Photon polarization and geometric phase in general relativity

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Rotation of polarization in an external gravitational field is one of the effects of general relativity that can serve as a basis for its precision tests. A careful analysis of reference frames is crucial for a proper evaluation of this effect. We introduce an operationally-motivated local reference frame that allows for a particularly simple description. We present a solution of null geodesics in Kerr space-time that is organized around a new expansion parameter, allowing a better control of the series, and use it to calculate the resulting polarization rotation. While this rotation depends on the reference-frame convention, we demonstrate a gauge-independent geometric phase for closed paths in general space-times.

I. INTRODUCTION

Electromagnetic waves—visible light and other bands of the spectrum—are our prime source of information about the Universe [1]. Since measurements of the light deflection near the Sun [2] were made in 1919 wave propagation is used to test general relativity (GR). Two of the "classical tests" of GR, light deflection and time delay, can be understood in terms of geometric optics [3]. The first post-eikonal approximation [4] allows to track the evolution of electric and magnetic fields along the light rays and thus discuss polarization.

In this approximation we speak about photons with a null four-momentum k and a transversal four-vector polarization f. Both vectors are parallel-transported along the trajectory, which is a null geodesic [5,6]

$$\mathsf{k} \cdot \mathsf{k} = 0, \qquad \boldsymbol{\nabla}_{\mathsf{k}} \mathsf{k} = 0, \tag{1}$$

$$\mathsf{k} \cdot \mathsf{f} = 0, \qquad \boldsymbol{\nabla}_{\mathsf{k}} \mathsf{f} = 0, \tag{2}$$

where $\boldsymbol{\nabla}_{\mathsf{k}}$ is a covariant derivative along k.

In the last decades polarization has been yielding important astrophysical and cosmological data. Cosmic microwave background [7], blazar flares [8,9], astrophysical jets [9,10], and searches for dark matter [11] are just several examples where polarization conveys crucial information.

Photons are commonly used as physical carriers of quantum information. The abstract unit of quantum information is a qubit (a quantum bit) [12], and two linearly independent polarizations encode the two basis states of a qubit. One of the branches of quantum information is quantum metrology, which aims to improve precision measurements by using explicitly quantum effects, such as entanglement [13]. Relativistic properties of the information carriers [14,15] become important when quantum

technology is used in precision tests of relativity [16] or quantum information processing on the orbit [17].

Gravity causes polarization to rotate. This effect is known as a gravimagnetic/Faraday/Rytov-Skrotskiĭ rotation [18–21]. Helicity is invariant under rotations, but states of a definite helicity acquire phases $e^{\pm i \Delta \chi}$. Depending on the context we refer to $\Delta \chi$ either as a polarization rotation or as a phase. Once evaluated this phase can be encapsulated as a quantum gate [12] and incorporated into (quantum) communication protocols or metrology tasks.

In the Schwarzschild space-time, as well as at the leading order of the post-Newtonian approximation, this phase is known to be zero [18,19,22,23]. It is higher-order gravitational moments that are held responsible for the rotation of the polarization plane. In particular, the GR effects were shown to dramatically alter polarization of the X-ray radiation that is coming from the accretion disc of the (then presumed) black hole in Cyg $X - 1$ [24]. Numerous analytical and numerical studies of trajectories and polarizations in different models and astrophysical regimes were performed ([19–21,25–29], and references therein). The scenarios included fast-moving gravitating bodies, influence of gravitational lenses and propagation through gravitational waves.

The results are often contradictory. Some of the contradictions result from genuine differences in superfluously similar physical situations [20,28,29]. On the other hand, polarization rotation is operationally meaningful if the evolving polarization vector is compared with some standard polarization basis (two linear polarizations, right- and left-circular polarizations, etc.) at each point along the ray [30,31].

Setting up and aligning detectors requires alignment of local reference frames. Much work has been done recently on the role of reference frames in communications, especially in the context of quantum information [32]. Partial

knowledge of reference frames can lead to loss of communication capacity, and mistakes in identifying the information content of a physical system. The lack of definition for polarization standards and an *ad hoc* introduction of the angle adjustments is one reason for the variety of quoted values for the phase $\Delta\chi$. Even the consensual result of a zero phase in the Schwarzschild space-time should be qualified. Without specifying the appropriate reference frame it is either meaningless or wrong.

In this article we investigate the role of local reference frames in defining $\Delta\chi$. The result is obviously gauge-dependent, and additional considerations should be used to fix the gauge. Communicating reference frames is sometimes a difficult procedure which may involve a high communication cost. On a curved background of GR it may require some knowledge of the metric at each point along the trajectory. Using the method presented by us in [31] we build a local reference frame in stationary space-times, and then fix the standard polarizations in a single construction using what we call the *Newton gauge*. Our construction does not require communication between the parties, gives a precise meaning to the idea that there is no polarization rotation in the Schwarzschild space-time, and reproduces the absence of phase in the Minkowski space-time. From an operational point of view it allows us to set up detectors at any point in space based only on the local properties at that point. We illustrate the use of this gauge by studying polarization rotation in the Kerr space-time, and derive an explicit expression for $\Delta\chi$ in the scattering scenario that is described below. The calculations are based on the results of [6,21,25] with the added value of the Newton gauge. Careful bookkeeping is required with respect to the various coordinate systems used in different parts of the calculation, as well as different orders of the series expansion. To simplify the latter, we introduce a new expansion parameter.

Closed paths result in a gauge-invariant gravity-induced phase. We discuss them in Sec. IV. The rest of the paper is organized as follows: First we discuss null trajectories in the Kerr space-time. In Sec. III we review the Wigner's construction of setting standard polarizations in a local frame, set up the Newton gauge and explicitly calculate the rotation in the scattering scenario, where the light is emitted and observed far from the gravitating body (r_1, $r_2 \gg M$, a), but can pass close to it. Finally, we discuss gauge-dependent and gauge-invariant aspects of our results from mathematical and operational points of view (Sec. IV). Summary of the important facts about the Kerr space-time, as well as detailed calculations and special cases are presented in the appendices.

We use $-+++$ signature, set $G = c = 1$ and use Einstein's summation convention in all dimensions. Three-dimensional vectors are written in boldface and the unit vectors are distinguished by carets, such as $\hat{\mathbf{b}}$. Local tetrad components are written with carets on indices,

such as $k^{\hat{\mu}}$, and the four-vector itself as $\mathsf{k} = k^{\hat{\mu}}\mathsf{e}_{(\mu)}$, where $\mathsf{e}_{(\mu)}$ are vectors of a local orthonormal tetrad.

II. NULL GEODESICS

A. Null geodesics in the Kerr space-time

The Kerr metric in the Boyer-Lyndquist coordinates is given by [33]

$$ds^2 = -\left(1 - \frac{2Mr}{\rho^2}\right)dt^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2$$
$$+ \left(r^2 + a^2 + \frac{2Mra^2}{\rho^2}\sin^2\theta\right)\sin^2\theta d\phi^2$$
$$- \frac{4Mar}{\rho^2}\sin^2\theta dt d\phi, \tag{3}$$

where M is the mass and $a = J/M$ the angular momentum per unit mass of the gravitating body. We use the standard notations

$$\rho^2 = r^2 + a^2\cos^2\theta, \qquad \Delta = r^2 - 2Mr + a^2. \tag{4}$$

Thanks to the three conserved quantities—the energy $E = -k_0$, the z-component of the angular momentum L and the constant η [6,34]—the geodesic equations in Kerr space-time are integrable in quadratures. When dealing with photons it is convenient to set the energy to unity and to rescale other quantities [6]. Explicit form and the asymptotic expansions of $D = L/E$ and η are given in Appendix A. Fixing the energy and using the null vector condition leaves us with only two components of the four-momentum k. In specifying the initial data we usually take k_1^θ and k_1^ϕ as independent.

We discuss polarization from the point of view of static observers that are at rest in the "absolute" space $t = $ const. At every point (outside the ergosphere, if the model represents a black hole) we introduce a chronometric orthonormal frame [5,35] (Appendix A), which will be used to express the initial conditions and observed quantities.

The Hamilton-Jacobi equation for a null geodesic separates and the trajectories can be deduced from it [6]. We label their initial and final points as (r_1, θ_1, ϕ_1) and (r_2, θ_2, ϕ_2), respectively. The gauge convention that we adapt (Sec. III B) makes the knowledge of $\Delta\phi$ redundant for discussions of polarization rotation in the scattering scenario (for the calculation of ϕ see Appendix A). In particular we are interested in the integrals

$$\mathcal{R} := \int^r \frac{dr}{\pm\sqrt{R}} = \int^\theta \frac{d\theta}{\pm\sqrt{\Theta}}, \tag{5}$$

where

$$R = r^4 + (a^2 - D^2 - \eta)r^2$$
$$+ 2M(\eta + (D - a)^2)r - a^2\eta, \tag{6}$$

$$\Theta = \eta + a^2\cos^2\theta - D^2\cot^2\theta. \tag{7}$$

Null geodesics in Kerr space-time are classified according to the sign of the constant η. The scattering scenario corresponds to $\eta > 0$, and typically $\eta \sim r_1^2$. Following [25] we also assume that the angle θ reaches its extremal value (either maximum or minimum) only once, $\theta_1 \to \theta_{\max/\min} \to \theta_2$. Series expansions of the above integrals are most conveniently written with the help of a new constant

$$\Lambda^2 := D^2 + \eta. \tag{8}$$

We use the parametrization $D = \Lambda \cos\alpha$, and $\eta = \Lambda^2 \sin^2\alpha$, with $0 \le \alpha \le \pi$. In this notation the minimal coordinate distance from the center is

$$r_{\min} = \Lambda - M - \frac{3M^2}{2\Lambda} + \frac{2aM\cos\alpha}{\Lambda} - \frac{a^2\cos^2\alpha}{2\Lambda}$$
$$+ \mathcal{O}(\Lambda^{-2}). \tag{9}$$

We perform the integration by the methods of [6,25] (Appendix B). In the scattering scenario $r_1, r_2 \to \infty$, while the constants of motion are kept finite. Expansion in terms of Λ gives

$$\mathcal{R} = \frac{1}{\sqrt{\Lambda^2 - a^2}}\left(\psi_1 + \psi_2 - \frac{a^2\sin^2\alpha}{4\Lambda^2}(3(\psi_1 + \psi_2) \right.$$
$$\left. + \sin\psi_1\cos\psi_1 + \sin\psi_2\cos\psi_2)) + \mathcal{O}(\Lambda^{-4})\right), \tag{10}$$

where $\cos\psi = \cos\theta/\mu_+$ and μ_+ is defined in the appendix.

To separate conceptual issues from the computational details we make two simplifying assumptions. First, the initial and final points are taken to lie in the asymptotically flat regions, $r_i \to \infty$, while η and D are kept finite. Second, in these regions the polar angle θ (nearly) reaches its asymptotic values θ_{in} and θ_{out}, respectively.

The asymptotic form of momentum can be deduced from the equations of motion [6]. In the chronometric tetrad basis the outgoing momentum tends to

$$k^{\hat{\mu}} \to \left(1, 1, -\frac{s\beta_{\text{out}}}{r}, \frac{D}{r\sin\theta_{\text{out}}}\right), \tag{11}$$

where $\beta_{\text{out}} := +\sqrt{\Theta(\theta_{\text{out}})}$ and $s = \pm 1$. This expression is true in any scenario where an outgoing photon reaches the asymptotically flat region.

The asymptotic expression for the incoming momentum is similar,

$$k^{\hat{\mu}} \to \left(1, -1, \frac{s\beta_{\text{in}}}{r}, \frac{D}{r\sin\theta_{\text{in}}}\right), \tag{12}$$

where $\beta_{\text{in}} := +\sqrt{\Theta(\theta_{\text{in}})}$, and $s = \pm 1$ correspond to $\theta_{\text{in}} \to \theta_{\max}/\theta_{\min} \to \theta_{\text{out}}$, respectively.

Equating the radial and angular expressions for $\mathcal{R}$ we obtain

$$\theta_{\text{out}} = \pi - \theta_{\text{in}} \pm \frac{4M}{\Lambda}\frac{\sqrt{\sin^2\alpha - \cos^2\theta_{\text{in}}}}{\sin\theta_{\text{in}}} + \mathcal{O}(\Lambda^{-2}), \tag{13}$$

with the higher-order terms and special cases described in Appendix B. The plus sign corresponds to the trajectory in which θ first decreases with r and reaches $\theta_{\min}$ before increasing to θ_{out}.

B. Null geodesics in $1 + 3$ formalism

In stationary space-times a tetrad of a static observer is naturally related to Landau-Lifshitz $1 + 3$ formalism [33]. Static observers follow the congruence of timelike Killing vectors that defines a projection from the space-time manifold $\mathcal{M}$ onto a three-dimensional space Σ_3, π: $\mathcal{M} \to \Sigma_3$.

In practice it is performed by dropping the timelike coordinate of an event, and vectors are projected by a push-forward map $\pi_*\mathbf{k} = \mathbf{k}$ in the same way. Contravariant vector components satisfy

$$(\mathbf{k})^m \equiv (\pi_*\mathbf{k})^m = (\mathsf{k})^m = dx^m/d\lambda, \qquad m = 1, 2, 3, \tag{14}$$

where λ is the affine parameter.

The metric g on $\mathcal{M}$ can be written in terms of a three-dimensional scalar h, a vector $\mathbf{g}$, and a metric γ on Σ_3 as

$$ds^2 = -h(dx^0 - g_m dx^m)^2 + dl^2, \tag{15}$$

where the three-dimensional distance is given by $dl^2 = \gamma_{mn}dx^m dx^n$. The metric components are

$$\gamma_{mn} = \left(g_{mn} - \frac{g_{0m}g_{0n}}{g_{00}}\right), \tag{16}$$

and

$$h = -g_{00}, \qquad g_m = -g_{0m}/g_{00}. \tag{17}$$

The inner product of three-vectors will always refer to this metric, $\mathbf{k} \cdot \mathbf{f} = \gamma_{mn}k^m f^n$. Vector products and differential operators are defined as appropriate dual vectors [33]. Finally, the spatial projection of a null geodesic has a length l that is related to the affine parameter λ as

$$\left(\frac{dl}{d\lambda}\right)^2 = \frac{k_0^2}{h} = \mathbf{k}^2 =: k^2. \tag{18}$$

For a static observer the three spatial basis vectors of the local orthonormal tetrad are projected into an orthonormal triad, $\pi_*\mathbf{e}_{(m)} = \hat{\mathbf{e}}_{(m)}$, $\hat{\mathbf{e}}_{(m)} \cdot \hat{\mathbf{e}}_{(n)} = \delta_{mn}$. We adapt a gauge in which polarization is orthogonal to the observer's four-velocity, $\mathsf{u} \cdot \mathsf{f} = 0$. In a three-dimensional form this condition reads as $\mathbf{k} \cdot \mathbf{f} = 0$.

In stationary space-times the evolution equations Eq. (1) and (2) can be reduced into a convenient three-dimensional form [20]. Using the relations between four- and three-dimensional covariant derivatives, ∇_μ and D_m, respectively, the propagation Eqs. (1) and (2) are brought to a three-dimensional form [19,20],

$$\frac{D\mathbf{k}}{d\lambda} = \mathbf{\Omega} \times \mathbf{k} + \frac{\mathbf{E}_g \cdot \mathbf{k}}{k^2}\mathbf{k}, \qquad \frac{D\hat{\mathbf{f}}}{d\lambda} = \mathbf{\Omega} \times \hat{\mathbf{f}}. \quad (19)$$

The angular velocity of rotation $\mathbf{\Omega}$ is

$$\mathbf{\Omega} = 2\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \hat{\mathbf{k}})\hat{\mathbf{k}} - \mathbf{E}_g \times \mathbf{k}, \quad (20)$$

where $\mathbf{g}$ and h play the roles of a vector potential of a gravimagnetic field $\mathbf{B}_g$,

$$\boldsymbol{\omega} = -\frac{1}{2}k_0\mathrm{curl}\,\mathbf{g} \equiv -\frac{1}{2}k_0\mathbf{B}_g, \quad (21)$$

and a scalar potential of a gravielectric field,

$$\mathbf{E}_g = -\frac{\nabla h}{2h}, \quad (22)$$

respectively [36].

III. LOCAL REFERENCE FRAMES AND POLARIZATION ROTATION

To discuss how a photon's polarization rotates it is necessary to define the standard polarization directions along its trajectory. Only after physically defining two standard polarizations it is meaningful to talk about polarization rotation, and only in a particular gauge the Schwarzschild space-time induces a zero phase on an open trajectory.

A. Wigner phase—polarization convention

Transversality of electromagnetic waves makes the choice of two standard polarization directions momentum-dependent. On a curved background it also depends on the location. Wigner's construction of the massless representation of the Poincare group [37,38] is the basis for classification of states in quantum field theory. We use it at every space-time point to produce standard polarization vectors.

The construction—part of the induced representation of the Poincaré group [38]—consists of a choice of a standard reference momentum k_S and two polarizations, a standard Lorentz transformation $L(\mathsf{k})$ that takes k_S to an arbitrary momentum k, and a decomposition of an arbitrary Lorentz transformation Λ in terms of the standard transformations and Wigner's little group element W. The standard reference three-momentum is directed along the z-axis of an arbitrarily chosen reference frame, with its x and y axes defining the two linear polarization vectors $\hat{\mathbf{b}}_{1,2}^S$, respectively. Hence $\mathsf{k}_S = (1, 0, 0, 1)$, and imposing the polariza-

tion gauge relates 3- and 4- polarization vectors, $\mathsf{b} = (0, \hat{\mathsf{b}}_{1,2})1, 2$.

The standard Lorentz transformation can be taken as[1]

$$L(\mathsf{k}) = R(\hat{\mathbf{k}})B_z(k), \quad (23)$$

where $B_z(k)$ is a pure boost along the z-axis that takes k_S to $(k, 0, 0, k)$, and the standard rotation $R(\hat{\mathbf{k}})$ brings the z-axis to the desired direction $\hat{\mathbf{k}}(\theta, \phi)$ first by rotating by θ around the y-axis and then by ϕ around the z-axis,

$$R(\hat{\mathbf{k}}) = R_z(\phi)R_y(\theta). \quad (24)$$

The standard polarization vectors for an arbitrary momentum are defined as

$$\hat{\mathbf{b}}_i(\mathbf{k}) = R(\hat{\mathbf{k}})\hat{\mathbf{b}}_i^S, \quad (25)$$

and a general real four-vector of polarization can be written as

$$\mathsf{f} = \cos\chi\mathsf{b}_1(\mathsf{k}) + \sin\chi\mathsf{b}_2(\mathsf{k}). \quad (26)$$

Helicity is invariant under Lorentz transformations. The corresponding polarization vectors are

$$\hat{\mathbf{b}}_\pm(\mathbf{k}) = \frac{1}{\sqrt{2}}(\hat{\mathbf{b}}_1(\mathbf{k}) \mp i\hat{\mathbf{b}}_2(\mathbf{k})). \quad (27)$$

Under a Lorentz transformation Λ a state of a definite helicity acquires a phase $e^{\pm i\xi(\Lambda,\mathbf{k})}$, which can be read off from Wigner's little group element

$$W(\Lambda, \mathsf{k}) = L^{-1}(\Lambda\mathsf{k})\Lambda L(\mathsf{k}) = R_z(\xi)T(\alpha, \beta). \quad (28)$$

The little group element W leaves the standard momentum k_S-invariant. Here $T(\alpha, \beta)$ form a subgroup which is isomorphic to the translations of a Euclidean plane and $R_z(\xi)$ is a rotation around the origin of that plane, which in this case is also a rotation around the z-axis.

If the transformation in question is a pure rotation $\mathcal{R}$, then the little group element is a rotation $R_z(\xi)$, and the polarization three-vector is rotated by $\mathcal{R}$ itself [30]. The phase χ has a simple geometric interpretation: it is the angle between, say, $\hat{\mathbf{b}}_1(\mathcal{R}\mathbf{k})$ and $\mathcal{R}\hat{\mathbf{b}}_1(\mathbf{k})$. It is zero if the standard polarizations of the new momentum are the same as the rotated standard polarizations of the old momentum.

This is what happens if a rotation $R_2(\omega)$ is performed around the current $\hat{\mathbf{b}}_2(\mathbf{k})$: the resulting phase $\Delta\chi(R_2(\omega))$ is zero [31]. Indeed, if $\hat{\mathbf{k}} = \hat{\mathbf{k}}(\theta, \phi)$, then by setting $\hat{\mathbf{k}}' = R_2(\omega)\hat{\mathbf{k}}$ and using the decomposition

$$R_2(\omega) = R(\hat{\mathbf{k}})R_y(\omega)R^{-1}(\hat{\mathbf{k}}), \quad (29)$$

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[1]When it does not lead to confusion we use the same letter to label a four-dimensional object and its three-dimensional part. In particular, R stands both for a Lorentz transformation which is a pure rotation, and for the corresponding three-dimensional rotation matrix itself.

we find that $\hat{\mathbf{k}}' = \hat{\mathbf{k}}'(\theta + \omega, \phi)$, and the little group element is

$$W = R^{-1}(\hat{\mathbf{k}}')R_2(\omega)R(\hat{\mathbf{k}}) = R_y^{-1}(\theta + \omega)R_y(\theta)R_y(\omega) = \mathbb{1},$$

$$(30)$$

indicating the absence of rotation with respect to standard polarization basis.

In a curved space-time one has to provide the standard (xyz) directions at every point. The next section deals with this problem.

B. Newton gauge

The three-dimensional propagation Eqs. (19) in stationary space-times result in a joint rotation of polarization and unit tangent vectors

$$\frac{D\hat{\mathbf{k}}}{d\lambda} = \mathbf{\Omega} \times \hat{\mathbf{k}}, \qquad \frac{D\hat{\mathbf{f}}}{d\lambda} = \mathbf{\Omega} \times \hat{\mathbf{f}}. \qquad (31)$$

In the Schwarzschild space-time test particles move in the plane passing through the origin [5,6,33], $\mathbf{\Omega} = -\mathbf{E}_g \times \mathbf{k}$ and the covariant time-time component of the metric is related to the Newtonian gravitational potential φ as $h = g_{00} = 1 + 2\varphi(r) = 1 - 2M/r$. *Requiring* the resulting phase to be zero, as in [18,19,26], constrains a choice of local reference frames. We use the zero phase condition to make a physically motivated choice of standard polarizations that does not require references to a parallel transport or communication between the observers.

Orient the local z-axis along the direction of the free-fall acceleration, $\hat{\mathbf{z}} \parallel \nabla\varphi$ and the standard polarizations as

$$\hat{\mathbf{y}} \equiv \hat{\mathbf{b}}_2 := \mathbf{w} \times \hat{\mathbf{k}}/|\mathbf{w} \times \hat{\mathbf{k}}|, \qquad \hat{\mathbf{b}}_1 := \hat{\mathbf{k}} \times \hat{\mathbf{b}}_2, \quad (32)$$

where $\mathbf{w}$ is a free-fall acceleration in the frame of a static observer. Then $\mathbf{\Omega} = \Omega\hat{\mathbf{b}}_2$, and the propagation induces no phase.

In a general static space-time we take the z-axis along the local free-fall direction as seen by a static observer. In the Kerr space-time its components are [35]

$$w_{\hat{1}} = \frac{M(\rho^2 - 2r^2)\sqrt{\Delta}}{\rho^3(\rho^2 - 2Mr)} = -\frac{M}{r^2} - \frac{M^2}{r^3} + \mathcal{O}(r^{-4}),$$

$$w_{\hat{2}} = \frac{Mra^2 \sin2\theta}{\rho^3(\rho^2 - 2Mr)} = a^2 M \frac{\sin2\theta}{r^4} + \mathcal{O}(r^{-5}),$$

$$w_{\hat{3}} = 0.$$

$$(33)$$

This convention, that we will call the Newton gauge, is consistent: if we set $\hat{\mathbf{z}} = -\hat{\mathbf{r}}$ in the flat space-time, then no phase is accrued as a result of the propagation. In addition to being defined by local operations, the Newton gauge has two further advantages. First, it does not rely on a weak field approximation to define the reference direction. Second, if the trajectory is closed or self-intersecting, the reference direction $\hat{\mathbf{z}}$ is the same at the points of the intersection.

C. Kerr space-time examples

We impose the temporal gauge in local frames, hence

$$f^{\hat{0}} \equiv 0 \Leftrightarrow f^t \equiv -\frac{2Marf^\phi \sin^2\theta}{\rho^2 - 2Mr}$$

$$= -\frac{2Maf^{\hat{3}} \sin\theta}{r^2} + \mathcal{O}(r^{-3}), \qquad (34)$$

where we used $f^{\hat{0}} = f^\mu e^{\hat{0}}_{(\mu)}$. This transversality reduces to a familiar three-dimensional expression $\mathbf{k} \cdot \mathbf{f} \equiv \mathbf{k} \cdot \mathbf{f} = f^{\hat{1}}k^{\hat{1}} + f^{\hat{2}}k^{\hat{2}} + f^{\hat{3}}k^{\hat{3}} = 0$. This also implies

$$f_t = e_{(\mu)t}f^{\hat{\mu}} = e_{(0)t}f^{\hat{0}} = 0. \qquad (35)$$

In the Kerr space-time the Walker-Penrose quantity [39] $K_2 + iK_1$,

$$K_1 = rB - aA\cos\theta, \qquad K_2 = rA + aB\cos\theta, \quad (36)$$

where

$$A = (k^t f^r - k^r f^t) + a(k^r f^\phi - k^\phi f^r)\sin^2\theta, \qquad (37)$$

$$B = ((r^2 + a^2)(k^\phi k^\theta - k^\theta f^\phi) - a(k^t f^\theta - k^\theta f^t))\sin\theta, \qquad (38)$$

is conserved along null geodesics [6]. Transversality and the gauge (34) make the Walker-Penrose constants functions of only two polarization components, for example $f^{\hat{2}}$ and $f^{\hat{3}}$.

For a generic outgoing null geodesic in the asymptotic regime the constants become

$$K_1 = \gamma_{\text{out}}f^{\hat{2}}_{\text{out}} - s\beta_{\text{out}}f^{\hat{3}}_{\text{out}},$$
$$K_2 = s\beta_{\text{out}}f^{\hat{2}}_{\text{out}} + \gamma_{\text{out}}f^{\hat{3}}_{\text{out}},$$

$$(39)$$

where

$$\gamma_{\text{out}} := D\csc\theta_{\text{out}} - a\sin\theta_{\text{out}}, \qquad (40)$$

and the analogous expression gives the constant in terms of the initial data. Hence,

$$f^{\hat{1}}_{\text{out}} = 0, \qquad f^{\hat{3}}_{\text{out}} = -\frac{1}{\beta^2_{\text{out}} + \gamma^2_{\text{out}}}(-s\beta_{\text{out}}K_1 + \gamma_{\text{out}}K_2),$$

$$f^{\hat{2}}_{\text{out}} = -\frac{1}{\beta^2_{\text{out}} + \gamma^2_{\text{out}}}(-s\beta_{\text{out}}K_2 - \gamma_{\text{out}}K_1). \qquad (41)$$

To determine the polarization rotation χ we take the initial polarization to be, say, $\mathbf{f}_{\text{in}} = \hat{\mathbf{b}}^{\text{in}}_1$, which often considerably simplifies the expression for K_1 and K_2. Since K_1 and K_2 are linear functions of polarization, expressing the above result in the basis $(\hat{\mathbf{b}}^{\text{out}}_1, \hat{\mathbf{b}}^{\text{out}}_2)$ the desired rotation angle χ is observed from

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow T \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\chi \\ \sin\chi \end{pmatrix}, \tag{42}$$

where T is an orthogonal matrix that is described below.

We assume fixed D and η. Using Eq. (32) we find that at the limit $r_{\rm in} \rightarrow \infty$ the initial standard linear polarization directions are

$$\hat{\mathbf{b}}_1^{\rm in} = \frac{1}{\sqrt{D^2 + \beta_{\rm in}^2 \sin^2\theta_{\rm in}}} (0, s\beta_{\rm in} \sin\theta_{\rm in}, D),$$

$$\hat{\mathbf{b}}_2^{\rm in} = \frac{1}{\sqrt{D^2 + \beta_{\rm in}^2 \sin^2\theta_{\rm in}}} (0, D, -s\beta_{\rm in} \sin\theta_{\rm in}). \tag{43}$$

Similarly, the final standard polarizations are

$$\hat{\mathbf{b}}_1^{\rm out} = \frac{1}{\sqrt{D^2 + \beta_{\rm out}^2 \sin^2\theta_{\rm in}}} (0, s\beta_{\rm out} \sin\theta_{\rm out}, -D),$$

$$\hat{\mathbf{b}}_2^{\rm out} = \frac{1}{\sqrt{D^2 + \beta_{\rm out}^2 \sin^2\theta_{\rm in}}} (0, D, s\beta_{\rm out} \sin\theta_{\rm out}). \tag{44}$$

The rotation matrices $N_{\rm in}$ and $N_{\rm out}$ from the polarization bases to $(\hat{\theta}, \hat{\phi})$ basis are given in the Appendix C.

Following [21] using the asymptotic relationship between Walker-Penrose constants and the components of polarization one can introduce a transformation matrix R,

$$\begin{pmatrix} f_{\rm out}^{\hat{2}} \\ f_{\rm out}^{3} \end{pmatrix} = R \begin{pmatrix} f_{\rm in}^{\hat{2}} \\ f_{\rm in}^{3} \end{pmatrix}. \tag{45}$$

The transformation matrix has the form

$$R = \frac{1}{\sqrt{1 + x^2}} \begin{pmatrix} 1 & -x \\ -x & -1 \end{pmatrix}, \tag{46}$$

where the parameter x is given by

$$x = s \frac{\beta_{\rm in} \gamma_{\rm out} - \beta_{\rm out} \gamma_{\rm in}}{\gamma_{\rm in} \gamma_{\rm out} + \beta_{\rm in} \beta_{\rm out}}, \tag{47}$$

with $\gamma_{\rm in} := D \csc\theta_{\rm in} - a \sin\theta_{\rm in}$. Finally,

$$T = N_{\rm out} R N_{\rm in}^{-1}. \tag{48}$$

Expansion in the inverse powers of Λ results in

$$T_{21} = \sin\chi = -\frac{4Ma}{\Lambda^2} \cos\theta_{\rm in} + \mathcal{O}(\Lambda^{-3}). \tag{49}$$

This is our main new result. Few special cases are of interest. If the initial propagation is parallel to the z-axis and the impact parameter equals b, then

$$\Lambda^2 = b^2 - a^2, \qquad D = 0, \tag{50}$$

and the polarization is rotated by

$$\sin\chi = \frac{4Ma}{\Lambda^2} + \frac{15M^2 a}{4\Lambda^3} + \mathcal{O}(\Lambda^{-4}), \tag{51}$$

and the antiparallel initial direction gives the opposite sign.

Motion in the equatorial plane corresponds to $\eta = 0$ ($\sin\alpha = 0$) and is qualitatively similar to the motion in Schwarzschild space-time. If the trajectory starts there but eventually moves outside, then the polarization is rotated by

$$\sin\chi = s \frac{8M^2 a}{\Lambda^3} \sin\alpha + \mathcal{O}(\Lambda^{-4}). \tag{52}$$

In the case of initial propagation along the z-axis agrees with both [18,20], if we take into account the respective definitions of reference frames. On the other hand, in a generic setting our Eq. (49) differs by a power of Λ from [21] or [19] (both works predict $\chi \sim \Lambda^{-3}$ but disagree on the prefactors). The origin of this disagreement is in the following. In addition to using a more transparent series expansion, we use a different polarization basis from the one in [21]. In particular, we do not require the knowledge of $\phi_{\rm out} - \phi_{\rm in}$. Moreover, [19] considered only a Machian effect (Sec. IV), which is dominated by the reference-frame term [31].

V. GEOMETRIC PHASE

A more geometric perspective on polarization rotation is possible both in a static space-time, where we use the $1 + 3$ formalism, as well as in arbitrary space-times. While the phase $\Delta\chi$ is gauge-dependent for an open trajectory, we will show that it is gauge-invariant on a closed path.

Consider the differential equations for polarization rotation [31]. By setting $\mathsf{f} = \mathsf{b}1$ at the starting point and using the parallel transport equations Eqs. (1) and (2) we arrive at the equation

$$\frac{d\chi}{d\lambda} = \frac{1}{\cos\chi} \boldsymbol{\nabla}_{\mathsf{k}}(\mathsf{f} \cdot \mathsf{b}_2) = \frac{1}{\mathsf{f} \cdot \mathsf{b}1} \mathsf{f} \cdot \boldsymbol{\nabla}_{\mathsf{k}} \mathsf{b}_2. \tag{53}$$

In a static space-time projection of the polarization on Σ_3 results in

$$\frac{d\chi}{d\lambda} = \frac{1}{\cos\chi} \frac{D(\hat{\mathbf{f}} \cdot \hat{\mathbf{b}}_2)}{d\lambda}, \tag{54}$$

hence the desired equation is

$$\frac{d\chi}{d\lambda} = \frac{1}{\hat{\mathbf{f}} \cdot \hat{\mathbf{b}}_1} \left(\frac{D\hat{\mathbf{f}}}{d\lambda} \cdot \hat{\mathbf{b}}_2 + \hat{\mathbf{f}} \cdot \frac{D\hat{\mathbf{b}}_2}{d\lambda} \right)$$

$$= \boldsymbol{\omega} \cdot \hat{\mathbf{k}} + \frac{1}{\hat{\mathbf{f}} \cdot \hat{\mathbf{b}}_2} \hat{\mathbf{f}} \cdot \frac{D\hat{\mathbf{b}}_2}{d\lambda}. \tag{55}$$

The first term corresponds to the original Machian effect that was postulated in [23], but the observable quantity involves both the Machian and the reference-frame terms [31].

Now we discuss the geometric meaning of these equations. First consider a basis of one-forms $(\sigma^1, \sigma^2, \sigma^3)$ that is dual to the orthonormal polarization basis $(\hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2, \hat{\mathbf{k}})$ at every point of the trajectory. A matrix of connection

1-forms ω is written with the help of Ricci rotation coefficients $\omega^{\hat{i}}_{\hat{j}\hat{l}}$ as $\omega^{\hat{i}}_{\hat{l}} = \omega^{\hat{i}}_{\hat{j}\hat{l}}\sigma^j$ [5,40], and a linear polarization is written as

$$\hat{\mathbf{f}} = f^{\hat{1}}\hat{\mathbf{b}}_1 + f^{\hat{2}}\hat{\mathbf{b}}_2 = \cos\chi\hat{\mathbf{b}}_1 + \sin\chi\hat{\mathbf{b}}_2. \qquad (56)$$

Its covariant derivative equals to

$$\frac{D\hat{\mathbf{f}}}{d\lambda} = k\omega^{\hat{a}}_{\hat{3}\hat{c}}f^c\hat{\mathbf{b}}_a + \omega^{\hat{3}}_{\hat{3}\hat{c}}f^c\mathbf{k} + \mathbf{k}(f^{\hat{c}})\hat{\mathbf{b}}_c, \qquad a,c = 1,2. \tag{57}$$

Taking into account the antisymmetry of the connections $\omega^{\hat{i}}_{\hat{l}} = \omega_{\hat{i}\hat{l}} = -\omega_{\hat{l}\hat{i}}$, we find

$$\frac{D\mathbf{f}}{d\lambda} = (-\hat{\mathbf{b}}_1 f^{\hat{2}} + \hat{\mathbf{b}}_2 f^{\hat{1}})\left(\frac{d\chi}{d\lambda} - \omega^{\hat{1}}_{\hat{3}\hat{2}}k\right)$$
$$+ \hat{\mathbf{k}}k(\omega^{\hat{3}}_{\hat{3}\hat{1}}f^{\hat{1}} + \omega^{\hat{3}}_{\hat{3}\hat{2}}f^{\hat{2}}). \tag{58}$$

A comparison with Eq. (31) leads to the identification $\Omega^1 = \mathbf{\Omega}\cdot\hat{\mathbf{b}}_1 = k\omega^{\hat{3}}_{\hat{3}\hat{2}}$, $\Omega^2 = -k\omega^{\hat{3}}_{\hat{3}\hat{1}}$ and to an alternative equation for the polarization rotation,

$$\frac{d\chi}{d\lambda} = \mathbf{\omega}\cdot\hat{\mathbf{k}} + \omega^{\hat{1}}_{\hat{3}\hat{2}}k. \tag{59}$$

Given a trajectory with a tangent vector $\mathbf{k}$ one can define a SO(2) line bundle with the connection $\bar{\omega} = \omega^{\hat{1}}_{\hat{3}\hat{2}}kd\lambda$, similarly to the usual treatment of geometric phase [40,41]. Freedom of choosing the polarization frame $(\hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2)$ at every point of the trajectory is represented by a SO(2) rotation $R_{\hat{\mathbf{k}}}(\psi(\lambda))$. Under its action the connection transforms as $\omega \to R\omega R^{-1} + R^{-1}dR$ [40], so

$$\frac{d\chi}{d\lambda} \to \mathbf{\omega}\cdot\hat{\mathbf{k}} + \omega^{\hat{1}}_{\hat{3}\hat{2}}k + \frac{d\psi}{d\lambda}. \tag{60}$$

In a static space-time we can consider a closed trajectory in space. Then the resulting phase is gauge-invariant, since the last term above is a total differential and drops out upon the integration on a closed contour,

$$\Delta\chi = \oint \mathbf{\omega}\cdot\hat{\mathbf{k}}d\lambda + \oint \bar{\omega}. \tag{61}$$

We can formalize it with the help of a curvature 2-form is introduced as $\bar{\theta} := d\bar{\omega} + \bar{\omega}\wedge\bar{\omega}$. For the SO(2) bundle it reduces to $\bar{\theta} = d\bar{\omega}$, so by using Stokes' theorem the reference-frame term can be rewritten as a surface integral of the bundle curvature as

$$\oint \bar{\omega} = \iint \bar{\theta}. \tag{62}$$

A more practical expression follows from our previous discussion: $\Delta\chi = \arcsin\hat{\mathbf{f}}_{\text{out}}\cdot\hat{\mathbf{b}}_2$. Conservation of K_1 and K_2 in the Kerr space-time ensures that if a trajectory is closed as a result of the initial conditions, then $\hat{\mathbf{f}}_{\text{out}} = \hat{\mathbf{f}}_{\text{in}}$ and $\Delta\chi = 0$. The Newton gauge is designed to give a zero phase along any trajectory in the Schwarzschild

space-time. As a result of the gauge invariance of Eq. (61), no gravitationally-induced phase is accrued along a closed trajectory in the Schwarzschild space-time, regardless of the gauge convention.

In a general space-time we introduce an orthonormal tetrad such that $\mathbf{k} = k\mathbf{e}_0 + k\mathbf{e}_3$ at every point of the trajectory. We again impose a temporal gauge and set $\mathbf{e}_{1,2} = \mathbf{b}_{1,2}$, where the local polarization basis is chosen according to some procedure. Then from Eq. (26) it follows that

$$\mathbf{\nabla}_{\mathbf{k}}\mathbf{f} = k^{\hat{\mu}}\mathbf{e}_{(\nu)}\omega^{\hat{\nu}}_{\hat{\mu}\hat{\rho}}f^{\hat{\rho}} + \frac{df^{\hat{\nu}}}{d\lambda}\mathbf{e}_{(\nu)}$$
$$= (-f^{\hat{2}}\mathbf{e}_{(1)} + f^{\hat{1}}\mathbf{e}_{(2)})\left(\frac{d\chi}{d\lambda} - (\omega^{\hat{1}}_{\hat{0}\hat{2}} + \omega^{\hat{1}}_{\hat{3}\hat{2}})k\right)$$
$$+ (k(\omega^{\hat{0}}_{\hat{0}\hat{c}} + \omega^{\hat{0}}_{\hat{3}\hat{c}})f^{\hat{c}})\mathbf{e}_{(0)} + (k(\omega^{\hat{3}}_{\hat{0}\hat{c}} + \omega^{\hat{3}}_{\hat{3}\hat{c}})f^{\hat{c}})\mathbf{e}_{(3)}$$
$$c = 1, 2, \tag{63}$$

where we used again the antisymmetry of the connection. From the parallel transport condition $\mathbf{\nabla}_{\mathbf{k}}\mathbf{f} = 0$, we can see that

$$(\omega^{\hat{0}}_{\hat{0}\hat{c}} + \omega^{\hat{0}}_{\hat{3}\hat{c}})f^{\hat{c}} = (\omega^{\hat{3}}_{\hat{0}\hat{c}} + \omega^{\hat{3}}_{\hat{3}\hat{c}})f^{\hat{c}} = 0 \tag{64}$$

and

$$d\chi = (\omega^{\hat{1}}_{\hat{0}\hat{2}} + \omega^{\hat{1}}_{\hat{3}\hat{2}})kd\lambda := \bar{\Omega}. \tag{65}$$

We cannot have a closed trajectory in chronologically protected space-time, so to obtain a gauge-invariant result we consider two future-directed trajectories that begin and end in the same space-time points. This layout is similar to the two arms of a Mach-Zender interferometer [4]. We also align the initial and final propagation directions of the beams and use the same rules to define the standard polarizations. Similarly to the previous case the curvature is just $\bar{\Theta} = d\bar{\Omega}$, and the Stokes theorem gives the phase as

$$\Delta\chi = \int_{\gamma_1} \bar{\Omega} - \int_{\gamma_2} \bar{\Omega} = \oint_{\gamma} \bar{\Omega} = \iint \bar{\Theta}. \tag{66}$$

V. CONCLUSIONS AND OUTLOOK

Rotation one ascribes to polarization as light propagates on a curved background depends on the gauge conventions that are used along the way. However, the closed-loop phase is gauge-independent (assuming we use the same reference frame for transmission and detection). Similarly to other instances of a geometric phase, the phase $\Delta\chi$ is given by the integral of the (bundle) curvature over the surface that is bounded by the trajectory. The Newton gauge provides a convenient local definition of the standard polarizations. It is motivated by its relative simplicity and path-independence of the reference frames. It is also the gauge in which the statement of a zero accrued phase along an arbitrary path in the Schwarzschild space-time is correct.

Our results characterize the behavior of polarization qubits on curved backgrounds. Specific regimes and realistic scenarios, both on the Kerr background and beyond, are to be investigated farther. Of special interest are those scenarios that will be experimentally feasible in the near future, such as sending photons between satellites [17]. Once the expression for $\Delta\chi$ is obtained, it can be represent it as the action of a quantum gate [12], similarly to the the special-relativistic scenarios [14,42]. This will allow us to use the full toolbox of quantum optics to design experiments. Understanding of the polarization phase in GR opens new possibilities for optics-based precision measurements [31], both classical and quantum.

ACKNOWLEDGMENTS

APPENDIX A: SOME ASPECTS OF THE KERR SPACE-TIME

1. Chronometric tetrad

We fix an orthonormal tetrad at every space-time point (outside the static limit) by demanding that $\mathsf{e}0$ is the four-velocity of a static observer, and the vectors $\mathsf{e}1$ and $\mathsf{e}2$ are proportional to the tangent vectors ∂_r and ∂_θ. Their covariant components $e_{(\alpha)\mu}$ are

$$\mathsf{e}_{(0)} = \left(-\sqrt{1-\frac{2Mr}{\rho^2}}, 0, 0, -\frac{2Mar\sin^2\theta}{\rho\sqrt{\rho^2-2Mr}}\right) \quad (A1)$$

and

$$\begin{aligned}\mathsf{e}_{(1)} &= (0, \rho/\sqrt{\Delta}, 0, 0),\\ \mathsf{e}_{(2)} &= (0, 0, \rho, 0),\\ \mathsf{e}_{(3)} &= \left(0, 0, 0, \rho\sin\theta\sqrt{\frac{\Delta}{\Delta-a^2\sin^2\theta}}\right)\end{aligned} \quad (A2)$$

Setting $E = -k_0 = 1$ leads to

$$k^t = \frac{\rho^2 - 2Mrak^\phi\sin^2\theta}{\rho^2 - 2Mr}, \quad (A3)$$

and k^r is expressed from the null condition $\mathsf{k}^2 = 0$,

$$(k^r)^2 = \frac{\Delta[1 - (k^\theta)^2(\Delta - a^2\sin^2\theta) - (k^\phi)^2\sin^2\theta\Delta]}{\Delta - a^2\sin^2\theta}. \quad (69)$$

The spatial components of the momentum in the coordinate and tetrad bases are related to $k^{\hat{m}} = e^{(m)}{}_\mu k^\mu$ as

$$k^{\hat{1}} = k^r\rho/\sqrt{\Delta}, \quad (A5)$$

$$k^{\hat{2}} = k^\theta\rho, \quad (A6)$$

$$k^{\hat{3}} = k^\phi\rho\sin\theta\sqrt{\frac{\Delta}{\Delta-a^2\sin^2\theta}}, \quad (A7)$$

while $k^2 = \mathbf{k}^2 = k^{\hat{m}}k_{\hat{m}}$ satisfies

$$k^2 = 1 + \frac{2Mr}{\rho^2 - 2Mr} = 1 + \frac{2M}{r} + \mathcal{O}(r^{-2}). \quad (A8)$$

2. Constants of motion

The rescaled ($E = 1$) z-component of the angular momentum is

$$\begin{aligned}D = \frac{1}{\rho^2 - 2Mr}(k^\phi(a^2 + r^2)(\rho^2 - 2Mr)\\ - 2Mar(1 - k^\phi a\sin^2\theta)).\end{aligned} \quad (A9)$$

The rescaled Carter's constant

$$\eta := K - (D - a)^2, \quad (A10)$$

where the constant K is most conveniently expressed as

$$K := (a\sin\theta - D/\sin\theta)^2 + (\rho^2 k^\theta)^2. \quad (A11)$$

The asymptotic expressions when $r \to \infty$ and the momentum components are fixed are

$$D = k_{\hat{3}}r\sin\theta + \mathcal{O}(r^{-1}), \quad (A12)$$

$$\begin{aligned}\eta = (k_{\hat{3}}^2\cos^2\theta + k_{\hat{2}}^2)r^2 + a\cos^2\theta(a(k_{\hat{2}}^2 + k_{\hat{3}}^2 - 1)\\ - 4Mk_{\hat{3}}\sin\theta) + \mathcal{O}(r^{-1}),\end{aligned} \quad (A13)$$

$$\Lambda = r\sqrt{k_{\hat{2}}^2 + k_{\hat{3}}^2} + \mathcal{O}(r^{-1}), \quad (A14)$$

$$\cos\alpha = \frac{k_{\hat{3}}\sin\theta}{\sqrt{k_{\hat{2}}^2 + k_{\hat{3}}^2}} + \mathcal{O}(r^{-2}). \quad (A15)$$

Scattering with the impact parameter b that measures the coordinate distance $r\cos\theta$ from the z-axis is most conveniently described with the help of a fiducial Cartesian system. Its axes are "parallel" to the fictitious Cartesian axes of the Byer-Lindquist coordinates, and in the asymptotic region to the global Cartesian grid. We introduce the momentum components $\hat{p}_i$ that are related to the spherical components $\hat{k}_i$ the the usual relations. In this case for the initial momentum parallel to the z-axis we have

$$D = \mathcal{O}(r_1^{-3}), \qquad \Lambda = \sqrt{b^2 - a^2} + \mathcal{O}(r_1^{-2}). \quad (A16)$$

Some care is needed in treating $R(r)$ as a function of the initial conditions when $r_1 \to \infty$. Introducing the constants c_0, c_1, c_2, we write it as

$$R(r) =: r^4 - c_2^2 r_1^2 r^2 + c_1 r_1^2 r + c_0 r_1^2, \qquad (A17)$$

where

$$c_2^2 := (\Lambda^2 - a^2)/r_1^2 = (k_1^2)^2 + (k_1^3)^2 + \mathcal{O}(r_1^{-2}) \quad (A18)$$

$$c_1 := 2M(\Lambda^2 + a^2 - 2\Lambda a \cos\alpha)/r_1$$

$$= 2M\left(c_2^2 - \frac{2ak_1^2 \sin\theta_1}{r_1}\right) + \mathcal{O}(r_1^{-2}), \qquad (A19)$$

$$c_0 := -a^2 \Lambda^2 \sin^2\alpha$$

$$= -a^2((k_1^2)^2\cos^2\theta_1 + (k_1^3)^2) + \mathcal{O}(r_1^{-2}). \qquad (A20)$$

APPENDIX B: TRAJECTORIES

1. The r integral

The integral over r in Eq. (5) is split into three parts,

$$\mathcal{R} := \int^r \frac{dr}{\pm\sqrt{R}} = \mathcal{R}_\infty - \mathcal{R}_1 - \mathcal{R}_2, \qquad (B1)$$

where

$$\mathcal{R}_\infty := 2\int_{r_{\min}}^\infty \frac{dr}{\sqrt{R}}, \qquad \mathcal{R}_i := \int_{r_i}^\infty \frac{dr}{\sqrt{R}}. \qquad (B2)$$

We present a corrected expression for $\mathcal{R}_i$ and evaluate the term $\mathcal{R}_\infty$ to the fourth order in Λ. Starting from $\mathcal{R}_\infty$ in using the method of [25].

Factoring out $(r - r_{\min})$ and substituting $r = r_{\min}/x$ allows to write $R(r)$ as

$$R = \frac{r_{\min}^2}{x^4}\left(r_{\min}^2 + Ax^2 + \frac{Bx^3}{r_{\min}} + \frac{Cx^4}{r_{\min}^2}\right) =: \frac{r_{\min}^2}{x^4}\tilde{R}(x), \quad (B3)$$

where the constants A, B, C are determined by the polynomial division. Noting that $\tilde{R}(1) = 0$ we have

$$r_{\min}^2 = -A - \frac{B}{r_{\min}} - \frac{C}{r_{\min}^2} \qquad (B4)$$

which gives

$$\tilde{R}(x) = -A - \frac{B}{r_{\min}} - \frac{C}{r_{\min}} + Ax^2 + \frac{B}{r_{\min}}x^4 + \frac{Cx^4}{r_{\min}^2}. \quad (B5)$$

Rewriting the equation and inserting the values for A, B and C gives us

$$R(r) = \frac{r_{\min}^2}{x^4}(\Lambda^2 - a^2)(1 - x^2)\left(1 + \frac{a^2\eta}{r_{\min}^2(\Lambda^2 - a^2)}(1 + x^2)\right.$$

$$\left. - \frac{2M}{r_{\min}}\frac{(D - a)^2 + \eta}{\Lambda^2 - a^2}\frac{1 + x + x^2}{1 + x}\right). \qquad (B6)$$

Defining

$$f(x) := \frac{\eta(1 + x^2)}{(\Lambda^2 - a^2)},$$

$$g(x) := -2\frac{(D - a)^2 + \eta}{\Lambda^2 - a^2}\frac{1 + x + x^2}{1 + x}, \qquad (B7)$$

we finally get

$$\mathcal{R}_\infty = 2\int_{r_{\min}}^\infty \frac{dr}{\sqrt{R}}$$

$$= \frac{2}{\sqrt{\Lambda^2 - a^2}}\int_0^1 dx\Big((1 - x^2)$$

$$\times \left(1 + \frac{a^2}{r_{\min}^2}f(x) + \frac{M}{r_{\min}}g(x)\right)\Big)^{-1/2}. \quad (B8)$$

Expanding the integrand up to the third order in Λ gives Eq. (15).

$$\mathcal{R}_\infty = \frac{1}{\sqrt{\Lambda^2 - a^2}}\left(\pi + \frac{4M}{\Lambda} + \frac{\pi}{\Lambda^2}\left(\frac{15}{4}M^2 - \frac{3}{4}a^2\sin^2\alpha\right)\right.$$

$$- \frac{8aM}{\Lambda^2}\cos\alpha + \frac{M}{\Lambda^3}\left[\frac{128}{3}M^2 + a^2(10\cos^2\alpha\right.$$

$$\left.\left. - 6\sin^2\alpha) + 15\pi Ma\cos\alpha\right]\right) + \mathcal{O}(\Lambda^{-4}). \qquad (B9)$$

The second and third integrals can be performed as an approximation in powers of r_1, using Eq. (A17). Since

$$R(r) = (r^4 - c_2^2 r_1^2 r^2)\left(1 + \frac{(c_0 + c_1 r)r_1^2}{r^2(r^2 - c_2^2 r_1^2)}\right), \qquad (B10)$$

the leading terms in the expansion are

$$\int_{r_i}^\infty \frac{dr}{\sqrt{R}} = \frac{1}{c_2 r_1}\arcsin\frac{c_2 r_1}{r_i}$$

$$- \frac{c_1}{2}\left(\frac{2}{(c_2^2 r_1)^2} + \frac{2r_i^2 - c_2^2 r_1^2}{c_2^4 r_i r_1^2\sqrt{r_i^2 - c_2^2 r_1^2}}\right) + \mathcal{O}(r_1^{-5}),$$

$$(B11)$$

with r_i being either r_1 or r_2, where in the latter case we assume that $r_2 \gtrsim r_1$.

2. The θ integral

We calculate the integral $\int 1/\sqrt{\Theta}d\theta$ using the method of [6]. Noting that

$$\Theta = \eta + (a - D)^2 - (a\sin\theta - D\csc\theta)^2, \qquad (B12)$$

we perform a change of variables $\mu := \cos\theta$ and obtain

$$I_\theta = \int \frac{d\theta}{\sqrt{\Theta}} = \int \frac{d\mu}{\sqrt{\Theta_\mu}}, \qquad (B13)$$

where for $\eta > 0$

$$\Theta_\mu = a^2(\mu_-^2 + \mu^2)(\mu_+^2 - \mu^2), \qquad 0 \le \mu^2 \le \mu_+^2, \tag{B14}$$

and

$$\mu_\pm^2 := \frac{1}{2a^2}\left(\sqrt{(\Lambda^2 - a^2)^2 + 4a^2\eta} \mp (\Lambda^2 - a^2)\right). \tag{B15}$$

The calculation is performed using the approximations to the auxiliary integral

$$\int_{\mu_*}^{\mu_+} \frac{d\mu}{\sqrt{\Theta_\mu}} = \frac{1}{a\sqrt{\mu_+^2 + \mu_-^2}} F(\psi, k^2), \tag{B16}$$

where $F(\psi, k^2)$ is the elliptic integral of the first kind, and

$$k^2 := \mu_+^2/(\mu_-^2 + \mu_+^2), \qquad \cos\psi := \mu_*/\mu_+, \tag{B17}$$

with $0 \le \psi \le \pi$ and $\mu_+ = +\sqrt{\mu_+^2}$.

The asymptotic expansion in the powers of Λ gives

$$\mu_+^2 = \sin^2\alpha + \frac{a^2}{4\Lambda^2}\sin^2 2\alpha + \mathcal{O}(\Lambda^{-4}) \tag{B18}$$

$$\mu_-^2 = \frac{\Lambda^2}{a^2} - \cos^2\alpha + \frac{a^2}{2\Lambda^2}\sin^2 2\alpha + \mathcal{O}(\Lambda^{-4}) \tag{B19}$$

so

$$k^2 = \frac{a^2\sin^2\alpha}{\Lambda^2} + \mathcal{O}(\Lambda^{-4}), \tag{B20}$$

$$\frac{1}{a\sqrt{\mu_+^2 + \mu_-^2}} = \frac{1}{\sqrt{\Lambda^2 - a^2}}\left(1 - \frac{a^2}{\Lambda^2}\sin^2\alpha + \mathcal{O}(\Lambda^{-4})\right). \tag{B21}$$

Trajectories of the type $\theta_1 \to \theta_{\min}/\theta_{\max} \to \theta_2$ correspond to $\cos\theta_{\min} = \mu_+$, and $\cos\theta_{\max} = -\mu_+ < 0$, respectively. In both cases the integration leads to

$$R = \frac{1}{a\sqrt{\mu_+^2 + \mu_-^2}}[F(\psi_1, k^2) + F(\psi_2, k^2)], \tag{B22}$$

and the expansion in powers of Λ leads to Eq. (10).

Using that at the zeroth order (the flat space-time) $\psi_1 + \psi_2 = \pi$, we expand $\psi_2 = \pi - \psi_1 + \delta_\psi$ as

$$\psi_2 = \pi - \psi_1 + \sum_k \frac{\xi_k}{\Lambda^k}. \tag{B23}$$

Equating the two expressions for $\mathcal{R}_\infty$ in the scattering scenario leads to

$$\xi_1 = 4M, \tag{B24}$$

$$\xi_2 = 15M^2\pi/4 - 8Ma\cos\alpha, \tag{B25}$$

$$\xi_3 = M(-6a^2 - 128M^2 + 45Ma\pi\cos\alpha \\ - 24a^2\cos 2\alpha + a^2(3 + \cos 2\psi_1)\sin^2\alpha). \tag{B26}$$

Similarly, setting $\theta_2 = \pi - \theta_1 + \delta_\theta$,

$$\delta_\theta = \sum_k \frac{\vartheta_k}{\Lambda^k}, \tag{B27}$$

and expanding the both sides of

$$\cos(\theta - \delta_\theta) = \mu_+ \cos\left(\arccos\frac{\cos\theta_1}{\mu_+} - \delta_\psi\right), \tag{B28}$$

we find

$$\vartheta_1 = 4M\frac{\sqrt{\sin^2\alpha - \cos^2\theta_1}}{\sin\theta_1}. \tag{B29}$$

In the special case of scattering with the initial momentum parallel (antiparallel) to the z-axis the angular momentum is zero, so $\mu_+ = 1$ and $\theta_{\rm in} = \psi_{\rm in} = 0, \pi$, and it is easy to see that $\theta_{\rm out} = \delta_\psi, \pi - \delta_\psi$, respectively.

3. The ϕ motion

The equation for ϕ is given by [6]

$$\phi = \pm\left(D\int^r \frac{dr}{\sqrt{R}} + 2Ma\int^r \frac{r\,dr}{\Delta\sqrt{R}} - a^2 D\int^r \frac{dr}{\Delta\sqrt{R}}\right) \\ \pm D\left(\int^\theta \frac{d\theta}{\sin^2\theta\sqrt{\Theta}} - \int^\theta \frac{d\theta}{\sqrt{\Theta}}\right). \tag{B30}$$

If $\theta \ne$ const. the first and the last terms cancel thanks to Eq. (5). For brevity we write the remaining terms as

$$\phi_2 - \phi_1 =: \pm\int^r R^\phi(r)dr \pm \int^\theta T(\theta)d\theta. \tag{B31}$$

We again decompose the radial integral as $\mathcal{R}^\phi = \mathcal{R}_\infty^\phi - \mathcal{R}_1^\phi - \mathcal{R}_2^\phi$, where

$$\mathcal{R}_\infty^\phi = 2\int_{r_{\min}}^\infty R^\phi(r)dr, \qquad \mathcal{R}_i^\phi = \int_{r_i}^\infty R^\phi(r)dr. \tag{B32}$$

Following the same procedure as in Sec. B 2 we find

$$\mathcal{R}_\infty^\phi = \frac{a}{\sqrt{\Lambda^2 - a^2}}\left(\frac{8M - a\pi\cos\alpha}{4\Lambda}\right. \\ \left. + \frac{M(3M\pi - 8a\cos\alpha)}{6\Lambda^2} + \mathcal{O}(\Lambda^{-3})\right). \tag{B33}$$

Expanding $R^\phi(r)$ for $r \gg r_1$ we obtain

$$R^\phi(r) = \frac{2aM}{\sqrt{(r^2 - c_2^2 r_1^2)}\left(\sqrt{1 + \frac{c_1 r_1^2}{r(r^2 - c_2^2 r_1^2)} + \frac{c_0 r_1^2}{r^2(r^2 - c_2^2 r_1^2)}}\right)(r^2 + a^2 - 2Mr)}$$

$$+ \frac{a^2 D}{r\sqrt{(r^2 - c_2^2 r_1^2)}\left(\sqrt{1 + \frac{c_1 r_1^2}{r(r^2 - c_2^2 r_1^2)} + \frac{c_0 r_1^2}{r^2(r^2 - c_2^2 r_1^2)}}\right)(r^2 + a^2 - 2Mr)}$$

$$= \frac{a}{r^2\sqrt{r^2 - c_2^2 r_1^2}}\left[2M\left(1 + \frac{2M}{r}\right) + \frac{aD}{r}\right] + \mathcal{O}(r^{-5}), \tag{B34}$$

where the coefficients c_i are defined in Appendix A 2. Hence

$$\int_{r_i}^{\infty} R^\phi(r) dr = \frac{2aM}{r_i^2 + r_i\sqrt{-c_2^2 r_1^2 + r_i^2}}$$

$$- \frac{a^2 D}{2c_2^3 r_1^3 r_i^3}\left(c_2 r_1\sqrt{-c_2^2 r_1^2 + r_i^2}\right.$$

$$\left. - r_i^2 \arctan\left(\frac{c_2 r_1}{\sqrt{-c_2^2 r_1^2 + r_i^2}}\right)\right) + \mathcal{O}(r_1^{-5}). \tag{B36}$$

Integral $\int T d\theta$ leads to the elliptic integral of the third kind. A standard change of variables $\cos\theta = \mu = \mu_+ \cos t$ and the identity

$$\sin^2\theta = (1 - \mu_+^2)(1 + p\sin^2 t), \tag{B37}$$

where

$$p := \frac{\mu_+^2}{1 - \mu_+^2} = \tan^2\alpha\left(1 + \frac{a^2}{\Lambda^2}\right) + \mathcal{O}(\Lambda^{-4}) \tag{B38}$$

lead to

$$\int_{\theta_{\min}}^{\theta_*} T(\theta) d\theta = \frac{1}{a\sqrt{\mu_+^2 + \mu_-^2}}\frac{D}{1 - \mu_+^2}$$

$$\times \int_0^{\psi_*} \frac{dt}{(1 + p\sin^2 t)\sqrt{1 - k^2\sin^2 t}}, \tag{B39}$$

with $\cos\psi_* = \cos\theta_*/\mu_+$ as in Appendix B 2. Hence

$$\int_{\theta_{\min}}^{\theta_*} T(\theta) d\theta = \frac{\Lambda\cos\alpha}{a\sqrt{\mu_+^2 + \mu_-^2}}\frac{1}{1 - \mu_+^2}\Pi(-p, \psi_*, k^2), \tag{B40}$$

where $\Pi(q, \psi, k^2)$ is the elliptic integral of the third kind. Expanding the prefactor in the powers of Λ gives

$$\frac{D}{a\sqrt{\mu_+^2 + \mu_-^2}}\frac{1}{1 - \mu_+^2} = \frac{\Lambda\sec\alpha}{\sqrt{\Lambda^2 - a^2}} + \mathcal{O}(\Lambda^{-4}), \tag{B41}$$

while

$$\Pi(-p, \psi, 0) = \frac{1}{\sqrt{1 + p}}\left(\frac{\pi}{2} - \text{arccot}(\sqrt{1 + p}\tan\psi)\right)$$

$$:= \Pi_0(p, \psi) \tag{B42}$$

and

$$\Pi(-p, \psi, k^2) = \Pi_0(p, \psi) + \frac{k^2}{2p}(\psi - \Pi_0(p, \psi))$$

$$+ \mathcal{O}(k^4). \tag{B43}$$

Taking into account that

$$\frac{1}{\sqrt{1 + p}} = |\cos\alpha|\left(1 - \frac{a^2}{\Lambda^2}\sin^2\alpha\right) + \mathcal{O}(\Lambda^{-4}), \tag{B44}$$

we get for the scattering scenario

$$\phi_{\text{out}} - \phi_{\text{in}} = \pi\frac{D}{|D|} + \sum_i \frac{\Phi_i}{\Lambda^i}, \tag{B45}$$

where the first two terms of the series are

$$\Phi_1 = \frac{\xi_1}{\cos^2\psi\cos\alpha + \sin^2\psi\sec\alpha}, \tag{B46}$$

and

$$\Phi_2 = \frac{\xi_2(3 + \cos 2\alpha)\sec^3\alpha + 4\sec\alpha(\xi_1^2\sin 2\psi_1 - \xi_2\cos 2\psi_1)\tan^2\alpha}{(1 + \sec^2\alpha - \cos 2\psi\tan^2\alpha)^2} + \frac{1}{4}a^2\cos\alpha + 2Ma, \tag{B47}$$

where ξ_i are given in Appendix B 2.

APPENDIX C: NEWTON GAUGE RELATIONSHIPS

In this Appendix we present a relationship between the components of polarization in the chronometric tetrad and the Newton gauge basis. This relationship allows to calculate the Walker-Penrose conserved quantity from the operationally meaningful polarization information and in the asymptotic regime gives the matrices N of Eq. (48).

The transversely of polarization $\hat{\mathbf{k}} \cdot \hat{\mathbf{f}} = 0$ ensures the linearity of the relationship between $(f^{\hat{2}}, f^{\hat{3}})$ and (f^x, f^y), where

$$\hat{\mathbf{f}} = f^x \hat{\mathbf{b}}_x + f^y \hat{\mathbf{b}}_y. \tag{C1}$$

Since $\mathbf{b}_y = \hat{\mathbf{w}} \times \mathbf{k} / |\hat{\mathbf{w}} \times \mathbf{k}|$ and $\mathbf{b}_x = -\mathbf{b}_y \times \hat{\mathbf{k}}$, by using Eq. (33) we find

$$\mathbf{b}_y = \frac{1}{\mathcal{N}} (w_{\hat{2}} k_{\hat{3}}, -w_{\hat{2}} k_{\hat{3}}, -w_{\hat{2}} k_{\hat{1}} + w_{\hat{1}} k_{\hat{2}}), \tag{C2}$$

$$\mathbf{b}_x = \frac{1}{\mathcal{N} k} (w_{\hat{2}} k_{\hat{1}} k_{\hat{2}} - w_{\hat{1}} (k_{\hat{3}}^2 + k_{\hat{2}}^2),$$
$$w_{\hat{2}} (k_{\hat{2}}^2 + k_{\hat{1}}^2) - w_{\hat{1}} k_{\hat{1}} k_{\hat{2}}, \quad -k_{\hat{3}} (w_{\hat{1}} k_{\hat{1}} + w_{\hat{2}} k_{\hat{3}})), \tag{C3}$$

$$\mathcal{N}^2 = (w_{\hat{2}} k_{\hat{3}})^2 + (w_{\hat{2}} k_{\hat{3}})^2 + (-w_{\hat{2}} k_{\hat{1}} + a_{\hat{1}} k_{\hat{2}})^2. \tag{C4}$$

Projecting $\hat{\mathbf{f}}$ on these directions gives

$$f^y = \frac{1}{\mathcal{N} k_{\hat{1}}} (f^{\hat{2}} (-w_{\hat{1}} k_{\hat{1}} k_{\hat{3}} - w_{\hat{2}} k_{\hat{3}} k_{\hat{2}})$$
$$+ f^{\hat{3}} (w_{\hat{1}} k_{\hat{1}} k_{\hat{2}} + a_{\hat{1}} (k_{\hat{3}}^2 + k_{\hat{1}}^2))), \tag{C5}$$

$$f^x = \frac{1}{\mathcal{N} k^{\hat{1}}} (f^{\hat{2}} (w_{\hat{2}} k_{\hat{1}} - w_{\hat{1}} k_{\hat{2}}) + f^{\hat{3}} (-w_{\hat{1}} k_{\hat{3}})). \tag{C6}$$

We define the transformation matrix N as

$$\begin{pmatrix} f^{\hat{2}} \\ f^{\hat{3}} \end{pmatrix} = N \begin{pmatrix} f^y \\ f^x \end{pmatrix}. \tag{C7}$$

In the asymptotic regime where $f^{\hat{1}} \to 0$ it becomes orthogonal. Taking $k^{\hat{m}} = (1, s\beta/r, D/r \sin\theta)$ we obtain in the leading order $1/r$

$$N = \frac{1}{\sqrt{\beta^2 + D^2/\sin^2\theta}} \begin{pmatrix} D/\sin\theta & s\beta \\ -s\beta & D/\sin\theta \end{pmatrix}. \tag{C8}$$

[1] T. Padmanabhan, *Theoretical Astrophysics* (Cambridge University Press, Cambridge, England, 2000), Vol. 1.

[2] A. Fowler, Observatory **42**, 241 (1919); F. W. Dyson, A. S. Eddington, and C. R. Davidson, Phil. Trans. R. Soc. A **220**, 291 (1920); D. Kennefick, Phys. Today **62**, 37 (2009).

[3] C. M. Will, *Theory and Experiment in Gravitational Physics* (Cambridge University Press, Cambridge, England, 1993); C. M. Will, Am. J. Phys. **78**, 1240 (2010).

[4] M. Born and E. Wolf, *Principles of Optics* (Cambridge University Press, Cambridge, England, 1999).

[5] C. W. Misner, K. S. Thorn, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).

[6] S. Chandrasekhar, *The Mathematical Theory of Black Holes* (Oxford University Press, New York, 1998).

[7] P. D. Naselsky, D. I. Novikov, and I. D. Novikov, *The Physics of the Cosmic Microwave Background* (Cambridge University Press, Cambridge, England, 2007).

[8] The Fermi-LAT collaboration and members of 3C 279 multi-band campaign, Nature (London) **463**, 919 (2010); P. Laurent, J. Rodriguez, J. Wilms, M. Cadolle Bel, K. Pottschmidt, and V. Grinberg, Science **332**, 438 (2011).

[9] P. F. Michelson, W. B. Atwood, and S. Ritz, Rep. Prog. Phys. **73**, 074901 (2010).

[10] M. Honda and Y. Honda, Astrophys. J. **569**, L39 (2002).

[11] R. Massey, T. Kitching, and J. Richard, Rep. Prog. Phys. **73**, 086901 (2010).

[12] D. Bruß and G. Leuchs, *Lectures on Quantum Information* (Wiley-VCH, Weinheim, 2007); M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University, Cambridge, 2000).

[13] M. Zwierz, Carlos A. Pérez-Delgado, and P. Kok, Phys. Rev. Lett. **105**, 180402 (2010); L. Maccone and V. Giovannetti, Nature Phys. **7**, 376 (2011).

[14] A. Peres and D. R. Terno, Rev. Mod. Phys. **76**, 93 (2004).

[15] P. M. Alsing and G. J. Stephenson, Jr, arXiv:0902.1399.

[16] C. Feiler *et al.*, Space Sci. Rev. **148**, 123 (2009).

[17] R. Ursin *et al.*, Europhysics News **40**, 3, 26 (2009); arXiv:0806.0945v1.

[18] G. V. Skrotskii, Sov. Phys. Dokl. **2**, 226 (1957).

[19] M. Nouri-Zonoz, Phys. Rev. D **60**, 024013 (1999); M. Sereno, Phys. Rev. D **69**, 087501 (2004).

[20] F. Fayos and J. Llosa, Gen. Relativ. Gravit. **14**, 865 (1982).

[21] H. Ishihara, M. Takahashi, and A. Tomimatsu, Phys. Rev. D **38**, 472 (1988).

[22] J. Plebanski, Phys. Rev. **118**, 1396 (1960).

[23] B. B. Godfrey, Phys. Rev. D **1**, 2721 (1970).

[24] R. F. Stark and P. A. Connors, Nature (London) **266**, 429 (1977); P. A. Connors, T. Piran, and R. F. Stark, Astrophys. J. **235**, 224 (1980).

[25] I. Bray, Phys. Rev. D **34**, 367 (1986).

[26] S. M. Kopeikin and G. Schäfer, Phys. Rev. D **60**, 124002 (1999); S. M. Kopeikin and B. Mashhoon, Phys. Rev. D **65**, 064025 (2002).

[27] P. P. Kronberg, C. C. Dyer, E. M. Burbidge, and V. T. Junkkarinen, Astrophys. J. **367**, L1 (1991); C. C. Dyer and E. G. Shaver, Astrophys. J. **390**, L5 (1992).

[28] C. R. Burns, C. C. Dyer, P. P. Kronberg, and H.-J. Röser, Astrophys. J. **613**, 672 (2004).

[29] V. Faraoni, New Astron **13**, 178 (2008) and the references therein.

[30] N. H. Lindner, A. Peres, and D. R. Terno, J. Phys. A **36**, L449 (2003).

[31] A. Brodutch and D. R. Terno, arXiv:1107.1274 [Phys. Rev. D (to be published).

[32] S. D. Bartlett, T. Rudolph, and R. W. Spekkens, Rev. Mod. Phys. **79**, 555 (2007).

[33] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Butterworth-Heinemann, Amsterdam, 1980).

[34] B. Carter, Phys. Rev. **174**, 1559 (1968).

[35] V. P. Frolov and I. D. Novikov, *Black Hole Physics* (Kluwer, Dordrecht, The Netherlands, 1998).

[36] I. Ciufolini and J. A. Wheeler, *Gravitation and Inertia* (Princeton University Press, Princeton, NJ, 1995).

[37] E. Wigner, Ann. Math. **40**, 149 (1939); S. Weinberg, *The Quantum Theory of Fields* (Cambridge University Press, Cambridge, England, 1996), Vol. 1.

[38] W.-K. Tung, *Group Theory in Physics* (World Scientific, Singapore, 1985).

[39] M. Walker and R. Penrose, Commun. Math. Phys. **18**, 265 (1970).

[40] T. Frankel, *The Geometry of Physics* (Cambridge University Press, Cambridge, England, 1999).

[41] M. V. Berry, Nature (London) **326**, 277 (1987); R. Ołkiewicz, Rep. Math. Phys. **33**, 325 (1993).

[42] S. D. Bartlett and D. R. Terno, Phys. Rev. A **71**, 012302 (2005).