the S matrix has a cut from the s -channel threshold $s = 4m^2$ to $s = \infty$, and another cut from the t -channel threshold $s = 4m^2 - t = 0$ to $s = -\infty$. Another exceptional value where the S matrix is not unity as $N \rightarrow \infty$ is $\theta = 2\pi i/N$, where the $r = 2$ bound-state occurs. In the complex θ plane, the first cut is the image of the line $\text{Im } \theta = 0$, and the other cut is the image of the line $\text{Im } \theta = 2\pi$ [20]. Between these two lines, in the interior of the so-called physical strip, excluding bound-state poles, the expansion (2.6) is valid, which is sufficient for the remaining discussion in this paper.

The basic properties of particle states are encoded in the Zamolodchikov algebra. Let us introduce particle-creation operators $\mathfrak{A}_P^\dagger(\theta)_{ab}$ and antiparticle creation operators $\mathfrak{A}_A^\dagger(\theta)_{ba}$. This algebra is essentially a non-Abelian particle-statistics relation:

$$\begin{aligned} \mathfrak{A}_P^\dagger(\theta_1)_{a_1 b_1} \mathfrak{A}_P^\dagger(\theta_2)_{a_2 b_2} &= S_{PP}(\theta_{12})_{a_1 b_1; a_2 b_2}^{c_2 d_2; c_1 d_1} \mathfrak{A}_P^\dagger(\theta_2)_{c_2 d_2} \mathfrak{A}_P^\dagger(\theta_1)_{c_1 d_1}, \\ \mathfrak{A}_A^\dagger(\theta_1)_{b_1 a_1} \mathfrak{A}_A^\dagger(\theta_2)_{b_2 a_2} &= S_{AA}(\theta_{12})_{b_1 a_1; b_2 a_2}^{d_2 c_2; d_1 c_1} \mathfrak{A}_A^\dagger(\theta_2)_{d_2 c_2} \mathfrak{A}_A^\dagger(\theta_1)_{d_1 c_1}, \\ \mathfrak{A}_P^\dagger(\theta_1)_{a_1 b_1} \mathfrak{A}_A^\dagger(\theta_2)_{b_2 a_2} &= S_{AP}(\theta_{12})_{a_1 b_1; b_2 a_2}^{d_2 c_2; c_1 d_1} \mathfrak{A}_A^\dagger(\theta_2)_{d_2 c_2} \mathfrak{A}_P^\dagger(\theta_1)_{c_1 d_1}. \end{aligned} \quad (2.7)$$

The Yang-Baxter relation is necessary as a consistency condition for (2.7). That is one way to understand why the absence of particle production implies integrability.

An in-state is defined as a product of creation operators in the order of increasing rapidity, from right to left, acting on the vacuum, e.g.,

$$|P, \theta_1, a_1, b_1; A, \theta_2, b_2, a_2, \dots\rangle_{\text{in}} = \mathfrak{A}_P^\dagger(\theta_1)_{a_1 b_1} \mathfrak{A}_A^\dagger(\theta_2)_{b_2 a_2} \cdots |0\rangle,$$

where $\theta_1 > \theta_2 > \cdots$. (2.8)

Similarly, an out-state is a product of creation operators in the order of decreasing rapidity, from right to left, acting on the vacuum.

The expression (2.6) becomes unity as $N \rightarrow \infty$, as we would expect. The algebra (2.7) thereby trivializes. Consider the field

$$M(x) = \int \frac{d\theta}{4\pi} [\mathfrak{A}_P(\theta) e^{imx^0 \cosh\theta - imx^1 \sinh\theta} + \mathfrak{A}_A^\dagger(\theta) e^{-imx^0 \cosh\theta + imx^1 \sinh\theta}], \quad (2.9)$$

where \mathfrak{A}_A is the destruction operator of an antiparticle. It is simply the adjoint of the operator \mathfrak{A}_A^\dagger . In the limit $N \rightarrow \infty$, $[\mathfrak{A}_{A,P}(\theta), \mathfrak{A}_{A,P}^\dagger(\theta')] \rightarrow 4\pi\delta(\theta - \theta')$, with all other commutators approaching zero (the commutators are more complicated for finite N). The $N \times N$ -matrix-valued field operator $M(x)$ is a massive free field. The form factors give the coefficients of an expansion of the renormalized field $\Phi(x)$ in terms of this field.

The form factors are matrix elements between the vacuum and multiparticle in-states of the field operator Φ . The

action of the global-symmetry transformation on Φ and the creation operators is

$$\Phi(x) \rightarrow V_L \Phi(x) V_R, \quad \mathfrak{A}_P^\dagger(\theta) \rightarrow V_R^\dagger \mathfrak{A}_P^\dagger(\theta) V_L, \\ \mathfrak{A}_A^\dagger(\theta) \rightarrow V_L \mathfrak{A}_A^\dagger(\theta) V_R. \quad (2.10)$$

Thus we expect that, for large N , the condition

$$\langle 0 | \Phi(0) | \Psi \rangle \neq 0,$$

on an in-state $|\Psi\rangle$, which is an eigenstate of particle number, holds only if $|\Psi\rangle$ contains m particles and $m - 1$ antiparticles, for some $m = 1, 2, \dots$. In the next section, we will find these matrix elements for $m = 2$ (the $m = 1$ case has already been discussed above).

III. MAXIMALLY ANALYTIC FORM FACTORS

In this section we will study matrix elements of the form $\langle 0 | \Phi(0) | \Psi \rangle$, where $|\Psi\rangle$ is an in-state with two elementary particles and one antiparticle, i.e., $m = 2$. This matrix element is defined for general choices of rapidity. Here are the form factors corresponding to different orderings of rapidities:

$$\begin{aligned} & \langle 0 | \Phi(0)_{b_0 a_0} | A, \theta_1, b_1, a_1; P, \theta_2, a_2, b_2; P, \theta_3, a_3, b_3 \rangle_{\text{in}} \\ &= \langle 0 | \Phi(0)_{b_0 a_0} \mathfrak{A}_A^\dagger(\theta_1)_{b_1 a_1} \mathfrak{A}_P^\dagger(\theta_2)_{a_2 b_2} \mathfrak{A}_P^\dagger(\theta_3)_{a_3 b_3} | 0 \rangle \\ &= \frac{1}{N^{3/2}} F_1(\theta_1, \theta_2, \theta_3) \delta_{a_0 a_2} \delta_{b_0 b_3} \delta_{b_1 b_2} \delta_{a_1 a_3} + \frac{1}{N^{3/2}} F_2(\theta_1, \theta_2, \theta_3) \delta_{a_0 a_3} \delta_{b_0 b_2} \delta_{a_1 a_2} \delta_{b_1 b_3} \\ &+ \frac{1}{N^{3/2}} F_3(\theta_1, \theta_2, \theta_3) \delta_{a_0 a_2} \delta_{b_0 b_2} \delta_{a_1 a_3} \delta_{b_1 b_3} + \frac{1}{N^{3/2}} F_4(\theta_1, \theta_2, \theta_3) \delta_{a_0 a_3} \delta_{b_0 b_3} \delta_{b_1 b_2} \delta_{a_1 a_2}, \end{aligned} \quad (3.1)$$

for $\theta_1 > \theta_2 > \theta_3$,

$$\begin{aligned} & \langle 0 | \Phi(0)_{b_0 a_0} | P, \theta_1, a_1, b_1; A, \theta_2, b_2, a_2; P, \theta_3, a_3, b_3 \rangle_{\text{in}} \\ &= \langle 0 | \Phi(0)_{b_0 a_0} \mathfrak{A}_P^\dagger(\theta_2)_{a_2 b_2} \mathfrak{A}_A^\dagger(\theta_1)_{b_1 a_1} \mathfrak{A}_P^\dagger(\theta_3)_{a_3 b_3} | 0 \rangle \\ &= \frac{1}{N^{3/2}} \tilde{F}_1(\theta_1, \theta_2, \theta_3) \delta_{a_0 a_2} \delta_{b_0 b_3} \delta_{b_1 b_2} \delta_{a_1 a_3} + \frac{1}{N^{3/2}} \tilde{F}_2(\theta_1, \theta_2, \theta_3) \delta_{a_0 a_3} \delta_{b_0 b_2} \delta_{a_1 a_2} \delta_{b_1 b_3} \\ &+ \frac{1}{N^{3/2}} \tilde{F}_3(\theta_1, \theta_2, \theta_3) \delta_{a_0 a_2} \delta_{b_0 b_2} \delta_{a_1 a_3} \delta_{b_1 b_3} + \frac{1}{N^{3/2}} \tilde{F}_4(\theta_1, \theta_2, \theta_3) \delta_{a_0 a_3} \delta_{b_0 b_3} \delta_{b_1 b_2} \delta_{a_2 a_1}, \end{aligned} \quad (3.2)$$

for $\theta_2 > \theta_1 > \theta_3$, and

$$\begin{aligned} & \langle 0 | \Phi(0)_{b_0 a_0} | P, \theta_1, a_1, b_1; P, \theta_2, a_2, b_2; A, \theta_3, b_3, a_3 \rangle_{\text{in}} \\ &= \langle 0 | \Phi(0)_{b_0 a_0} \mathfrak{A}_P^\dagger(\theta_2)_{a_2 b_2} \mathfrak{A}_P^\dagger(\theta_3)_{a_3 b_3} \mathfrak{A}_A^\dagger(\theta_1)_{b_1 a_1} | 0 \rangle \\ &= \frac{1}{N^{3/2}} \tilde{\tilde{F}}_1(\theta_1, \theta_2, \theta_3) \delta_{a_0 a_2} \delta_{b_0 b_3} \delta_{b_1 b_2} \delta_{a_1 a_3} + \frac{1}{N^{3/2}} \tilde{\tilde{F}}_2(\theta_1, \theta_2, \theta_3) \delta_{a_0 a_3} \delta_{b_0 b_2} \delta_{a_1 a_2} \delta_{b_1 b_3} \\ &+ \frac{1}{N^{3/2}} \tilde{\tilde{F}}_3(\theta_1, \theta_2, \theta_3) \delta_{a_0 a_2} \delta_{b_0 b_2} \delta_{a_1 a_3} \delta_{b_1 b_3} + \frac{1}{N^{3/2}} \tilde{\tilde{F}}_4(\theta_1, \theta_2, \theta_3) \delta_{a_0 a_3} \delta_{b_0 b_3} \delta_{b_1 b_2} \delta_{a_2 a_1}, \end{aligned} \quad (3.3)$$

for $\theta_3 > \theta_1 > \theta_2$. We note that (3.1) is equivalent to

$$\begin{aligned}
 & \langle 0 | \Phi(0)_{b_0 a_0} | A, \theta_1, b_1, a_1; P, \theta_3, a_3, b_3; P, \theta_2, a_2, b_2 \rangle_{\text{in}} \\
 &= \langle 0 | \Phi(0)_{b_0 a_0} \mathfrak{Y}_A^\dagger(\theta_1)_{b_1 a_1} \mathfrak{Y}_P^\dagger(\theta_3)_{a_3 b_3} \mathfrak{Y}_P^\dagger(\theta_2)_{a_2 b_2} | 0 \rangle \\
 &= \frac{1}{N^{3/2}} F_2(\theta_1, \theta_3, \theta_2) \delta_{a_0 a_2} \delta_{b_0 b_3} \delta_{b_1 b_2} \delta_{a_1 a_3} + \frac{1}{N^{3/2}} F_1(\theta_1, \theta_3, \theta_2) \delta_{a_0 a_3} \delta_{b_0 b_2} \delta_{a_1 a_2} \delta_{b_1 b_3} \\
 &+ \frac{1}{N^{3/2}} F_4(\theta_1, \theta_3, \theta_2) \delta_{a_0 a_2} \delta_{b_0 b_2} \delta_{a_1 a_3} \delta_{b_1 b_3} + \frac{1}{N^{3/2}} F_3(\theta_1, \theta_3, \theta_2) \delta_{a_0 a_3} \delta_{b_0 b_3} \delta_{b_1 b_2} \delta_{a_1 a_2}, \tag{3.4}
 \end{aligned}$$

for $\theta_1 > \theta_3 > \theta_2$.

We generalize the form factor [23], so that (3.1), (3.2), (3.3), and (3.4) are valid without the inequalities on the arguments $\theta_{1,2,3}$.

In each of the expressions (3.1), (3.2), (3.3), and (3.4), we have written the quantity on the right in a similar way. Each of the products of Kronecker deltas is a possible covariant tensor of the global color symmetry. No other combinations are allowed for $N > 3$, by Eq. (2.10).

Notice that Lorentz invariance implies that the scalar functions F , G , and H are unchanged under an overall boost $\theta_j \rightarrow \theta_j + \Delta\theta$, $j = 1, 2, 3$. This means that the form factors depend only on differences of the rapidities.

If we examine the contribution of these form factors to the Wightman function $C(x)$, defined in (2.1), we see that F , \tilde{F} , and $\tilde{\tilde{F}}$ must be multiplied by $N^{-3/2}$, as we have in (3.1), (3.2), (3.3), and (3.4). We will eventually show in this section that $F_{3,4}$, $\tilde{F}_{3,4}$, and $\tilde{\tilde{F}}_{3,4}$ are down by a further power

of N . This means we could have written (3.1), (3.2), (3.3), and (3.4) with the coefficient $1/N^{5/2}$ in front of the last two entries, instead of $1/N^{3/2}$. These are the coefficients of tensors where the both quantum numbers of the antiparticle coincide with both of those for the one of the particles. For the time being, however, we will treat $F_{3,4}$, $\tilde{F}_{3,4}$, and $\tilde{\tilde{F}}_{3,4}$ just like the other functions.

First we apply the scattering form-factor axiom, also called Watson's theorem. This axiom can be most simply understood as the application of the Zamolodchikov algebra to the vacuum expectation values in the first lines of Eqs. (3.1), (3.2), and (3.3) above. It is essentially the assumption that we can continue the functions F , G , and H outside the domain $\theta_1 < \theta_2 < \theta_3$, in such a way that the Zamolodchikov algebra is satisfied. For example, if we apply Watson's theorem on the incoming antiparticle with rapidity θ_1 and the incoming particle with rapidity θ_2 , on the left-hand side of (3.1) we find

$$\langle 0 | \Phi(0)_{b_0 a_0} \mathfrak{Y}_P^\dagger(\theta_1)_{a_1 b_1} \mathfrak{Y}_A^\dagger(\theta_2)_{b_2 a_2} \mathfrak{Y}_P^\dagger(\theta_3)_{a_3 b_3} | 0 \rangle = S_{AP}(\theta_{12})_{a_1 b_1; b_2 a_2}^{d_2 c_2; c_1 d_1} \langle 0 | \Phi(0)_{b_0 a_0} \mathfrak{Y}_A^\dagger(\theta_2)_{d_2 c_2} \mathfrak{Y}_P^\dagger(\theta_1)_{c_1 d_1} \mathfrak{Y}_P^\dagger(\theta_3)_{a_3 b_3} | 0 \rangle. \tag{3.5}$$

The $1/N$ expansion of the S -matrix element in (3.5) is

$$\begin{aligned}
 S_{AP}(\theta_{12})_{a_1 b_1; b_2 a_2}^{d_2 c_2; c_1 d_1} &= [1 + O(1/N^2)] \left[\delta_{b_2}^{d_2} \delta_{a_2}^{c_2} \delta_{a_1}^{c_1} \delta_{b_1}^{d_1} - \frac{2\pi i}{N \hat{\theta}_{12}} (\delta_{a_1 a_2} \delta^{c_1 c_2} \delta_{b_2}^{d_2} \delta_{b_1}^{d_1} + \delta_{a_2}^{c_2} \delta_{a_1}^{c_1} \delta_{b_1 b_2} \delta^{d_1 d_2}) \right. \\
 &\quad \left. - \frac{4\pi^2}{N^2 \hat{\theta}_{12}^2} \delta_{a_1 a_2} \delta^{c_1 c_2} \delta_{b_1 b_2} \delta^{d_1 d_2} \right], \tag{3.6}
 \end{aligned}$$

where $\hat{\theta}_{12} = \pi i - \theta_{12}$ is the rapidity difference after crossing from the s channel to the t channel. Inserting the explicit expressions on the right-hand sides of (3.1) and (3.2) into (3.5) and after some work, we find

$$\tilde{F}(\theta_1, \theta_2, \theta_3) = \begin{pmatrix} 1 - \frac{2\pi i}{\hat{\theta}_{12}} & 0 & -\frac{2\pi i}{N \hat{\theta}_{12}} & 0 \\ 0 & 1 - \frac{2\pi i}{\hat{\theta}_{12}} & -\frac{2\pi i}{N \hat{\theta}_{12}} & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{2\pi i}{N \hat{\theta}_{12}} \left(1 - \frac{2\pi i}{\hat{\theta}_{12}}\right) & -\frac{2\pi i}{N \hat{\theta}_{12}} \left(1 - \frac{2\pi i}{\hat{\theta}_{12}}\right) & 0 & \left(1 - \frac{2\pi i}{\hat{\theta}_{12}}\right)^2 \end{pmatrix} F(\theta_1, \theta_2, \theta_3) + O\left(\frac{1}{N^2}\right), \tag{3.7}$$

where we have denoted the four-component vectors in the obvious way, e.g.,

$$F(\theta_1, \theta_2, \theta_3) = \begin{pmatrix} F_1(\theta_1, \theta_2, \theta_3) \\ F_2(\theta_1, \theta_2, \theta_3) \\ F_3(\theta_1, \theta_2, \theta_3) \\ F_4(\theta_1, \theta_2, \theta_3) \end{pmatrix}.$$

In finding (3.7) some factors of N appeared as a result of contracting indices. These factors of N canceled some factors of $1/N$ in the second and third terms of the S -matrix element in (3.6).

There are two more useful relations following from the scattering axiom. These are

$$\langle 0 | \Phi(0)_{b_0 a_0} \mathfrak{A}_P^\dagger(\theta_2)_{a_2 b_2} \mathfrak{A}_P^\dagger(\theta_3)_{a_3 b_3} \mathfrak{A}_A^\dagger(\theta_1)_{b_1 a_1} | 0 \rangle = S_{AP}(\theta_{13})_{a_3 b_3; b_1 a_1}^{d_1 c_1; c_3 d_3} \langle 0 | \Phi(0)_{b_0 a_0} \mathfrak{A}_P^\dagger(\theta_2)_{a_2 b_2} \mathfrak{A}_A^\dagger(\theta_1)_{d_1 c_1} \mathfrak{A}_P^\dagger(\theta_3)_{c_3 d_3} | 0 \rangle,$$

which may be reexpressed as

$$\tilde{F}(\theta_1, \theta_2, \theta_3) = \begin{pmatrix} 1 - \frac{2\pi i}{\theta_{13}} & 0 & 0 & -\frac{2\pi i}{N\theta_{13}} \\ 0 & 1 - \frac{2\pi i}{\theta_{13}} & 0 & -\frac{2\pi i}{N\theta_{13}} \\ -\frac{2\pi i}{N\theta_{13}} \left(1 - \frac{2\pi i}{\theta_{13}}\right) & -\frac{2\pi i}{N\theta_{13}} \left(1 - \frac{2\pi i}{\theta_{13}}\right) & \left(1 - \frac{2\pi i}{\theta_{13}}\right)^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tilde{F}(\theta_1, \theta_2, \theta_3) + O\left(\frac{1}{N^2}\right), \quad (3.8)$$

and finally

$$\langle 0 | \Phi(0)_{b_0 a_0} \mathfrak{A}_A^\dagger(\theta_1)_{b_1 a_1} \mathfrak{A}_P^\dagger(\theta_2)_{a_2 b_2} \mathfrak{A}_P^\dagger(\theta_3)_{a_3 b_3} | 0 \rangle = S_{PP}(\theta_{23})_{a_2 b_2; a_3 b_3}^{c_2 d_2; c_3 d_3} \langle 0 | \Phi(0)_{b_0 a_0} \mathfrak{A}_A^\dagger(\theta_1)_{b_1 a_1} \mathfrak{A}_P^\dagger(\theta_3)_{c_3 d_3} \mathfrak{A}_P^\dagger(\theta_2)_{c_2 d_2} | 0 \rangle,$$

which reduces to

$$F(\theta_1, \theta_2, \theta_3) = \begin{pmatrix} 0 & 1 & -\frac{2\pi i}{N\theta_{23}} & -\frac{2\pi i}{N\theta_{23}} \\ 1 & 0 & -\frac{2\pi i}{N\theta_{23}} & -\frac{2\pi i}{N\theta_{23}} \\ -\frac{2\pi i}{N\theta_{23}} & -\frac{2\pi i}{N\theta_{23}} & 0 & 1 \\ -\frac{2\pi i}{N\theta_{23}} & -\frac{2\pi i}{N\theta_{23}} & 1 & 0 \end{pmatrix} F(\theta_1, \theta_3, \theta_2) + O\left(\frac{1}{N^2}\right). \quad (3.9)$$

Now in (3.8), some factors of $1/N$ in S -matrix elements were canceled after summing over indices, as we noted above for (3.7). This did not happen in obtaining (3.9). The reason is that the particle-particle S matrix (2.6) does not contract colors of incoming particles; colors can only be exchanged.

Another of Smirnov's axioms is the periodicity condition. This axiom is an application of crossing. Explicitly:

$$\langle 0 | \Phi(0)_{b_0 a_0} \mathfrak{A}_{I_1}^\dagger(\theta_1)_{C_1} \mathfrak{A}_{I_2}^\dagger(\theta_2)_{C_2} \cdots \mathfrak{A}_{I_M}^\dagger(\theta_M)_{C_M} | 0 \rangle = \langle 0 | \Phi(0)_{b_0 a_0} \mathfrak{A}_{I_M}^\dagger(\theta_M - 2\pi i)_{C_M} \mathfrak{A}_{I_1}^\dagger(\theta_1)_{C_1} \cdots \mathfrak{A}_{I_{M-1}}^\dagger(\theta_{M-1})_{C_{M-1}} | 0 \rangle, \quad (3.10)$$

where I_k , $k = 1, \dots, M$ is P or A (particle or antiparticle) and C_k denotes a pair of indices (which may be written $a_k b_k$, for $C_k = P$, and $b_k a_k$, for $C_k = A$). A brief explanation of (3.10) follows. For more details, see Ref. [23]. Consider what happens when a creation operator in front of the ket is replaced by an annihilation operator behind the bra by crossing. Consider the vacuum expectation value of creation operators and $\Phi(0)_{b_0 a_0}$:

$$\begin{aligned} & \langle 0 | \mathfrak{A}_{I_1}(\theta_1)_{C_1} \Phi(0)_{b_0 a_0} \mathfrak{A}_{I_M}^\dagger(\theta_M)_{C_M} \mathfrak{A}_{I_{M-1}}^\dagger(\theta_{M-1})_{C_{M-1}} \cdots \mathfrak{A}_{I_2}^\dagger(\theta_2)_{C_2} | 0 \rangle_{\text{connected}} \\ &= \langle 0 | \mathfrak{A}_{I_1}(\theta_1)_{C_1} \Phi(0)_{b_0 a_0} \mathfrak{A}_{I_M}^\dagger(\theta_M)_{C_M} \mathfrak{A}_{I_{M-1}}^\dagger(\theta_{M-1})_{C_{M-1}} \cdots \mathfrak{A}_{I_2}^\dagger(\theta_2)_{C_2} | 0 \rangle - \langle 0 | \mathfrak{A}_{I_1}(\theta_1)_{C_1} \Phi(0)_{b_0 a_0} | 0 \rangle \\ & \quad \times \langle 0 | \mathfrak{A}_{I_M}^\dagger(\theta_M)_{C_M} \mathfrak{A}_{I_{M-1}}^\dagger(\theta_{M-1})_{C_{M-1}} \cdots \mathfrak{A}_{I_2}^\dagger(\theta_2)_{C_2} | 0 \rangle. \end{aligned}$$

The subscript ‘‘connected’’ is included because the vacuum intermediate channel is subtracted [23]. This expression means $M - 1$ incoming particles are absorbed by a ‘‘probe,’’ corresponding to the operator $\Phi(0)_{b_0 a_0}$. This probe then emits a single particle. Consider the pair of particles, with labels 1 (the outgoing particle) and M . Under crossing, these both become incoming particles, but with θ_1 replaced by $\theta_1 - \pi i$. The reason is that $\theta_1 \rightarrow \theta_1 - \pi i$ preserves the relativistic invariants $s_{jj+1} = (p_j + p_{j+1})^2$, and $t_{jj+1} = (p_j - p_{j+1})^2$, where $j = 2, \dots, M - 1$, while interchanging the two invariants $s_{1M} = (p_1 + p_M)^2$ and $t_{1M} = (p_1 - p_M)^2$. Thus

$$\begin{aligned} & \langle 0 | \mathfrak{A}_{I_1}(\theta_1)_{C_1} \Phi(0)_{b_0 a_0} \mathfrak{A}_{I_M}^\dagger(\theta_M)_{C_M} \mathfrak{A}_{I_{M-1}}^\dagger(\theta_{M-1})_{C_{M-1}} \cdots \mathfrak{A}_{I_2}^\dagger(\theta_2)_{C_2} | 0 \rangle_{\text{connected}} \\ &= \langle 0 | \Phi(0)_{b_0 a_0} \mathfrak{A}_{I_1}^\dagger(\theta_1 - \pi i)_{C_1} \mathfrak{A}_{I_2}^\dagger(\theta_2)_{C_2} \cdots \mathfrak{A}_{I_M}^\dagger(\theta_M)_{C_M} | 0 \rangle. \end{aligned} \quad (3.11)$$

Suppose that instead of interchanging the invariants s_{1M} and t_{1M} , we interchange the invariants $s_{12} = (p_1 + p_2)^2$ and $t_{12} = (p_1 - p_2)^2$. Then we find

$$\begin{aligned} & \langle 0 | \mathfrak{U}_{I_1}(\theta_1)_{C_1} \Phi(0)_{b_0 a_0} \mathfrak{U}_{I_M}(\theta_M)_{C_M}^\dagger \mathfrak{U}_{I_{M-1}}(\theta_{M-1})_{C_{M-1}}^\dagger \cdots \mathfrak{U}_{I_2}(\theta_2)_{C_2}^\dagger | 0 \rangle_{\text{connected}} \\ & = \langle 0 | \Phi(0)_{b_0 a_0} \mathfrak{U}_{I_2}^\dagger(\theta_2)_{C_2} \mathfrak{U}_{I_3}^\dagger(\theta_3)_{C_3} \cdots \mathfrak{U}_{I_M}^\dagger(\theta_M)_{C_M} \mathfrak{U}_{I_1}^\dagger(\theta_1 + \pi i)_{C_1} | 0 \rangle. \end{aligned} \quad (3.12)$$

The periodicity axiom (3.10) follows from (3.11) and (3.12).

Notice that integrability was not used to justify (3.10). The periodicity axiom follows from very general considerations in $1 + 1$ dimensions [25].

The periodicity axiom implies the three relations

$$\begin{aligned} & \langle 0 | \Phi(0)_{b_0 a_0} \mathfrak{U}_A^\dagger(\theta_1 - 2\pi i)_{b_1 a_1} \mathfrak{U}_P^\dagger(\theta_2)_{a_2 b_2} \mathfrak{U}_P^\dagger(\theta_3)_{a_3 b_3} | 0 \rangle = \langle 0 | \Phi(0)_{b_0 a_0} \mathfrak{U}_P^\dagger(\theta_2)_{a_2 b_2} \mathfrak{U}_P^\dagger(\theta_3)_{a_3 b_3} \mathfrak{U}_A^\dagger(\theta_1)_{b_1 a_1} | 0 \rangle, \\ & \langle 0 | \Phi(0)_{b_0 a_0} \mathfrak{U}_P^\dagger(\theta_2 - 2\pi i)_{a_2 b_2} \mathfrak{U}_A^\dagger(\theta_1)_{b_1 a_1} \mathfrak{U}_P^\dagger(\theta_3)_{a_3 b_3} | 0 \rangle = \langle 0 | \Phi(0)_{b_0 a_0} \mathfrak{U}_A^\dagger(\theta_1)_{b_1 a_1} \mathfrak{U}_P^\dagger(\theta_3)_{a_3 b_3} \mathfrak{U}_P^\dagger(\theta_2)_{a_2 b_2} | 0 \rangle, \\ & \langle 0 | \Phi(0)_{b_0 a_0} \mathfrak{U}_P^\dagger(\theta_2 - 2\pi i)_{a_2 b_2} \mathfrak{U}_P^\dagger(\theta_3)_{a_3 b_3} \mathfrak{U}_A^\dagger(\theta_1)_{b_1 a_1} | 0 \rangle = \langle 0 | \Phi(0)_{b_0 a_0} \mathfrak{U}_P^\dagger(\theta_3)_{a_3 b_3} \mathfrak{U}_A^\dagger(\theta_1)_{b_1 a_1} \mathfrak{U}_P^\dagger(\theta_2)_{a_2 b_2} | 0 \rangle, \end{aligned}$$

which may be written as

$$F(\theta_1 - 2\pi i, \theta_2, \theta_3) = \tilde{F}(\theta_1, \theta_2, \theta_3), \quad (3.13)$$

$$\tilde{F}(\theta_1, \theta_2 - 2\pi i, \theta_3) = F(\theta_1, \theta_3, \theta_2), \quad (3.14)$$

$$\tilde{F}(\theta_1, \theta_2 - 2\pi i, \theta_3) = \tilde{F}(\theta_1, \theta_3, \theta_2), \quad (3.15)$$

respectively.

Our work is simplified by expanding the form factors in powers of $1/N$,

$$F(\theta_1, \theta_2, \theta_3) = F^0(\theta_1, \theta_2, \theta_3) + \frac{1}{N} F^1(\theta_1, \theta_2, \theta_3) + \cdots, \quad (3.16)$$

and similarly for $\tilde{F}(\theta_1, \theta_2, \theta_3)$ and $\tilde{\tilde{F}}(\theta_1, \theta_2, \theta_3)$. We truncate this expansion to leading order, keeping only $F^0(\theta_1, \theta_2, \theta_3)$, $\tilde{F}^0(\theta_1, \theta_2, \theta_3)$, and $\tilde{\tilde{F}}^0(\theta_1, \theta_2, \theta_3)$.

Combining (3.7) and (3.8) with (3.13), we find

$$\begin{aligned} F_1^0(\theta_1 - 2\pi i, \theta_2, \theta_3) &= \frac{\theta_{12} + \pi i}{\theta_{12} - \pi i} \frac{\theta_{13} + \pi i}{\theta_{13} - \pi i} F_1^0(\theta_1, \theta_2, \theta_3), \\ F_2^0(\theta_1 - 2\pi i, \theta_2, \theta_3) &= \frac{\theta_{12} + \pi i}{\theta_{12} - \pi i} \frac{\theta_{13} + \pi i}{\theta_{13} - \pi i} F_2^0(\theta_1, \theta_2, \theta_3), \\ F_3^0(\theta_1 - 2\pi i, \theta_2, \theta_3) &= \left(\frac{\theta_{13} + \pi i}{\theta_{13} - \pi i} \right)^2 F_3^0(\theta_1, \theta_2, \theta_3), \\ F_4^0(\theta_1 - 2\pi i, \theta_2, \theta_3) &= \left(\frac{\theta_{12} + \pi i}{\theta_{12} - \pi i} \right)^2 F_4^0(\theta_1, \theta_2, \theta_3). \end{aligned} \quad (3.17)$$

Thus the components of the form factor are periodic, except for phases. Furthermore, (3.9) implies that

$$\begin{aligned} F_1^0(\theta_1, \theta_2, \theta_3) &= F_2^0(\theta_1, \theta_3, \theta_2), \\ F_3^0(\theta_1, \theta_2, \theta_3) &= F_4^0(\theta_1, \theta_3, \theta_2). \end{aligned} \quad (3.18)$$

The general solution of (3.17) and (3.18) is

$$\begin{aligned} F_1^0(\theta_1, \theta_2, \theta_3) &= (\theta_{12} + \pi i)^{-1} (\theta_{13} + \pi i)^{-1} g_1(\theta_1, \theta_2, \theta_3), \\ F_2^0(\theta_1, \theta_2, \theta_3) &= (\theta_{12} + \pi i)^{-1} (\theta_{13} + \pi i)^{-1} g_1(\theta_1, \theta_3, \theta_2), \\ F_3^0(\theta_1, \theta_2, \theta_3) &= (\theta_{13} + \pi i)^{-2} g_3(\theta_1, \theta_2, \theta_3), \\ F_4^0(\theta_1, \theta_2, \theta_3) &= (\theta_{12} + \pi i)^{-2} g_3(\theta_1, \theta_3, \theta_2), \end{aligned} \quad (3.19)$$

where the functions g_1 and g_3 are periodic in θ_1 :

$$\begin{aligned} g_1(\theta_1 - 2\pi i, \theta_2, \theta_3) &= g_1(\theta_1, \theta_2, \theta_3), \\ g_3(\theta_1 - 2\pi i, \theta_2, \theta_3) &= g_3(\theta_1, \theta_2, \theta_3). \end{aligned}$$

We now turn to the remaining periodicity conditions (3.14) and (3.15). Combining (3.8) with (3.14), we find

$$\begin{aligned} \frac{\theta_{12} + 3\pi i}{\theta_{12} + \pi i} F_{1,2}^0(\theta_1, \theta_2 - 2\pi i, \theta_3) &= F_{2,1}^0(\theta_1, \theta_2, \theta_3), \\ F_3^0(\theta_1, \theta_2 - 2\pi i, \theta_3) &= F_4^0(\theta_1, \theta_2, \theta_3), \\ \left(\frac{\theta_{12} + 3\pi i}{\theta_{12} + \pi i} \right)^2 F_4^0(\theta_1, \theta_2 - 2\pi i, \theta_3) &= F_3^0(\theta_1, \theta_2, \theta_3), \end{aligned} \quad (3.20)$$

and combining (3.7) and (3.8) with (3.15) yields

$$\begin{aligned} \frac{\theta_{12} + 3\pi i}{\theta_{12} + \pi i} \frac{\theta_{13} + \pi i}{\theta_{13} - \pi i} F_{1,2}^0(\theta_1, \theta_2 - 2\pi i, \theta_3) &= \frac{\theta_{13} + \pi i}{\theta_{13} - \pi i} F_{2,1}^0(\theta_1, \theta_3, \theta_2), \left(\frac{\theta_{13} + 3\pi i}{\theta_{13} + \pi i} \right)^2 F_3^0(\theta_1, \theta_2 - 2\pi i, \theta_3) \\ &= \left(\frac{\theta_{12} + \pi i}{\theta_{12} - \pi i} \right)^2 F_4^0(\theta_1, \theta_2, \theta_3), \left(\frac{\theta_{12} + 3\pi i}{\theta_{12} + \pi i} \right)^2 F_4^0(\theta_1, \theta_2 - 2\pi i, \theta_3) \\ &= F_3^0(\theta_1, \theta_2, \theta_3). \end{aligned} \quad (3.21)$$

The first of (3.20) and the first of (3.21) are the same equation. The last of (3.20) and the last of (3.21) are the same equation. The second of (3.20) and the second of (3.21) are inconsistent unless

$$F_3^0(\theta_1, \theta_2, \theta_3) = F_4^0(\theta_1, \theta_2, \theta_3) = 0, \quad (3.22)$$

which we claimed at the beginning of this section. Thus the double poles in (3.19) are absent. The conditions (3.20) and (3.21) imply

$$g_1(\theta_1, \theta_2 - 2\pi i, \theta_3) = g_1(\theta_1, \theta_3, \theta_2).$$

The minimal choice of the form factor, with no unnecessary poles or zeros, satisfying both Watson's theorem and the periodicity axiom, is obtained by setting the function $g_1(\theta_1, \theta_3, \theta_2)$ equal to a constant:

$$F_1^0(\theta_1, \theta_2, \theta_3) = \frac{g_1}{(\theta_{12} + \pi i)(\theta_{13} + \pi i)},$$

$$F_2^0(\theta_1, \theta_2, \theta_3) = \frac{g_1}{(\theta_{12} + \pi i)(\theta_{13} + \pi i)}.$$

We fix the constant with the annihilation-pole axiom.

The annihilation-pole axiom concerns the residues of form factors at singularities. This axiom follows from the LSZ reduction formula. The derivation can be found in Ref. [23], but some clarification may be helpful to the reader. We take the field Φ in the left-hand side of (3.1) on the mass shell, and compare with the S matrix. We first cross the antiparticle: $\theta_1 \rightarrow \theta_1 - \pi i$. So now we are considering two particles, of rapidities θ_2 and θ_3 , in the initial state. These scatter and there is a particle (not antiparticle) of rapidity θ_1 in the final state. There must also be a second particle in the final state, which corresponds to taking Φ on shell; we denote its rapidity by θ_0 . The reduction formula is

$$\begin{aligned} & \text{out}\langle P, \theta_1, a_1, b_1; P, \theta_0, a_0, b_0 | P, \theta_2, a_2, b_2; P, \theta_3, a_3, b_3 \rangle_{\text{in}} \\ &= \text{out}\langle P, \theta_1, a_1, b_1 | P, \theta_2, a_2, b_2 \rangle_{\text{in}} \text{out}\langle P, \theta_0, a_0, b_0 | P, \theta_3, a_3, b_3 \rangle_{\text{in}} + \text{out}\langle P, \theta_1, a_1, b_1 | P, \theta_3, a_3, b_3 \rangle_{\text{in}} \text{out}\langle P, \theta_0, a_0, b_0 | P, \theta_2, a_2, b_2 \rangle_{\text{in}} \\ &+ i\sqrt{N} \int d^2x e^{imx^0 \cosh\theta_0 - imx^1 \sinh\theta_0} \text{out}\langle P, \theta_1, a_1, b_1 | (\partial_0^2 - \partial_1^2 + m^2)\Phi(x)_{b_0 a_0} | P, \theta_2, a_2, b_2; P, \theta_3, a_3, b_3 \rangle_{\text{in}}, \end{aligned} \quad (3.23)$$

where the factor \sqrt{N} comes from the normalization of Φ (2.3). The second term on the right-hand side of (3.23) vanishes if $\theta_1 < \theta_0$. The right-hand side is the particle-particle S -matrix element, which can be directly compared with (2.6).

To evaluate the right-hand side of (3.23), we use the formulas for the Klein-Gordon operator [12]

$$(p_1 - p_2 - p_3)^2 - m^2 = -8m^2 \sinh\frac{\theta_{12}}{2} \sinh\frac{\theta_{13}}{2} \cosh\frac{\theta_{23}}{2}, \quad (3.24)$$

and for the covariant delta function

$$\begin{aligned} \delta^2(p_1 + p_0 - p_2 - p_3) &= \delta[(p_1)_+ + (p_0)_+ - (p_2)_+ - (p_3)_+] \delta[(p_1)_- + (p_0)_- - (p_2)_- - (p_3)_-] \\ &= \frac{2}{m^2} \delta[(p_1)_+ + (p_0)_+ - (p_2)_+ - (p_3)_+] \delta[(p_1)_+^{-1} + (p_0)_+^{-1} - (p_2)_+^{-1} - (p_3)_+^{-1}] \\ &= \frac{2}{m^2} \left| \frac{1}{(p_3)_+^2} - \frac{1}{(p_2)_+^2} \right|^{-1} \delta[(p_1)_+ - (p_2)_+] \delta[(p_3)_+ - (p_0)_+] \\ &+ \frac{2}{m^2} \left| \frac{1}{(p_3)_+^2} - \frac{1}{(p_2)_+^2} \right|^{-1} \delta[(p_1)_+ - (p_3)_+] \delta[(p_2)_+ - (p_0)_+] \\ &= \frac{\delta(\theta_{12})\delta(\theta_{30})}{m^2 |\sinh\theta_{13}|} + \frac{\delta(\theta_{13})\delta(\theta_{20})}{m^2 |\sinh\theta_{12}|}, \end{aligned} \quad (3.25)$$

where the components of each of the momenta along the light cone are $p_{\pm} = 2^{-1/2}(p_0 \pm p_1) = 2^{-1/2}e^{\pm\theta}$. We hope the indices cause no confusion; we have written $(p_i)_{\mu}$ for the μ th component of the momentum of the i th particle.

Inserting (3.24) and (3.25) into (3.23), finally crossing the out-particle with rapidity θ_1 back to an in-antiparticle with $\theta_1 \rightarrow \theta_1 + \pi i$, gives the annihilation-pole axiom for the problem in this section. Explicitly:

$$\begin{aligned} \text{Res}_{|\theta_{12} = -\pi i} \langle 0 | \Phi(0)_{b_0 a_0} | A, \theta_1, b_1, a_1; P, \theta_2, a_2, b_2; P, \theta_3, a_3, b_3 \rangle &= 2i \langle 0 | \Phi(0)_{b_0 a_0} | P, \theta_3, a_3, b_3 \rangle [\delta_{a_1 a_2} \delta_{b_1 b_2} - S_{a_2 b_2}^{a_1 b_1}(\theta_{23})], \\ \text{Res}_{|\theta_{13} = -\pi i} \langle 0 | \Phi(0)_{b_0 a_0} | A, \theta_1, b_1, a_1; P, \theta_2, a_2, b_2; P, \theta_3, a_3, b_3 \rangle &= 2i \langle 0 | \Phi(0)_{b_0 a_0} | P, \theta_2, a_2, b_2 \rangle [\delta_{a_1 a_3} \delta_{b_1 b_3} - S_{a_3 b_3}^{a_1 b_1}(\theta_{23})]. \end{aligned} \quad (3.26)$$

The leading terms of each side of (3.26) are both of order $N^{-3/2}$. Our final Lorentz-invariant expression for the large- N limit of the one-antiparticle, two-particle form factor is

$$F_1^0(\theta_1, \theta_2, \theta_3) = F_2^0(\theta_1, \theta_2, \theta_3) = -\frac{4\pi}{(\theta_{12} + \pi i)(\theta_{13} + \pi i)}, \quad F_3^0(\theta_1, \theta_2, \theta_3) = F_4^0(\theta_1, \theta_2, \theta_3) = 0. \quad (3.27)$$

The other functions $\tilde{F}_j^0(\theta_1, \theta_2, \theta_3)$ and $\tilde{\tilde{F}}_j^0(\theta_1, \theta_2, \theta_3)$ are the same as $F_j^0(\theta_1, \theta_2, \theta_3)$, up to irrelevant phases (these phases disappear upon evaluation of Wightman functions).

IV. THE WIGHTMAN FUNCTION IN THE 'T HOOFT LIMIT

We can use the result of the previous section to find an improved expression for the $N = \infty$ two-point Wightman function (2.1):

$$\begin{aligned} \mathcal{W}(x) &= \frac{1}{N} \int \frac{d\theta}{4\pi} e^{im(x^0 \cosh\theta - x^1 \sinh\theta)} \langle 0 | \Phi(0)_{b_0 a_0} | P, \theta, a_1, b_1 \rangle_{\text{in in}} \langle P, \theta, a_1, b_1 | \Phi(0)_{b_0 a_0}^* | 0 \rangle \\ &+ \frac{1}{N} \int \frac{d\theta_1}{4\pi} \int \frac{d\theta_2}{4\pi} \int \frac{d\theta_3}{4\pi} \frac{1}{2!} e^{im \sum_{j=1}^3 (x^0 \cosh\theta_j - x^1 \sinh\theta_j)} \langle 0 | \Phi(0)_{b_0 a_0} | A, \theta_1, a_1, b_1; P, \theta_2, a_2, b_2; P, \theta_3, a_3, b_3 \rangle_{\text{in}} \\ &\times \langle A, \theta_1, a_1, b_1; P, \theta_2, a_2, b_2; P, \theta_3, a_3, b_3 | \Phi(0)_{b_0 a_0}^* | 0 \rangle + \dots, \end{aligned} \quad (4.1)$$

where, as in (2.3), we sum over repeated color indices.

All of the one-antiparticle, two-particle form factors are given by (3.27) up to an irrelevant phase. When summing over color indices, we find that contributions quadratic in either F_1^0 or F_2^0 are of order one. The mixed contributions, linear in both F_1^0 and F_2^0 , are down by a power of $1/N$. We therefore drop the latter contributions. Thus the expansion (4.1) is

$$\mathcal{W}(x) = \frac{1}{4\pi} \int d\theta e^{im(x^0 \cosh\theta - x^1 \sinh\theta)} + \frac{1}{4\pi} \int d^3\theta e^{im \sum_{j=1}^3 (x^0 \cosh\theta_j - x^1 \sinh\theta_j)} (\theta_{12}^2 + \pi^2)^{-1} (\theta_{13}^2 + \pi^2)^{-1} + \dots \quad (4.2)$$

The first term on the right-hand side is the free-field approximation, discussed in Sec. II. The result (4.2) should be extremely good at large distances, as contributions from more intermediate particles fall off more quickly. Unfortunately, we cannot recover the short-distance behavior predicted by perturbation theory. It is necessary to sum over all intermediate states to obtain the Wightman function for small x . In other words, all the form factors of Φ are needed to compare with the perturbative result.

V. THE CORRESPONDENCE WITH A FREE FIELD

The renormalized field can be written in terms of the Zamolodchikov particle-creation operators, and their adjoints (together these form the Faddeev-Zamolodchikov algebra, which we do not discuss here). At large N , these are the standard operators used to build a free complex $(\infty \times \infty)$ -matrix field $M(x)$ in (2.9).

Examining the definitions of the functions F , \tilde{F} , and $\tilde{\tilde{F}}$ gives an expansion for $\Phi(x)$:

$$\begin{aligned} \Phi(x)_{b_0 a_0} &= \frac{1}{N^{1/2}} M(x)_{b_0 a_0} - \frac{1}{N^{3/2}} \int \frac{d^3\theta}{(4\pi)^3} [\mathfrak{A}_A(\theta_1)_{a_1 b_1} e^{imx^0 \cosh\theta_1 - imx^1 \sinh\theta_1} + \mathfrak{A}_P^\dagger(\theta_1)_{a_1 b_1} e^{-imx^0 \cosh\theta_1 + imx^1 \sinh\theta_1}] \\ &\times \frac{1}{2!} [\mathfrak{A}_P(\theta_2)_{b_2 a_2} e^{imx^0 \cosh\theta_2 - imx^1 \sinh\theta_2} + \mathfrak{A}_A^\dagger(\theta_2)_{b_2 a_2} e^{-imx^0 \cosh\theta_2 + imx^1 \sinh\theta_2}] [\mathfrak{A}_P(\theta_3)_{b_3 a_3} e^{imx^0 \cosh\theta_3 - imx^1 \sinh\theta_3} \\ &+ \mathfrak{A}_A^\dagger(\theta_3)_{b_3 a_3} e^{-imx^0 \cosh\theta_3 + imx^1 \sinh\theta_3}] \frac{4\pi}{(\theta_{12} + \pi i)(\theta_{13} + \pi i)} \left(\frac{\theta_{12} + \pi i}{\theta_{12} - \pi i} \right)^{\Theta(\theta_{12})} \left(\frac{\theta_{13} + \pi i}{\theta_{13} - \pi i} \right)^{\Theta(\theta_{13})} \\ &\times (\delta_{a_0 a_2} \delta_{b_0 b_3} \delta_{a_1 a_3} \delta_{b_1 b_2} + \delta_{a_0 a_3} \delta_{b_0 b_2} \delta_{a_1 a_2} \delta_{b_1 b_3}) + \dots, \end{aligned} \quad (5.1)$$

where Θ is the step function, $\Theta(\theta) = 0$ for $\theta < 0$, and $\Theta(\theta) = 1$ for $\theta > 0$, and the operators \mathfrak{A} and \mathfrak{A}^\dagger are expressed in terms of the free field as

$$\begin{aligned} \mathfrak{A}_A^\dagger(\theta)_{ba} &= (2mi \cosh\theta)^{-1} \int dx^1 e^{imx^0 \cosh\theta_1 - imx^1 \sinh\theta_1} \overleftrightarrow{\partial}_0 M(x)_{ba}, \\ \mathfrak{A}_P^\dagger(\theta)_{ab} &= (2mi \cosh\theta)^{-1} \int dx^1 e^{imx^0 \cosh\theta_1 - imx^1 \sinh\theta_1} \overleftrightarrow{\partial}_0 [M(x)^\dagger]_{ab}, \end{aligned} \quad (5.2)$$

and their adjoints. The matrix elements of this expression (5.1) between the vacuum bra and an in-state ket are unchanged if we suppress the creation operators. The creation operators are needed, however, for matrix elements of (5.1) to satisfy crossing.

VI. CONCLUSIONS

To summarize, we found exact form factors for the $(1 + 1)$ -dimensional principal chiral model at large N . We expanded the two-point Wightman function in terms of these form factors. Finally, we identified an underlying free matrix-field operator $M(x)$, and discussed how the renormalized field can be obtained from $M(x)$.

The $1/N$ expansion of the principal chiral model is quite different from the expansion of vector models, such as the $O(N)$ sigma model. The renormalized field of a vector model is a free field, as $N \rightarrow \infty$.

There is little difference between the free massive field $M(x)$ and the classical master field of the large- N limit. The response of this field to a source is the same, whether or not it is quantized.

The ingredients to find higher-order corrections in the $1/N$ expansion (3.16) are already in Sec. III. This problem is under investigation.

It would be interesting to understand form factors for in-states with more particles. The number of functions rapidly

increases with more particles. Nonetheless, two-antiparticle, three-particle form factors seem possible to obtain. It may be that all the form factors can be found. This would yield the complete sum of planar diagrams and a direct comparison with perturbation theory could be made.

We have not discussed operators other than the renormalized field in this paper. It seems possible to find the form factors of currents and the energy-momentum tensor by similar methods.

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