Transport equation approach to calculations of Hadamard Green functions and non-coincident DeWitt coefficients

Adrian C. Ottewill[*]

*Complex and Adaptive Systems Laboratory and School of Mathematical Sciences,
University College Dublin, Belfield, Dublin 4, Ireland*

Barry Wardell[†]

Max-Planck-Institut für Gravitationsphysik, Albert-Einstein-Institut, 14476 Potsdam, Germany

Building on an insight due to Avramidi, we provide a system of transport equations for determining key fundamental bitensors, including derivatives of the world function, $\sigma(x, x')$, the square root of the Van Vleck determinant, $\Delta^{1/2}(x, x')$, and the tail term, $V(x, x')$, appearing in the Hadamard form of the Green function. These bitensors are central to a broad range of problems from radiation reaction to quantum field theory in curved spacetime and quantum gravity. Their transport equations may be used either in a semi-recursive approach to determining their covariant Taylor series expansions, or as the basis of numerical calculations. To illustrate the power of the semi-recursive approach, we present an implementation in MATHEMATICA, which computes very high order covariant series expansions of these objects. Using this code, a moderate laptop can, for example, calculate the coincidence limit $[a_7(x, x)]$ and $V(x, x')$ to order $(\sigma^a)^{20}$ in a matter of minutes. Results may be output in either a compact notation or in XTENSOR form. In a second application of the approach, we present a scheme for numerically integrating the transport equations as a system of coupled ordinary differential equations. As an example application of the scheme, we integrate along null geodesics to solve for $V(x, x')$ in Nariai and Schwarzschild spacetimes.

I. INTRODUCTION

In a recent paper [1], we presented methods for obtaining coordinate expansions for the (tail part of the) retarded Green function in spherically symmetric spacetimes. By using computer algebra to obtain high order Taylor series [of order $(\Delta x^\alpha)^{50}$] and applying the theory of Padé approximants we were able to obtain accurate expressions in remarkably large regions. Using these expressions, we were able to present the first complete matched expansion calculation of the self-force in a model ''black hole'' spacetime, the Nariai spacetime [2], and are currently applying the method to Schwarzschild spacetime. Our ultimate goal in this program is to work in more general spacetimes, especially Kerr spacetime. A key component of the matched expansion approach is knowledge of the Green function for points close together (i.e., in a *quasilocal* region). As we move away from specific symmetry conditions, we can no longer rely on methods based on a special choice of coordinates in the construction of our quasilocal solution and are led instead to consider other techniques such as transport equations and covariant expansion methods.

Covariant methods for calculating the Green function of the wave operator and the corresponding heat kernel, briefly reviewed in Sec. II below, are central to a broad range of problems from radiation reaction to quantum field theory in curved spacetime and quantum gravity. There is extremely extensive literature on this topic; here, we provide only a very brief overview referring the reader to the reviews by Vassilevich [3] and Poisson [4] and references therein for a more complete discussion. These methods have evolved from pioneering work by Hadamard [5] on the classical theory and DeWitt [6,7] on the quantum theory. The central objects in the Hadamard and DeWitt covariant expansions are geometrical bitensor coefficients $a_n^{AB'}(x, x')$ which are commonly called DeWitt[1] coefficients in the physics literature. These coefficients are closely related to the short proper-time asymptotic expansion of the heat kernel of an elliptic operator in a Riemannian space and so are commonly called heat kernel coefficients in the mathematics literature. Traditionally, most attention has focused on the *diagonal value* of the heat kernel $K^A{}_A(x, x; s)$, since the coincidence limits

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[1]The Hadamard and DeWitt coefficients also appear in the literature under several other guises. They may be called DeWitt, Gilkey, Minakshisundaram, Schwinger, or Seeley coefficients, or any combination thereof (yielding acronyms such as DWSC, DWSG and HDMS). In the coincidence limit, it has been proposed that they be called Hadamard-Minakshisundaram-DeWitt (HaMiDeW) [8] coefficients. For the remainder of this paper, we will refer to them as either DeWitt (for the coefficients $a_k{}^A{}_{B'}$) or Hadamard (for the coefficients $V_r{}^A{}_{B'}$) coefficients.

[*]adrian.ottewill@ucd.ie
[†]barry.wardell@aei.mpg.de

$a^A_{n\,A}(x, x)$ play a central role in the classical theory of spectral invariants [9] and in the quantum theory of the effective action and trace anomalies [10]. By contrast, for the quasilocal part of the matched expansion approach to radiation reaction [11,12], we seek expansions valid for x and x' as far apart as geometrical methods permit.

The classical approach to the calculation of these coefficients in the physics literature was to use a recursive approach developed by DeWitt [7] in the 1960s. Although these recursive methods work well for the first few terms in the expansion [13,14], and may be implemented in a tensor software package [15], the amount of calculation required to compute subsequent terms quickly becomes prohibitively long, even when implemented as a computer program. An alternative approach, more common in the mathematics literature, is to use pseudodifferential operators and invariance theory [9], where a basis of curvature invariants of the appropriate structure is constructed [16] and then their coefficients determined by explicit evaluation in simple spacetimes. However, here too, the size of the basis grows rapidly and there seems little prospect of reaching orders comparable to those we obtained in the highly symmetric configurations previously studied.

An extremely elegant, nonrecursive approach to the calculation of DeWitt coefficients has been given by Avramidi [17,18]. As his motivation was to study the effective action in quantum gravity he was primarily interested in the coincidence limit of the DeWitt coefficients, while in the self-force problem, as noted above, we require point-separated expressions. In addition, Avramidi introduced his method in the language of quantum mechanics, quite distinct from the language of transport equations, such as the Raychaudhuri equation, more familiar to discussions of geodesics among relativists. In this paper, we present Avramidi's approach in the language of transport equations and show that it is ideal for numerical and symbolic computation. In so doing, we are building on the work of Décanini and Folacci [19] who wrote many of the equations we present (we indicate below where we deviate from their approach) and implemented them explicitly by hand. However, calculations by hand are long and inevitably prone to error, particularly for higher spin and for higher order terms in the series and are quite impractical for the very high order expansions we would like for radiation reaction calculations. Instead, we use the transport equations as the basis for MATHEMATICA code for algebraic calculations and C code for numerical calculations. Rather than presenting our higher order results in excessively long equations [our noncanonical expression for $a_7(x, x)$ for a scalar field contains 2 987 366 terms], we have made these codes freely available online [20,21].

In Sec. II, we provide a brief review of Green functions, bitensors and covariant expansions, outlining the relations between the classical and quantum theories.

In Sec. III, we detail the principles that we consider to encapsulate the key insights of the Avramidi approach and use these to write down a set of transport equations for the key bitensors of the theory. These provide an adaptation of the Avramidi approach, which is ideally suited to implementation on a computer either numerically or symbolically.

In Sec. IV, we describe a semirecursive approach to solving for covariant expansions and briefly describe our MATHEMATICA implementation of it and its interface with the tensor software package XTENSOR [22].

In Sec. V, we present a numerical implementation of the transport equation approach to the calculation of the biscalar $V(x, x')$ appearing in the Hadamard form of the Green function along null geodesics.

In the Appendix, we give canonical expressions for the coincidence limits of the first five terms in the Hadamard expansion of $V(x, x')$.

Given our motivation in studying the radiation reaction problem, we shall phrase all the discussion of this paper in 4-dimensional spacetime. The reader is referred to the work by Décanini and Folacci [19] for a discussion of the corresponding situation in spacetimes of more general (integer) dimension. We do note, however, that the DeWitt coefficients are purely geometric bitensors, formally independent of the spacetime dimension.

Throughout this paper, we use units in which $G = c = 1$ and adopt the sign conventions of [23]. We denote symmetrization of indices using brackets [e.g., $(\alpha\beta)$] and exclude indices from symmetrization by surrounding them by vertical bars [e.g., $(\alpha|\beta|\gamma)$]. Roman letters are used for free indices and Greek letters for indices summed over all spacetime dimensions. Capital letters are used to denote the spinorial/tensorial indices appropriate to the field being considered.

II. A BRIEF REVIEW OF GREEN FUNCTIONS, BITENSORS, AND COVARIANT EXPANSIONS

A. Classical Green functions

We take an arbitrary field $\varphi^A(x)$ and consider wave operators, which are second order partial differential operators of the form [18]

$$\mathcal{D}^A_{\ B} = \delta^A_{\ B}(\Box - m^2) - P^A_{\ B}, \qquad (2.1)$$

where $\Box \equiv g^{\alpha\beta}\nabla_\alpha\nabla_\beta$, $g^{\alpha\beta}$ is the (contravariant) metric tensor, ∇_α is the covariant derivative defined by a connection $\mathcal{A}^A_{\ B\alpha}$: $\nabla_\alpha\varphi^A = \partial_\alpha\varphi^A + \mathcal{A}^A_{\ B\alpha}\varphi^B$, m is the mass of the field and $P^A_{\ B}(x)$ is a possible potential term.

In the classical theory of wave propagation in curved spacetime, a fundamental object is the retarded Green function, $G_{\text{ret}}{}^B_{\ C'}(x, x')$. It is a solution of the inhomogeneous wave equation,

$$\mathcal{D}^A_{\ B}G_{\text{ret}}{}^B_{\ C'}(x, x') = -4\pi\delta^A_{\ C'}\delta(x, x'), \qquad (2.2)$$

with support on and within the past light cone of the field point. (The factor of 4π is a matter of convention, our choice here is consistent with Ref. [4].) Finding the retarded Green function globally can be extremely hard. However, provided x and x' are sufficiently close (within a normal neighborhood[2]), we can use the Hadamard form for the retarded Green function solution [5,24], which in 4 spacetime dimensions takes the form

$$G_{\mathrm{ret}}{}^A{}_{B'}(x, x') = \theta_-(x, x')\{U^A{}_{B'}(x, x')\delta(\sigma(x, x'))$$
$$- V^A{}_{B'}(x, x')\theta(-\sigma(x, x'))\}, \qquad (2.3)$$

where $\theta_-(x, x')$ is analogous to the Heaviside step function, being 1 when x' is in the causal past of x, and 0 otherwise; $\delta(\sigma(x, x'))$ is the covariant form of the Dirac delta function, and $U^{AB'}(x, x')$ and $V^{AB'}(x, x')$ are symmetric bispinors/tensors and are regular for $x' \to x$. The biscalar $\sigma(x, x')$ is the Synge [4] world function, which is equal to one half of the squared geodesic distance between x and x'. The first term, involving $U^A{}_{B'}(x, x')$, in Eq. (2.3) represents the *direct* part of the Green function, while the second term, involving $V^A{}_{B'}(x, x')$, is known as the *tail* part of the Green function. This tail term represents backscattering off the spacetime geometry and is, for example, responsible for the quasilocal contribution to the self-force.

Within the Hadamard approach, the symmetric biscalar $V^{AB'}(x, x')$ is expressed in terms of a formal expansion in increasing powers of σ [19]:

$$V^{AB'}(x, x') = \sum_{r=0}^{\infty} V_r{}^{AB'}(x, x')\sigma^r(x, x'). \qquad (2.4)$$

The coefficients $U^{AB'}$ and $V_r{}^{AB'}$ are determined by imposing the wave equation, using the identity $\sigma_{;\alpha}\sigma^{;\alpha} = 2\sigma = \sigma_{;\alpha'}\sigma^{;\alpha'}$, and setting the coefficient of each *manifest* power of σ equal to zero. Since $V^A{}_{B'}$ is symmetric for self-adjoint wave operators, we are free to apply the wave equation either at x or at x'; here we choose to apply it at x'. We find that $U^{AB'}(x, x') = \Delta^{1/2}(x, x')g^{AB'}(x, x')$, where $\Delta(x, x')$ is the Van Vleck-Morette determinant defined as [4]

$$\Delta(x, x') = -[-g(x)]^{-1/2} \det(-\sigma_{;\alpha\beta'}(x, x'))[-g(x')]^{-1/2}$$
$$= \det(-g^{\alpha'}{}_\alpha(x, x')\sigma^{;\alpha}{}_{\beta'}(x, x')), \qquad (2.5)$$

with $g^{\alpha'}{}_\alpha(x, x')$ being the bivector of parallel transport (defined fully below) and where $g^{AB'}$ is the bitensor of

parallel transport appropriate to the tensorial nature of the field, e.g.,

$$g^{AB'} = \begin{cases} 1 & \text{(scalar)} \\ g^{ab'} & \text{(electromagnetic)} \\ g^{a'(a}g^{b)b'} & \text{(gravitational)}, \end{cases} \qquad (2.6)$$

where the higher spin fields are taken in Lorentz gauge. In making this identification, we have used the transport equation for the Van Vleck-Morette determinant:

$$\sigma^{;\alpha}\nabla_\alpha \ln\Delta = (4 - \Box\sigma). \qquad (2.7)$$

The coefficients $V_r^{AB'}(x, x')$ satisfy the recursion relations

$$\sigma^{;\alpha'}(\Delta^{-1/2}V_r^{AB'})_{;\alpha'} + (r + 1)\Delta^{-1/2}V_r^{AB'}$$
$$+ \frac{1}{2r}\Delta^{-1/2}\mathcal{D}^{B'}{}_{C'}V_{r-1}^{AC'} = 0 \qquad (2.8a)$$

for $r \in \mathbb{N}$ along with the "initial condition"

$$\sigma^{;\alpha'}(\Delta^{-1/2}V_0^{AB'})_{;\alpha'} + \Delta^{-1/2}V_0^{AB'}$$
$$+ \tfrac{1}{2}\Delta^{-1/2}\mathcal{D}^{B'}{}_{C'}(\Delta^{1/2}g^{AC'}) = 0. \qquad (2.8b)$$

These are transport equations which may be solved in principle within a normal neighborhood by direct integration along the geodesic from x to x'. The complication is that the calculation of $V_r^{AB'}$ requires the calculation of second derivatives of $V_{r-1}^{AB'}$ in directions off the geodesic; we address this issue below.

Finally, we emphasize that the Hadamard expansion (2.4) is an ansatz not a Taylor series. For example, in de Sitter spacetime for a conformally invariant scalar theory all the V_r's are nonzero while $V \equiv 0$.

B. The quantum theory

In curved spacetime, a fundamental object of interest is the Feynman Green function defined for a quantum field $\hat{\varphi}^A(x)$ in the state $|\Psi\rangle$ by

$$G_{\mathrm{f}}^{AB'}(x, x') = i\langle\Psi|\mathrm{T}[\hat{\varphi}^A(x)\hat{\varphi}^{B'}(x')]|\Psi\rangle, \qquad (2.9)$$

where T denotes time ordering. The Feynman Green function may be related to the advanced and retarded Green functions of the classical theory by the covariant commutation relations [7]

$$G_{\mathrm{f}}^{AB'}(x, x') = \frac{1}{8\pi}(G_{\mathrm{adv}}^{AB'}(x, x') + G_{\mathrm{ret}}^{AB'}(x, x'))$$
$$+ \frac{i}{2}\langle\Psi|\hat{\varphi}^A(x)\hat{\varphi}^{B'}(x') + \hat{\varphi}^{B'}(x')\hat{\varphi}^A(x)|\Psi\rangle.$$
$$(2.10)$$

The anticommutator function $\langle\Psi|\hat{\varphi}^A(x)\hat{\varphi}^{B'}(x') + \hat{\varphi}^{B'}(x')\hat{\varphi}^A(x)|\Psi\rangle$ clearly satisfies the homogeneous wave equation so that the Feynman Green function satisfies the equation

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[2]More precisely, the Hadamard form requires that x and x' lie within a *causal domain*, a *convex normal neighborhood* with a causality condition attached. This effectively requires that x and x' be connected by at most one nonspacelike geodesic, which stays within the causal domain. However, as we expect the term *normal neighborhood* to be more familiar to the reader, we will use it throughout this paper, with implied assumptions of convexity and a causality condition.

$$\mathcal{D}^A{}_B G_f{}^B{}_{C'}(x, x') = -\delta^A{}_{C'}\delta(x, x'). \quad (2.11)$$

Using the proper-time formalism [7], the identity

$$i\int_0^\infty ds\, e^{-\epsilon s}\exp(isx) = -\frac{1}{x + i\epsilon}, \qquad (\epsilon > 0), \quad (2.12)$$

allows the causal properties of the Feynman function to be encapsulated in the formal expression

$$G_f{}^A{}_{C'}(x, x') = i\int_0^\infty ds\, e^{-\epsilon s}\exp(is\mathcal{D})^A{}_B \delta^B{}_{C'}\delta(x, x'), \quad (2.13)$$

where the limit $\epsilon \to 0+$ is understood. The integrand

$$K^A{}_{C'}(x, x'; s) = \exp(is\mathcal{D})^A{}_B \delta^B{}_{C'}\delta(x, x') \quad (2.14)$$

clearly satisfies the Schrödinger/heat equation

$$\frac{1}{i}\frac{\partial K^A{}_{C'}}{\partial s}(x, x'; s) = \mathcal{D}^A{}_B K^B{}_{C'}(x, x'; s) \quad (2.15)$$

together with the initial condition $K^A{}_{B'}(x, x'; 0) = \delta^A{}_{B'}(x, x')$. The trivial way in which the mass m enters these equations allows it to be eliminated through the prescription

$$K^A{}_{C'}(x, x'; s) = e^{-im^2 s}K_0{}^A{}_{C'}(x, x'; s), \quad (2.16)$$

with the massless heat kernel satisfying the equation

$$\frac{1}{i}\frac{\partial K_0{}^A{}_{C'}}{\partial s}(x, x'; s) = (\delta^A{}_B\Box - P^A{}_B)K_0{}^B{}_{C'}(x, x'; s) \quad (2.17)$$

together with the initial condition $K_0{}^A{}_{B'}(x, x'; 0) = \delta^A{}_{B'}\delta(x, x')$.

In 4-dimensional Minkowski spacetime without potential, the massless heat kernel is readily obtained as

$$K_0{}^A{}_{B'}(x, x'; s) = \frac{1}{(4\pi s)^2}\exp\left(-\frac{\sigma}{2is}\right)\delta^A{}_{B'}$$

$$\text{(flat spacetime).} \quad (2.18)$$

This motivates the ansatz [7] that in general the massless heat kernel allows the representation

$$K_0{}^A{}_{B'}(x, x'; s) \sim \frac{1}{(4\pi s)^2}\exp\left(-\frac{\sigma}{2is}\right)\Delta^{1/2}(x, x')$$

$$\times \Omega_0{}^A{}_{B'}(x, x'; s), \quad (2.19)$$

where $\Omega^A{}_{B'}(x, x'; s)$ possesses the following asymptotic expansion as $s \to 0+$:

$$\Omega^A{}_{B'}(x, x'; s) \sim \sum_{r=0}^\infty a_r^A{}_{B'}(x, x')(is)^r, \quad (2.20)$$

where $a_0{}^A{}_{B'}(x, x) = \delta^A{}_{B'}$ and $a_r{}^A{}_{B'}(x, x')$ has dimension $(\text{length})^{-2r}$. The inclusion of the explicit factor of $\Delta^{1/2}$ is simply a matter of convention; by including it we are following DeWitt, but many authors, including Décanini and Folacci, choose instead to include it in the series coefficients

$$A_r{}^A{}_{B'}(x, x') = \Delta^{1/2}a_r{}^A{}_{B'}(x, x'). \quad (2.21)$$

It is clearly trivial to convert between the two conventions and, in any case, the coincidence limits agree.

Now, requiring our expansion to satisfy Eq. (2.17) and using the symmetry of $\Omega^A{}_{B'}(x, x'; s)$ to allow operators to act at x', we find that $\Omega^A{}_{B'}(x, x'; s)$ must satisfy

$$\frac{1}{i}\frac{\partial \Omega^{AB'}}{\partial s} + \frac{1}{is}\sigma^{;\alpha'}\Omega^{AB'}{}_{;\alpha'}$$

$$= \Delta^{-1/2}(\delta^{B'}{}_{C'}\Box - P^{B'}{}_{C'})(\Delta^{1/2}\Omega^{AC'}(x, x'; s)). \quad (2.22)$$

Inserting the expansion Eq. (2.20), the coefficients $a_n^{AB'}(x, x')$ satisfy the recursion relations

$$\sigma^{;\alpha'}a_{r+1}{}^{AB'}{}_{;\alpha'} + (r + 1)a_{r+1}{}^{AB'}$$

$$- \Delta^{-1/2}(\delta^{B'}{}_{C'}\Box - P^{B'}{}_{C'})(\Delta^{1/2}a_r{}^{AC'}) = 0 \quad (2.23a)$$

for $r \in \mathbb{N}$ along with the initial condition

$$\sigma^{;\alpha'}a_0{}^{AB'}{}_{;\alpha'} = 0, \quad (2.23b)$$

with the implicit requirement that they be regular as $x' \to x$.

To compare the DeWitt approach to the Hadamard approach we may start by rewriting the Hadamard recursion relations (2.8) as

$$\sigma^{;\alpha'}((-2)^{r+1}r!\Delta^{-1/2}V_r^{AB'})_{;\alpha'} + (r + 1)((-2)^{r+1}r!\Delta^{-1/2}V_r^{AB'}) - \Delta^{-1/2}(\delta^{B'}{}_{C'}\Box' - P^{B'}{}_{C'})(\Delta^{1/2}(-2)^r(r - 1)!\Delta^{-1/2}V_{r-1}^{AC'})$$

$$+ m^2((-2)^r(r - 1)!\Delta^{-1/2}V_{r-1}^{AB'}) = 0, \quad (2.24)$$

which can be taken to include $r = 0$ with the formal identification $(-1)!\Delta^{-1/2}V_{-1}^{AB'} = g^{AB'} = a_0{}^{AB'}$. Comparing (2.24) and (2.23a), one can see that the massless Hadamard and (mass-independent) DeWitt coefficients are related by

$$a_{r+1}{}^{A}{}_{B'}(x, x') = (-2)^{r+1} r! \Delta^{-1/2}(x, x') V_r^{(m^2=0)A}{}_{B'}(x, x'),$$
(2.25a)

$$V_r^{(m^2=0)A}{}_{B'}(x, x') = (-1)^{r+1} \frac{\Delta^{1/2}(x, x')}{2^{r+1} r!} a_{r+1}{}^{A}{}_{B'}(x, x').$$
(2.25b)

We can also relate the Hadamard coefficients for a theory of mass m and the (mass-independent) DeWitt coefficients. We start by noting that from (2.16), (2.19), and (2.20) the massive heat kernel has the asymptotic expansion

$$K^{A}{}_{B'}(x, x'; s) \sim \frac{1}{(4\pi s)^2} \exp\left(-\frac{\sigma}{2is}\right) \Delta^{1/2}(x, x')$$
$$\times \sum_{r=0}^{\infty} \left(\sum_{k=0}^{r} \frac{(-m^2)^{r-k}}{(r-k)!} a_{k}^{A}{}_{B'}(x, x') \right) (is)^r.$$
(2.26)

It follows from linearity that the massive Hadamard coefficients may be obtained from (2.25b) with the replacement

$$a_r{}^{A}{}_{B'}(x, x') \to \sum_{k=0}^{r} \frac{(-m^2)^{r-k}}{(r-k)!} a_r{}^{A}{}_{B'}(x, x')$$
(2.27)

yielding

$$V_r{}^{A}{}_{B'}(x, x') = (-1)^{r+1} \frac{\Delta^{1/2}(x, x')}{2^{r+1} r!}$$
$$\times \sum_{k=0}^{r+1} \frac{(-m^2)^{r+1-k}}{(r+1-k)!} a_k{}^{A}{}_{B'}(x, x')$$
(2.28)

with inverse

$$a_{r+1}{}^{A}{}_{B'}(x, x') = \Delta^{-1/2} \sum_{k=0}^{r} (-2)^{k+1} \frac{k!}{(r-k)!}$$
$$\times (m^2)^{r-k} V_k{}^{A}{}_{B'}(x, x') + \frac{(m^2)^{r+1}}{(r+1)!}.$$
(2.29)

These relations enable us to relate the "tail term" of the massive theory to that of the massless theory by

$$V(x, x')^{A}{}_{B'} = \sum_{r=0}^{\infty} V_r^{(m^2=0)A}{}_{B'}(x, x') \frac{(2\sigma)^r r! J_r((-2m^2\sigma)^{1/2})}{(-2m^2\sigma)^{r/2}}$$
$$+ m^2 \Delta^{1/2} \frac{J_1((-2m^2\sigma)^{1/2})}{(-2m^2\sigma)^{1/2}} \delta^{A}{}_{B'},$$
(2.30)

where $J_r(x)$ are Bessel functions of the first kind. This last expression is obtained by using (2.25a) in (2.28), substituting the result into (2.4) and interchanging the order of summation (upon doing so, the sum over k yields the Bessel functions).

C. Classical approach to covariant expansion calculations

The Synge world function, $\sigma(x, x')$ is a biscalar (i.e., a scalar at x and at x') defined to be equal to half the square of the geodesic distance between x and x'. The world function is defined through the fundamental identity

$$\sigma_\alpha \sigma^\alpha = 2\sigma = \sigma_{\alpha'} \sigma^{\alpha'},$$
(2.31)

together with the initial conditions $\lim_{x'\to x} \sigma(x, x') = 0$ and $\lim_{x'\to x} \sigma_{ab}(x, x') = g_{ab}(x)$. Here, we indicate derivatives at the (un-)primed point by (un-)primed indices:

$$\sigma^a \equiv \nabla^a \sigma \qquad \sigma_a \equiv \nabla_a \sigma$$
$$\sigma^{a'} \equiv \nabla^{a'} \sigma \qquad \sigma_{a'} \equiv \nabla_{a'} \sigma.$$
(2.32)

σ^a is a vector at x of length equal to the geodesic distance between x and x', tangent to the geodesic at x and oriented in the direction $x' \to x$, while $\sigma^{a'}$ is a vector at x' of length equal to the geodesic distance between x and x', tangent to the geodesic at x' and oriented in the opposite direction.

The covariant derivatives of σ may be written as

$$\sigma^a(x, x') = (s - s')u^a \qquad \sigma^{a'}(x, x') = (s' - s)u^{a'},$$
(2.33)

where s is an affine parameter and u^a is tangent to the geodesic. For timelike geodesics, s may be taken as the proper time along the geodesic, while u^a is the 4-velocity tangent to the geodesic and

$$\sigma(x, x') = -\tfrac{1}{2}(s - s')^2.$$
(2.34)

Similarly, for spacelike geodesics, s may be taken as the spatial geodesic distance along the geodesic and

$$\sigma(x, x') = +\tfrac{1}{2}(s - s')^2.$$
(2.35)

For null geodesics, u^a is null and $\sigma(x, x') = 0$.

Another bitensor of frequent interest is the bivector of parallel transport, $g_{ab'}$ defined by the transport equation

$$\sigma^\alpha g_{ab';\alpha} = 0 = \sigma^{\alpha'} g_{ab';\alpha'}$$
(2.36)

with initial condition $\lim_{x'\to x} g_{ab'}(x, x') = g_{ab}(x)$. From the definition of a geodesic, it follows that

$$g_{a\alpha'}\sigma^{\alpha'} = -\sigma_a \quad \text{and} \quad g_{\alpha a'}\sigma^\alpha = -\sigma_{a'}.$$
(2.37)

Given a bitensor T_a at x, the parallel transport bivector allows us to define $\bar{T}_{a'}$, a bitensor at x', obtained by parallel transporting T_a along the geodesic from x to x' and vice versa,

$$T_\alpha g^{\alpha}{}_{a'} = \bar{T}_{a'} \qquad \bar{T}_{\alpha'} g^{\alpha'}{}_a = T_a.$$
(2.38)

These are consistent as $g^{a}{}_{\alpha'} g_b{}^{\alpha'} = \delta^{a}{}_b$ and $g_\alpha{}^{a'} g^{\alpha}{}_{b'} = \delta^{a'}{}_{b'}$.

Any sufficiently smooth bitensor $T_{a_1 \cdots a_m a'_1 \cdots a'_n}$ may be expanded in a local covariant Taylor series about the

point x. To do so, it is convenient and conventional to first use the bivector of parallel transport to transport all tensor indices to x, for example:

$$g_{b_1}{}^{b_1'} \cdots g_{b_n}{}^{b_n'} T_{a_1\cdots a_m b_1'\cdots b_n'}(x, x')$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} t_{a_1\cdots a_m b_1\cdots b_n \alpha_1\cdots \alpha_k}(x)\sigma^{\alpha_1}\cdots\sigma^{\alpha_k}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} T_{a_1\cdots a_m b_1\cdots b_n(k)}, \qquad (2.39)$$

where the $t_{a_1\cdots a_m b_1\cdots b_n \alpha_1\cdots \alpha_k}$ are the coefficients of the series and are local tensors at x, and $T_{a_1\cdots a_m b_1\cdots b_n(k)}$ is defined as this coefficient contracted with the corresponding σ^{α_i}. Similarly, we can also expand about x':

$$g^{a_1}{}_{a_1'} \cdots g^{a_m}{}_{a_m'} T_{a_1\cdots a_m b_1'\cdots b_n'}(x, x')$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} t_{a_1'\cdots a_m' b_1'\cdots b_n' \alpha_1'\cdots \alpha_k'}(x')\sigma^{\alpha_1'}\cdots\sigma^{\alpha_k'}. \quad (2.40)$$

For many fundamental bitensors, one would typically use the DeWitt approach [6] to determine the coefficients in these expansions as follows:

(1) Take covariant derivatives of the defining equation for the bitensor (the number of derivatives required depends on the order of the term to be found).
(2) Replace all known terms with their coincidence limit, $x \rightarrow x'$.
(3) Sort covariant derivatives, introducing Riemann tensor terms in the process.
(4) Take the coincidence limit $x' \rightarrow x$ of the result.

This method allows all coefficients to be determined recursively in terms of lower order coefficients and Riemann tensor polynomials. Although this method proves effective for determining the lowest few order terms by hand and can be readily implemented in software, it does not scale well and it is not long before the computation time required to calculate the next term is prohibitively large. This issue can be understood from the fact that the calculation yields extremely large intermediate expressions, which simplify tremendously in the end. It is therefore desirable to find an alternative approach, which is more efficient and better suited to implementation in software. In the following sections, we will describe one such approach, which proves to be highly efficient.

III. AVRAMIDI APPROACH TO COVARIANT EXPANSION CALCULATIONS

The traditional DeWitt [6] approach to the calculation of covariant expansions of fundamental bitensors is to derive a set of recursion relations for the coefficients of the series. Avramidi [17] has proposed an alternative, extremely elegant nonrecursive method for the calculation of these coefficients. Translated into the language of transport

equations, this approach emphasizes two fundamental principles when doing calculations:

(1) When expanding about x, always try to take derivatives at x'. The result is that derivatives only act on the σ^a's and not on the coefficients.
(2) Where possible, whenever taking a covariant derivative, $\nabla_{a'}$, contract the derivative with $\sigma^{a'}$.

Applying these two principles, Avramidi has derived non-recursive[3] expressions for the coefficients of covariant expansions of several bitensors. As Avramidi's derivations use a rather abstract notation, we will now briefly review his technique in a more explicit notation. We will also extend the derivation to include several other bitensors and note that Eqs. (3.11), (3.13), (3.15), (3.17), (3.34), (3.35), (3.46), and (3.49) were previously written down and used by Décanini and Folacci [19].

Throughout this section, we fix the base point x and allow it to be connected to any other point x' by a geodesic. In all cases, we expand about the fixed point, x.

Defining the transport operators D and D' as

$$D \equiv \sigma^\alpha \nabla_\alpha \qquad D' \equiv \sigma^{\alpha'} \nabla_{\alpha'}, \qquad (3.1)$$

we can rewrite Eq. (2.31) as

$$(D - 2)\sigma = 0 \qquad (D' - 2)\sigma = 0. \qquad (3.2)$$

Differentiating these equations at x and at x', we get

$$(D - 1)\sigma^a = 0 \qquad (D - 1)\sigma^{a'} = 0$$
$$(D' - 1)\sigma^a = 0 \qquad (D' - 1)\sigma^{a'} = 0. \qquad (3.3)$$

Defining

$$\eta^a{}_{b'} \equiv \sigma^a{}_{b'} \qquad \xi^{a'}{}_{b'} \equiv \sigma^{a'}{}_{b'}, \qquad (3.4)$$

the second pair of these equations can be rewritten as

$$\sigma^a = \eta^a{}_{\alpha'}\sigma^{\alpha'} \qquad \sigma^{a'} = \xi^{a'}{}_{\alpha'}\sigma^{\alpha'}. \qquad (3.5)$$

Finally, we define $\gamma^{a'}{}_b$, the inverse of $\eta^a{}_{b'}$,

$$\gamma^{a'}{}_b \equiv (\eta^b{}_{a'})^{-1}, \qquad (3.6)$$

and also introduce the definition

$$\lambda^a{}_b \equiv \sigma^a{}_b. \qquad (3.7)$$

We will now derive transport equations for each of these newly introduced quantities along with some others, which will be defined as required. Many of these derivations involve considerable index manipulations and are most easily (and accurately) done using a tensor software package such as XTENSOR [22].

The transport equations of this section may be derived in a recursive manner, making use of the identities

$$D'(\sigma_{a_1'...a_n'a_{n+1}'}) = \nabla_{a_{n+1}'}(D'\sigma_{a_1'...a_n'}) - \xi^{\alpha'}{}_{a_{n+1}'}\nabla_{\alpha'}\sigma_{a_1'...a_n'}$$

$$+ \sigma^{\alpha'}R^{c'}{}_{a_1'a_{n+1}'\alpha'}\sigma_{c'...a_n'} + ...$$

$$+ \sigma^{\alpha'}R^{c'}{}_{a_n'a_{n+1}'\alpha'}\sigma_{a_1'...c'} \qquad (3.8)$$

$$D'(\sigma^b{}_{a_1'...a_n'}) = \nabla^b(D'\sigma_{a_1'...a_n'}) - \eta^b{}_{\alpha'}\nabla^{\alpha'}\sigma_{a_1'...a_n'}, \qquad (3.9)$$

and its generalization, given below. This method is naturally algorithmic and well suited to implementation on a computer, thus allowing for the automated derivation of a transport equation for an arbitrary number of derivatives of a bitensor.

A. Transport equation for $\xi^{a'}{}_{b'}$

Taking a primed derivative of the second equation in (3.5), we obtain

$$\xi^{a'}{}_{b'} = \xi^{a'}{}_{\alpha'b'}\sigma^{\alpha'} + \xi^{a'}{}_{\alpha'}\xi^{\alpha'}{}_{b'}. \qquad (3.10)$$

We now commute the last two covariant derivatives in the first term on the right-hand side of this equation and rearrange to obtain

$$D'\xi^{a'}{}_{b'} + \xi^{a'}{}_{\alpha'}\xi^{\alpha'}{}_{b'} - \xi^{a'}{}_{b'} + R^{a'}{}_{\alpha'b'\beta'}\sigma^{\alpha'}\sigma^{\beta'} = 0. \qquad (3.11)$$

B. Transport equation for $\eta^a{}_{b'}$

Taking a primed derivative of the first equation in (3.5), we obtain

$$\eta^a{}_{b'} = \eta^a{}_{\alpha'b'}\sigma^{\alpha'} + \eta^a{}_{\alpha'}\xi^{\alpha'}{}_{b'}. \qquad (3.12)$$

In this case, since σ^a is a scalar at x', we can commute the two primed covariant derivatives in the first term on the right-hand side of this equation without introducing a Riemann term. Rearranging, we obtain

$$D'\eta^a{}_{b'} + \eta^a{}_{\alpha'}\xi^{\alpha'}{}_{b'} - \eta^a{}_{b'} = 0. \qquad (3.13)$$

C. Transport equation for $\gamma^{a'}{}_b$

Solving Eq. (3.13) for $\xi^{a'}{}_{b'}$ and using (3.6), we obtain

$$\xi^{a'}{}_{b'} = \delta^{a'}{}_{b'} - \gamma^{a'}{}_\alpha(D'\eta^\alpha{}_{b'}) = \delta^{a'}{}_{b'} + (D'\gamma^{a'}{}_\alpha)\eta^\alpha{}_{b'}. \qquad (3.14)$$

Next, substituting Eq. (3.14) into Eq. (3.11) and rearranging, we obtain a transport equation for $\gamma^{a'}{}_b$:

$$(D')^2\gamma^{a'}{}_b + D'\gamma^{a'}{}_b + R^{a'}{}_{\alpha'\gamma'\beta'}\gamma^{\gamma'}{}_b\sigma^{\alpha'}\sigma^{\beta'} = 0. \qquad (3.15)$$

D. Equation for $\lambda^a{}_b$

Differentiating Eq. (2.31) at x and x', we obtain

$$\eta^a{}_{b'} = \lambda^a{}_\alpha\eta^\alpha{}_{b'} + D\eta^a{}_{b'}, \qquad (3.16)$$

which is easily rearranged to give an equation for $\lambda^a{}_b$:

$$\lambda^a{}_b = \delta^a{}_b - (D\eta^a{}_{\alpha'})\gamma^{\alpha'}{}_b. \qquad (3.17)$$

E. Transport equation for $\sigma^{a'}{}_{b'c'}$

Applying the identity (3.8) to (3.11) and simplifying the resulting expression, we obtain

$$(D'-1)\sigma^{a'}{}_{b'c'} + \sigma^{a'}{}_{c'}\sigma^a{}_{\alpha'b'} + \sigma^{\alpha'}{}_{b'}\sigma^{a'}{}_{\alpha'c'} + \sigma^{a'}{}_{\alpha'}\sigma^{\alpha'}{}_{b'c'} + R^{a'}{}_{\alpha'b'\beta';c'}\sigma^{\alpha'}\sigma^{\beta'} - R^{a'}{}_{\alpha'\beta'b'}\sigma^{\beta'}\sigma^{\alpha'}{}_{c'}$$

$$- R^{a'}{}_{\alpha'\beta'c'}\sigma^{\beta'}\sigma^{\alpha'}{}_{b'} + R^{a'}{}_{b'\beta'c'}\sigma^{\beta'}\sigma^a{}_{\alpha'} = 0. \qquad (3.18)$$

F. Transport equation for $\sigma^a{}_{b'c'}$

Applying the identity (3.9) to (3.11) and simplifying the resulting expression, we obtain

$$(D'-1)\sigma^a{}_{b'c'} + \sigma^{\alpha'}{}_{b'}\sigma^a{}_{\alpha'c'} + \sigma^{\alpha'}{}_{c'}\sigma^a{}_{\alpha'b'} + \sigma^a{}_{\alpha'}\sigma^{\alpha'}{}_{b'c'} + R^{\alpha'}{}_{b'\beta'c'}\sigma^a{}_{\alpha'}\sigma^{\beta'} = 0. \qquad (3.19)$$

G. Transport equation for $\sigma^{a'}{}_{b'c'd'}$

Applying the identity (3.8) to (3.18) and simplifying the resulting expression, we obtain

$$(D'-1)\sigma^{a'}{}_{b'c'd'} + \sigma^{a'}{}_{\alpha'b'c'}\sigma^{\alpha'}{}_{d'} + \sigma^{a'}{}_{\alpha'b'd'}\sigma^{\alpha'}{}_{c'} + \sigma^{a'}{}_{\alpha'c'd'}\sigma^{\alpha'}{}_{b'} + \sigma^{a'}{}_{\alpha'b'}\sigma^{\alpha'}{}_{c'd'} + \sigma^{a'}{}_{\alpha'c'}\sigma^{\alpha'}{}_{b'd'} + \sigma^{a'}{}_{\alpha'd'}\sigma^{\alpha'}{}_{b'c'}$$

$$+ \sigma^{a'}{}_{\alpha'}\sigma^{\alpha'}{}_{b'c'd'} + R^{a'}{}_{\alpha'\beta'c'}R^{\alpha'}{}_{d'\gamma'b'}\sigma^{\beta'}\sigma^{\gamma'} + R^{a'}{}_{\alpha'\beta'b'}R^{\alpha'}{}_{d'\gamma'c'}\sigma^{\beta'}\sigma^{\gamma'} + R^{a'}{}_{\alpha'\beta'd'}R^{\alpha'}{}_{c'\gamma'b'}\sigma^{\beta'}\sigma^{\gamma'}$$

$$- R^{a'}{}_{\beta'\alpha'd'}R^{\alpha'}{}_{b'\gamma'c'}\sigma^{\beta'}\sigma^{\gamma'} - R^{a'}{}_{\beta'\alpha'c'}R^{\alpha'}{}_{b'\gamma'd'}\sigma^{\beta'}\sigma^{\gamma'} + R^{a'}{}_{b'b'\gamma';c'd'}\sigma^{\beta'}\sigma^{\gamma'} + R^{a'}{}_{b'\beta'c';d'}\sigma^{\beta'}\sigma^{a'}{}_{\alpha'}$$

$$- R^{a'}{}_{\alpha'\beta'c';d'}\sigma^{\beta'}\sigma^{\alpha'}{}_{b'} - R^{a'}{}_{\alpha'\beta'b';d'}\sigma^{\beta'}\sigma^{\alpha'}{}_{c'} - R^{a'}{}_{\alpha'\beta'b';c'}\sigma^{\beta'}\sigma^{\alpha'}{}_{d'} + R^{\alpha'}{}_{c'\beta'd'}\sigma^{\beta'}\sigma^{a'}{}_{b'\alpha'} + R^{\alpha'}{}_{b'\beta'd'}\sigma^{\beta'}\sigma^{a'}{}_{c'\alpha'}$$

$$+ R^{\alpha'}{}_{b'\beta'c'}\sigma^{\beta'}\sigma^{a'}{}_{d'\alpha'} - R^{a'}{}_{\alpha'\beta'd'}\sigma^{\beta'}\sigma^{\alpha'}{}_{b'c'} - R^{a'}{}_{\alpha'\beta'c'}\sigma^{\beta'}\sigma^{\alpha'}{}_{b'd'} - R^{a'}{}_{\alpha'\beta'b'}\sigma^{\beta'}\sigma^{\alpha'}{}_{c'd'} = 0. \qquad (3.20)$$

H. Transport equation for $\sigma^a{}_{b'c'd'}$

Applying the identity (3.9) to (3.18) and simplifying the resulting expression, we obtain

$$(D'-1)\sigma^a{}_{b'c'd'} + \sigma^a{}_{\alpha'b'c'}\sigma^{\alpha'}{}_{d'} + \sigma^a{}_{\alpha'b'd'}\sigma^{\alpha'}{}_{c'} + \sigma^a{}_{\alpha'c'd'}\sigma^{\alpha'}{}_{b'} + \sigma^a{}_{\alpha'b'}\sigma^{\alpha'}{}_{c'd'} + \sigma^a{}_{\alpha'c'}\sigma^{\alpha'}{}_{b'd'} + \sigma^a{}_{\alpha'd'}\sigma^{\alpha'}{}_{b'c'}$$

$$+ \sigma^a{}_{\alpha'}\sigma^{\alpha'}{}_{b'c'd'} + R^{\alpha'}{}_{b'\beta'c';d'}\sigma^{\beta'}\sigma^a{}_{\alpha'} + R^{\alpha'}{}_{b'\beta'c'}\sigma^{\beta'}\sigma^a{}_{d'\alpha'} + R^{\alpha'}{}_{b'\beta'd'}\sigma^{\beta'}\sigma^a{}_{c'\alpha'} + R^{\alpha'}{}_{c'\beta'd'}\sigma^{\beta'}\sigma^a{}_{b'\alpha'} = 0. \qquad (3.21)$$

I. Transport equation for $g_{a'}{}^b$

The bivector of parallel transport is defined by the transport equation

$$D'g_{a'}{}^b = \sigma^{\alpha'}g_{a'}{}^b{}_{;\alpha'} \equiv 0. \qquad (3.22)$$

J. Transport equation for $g_{ab';c'}$

Let

$$A_{abc} = g_b{}^{\alpha'}g_c{}^{\beta'}g_{a\alpha';\beta'}. \qquad (3.23)$$

Applying D' and commuting covariant derivatives, we obtain a transport equation for A_{abc}:

$$D'A_{abc} + A_{ab\alpha}\xi^{\beta'}{}_{\gamma'}g_{\beta'}{}^{\alpha}g_c{}^{\gamma'}$$
$$+ g_a{}^{\alpha'}g_b{}^{\beta'}g_c{}^{\gamma'}R_{\alpha'\beta'\gamma'\delta'}\sigma^{\delta'} = 0. \qquad (3.24)$$

K. Transport equation for $g_{ab';c}$

Let

$$B_{abc} = g_b{}^{\beta'}g_{a\beta';c}. \qquad (3.25)$$

Applying D' and rearranging, we obtain a transport equation for $B_{\alpha\beta\gamma}$:

$$D'B_{abc} = -A_{ab\alpha}\eta^{\alpha}{}_{\beta'}g_c{}^{\beta'}. \qquad (3.26)$$

L. Transport equation for $g_a{}^{b'}{}_{;c'd'}$

Applying D' to $g_a{}^{b'}{}_{;c'd'}$, we obtain

$$D'g_a{}^{b'}{}_{;c'd'} = \sigma^{\alpha'}g_a{}^{b'}{}_{;c'd'\alpha'}. \qquad (3.27)$$

Commuting covariant derivatives on the right-hand side, this becomes

$$D'g_a{}^{b'}{}_{;c'd'} = \sigma^{\beta'}(g_a{}^{b'}{}_{;\beta'c'd'} + R^{b'}{}_{\alpha'\beta'd'}g_a{}^{\alpha'}{}_{;c'} + R^{b'}{}_{\alpha'\beta'c'}g_a{}^{\alpha'}{}_{;d'} - R^{\alpha'}{}_{c'\beta'd'}g_a{}^{b'}{}_{;\alpha'} + R^{b'}{}_{\alpha'\beta'c';d'}g_a{}^{\alpha'}). \qquad (3.28)$$

Bringing $\sigma^{\beta'}$ inside the derivative in the first time on the right-hand side, and noting that $\sigma^{\beta'}g_a{}^{b'}{}_{;\beta'} = 0$, this then yields a transport equation for $g_a{}^{b'}{}_{;c'd'}$:

$$D'g_a{}^{b'}{}_{;c'd'} = -\sigma^{\beta'}{}_{c'}g_a{}^{b'}{}_{;\beta'd'} - \sigma^{\beta'}{}_{d'}g_a{}^{b'}{}_{;\beta'c'} - \sigma^{\beta'}{}_{c'd'}g_a{}^{b'}{}_{;\beta'} + R^{b'}{}_{\alpha'\beta'd'}\sigma^{\beta'}g_a{}^{\alpha'}{}_{;c'} + R^{b'}{}_{\alpha'\beta'c'}\sigma^{\beta'}g_a{}^{\alpha'}{}_{;d'}$$
$$- R^{\alpha'}{}_{c'\beta'd'}\sigma^{\beta'}g_a{}^{b'}{}_{;\alpha'} + R^{b'}{}_{\alpha'\beta'c';d'}\sigma^{\beta'}g_a{}^{\alpha'}. \qquad (3.29)$$

M. Transport equation for $\zeta = \ln\Delta^{1/2}$

The Van Vleck-Morette determinant, Δ is a biscalar defined by

$$\Delta(x, x') \equiv \det[\Delta^{\alpha'}{}_{\beta'}],$$
$$\Delta^{\alpha'}{}_{\beta'} \equiv -g^{\alpha'}{}_\alpha\sigma^\alpha{}_{\beta'} = -g^{\alpha'}{}_\alpha\eta^\alpha{}_{\beta'}. \qquad (3.30)$$

By Eq. (3.13), we can write the second equation here as

$$\Delta^{\alpha'}{}_{\beta'} = -g^{\alpha'}{}_\alpha(D'\eta^\alpha{}_{\beta'} + \eta^\alpha{}_{\gamma'}\xi^{\gamma'}{}_{\beta'}). \qquad (3.31)$$

Since $D'g^{\alpha'}{}_\alpha = g^{\alpha'}{}_{\alpha;\beta'}\sigma^{\beta'} = 0$, we can rewrite this as

$$\Delta^{\alpha'}{}_{\beta'} = D'\Delta^{\alpha'}{}_{\beta'} + \Delta^{\alpha'}{}_{\gamma'}\xi^{\gamma'}{}_{\beta'}. \qquad (3.32)$$

Introducing the inverse $(\Delta^{-1})^{\alpha'}{}_{\beta'}$ and multiplying it by the above, we obtain

$$4 = \xi^{\alpha'}{}_{\alpha'} + D'(\ln\Delta), \qquad (3.33)$$

where we have used the matrix identity $\delta\ln\det\mathbf{M} = \mathrm{Tr}\mathbf{M}^{-1}\delta\mathbf{M}$ to convert the trace to a determinant. This can also be written in terms of $\Delta^{1/2}$:

$$D'\zeta = \tfrac{1}{2}(4 - \xi^{\alpha'}{}_{\alpha'}). \qquad (3.34)$$

N. Transport equation for the Van Vleck-Morette determinant, $\Delta^{1/2}$

By the definition of ζ, the Van Vleck-Morette determinant is given by

$$\Delta^{1/2} = e^\zeta, \qquad (3.35)$$

and so satisfies the transport equation

$$D'\Delta^{1/2} = \tfrac{1}{2}\Delta^{1/2}(4 - \xi^{\alpha'}{}_{\alpha'}). \qquad (3.36)$$

O. Equation for $\Delta^{-1/2}D(\Delta^{1/2})$

Defining $\tau = \Delta^{-1/2}D(\Delta^{1/2})$, it is immediately clear that

$$\tau = \Delta^{-1/2}D(\Delta^{1/2}) = D\zeta. \tag{3.37}$$

P. Equation for $\Delta^{-1/2}D'(\Delta^{1/2})$

Defining $\tau' = \Delta^{-1/2}D'(\Delta^{1/2})$, it is immediately clear that

$$\tau' = \Delta^{-1/2}D'(\Delta^{1/2}) = D'\zeta. \tag{3.38}$$

Q. Equation for $\nabla_{a'}\Delta$

To derive an equation for $\nabla_{a'}\Delta$, we note that

$$\Delta \equiv \det[-g^{a'}{}_{\alpha}\eta^{\alpha}{}_{b'}] = -\det[\eta^{\alpha}{}_{b'}]\det[g_{\alpha}{}^{a'}], \tag{3.39}$$

and make use of Jacobi's matrix identity

$$d(\det\mathbf{A}) = \operatorname{tr}(\operatorname{adj}(\mathbf{A})d\mathbf{A}) = (\det\mathbf{A})\operatorname{tr}(\mathbf{A}^{-1}d\mathbf{A}), \tag{3.40}$$

where the operator d indicates a derivative. Applying (3.40) to (3.39), we obtain an equation for $\nabla_{a'}\Delta$:

$$\nabla_{a'}\Delta = -\Delta[g_{\alpha'}{}^{\alpha}g_{\alpha}{}^{\alpha'}{}_{;a'} + \gamma^{\alpha'}{}_{\alpha}\sigma^{\alpha}{}_{\alpha'a'}]. \tag{3.41}$$

As a consistency check, we note that contracting with $\sigma^{a'}$ and using Eq. (3.13) we recover Eq. (3.33).

R. Equation for $\Box'\Delta$

Applying Jacobi's identity twice, together with $d(\mathbf{A}^{-1}) = -\mathbf{A}^{-1}(d\mathbf{A})\mathbf{A}^{-1}$, we find an identity for the second derivative of the determinant of a matrix:

$$d^2(\det\mathbf{A}) = (\det\mathbf{A})(\operatorname{tr}(\mathbf{A}^{-1}d\mathbf{A})\operatorname{tr}(\mathbf{A}^{-1}d\mathbf{A})$$
$$- \operatorname{tr}(\mathbf{A}^{-1}d\mathbf{A}\mathbf{A}^{-1}d\mathbf{A}) + \operatorname{tr}(\mathbf{A}^{-1}d^2\mathbf{A})). \tag{3.42}$$

Using this identity in Eq. (3.39), we obtain an equation for $\Box'\Delta$,

$$\Box'\Delta = \Delta\left[\left(g_{\alpha'}{}^{\alpha}g_{\alpha}{}^{\alpha'}{}_{;\mu'} + \gamma^{\alpha'}{}_{\alpha}\sigma^{\alpha}{}_{\alpha'\mu'}\right)\left(g_{\alpha'}{}^{\alpha}g_{\alpha}{}^{\alpha';\mu'} + \gamma^{\alpha'}{}_{\alpha}\sigma^{\alpha}{}_{\alpha'}{}^{\mu'}\right) - \left(g_{\alpha'}{}^{\alpha}g_{\alpha}{}^{\beta'}{}_{;\mu'}g_{\beta'}{}^{\beta}g_{\beta}{}^{\alpha';\mu'}\right)\right.$$
$$\left. - \left(\gamma^{\alpha'}{}_{\alpha}\sigma^{\alpha}{}_{\beta'\mu'}\gamma^{\beta'}{}_{\beta}\sigma^{\beta}{}_{\alpha'}{}^{\mu'}\right) + \left(g_{\alpha'}{}^{\alpha}g_{\alpha}{}^{\alpha'}{}_{;\mu'}{}^{\mu'}\right) + \left(\gamma^{\alpha'}{}_{\alpha}\sigma^{\alpha}{}_{\alpha'}{}^{\mu'}{}_{\mu'}\right)\right]. \tag{3.43}$$

S. Equation for $\Box'\Delta^{1/2}$

Noting that

$$\Box'\Delta^{1/2} = (\tfrac{1}{2}\Delta^{-1/2}\Delta_{;\mu'})^{;\mu'} = \tfrac{1}{2}\Delta^{-1/2}\Box'\Delta - \tfrac{1}{4}\Delta^{-3/2}\Delta^{;\mu'}\Delta_{;\mu'}, \tag{3.44}$$

it is straightforward to use Eqs. (3.41) and (3.43) to find an equation for $\Box'\Delta^{1/2}$:

$$\Box'\Delta^{1/2} = \tfrac{1}{2}\Delta^{1/2}\left[\tfrac{1}{2}\left(g_{\alpha'}{}^{\alpha}g_{\alpha}{}^{\alpha'}{}_{;\mu'} + \gamma^{\alpha'}{}_{\alpha}\sigma^{\alpha}{}_{\alpha'\mu'}\right)\left(g_{\alpha'}{}^{\alpha}g_{\alpha}{}^{\alpha';\mu'} + \gamma^{\alpha'}{}_{\alpha}\sigma^{\alpha}{}_{\alpha'}{}^{;\mu'}\right) - \left(g_{\alpha'}{}^{\alpha}g_{\alpha}{}^{\beta'}{}_{;\mu'}g_{\beta'}{}^{\beta}g_{\beta}{}^{\alpha';\mu'}\right)\right.$$
$$\left. - \left(\gamma^{\alpha'}{}_{\alpha}\sigma^{\alpha}{}_{\beta'\mu'}\gamma^{\beta'}{}_{\beta}\sigma^{\beta}{}_{\alpha'}{}^{\mu'}\right) + \left(g_{\alpha'}{}^{\alpha}g_{\alpha}{}^{\alpha'}{}_{;\mu'}{}^{\mu'}\right) + \left(\gamma^{\alpha'}{}_{\alpha}\sigma^{\alpha}{}_{\alpha'}{}^{\mu'}{}_{\mu'}\right)\right]. \tag{3.45}$$

T. Transport equation for V_0

As is given in Eq. (2.8b), V_0 satisfies the transport equation

$$(D'+1)V_0^{AB'} + \tfrac{1}{2}V_0^{AB'}(\xi^{\mu'}{}_{\mu'} - 4) + \tfrac{1}{2}\mathcal{D}^{B'}{}_{C'}(\Delta^{1/2}g^{AC'}) = 0, \tag{3.46}$$

or equivalently

$$(D'+1)(\Delta^{-1/2}V_0^{AB'}) + \tfrac{1}{2}\Delta^{-1/2}\mathcal{D}^{B'}{}_{C'}(\Delta^{1/2}g^{AC'}) = 0. \tag{3.47}$$

In particular, for a scalar field

$$(D'+1)V_0 + \tfrac{1}{2}V_0(\xi^{\mu'}{}_{\mu'} - 4) + \tfrac{1}{2}(\Box' - m^2 - P')\Delta^{1/2} = 0, \tag{3.48}$$

where $P' \equiv P(x')$ is frequently taken to be proportional to the Ricci scalar: $P = \xi R$.

U. Transport equations for V_r

As is given in Eq. (2.8a), V_r satisfies the transport equation

$$(D'+r+1)V_r^{AB'} + \frac{1}{2}V_r^{AB'}(\xi^{\mu'}{}_{\mu'} - 4) + \frac{1}{2r}\mathcal{D}^{B'}{}_{C'}V_{r-1}^{AC'} = 0, \tag{3.49}$$

or equivalently

$$(D'+r+1)(\Delta^{-1/2}V_r^{AB'}) + \frac{1}{2r}\Delta^{-1/2}\mathcal{D}^{B'}{}_{C'}V_{r-1}^{AC'} = 0. \tag{3.50}$$

Comparing with Eq. (3.47), it is clear that Eq. (3.50) may be taken to include $r = 0$ if we replace $V_{r-1}^{AC'}/r$ by $\Delta^{1/2}g^{AC'}$.

In particular, for a scalar field,

$$(D' + r + 1)V_r + \frac{1}{2}V_r(\xi^{\mu'}{}_{\mu'} - 4) + \frac{1}{2r}(\Box' - m^2 - P')V_{r-1}$$

$$= 0. \tag{3.51}$$

Together with the earlier equations, the transport equation Eq. (3.46) allows us to immediately solve for $V_0^{AB'}$

along a geodesic. To obtain the higher order V_r, we also need to determine $\Box' V_{r-1}^{AB'}$. At first sight, this appears to require integrating along a family of neighboring geodesics but, in fact, again we can write transport equations for it. First, we note the identity

$$\nabla_{a'}(D'T^{AB'}{}_{a'_1...a'_n}) = D'(\nabla_{a'}T^{AB'}{}_{a'_1...a'_n}) + \xi^{\alpha'}{}_{a'}\nabla_{\alpha'}T^{AB'}{}_{a'_1...a'_n} + \sigma^{\alpha'}\mathcal{R}^{B'}{}_{C'a'\alpha'}T^{AC'}{}_{a'_1...a'_n}$$

$$- \sigma^{\alpha'}R^{c'}{}_{a'_1a'\alpha'}T^{AC'}{}_{c'...a'_n} - \ldots - \sigma^{\alpha'}R^{c'}{}_{a'_na'\alpha'}T^{AC'}{}_{a'_1...c'}, \tag{3.52}$$

where $\mathcal{R}^A{}_{Bcd} = \partial_c\mathcal{A}^A{}_{Bd} - \partial_c\mathcal{A}^A{}_{Bd} + \mathcal{A}^A{}_{Cd}\mathcal{A}^C{}_{Bc} - \mathcal{A}^A{}_{Cd}\mathcal{A}^C{}_{Bc}$. Working with $\tilde{V}_r^{AB'} = \Delta^{-1/2}V_r^{AB'}$, on differentiating Eq. (3.50) we obtain a transport equation for the first derivative of $V_r^{AB'}$

$$(D' + r + 1)(\tilde{V}_r^{AB'}{}_{;a'}) + \xi^{\alpha'}{}_{a'}\tilde{V}_r^{AB'}{}_{;\alpha'} + \sigma^{\alpha'}\mathcal{R}^{B'}{}_{C'a'\alpha'}\tilde{V}_r^{AC'} + \frac{1}{2r}(\Delta^{-1/2}\mathcal{D}^{B'}{}_{C'}(\Delta^{1/2}\tilde{V}_{r-1}^{AC'}))_{;a'} = 0. \tag{3.53}$$

As noted above, this equation also includes $r = 0$ if we replace $\tilde{V}_{r-1}^{AC'}/r$ in this case by $g^{AC'}$:

$$(D' + 1)(\tilde{V}_0^{AB'}{}_{;a'}) + \xi^{\alpha'}{}_{a'}\tilde{V}_0^{AB'}{}_{;\alpha'} + \sigma^{\alpha'}\mathcal{R}^{B'}{}_{C'a'\alpha'}\tilde{V}_0^{AC'} + \frac{1}{2}(\Delta^{-1/2}\mathcal{D}^{B'}{}_{C'}(\Delta^{1/2}g^{AC'}))_{;a'} = 0. \tag{3.54}$$

Repeating the process

$$(D' + r + 1)(\tilde{V}_r^{AB'}{}_{;a'b'}) + \xi^{\alpha'}{}_{b'}\tilde{V}_r^{AB'}{}_{;a'\alpha'} + \xi^{\alpha'}{}_{a'}\tilde{V}_r^{AB'}{}_{;\alpha'b'}$$

$$+ \sigma^{\alpha'}\mathcal{R}^{B'}{}_{C'b'\alpha'}V_r^{AC'}{}_{;a'} + \sigma^{\alpha'}\mathcal{R}^{B'}{}_{C'a'\alpha'}\tilde{V}_r^{AC'}{}_{;b'} + \xi^{\alpha'}{}_{a';b'}\tilde{V}_r^{AB'}{}_{;\alpha'} - \sigma^{\alpha'}R^{\beta'}{}_{a'b'\alpha'}\tilde{V}_r^{AC'}{}_{;\beta'} + \xi^{\alpha'}{}_{b'}\mathcal{R}^{B'}{}_{C'a'\alpha'}\tilde{V}_r^{AC'}$$

$$+ \sigma^{\alpha'}\mathcal{R}^{B'}{}_{C'a'\alpha';b'}\tilde{V}_r^{AC'} + \frac{1}{2r}(\Delta^{-1/2}\mathcal{D}^{B'}{}_{C'}(\Delta^{1/2}\tilde{V}_{r-1}^{AC'}))_{;a'b'} = 0, \tag{3.55}$$

with the $\tilde{V}_0^{AB'}{}_{;a'b'}$ equation given by the same replacement as above.

Clearly, this process may be repeated as many times as necessary. At each stage, we require two more derivatives on $\tilde{V}_{r-1}^{AC'}$ than on $\tilde{V}_r^{AC'}$, but this may be obtained by a bootstrap process grounded by the $\tilde{V}_0^{AC'}$ equation, which involves only the fundamental objects $\Delta^{1/2}$ and $g^{AC'}$, which we have explored above. As with our previous equations, while this process quickly becomes too tedious to follow by hand, it is straightforward to program.

For example, to determine V_1 for a scalar field, we first need to solve the two transport equations

$$(D' + 1)(\tilde{V}_{0;a'}) + \xi^{c'}{}_{a'}\tilde{V}_{0;c'}$$

$$+ \frac{1}{2}(\Delta^{-1/2}(\Box' - m^2 - P')\Delta^{1/2})_{;a'} = 0, \tag{3.56}$$

and

$$(D' + 1)(\tilde{V}_{0;a'b'}) + \xi^{c'}{}_{b'}\tilde{V}_{0;a'c'} + \xi^{c'}{}_{a'}\tilde{V}_{0;c'b'}$$

$$+ \xi^{c'}{}_{a';b'}\tilde{V}_{0;c'} - \sigma^{c'}R^{d'}{}_{a'b'c'}\tilde{V}_{0;d'}$$

$$+ \frac{1}{2}(\Delta^{-1/2}(\Box' - m^2 - P')\Delta^{1/2})_{;a'b'} = 0. \tag{3.57}$$

In the next two sections, we show how the above system of transport equations can be solved either as a series expansion or numerically. For sufficiently simple spacetimes, it is also possible to find closed form solutions, which provide a useful check on our results.

IV. SEMIRECURSIVE APPROACH TO COVARIANT EXPANSIONS

In this section, we will investigate solutions to the transport equations of Sec. III in the form of covariant series expansions. The goal is to find covariant series expressions for the Hadamard and DeWitt coefficients. Several methods have been previously applied for doing such calculations, both by hand and using computer algebra [25–44]. However, this effort has been focused

primarily on the calculation of the *diagonal* coefficients. To our knowledge, only the work of Décanini and Folacci [19], upon which our method is based, has been concerned with the off-diagonal coefficients.

Before proceeding further, it is helpful to see how covariant expansions behave under certain operations. First, applying the operator D' to the covariant expansion of any bitensor $T_{a_1 \cdots a_m b'_1 \cdots b'_n}$ about the point x, we obtain

$$g_{b_1}{}^{b'_1} \cdots g_{b_n}{}^{b'_n} D' T_{a_1 \cdots a_m b'_1 \cdots b'_n}(x, x')$$
$$= D' g_{b_1}{}^{b'_1} \cdots g_{b_n}{}^{b'_n} T_{a_1 \cdots a_m b'_1 \cdots b'_n}(x, x')$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} k t_{a_1 \cdots a_m b_1 \cdots b_n \beta_1 \cdots \beta_k}(x) \sigma^{\beta_1} \cdots \sigma^{\beta_k}{}_{\alpha'} \sigma^{\alpha'}$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} k t_{a_1 \cdots a_m b_1 \cdots b_n \beta_1 \cdots \beta_k}(x) \sigma^{\beta_1} \cdots \sigma^{\beta_k}, \quad (4.1)$$

where the last equality is obtained by applying Eq. (3.5). In other words, applying D' to the kth term in the series is equivalent to multiplying that term by k:

$$(D'T)_{(k)} = k T_{(k)}.$$

Next, we consider applying the operator D to the covariant expansion of any bitensor $T_{a_1 \cdots a_m a'_1 \cdots a'_l}$ about the point x. In this case, there will also be a term involving the derivative of the series coefficient, giving

$$g_{b_1}{}^{b'_1} \cdots g_{b_n}{}^{b'_n} D T_{a_1 \cdots a_m b'_1 \cdots b'_n}(x, x')$$
$$= D g_{b_1}{}^{b'_1} \cdots g_{b_n}{}^{b'_n} T_{a_1 \cdots a_m b'_1 \cdots b'_n}(x, x')$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} [k t_{a_1 \cdots a_m b_1 \cdots b_n \beta_1 \cdots \beta_k}(x) \sigma^{\beta_1} \cdots \sigma^{\beta_k}{}_{\alpha} \sigma^{\alpha}$$
$$+ t_{a_1 \cdots a_m b_1 \cdots b_n \beta_1 \cdots \beta_k; \alpha}(x) \sigma^{\beta_1} \cdots \sigma^{\beta_k} \sigma^{\alpha}]$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} [k t_{a_1 \cdots a_m b_1 \cdots b_n \beta_1 \cdots \beta_k}(x) \sigma^{\beta_1} \cdots \sigma^{\beta_k}$$
$$+ t_{a_1 \cdots a_m b_1 \cdots b_n \beta_1 \cdots \beta_k; \alpha}(x) \sigma^{\beta_1} \cdots \sigma^{\beta_k} \sigma^{\alpha}]. \quad (4.2)$$

We can also consider multiplication of covariant expansions. For example, for any two tensors, $S^a{}_b$ and $T^a{}_b$, with product $U^a{}_b \equiv S^a{}_\alpha T^\alpha{}_b$, for example, we can relate their covariant expansions by

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} u^a{}_{b \beta_1 \cdots \beta_n} \sigma^{\beta_1} \cdots \sigma^{\beta_n}$$
$$= \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} s^a{}_{\alpha \beta_1 \cdots \beta_k} \sigma^{\beta_1} \cdots \sigma^{\beta_k} \right)$$
$$\times \left(\sum_{l=0}^{\infty} \frac{(-1)^l}{l!} t^\alpha{}_{b \beta_1 \cdots \beta_l} \sigma^{\beta_1} \cdots \sigma^{\beta_l} \right)$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \sum_{k=0}^{n} \binom{n}{k} s^a{}_{\alpha \beta_1 \cdots \beta_k} t^\alpha{}_{b \beta_{k+1} \cdots \beta_n} \sigma^{\beta_1} \cdots \sigma^{\beta_n}, \quad (4.3)$$

or equivalently

$$U_{(n)} = \sum_{k=0}^{n} \binom{n}{k} S_{(k)} T_{(n-k)}.$$

Finally, many of the equations derived in the previous section contain terms involving the Riemann tensor at x', $R^{a'}{}_{b'c'd'}$. As all other quantities are expanded about x rather than x', we will also need to rewrite these Riemann terms in terms of their expansion about x:

$$g^a{}_{a'} g_b{}^{b'} R^{a'}{}_{\alpha' b' \beta'} \sigma^{\alpha'} \sigma^{\beta'}$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} R^a{}_{(\alpha|b|\beta; \gamma_1 \cdots \gamma_k)} \sigma^\alpha \sigma^\beta \sigma^{\gamma_1} \cdots \sigma^{\gamma_k}$$
$$= \sum_{k=2}^{\infty} \frac{(-1)^k}{(k-2)!} \mathcal{K}^a{}_{b(k)}, \quad (4.4)$$

where we follow Avramidi [17] in introducing the definition

$$\mathcal{K}^a{}_{b(n)} \equiv R^a{}_{(\alpha_1|b|\alpha_2; \alpha_3 \cdots \alpha_n)} \sigma^{\alpha_1} \cdots \sigma^{\alpha_n}$$
$$\equiv \bar{\mathcal{K}}^a{}_{b(n)} \sigma^{\alpha_1} \cdots \sigma^{\alpha_n}. \quad (4.5)$$

These four considerations will now allow us to rewrite the transport equations of Sec. III as recursion relations for the coefficients of the covariant expansions of the tensors involved.

A. Recursion relation for coefficients of the covariant expansion of $\gamma^{a'}{}_b$

Rewriting Eq. (3.15) in terms of covariant expansions, we find

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (n^2 + n) g^a{}_{a'} \gamma^{a'}{}_{b \alpha_1 \cdots \alpha_n}(x) \sigma^{\alpha_1} \cdots \sigma^{\alpha_n} + \left(\sum_{k=2}^{\infty} \frac{(-1)^k}{(k-2)!} \mathcal{K}^a{}_{\beta(k)} \right) \left(\sum_{l=0}^{\infty} \frac{(-1)^l}{l!} g^\beta{}_{\beta'} \gamma^{\beta'}{}_{b \alpha_1 \cdots \alpha_l}(x) \sigma^{\alpha_1} \cdots \sigma^{\alpha_l} \right) = 0. \quad (4.6)$$

From this, the nth term in the covariant series expansion of $g^a{}_{a'} \gamma^{a'}{}_b$,

$$g^a{}_{a'} \gamma^{a'}{}_b = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma^a{}_{b(n)},$$

can be written recursively in terms of products of lower order terms in the series with $\mathcal{K}$:

$$\gamma^a{}_{b(0)} = -\delta^a{}_b, \qquad \gamma^a{}_{b(1)} = 0,$$

$$\gamma^a{}_{b(n)} = -\left(\frac{n-1}{n+1}\right)\sum_{k=0}^{n-2}\binom{n-2}{k}\mathcal{K}^a{}_{\alpha(n-k)}\gamma^\alpha{}_{b(k)}. \tag{4.7}$$

Many of the following recursion relations will make use of these coefficients; to illustrate their structure we write the next five explicitly,

$$\gamma^a{}_{b(2)} = \tfrac{1}{3}\mathcal{K}^a{}_{b(2)},$$

$$\gamma^a{}_{b(3)} = \tfrac{1}{2}\mathcal{K}^a{}_{b(3)},$$

$$\gamma^a{}_{b(4)} = \tfrac{3}{5}\mathcal{K}^a{}_{b(4)} - \tfrac{1}{5}\mathcal{K}^a{}_{\alpha(2)}\mathcal{K}^\alpha{}_{b(2)},$$

$$\gamma^a{}_{b(5)} = \tfrac{2}{3}\mathcal{K}^a{}_{b(5)} - \tfrac{2}{3}\mathcal{K}^a{}_{\alpha(3)}\mathcal{K}^\alpha{}_{b(2)} - \tfrac{1}{3}\mathcal{K}^a{}_{\alpha(2)}\mathcal{K}^\alpha{}_{b(3)},$$

$$\gamma^a{}_{b(6)} = \tfrac{5}{7}\mathcal{K}^a{}_{b(6)} - \tfrac{10}{7}\mathcal{K}^a{}_{\alpha(4)}\mathcal{K}^\alpha{}_{b(2)} - \tfrac{10}{7}\mathcal{K}^a{}_{\alpha(3)}\mathcal{K}^\alpha{}_{b(3)}$$
$$- \tfrac{3}{7}\mathcal{K}^a{}_{\alpha(2)}\mathcal{K}^\alpha{}_{b(4)} + \tfrac{1}{7}\mathcal{K}^a{}_{\alpha(2)}\mathcal{K}^\alpha{}_{\beta(2)}\mathcal{K}^\beta{}_{b(2)}.$$

While one can give a closed form combinatoric expression, the recursive formula (4.7) is best suited to our needs.

B. Recursion relation for coefficients of the covariant expansion of $\eta^a{}_{b'}$

Since $\gamma^{a'}{}_b$ is the inverse of $\eta^a{}_{b'}$, we have

$$(g^a{}_{a'}\gamma^{a'}{}_\alpha)(g_b{}^{b'}\eta^\alpha{}_{b'}) = \delta^a{}_b. \tag{4.8}$$

Substituting in covariant expansion expressions for $g^a{}_{a'}\gamma^{a'}{}_\alpha$ and $g_b{}^{b'}\eta^\alpha{}_{b'}$, we find, using our standard notation, that the nth term in the covariant series expansion of $g_b{}^{b'}\eta^a{}_{b'}$ is

$$\eta^a{}_{b(0)} = -\delta^a{}_b, \qquad \eta^a{}_{b(1)} = 0,$$

$$\eta^a{}_{b(n)} = \sum_{k=2}^{n}\binom{n}{k}\gamma^\alpha{}_{\beta(k)}\eta^\beta{}_{b(n-k)}. \tag{4.9}$$

Again, to illustrate their structure, we write the next five explicitly,

$$\eta^a{}_{b(2)} = -\tfrac{1}{3}\mathcal{K}^a{}_{b(2)},$$

$$\eta^a{}_{b(3)} = -\tfrac{1}{2}\mathcal{K}^a{}_{b(3)},$$

$$\eta^a{}_{b(4)} = -\tfrac{3}{5}\mathcal{K}^a{}_{b(4)} - \tfrac{7}{15}\mathcal{K}^a{}_{\alpha(2)}\mathcal{K}^\alpha{}_{b(2)},$$

$$\eta^a{}_{b(5)} = -\tfrac{2}{3}\mathcal{K}^a{}_{b(5)} - \mathcal{K}^a{}_{\alpha(3)}\mathcal{K}^\alpha{}_{b(2)} - \tfrac{4}{3}\mathcal{K}^a{}_{\alpha(2)}\mathcal{K}^\alpha{}_{b(3)},$$

$$\eta^a{}_{b(6)} = -\tfrac{5}{7}\mathcal{K}^a{}_{b(6)} - \tfrac{11}{7}\mathcal{K}^a{}_{\alpha(4)}\mathcal{K}^\alpha{}_{b(2)} - \tfrac{25}{7}\mathcal{K}^a{}_{\alpha(3)}\mathcal{K}^\alpha{}_{b(3)}$$
$$- \tfrac{18}{7}\mathcal{K}^a{}_{\alpha(2)}\mathcal{K}^\alpha{}_{b(4)} - \tfrac{31}{21}\mathcal{K}^a{}_{\alpha(2)}\mathcal{K}^\alpha{}_{\beta(2)}\mathcal{K}^\beta{}_{b(2)}.$$

C. Recursion relation for coefficients of the covariant expansion of $\xi^{a'}{}_{b'}$

Writing Eq. (3.14) in terms of covariant series, it is immediately apparent that the nth term in the covariant expansion of $g^a{}_{a'}g_b{}^{b'}\xi^{a'}{}_{b'}$ is

$$\xi^a{}_{b(0)} = \delta^a{}_b, \qquad \xi^a{}_{b(1)} = 0$$

$$\xi^a{}_{b(n)} = n\eta^a{}_{b(n)} - \sum_{k=2}^{n-2}\binom{n}{k}k\gamma^a{}_{\alpha(n-k)}\eta^\alpha{}_{b(k)}. \tag{4.10}$$

Once more, to illustrate their structure, we write the next five explicitly,

$$\xi^a{}_{b(2)} = -\tfrac{2}{3}\mathcal{K}^a{}_{b(2)},$$

$$\xi^a{}_{b(3)} = -\tfrac{3}{2}\mathcal{K}^a{}_{b(3)},$$

$$\xi^a{}_{b(4)} = -\tfrac{12}{5}\mathcal{K}^a{}_{b(4)} - \tfrac{8}{15}\mathcal{K}^a{}_{\alpha(2)}\mathcal{K}^\alpha{}_{b(2)},$$

$$\xi^a{}_{b(5)} = -\tfrac{10}{3}\mathcal{K}^a{}_{b(5)} - \tfrac{5}{3}\mathcal{K}^a{}_{\alpha(3)}\mathcal{K}^\alpha{}_{b(2)} - \tfrac{5}{3}\mathcal{K}^a{}_{\alpha(2)}\mathcal{K}^\alpha{}_{b(3)},$$

$$\xi^a{}_{b(6)} = -\tfrac{30}{7}\mathcal{K}^a{}_{b(6)} - \tfrac{24}{7}\mathcal{K}^a{}_{\alpha(4)}\mathcal{K}^\alpha{}_{b(2)} - \tfrac{45}{7}\mathcal{K}^a{}_{\alpha(3)}\mathcal{K}^\alpha{}_{b(3)}$$
$$- \tfrac{24}{7}\mathcal{K}^a{}_{\alpha(2)}\mathcal{K}^\alpha{}_{b(4)} - \tfrac{32}{21}\mathcal{K}^a{}_{\alpha(2)}\mathcal{K}^\alpha{}_{\beta(2)}\mathcal{K}^\beta{}_{b(2)}.$$

D. Recursion relation for coefficients of the covariant expansion of $\lambda^a{}_b$

Using Eq. (3.17), $\lambda^a{}_b = \delta^a{}_b - (D\eta^a{}_{\alpha'})\gamma^{\alpha'}{}_b$, we can write an equation for the nth order coefficient of the covariant expansion of $\lambda^a{}_b$. However, the expression involves the operator D acting on the covariant series expansion of $g_b{}^{b'}\eta^a{}_{b'}$, so we will first need to find an expression for that. As discussed in the beginning of this section, the derivative in D will affect both the coefficient and the σ^a's. When acting on the σ^a's, it has the effect of multiplying the term by n as was previously the case with D'. When acting on the coefficient, it will add a derivative to it and increase the order of the term (since we will then be adding a σ^a). In particular, given our definition (4.5)

$$\mathcal{K}^a{}_{b(n)} \equiv R^a{}_{(\alpha_1|b|\alpha_2;\alpha_3\cdots\alpha_n)}\sigma^{\alpha_1}\cdots\sigma^{\alpha_n},$$

we have

$$D\mathcal{K}^a{}_{b(n)} = \sigma^{\alpha_{n+1}}\nabla_{\alpha_{n+1}}\left(R^a{}_{(\alpha_1|b|\alpha_2;\alpha_3\cdots\alpha_n)}\sigma^{\alpha_1}\cdots\sigma^{\alpha_n}\right)$$
$$= R^a{}_{(\alpha_1|b|\alpha_2;\alpha_3\cdots\alpha_n)\alpha_{n+1}}\sigma^{\alpha_1}\cdots\sigma^{\alpha_n}\sigma^{\alpha_{n+1}}$$
$$+ nR^a{}_{(\alpha_1|b|\alpha_2;\alpha_3\cdots\alpha_n)}\sigma^{\alpha_1}\cdots\sigma^{\alpha_n}$$
$$= \mathcal{K}^a{}_{b(n+1)} + n\mathcal{K}^a{}_{b(n)}.$$

Here, the first term is one order higher while the second keeps the order the same.

We now appeal to the fact that, from Eqs. (4.7) and IV C, the terms in the expansion of $g_b{}^{b'}\eta^a{}_{b'}$ consist solely of products of $\mathcal{K}^a{}_{b(n)}$. This means that we can apply the preceding rules when D acts on $\mathcal{K}^a{}_{b(n)}$, and when encountering compound expressions (i.e., consisting of more than a single $\mathcal{K}^a{}_{b(n)}$), use the normal rules for differentiation (product rule, distributivity, etc.). To illustrate this explicitly,

$$g_b{}^{b'}D\eta^a{}_{b'} = D(g_b{}^{b'}\eta^a{}_{b'}) = D\left(\eta^a{}_{b(0)} - \eta^a{}_{b(1)} + \frac{1}{2!}\eta^a{}_{b(2)} - \frac{1}{3!}\eta^a{}_{b(3)} + \frac{1}{4!}\eta^a{}_{b(4)} + \cdots\right)$$

$$= D\left(-\delta^a{}_b + \frac{1}{2!}\left(-\frac{1}{3}\mathcal{K}^a{}_{b(2)}\right) - \frac{1}{3!}\left(-\frac{1}{2}\mathcal{K}^a{}_{b(3)}\right) + \frac{1}{4!}\left(-\frac{3}{5}\mathcal{K}^a{}_{b(4)} - \frac{7}{15}\mathcal{K}^a{}_{\alpha(2)}\mathcal{K}^\alpha{}_{b(2)}\right) + \cdots\right)$$

$$= \frac{1}{2!}\left(-\frac{1}{3}\right)(\mathcal{K}^a{}_{b(3)} + 2\mathcal{K}^a{}_{b(2)}) - \frac{1}{3!}\left(-\frac{1}{2}\right)(\mathcal{K}^a{}_{b(4)} + 3\mathcal{K}^a{}_{b(3)}) + \frac{1}{4!}\left(-\frac{3}{5}\right)(\mathcal{K}^a{}_{b(5)} + 4\mathcal{K}^a{}_{b(4)})$$

$$+ \frac{1}{4!}\left(-\frac{7}{15}\right)((\mathcal{K}^a{}_{\alpha(3)} + 2\mathcal{K}^a{}_{\alpha(2)})\mathcal{K}^\alpha{}_{b(2)} + \mathcal{K}^a{}_{\alpha(2)}(\mathcal{K}^\alpha{}_{b(3)} + 2\mathcal{K}^\alpha{}_{b(2)})) + \cdots$$

$$= \frac{1}{2!}\left(-\frac{2}{3}\mathcal{K}^a{}_{b(2)}\right) - \frac{1}{3!}\left(-\frac{1}{2}\mathcal{K}^a{}_{b(3)}\right) + \frac{1}{4!}\left(-\frac{2}{5}\mathcal{K}^a{}_{b(4)} - \frac{28}{15}\mathcal{K}^a{}_{\alpha(2)}\mathcal{K}^\alpha{}_{b(2)}\right) + \cdots.$$

We can then write the general nth term in the covariant series expansion of $g_b{}^{b'}D\eta^a{}_{b'}$ symbolically as

$$(D\eta^a{}_b)_{(n)} = D^0\eta^a{}_{b(n)} - nD^+\eta^a{}_{b(n-1)}, \qquad (4.11)$$

where D^+ signifies the contribution that raises the order by one and D^0 signifies the contribution that keeps the order the same. For example,

$$D^+\eta^a{}_{b(2)} = -\tfrac{1}{3}\mathcal{K}^\alpha{}_{b(3)}, \qquad D^0\eta^a{}_{b(3)} = -\tfrac{3}{2}\mathcal{K}^\alpha{}_{b(3)}$$

and so

$$(D\eta^a{}_b)_{(3)} = -\tfrac{3}{2}\mathcal{K}^\alpha{}_{b(3)} - 3\left(-\tfrac{1}{3}\mathcal{K}^\alpha{}_{b(3)}\right) = -\tfrac{1}{2}\mathcal{K}^\alpha{}_{b(3)}.$$

It is then straightforward to write an expression for the nth term in the covariant series expansion of $\lambda^a{}_b$:

$$\lambda^a{}_{b(0)} = \delta^a{}_b,$$

$$\lambda^a{}_{b(1)} = 0$$

$$\lambda^a{}_{b(n)} = -\sum_{k=0}^{n-2}\binom{n}{k}(D^0\eta^a{}_{\alpha(n-k)}$$
$$- (n-k)D^+\eta^a{}_{\alpha(n-k-1)})\gamma^\alpha{}_{b(k)}. \qquad (4.12)$$

The next five terms are given explicitly by

$$\lambda^a{}_{b(2)} = -\tfrac{2}{3}\mathcal{K}^a{}_{b(2)},$$

$$\lambda^a{}_{b(3)} = -\tfrac{1}{2}\mathcal{K}^a{}_{b(3)},$$

$$\lambda^a{}_{b(4)} = -\tfrac{2}{5}\mathcal{K}^a{}_{b(4)} - \tfrac{8}{15}\mathcal{K}^a{}_{\alpha(2)}\mathcal{K}^\alpha{}_{b(2)},$$

$$\lambda^a{}_{b(5)} = -\tfrac{1}{3}\mathcal{K}^a{}_{b(5)} - \mathcal{K}^a{}_{\alpha(3)}\mathcal{K}^\alpha{}_{b(2)} - \mathcal{K}^a{}_{\alpha(2)}\mathcal{K}^\alpha{}_{b(3)},$$

$$\lambda^a{}_{b(6)} = -\tfrac{2}{7}\mathcal{K}^a{}_{b(6)} - \tfrac{10}{7}\mathcal{K}^a{}_{\alpha(4)}\mathcal{K}^\alpha{}_{b(2)} - \tfrac{17}{7}\mathcal{K}^a{}_{\alpha(3)}\mathcal{K}^\alpha{}_{b(3)}$$
$$- \tfrac{10}{7}\mathcal{K}^a{}_{\alpha(2)}\mathcal{K}^\alpha{}_{b(4)} - \tfrac{32}{21}\mathcal{K}^a{}_{\alpha(2)}\mathcal{K}^\alpha{}_{\beta(2)}\mathcal{K}^\beta{}_{b(2)}.$$

E. Recursion relation for coefficients of the covariant expansion of A_{abc}

We can rewrite Eq. (3.24) as

$$(D' + 1)(A_{ab\alpha}g^\alpha{}_{\alpha'}\gamma^{\alpha'}{}_c) + g_a{}^{\alpha'}g_b{}^{\beta'}R_{\alpha'\beta'\alpha\beta}\sigma^\alpha\gamma^\beta{}_c = 0, \qquad (4.13)$$

which when rewritten in terms of covariant series becomes

$$A_{abc(k)} = -\frac{1}{n+1}\sum_{k=0}^{n}\binom{n}{k}k\mathcal{R}_{ab\alpha(k)}\gamma^\alpha{}_{c(n-k)}$$
$$+ \sum_{k=0}^{n-2}\binom{n}{k}A_{ab\alpha(k)}\gamma^\alpha{}_{c(n-k)}, \qquad (4.14)$$

where we follow Avramidi [17,18] in defining

$$\mathcal{R}_{abc(n)} \equiv R_{ab(\alpha_1|c|;\alpha_2\cdots\alpha_n)}\sigma^{\alpha_1}\cdots\sigma^{\alpha_n}. \qquad (4.15)$$

Alternatively, writing Eq. (3.24) directly in terms of covariant series, we obtain

$$A_{abc(k)} = \frac{n}{n+1}\mathcal{R}_{abc(n)} - \frac{1}{n+1}\sum_{k=0}^{n-2}\binom{n}{k}A_{ab\alpha(k)}\xi^\alpha{}_{c(n-k)}, \qquad (4.16)$$

which has the benefit of requiring half as much computation as the previous expression.

F. Recursion relation for coefficients of the covariant expansion of B_{abc}

By Eq. (3.26), we can immediately write an equation for the coefficients of the covariant expansion of B_{abc}:

$$B_{abc(n)} = \frac{1}{n}\sum_{k=0}^{n}\binom{n}{k}A_{ab\alpha(k)}\eta^\alpha{}_{c(n-k)}. \qquad (4.17)$$

G. Covariant expansion of ζ

From Eq. (3.34), we immediately obtain expressions for the coefficients of the covariant series of ζ:

$$\zeta_{(0)} = 0 \qquad \zeta_{(1)} = 0 \qquad \zeta_{(n)} = -\frac{1}{2n}\xi^\rho{}_{\rho(n)}. \quad (4.18)$$

H. Recursion relation for $\Delta^{1/2}$, $\Delta^{-1/2}$

Since $\zeta = \ln\Delta^{1/2}$, we can write

$$\Delta^{1/2}D'\zeta = D'\Delta^{1/2}. \quad (4.19)$$

This allows us to write down a recursive equation for the coefficients of the covariant series expansion of $\Delta^{1/2}$,

$$\Delta^{1/2}_{(n)} = \frac{1}{n}\sum_{k=2}^{n}\binom{n}{k}k\zeta_{(k)}\Delta^{1/2}_{(n-k)}. \quad (4.20)$$

Similarly, the equation

$$-\Delta^{-1/2}D'\zeta = D'\Delta^{-1/2} \quad (4.21)$$

allows us to write down a recursive equation for the coefficients of the covariant series expansion of $\Delta^{-1/2}$,

$$\Delta^{-1/2}_{(n)} = -\frac{1}{n}\sum_{k=2}^{n}\binom{n}{k}k\zeta_{(k)}\Delta^{-1/2}_{(n-k)}. \quad (4.22)$$

I. Covariant expansion of τ and τ'

Equations (3.37) and (3.38) may be immediately written as covariant series equations,

$$\tau_{(n)} = D^0\zeta_{(n)} - nD^+\zeta_{(n-1)}, \qquad \tau'_{(n)} = n\zeta_{(n)}. \quad (4.23)$$

J. Covariant expansion of covariant derivative at x' of a biscalar

Let $T(x, x')$ be a general biscalar. Writing T as a covariant series,

$$T(x, x') = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}T_{(n)}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}t_{\alpha_1\cdots\alpha_n}(x)\sigma^{\alpha_1}\cdots\sigma^{\alpha_n}, \quad (4.24)$$

and applying a covariant derivative at x', we obtain

$$g_a{}^{\alpha'}T_{;\alpha'} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}g_a{}^{\alpha'}T_{(n);\alpha'}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}nt_{(\alpha_1\cdots\alpha_n)}\sigma^{\alpha_1}\cdots\sigma^{\alpha_{n-1}}g_a{}^{\alpha'}\sigma^{\alpha_n}{}_{\alpha'}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}nt_{(\alpha_1\cdots\alpha_{n-1}\rho)}\sigma^{\alpha_1}\cdots\sigma^{\alpha_{n-1}}g_a{}^{\alpha'}\eta^\rho{}_{\alpha'}$$

$$= -\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\sum_{k=0}^{n}\binom{n}{k}(T_{(k+1)})_{(-1)\rho}\eta^\rho{}_{a(n-k)}, \quad (4.25)$$

where we have introduced the notation

$$(T_{(n)})_{(-k)\alpha_{(n-k+1)}\cdots\alpha_n} \equiv t_{(\alpha_1\cdots\alpha_{(n-k)}\alpha_{(n-k+1)}\cdots\alpha_n)}\sigma^{\alpha_1}\cdots\sigma^{\alpha_{n-k}}. \quad (4.26)$$

K. Covariant expansion of d'Alembertian at x' of a biscalar

Let $T(x, x')$ be a general biscalar as in the previous section. Applying (4.25) twice and taking care to include the term involving $g_a{}^{b'}$, we can then write the d'Alembertian, $\Box'T(x, x')$ at x' in terms of a covariant series,

$$(\Box'T)_{(n)} = -\sum_{k=0}^{n}\binom{n}{k}((g_\alpha{}^{\alpha'}T_{;\alpha'})_{(k+1)})_{(-1)\rho}\eta^{\rho\alpha}{}_{(n-k)}$$

$$- \sum_{k=1}^{n}\binom{n}{k}A^{\alpha\rho}{}_{\rho(k)}(g_\alpha{}^{\alpha'}T_{;\alpha'})_{(n-k)}, \quad (4.27)$$

where $A_{abc(n)}$ is the nth term in the covariant series of the tensor defined in (3.24).

L. Covariant expansion of $\nabla_{a'}\Delta^{1/2}$

Applying Eq. (4.25) to the case $T = \Delta^{1/2}$, we obtain

$$(g_a{}^{a'}\Delta^{1/2}_{;a'})_{(n)} = -\sum_{k=0}^{n}\binom{n}{k}(\Delta^{1/2}_{(k+1)})_{(-1)\rho}\eta^\rho{}_{a(n-k)}. \quad (4.28)$$

M. Covariant expansion of $\Box'\Delta^{1/2}$

Applying Eq. (4.27) to the case $T = \Delta^{1/2}$, we obtain

$$(\Box'\Delta^{1/2})_{(n)} = -\sum_{k=0}^{n}\binom{n}{k}((g_\alpha{}^{\alpha'}\Delta^{1/2}_{;\alpha'})_{(k+1)})_{(-1)\rho}\eta^{\rho\alpha}{}_{(n-k)}$$

$$- \sum_{k=1}^{n}\binom{n}{k}A^{\alpha\rho}{}_{\rho(k)}(g_\alpha{}^{\alpha'}\Delta^{1/2}_{;\alpha'})_{(n-k)}, \quad (4.29)$$

N. Covariant expansion of V_0

The transport equation for V_0, Eq. (3.46), can be written in the alternative form

$$(D' + 1)V_0 - V_0\tau' + \tfrac{1}{2}(\Box' - m^2 - P')\Delta^{1/2} = 0. \quad (4.30)$$

This equation is then easily written in terms of covariant expansion coefficients,

$$V_{0(n)} = \frac{1}{n+1}\Bigg(\sum_{k=0}^{n-2}\binom{n}{k}V_{0(k)}\tau'_{(n-k)}$$
$$- \frac{1}{2}\Big((\Box'\Delta^{1/2})_{(n)} - m^2\Delta^{1/2}_{(n)} - \sum_{k=0}^{n}\binom{n}{k}P_{(k)}\Delta^{1/2}_{(n-k)}\Big)\Bigg).$$
$$(4.31)$$

O. Covariant expansion of V_r

The transport equation for V_r, Eq. (3.49) can also be written in the alternative form

$$(D' + r + 1)V_r - V_r\tau' + \frac{1}{2r}(\Box' - m^2 - P')V_{r-1} = 0. \quad (4.32)$$

Again, this is easily written in terms of covariant expansion coefficients,

$$V_{r(n)} = \frac{1}{r+n+1}\Bigg(\sum_{k=0}^{n-2}\binom{n}{k}V_{r(k)}\tau'_{(n-k)} - \frac{1}{2r}\Big((\Box'V_{r-1})_{(n)} - m^2V_{r-1(n)} - \sum_{k=0}^{n}\binom{n}{k}P_{(k)}V_{r-1(n-k)}\Big)\Bigg). \quad (4.33)$$

P. Results

We have implemented the semirecursive algorithm as a MATHEMATICA package, which we are making freely available online [20]. It serves as an efficient tool for easily computing high order covariant expansions. The high level of efficiency is illustrated in Tables I and II, where we show the performance of our implementation when run on a desktop computer (2.4 GHz processor). For each coefficient, we list the time, number of terms, and memory consumed in the calculation of that term. We also list the number of terms after reduction to canonical form by the INVAR [45,46] package. Note that INVAR currently only supports canonicalization of scalar invariants up to order 6. We have therefore not canonicalized our expressions for a_7, a_8, and a_9, nor our expressions for the nondiagonal coefficients given in Table II. We have also not canonicalized our expression for a_6, primarily due to memory constraints.

The expressions for the DeWitt coefficients produced by our code are valid for any spacetime of any dimension. These may in turn be used to construct the Hadamard coefficients in any dimension, although we have limited ourselves to the 4-dimensional case here. Given our motivation to study massless fields in vacuum spacetimes such as Schwarzschild and Kerr, it is possible to make further assumptions in order to reduce the number of terms which appear. It is straightforward to impose the fact that the field is massless and the Ricci tensor vanishes with the requirements

$$\mathcal{K}^{\alpha}{}_{\alpha(n)} = 0, \qquad \mathcal{R}_a{}^{\beta}{}_{\beta(n)} = 0, \qquad m = 0. \quad (4.34)$$

This is a conservative requirement: terms such as $\mathrm{tr}((\mathcal{K}^a{}_{b(3)})_{(-2)})$ will yield some terms involving a Ricci tensor after the symmetrization is explicitly expanded. However, as is shown in Tables I and II, it is sufficient to significantly reduce the number of terms in the expansions.

TABLE I. Calculation performance of our semirecursive implementation of the Avramidi method for computing the coincident (diagonal) DeWitt coefficients, $a_{r(0)}$ for both general and vacuum spacetimes. In each case, we list the computation time (in seconds), number of terms, memory consumed, and number of terms after canonicalization. In the general case, we also list the number of canonical terms when the potential P is set to 0. This would be the case, for example, for a minimally coupled scalar field.

DeWitt Coefficient	General					Vacuum			
	Time	Terms	Memory	Canonical	Canonical ($P = 0$)	Time	Terms	Memory	Canonical
a_0	0	1	16 B	1	1	0	1	16 B	1
a_1	0	2	432 B	2	1	0	0	16 B	0
a_2	0.003	10	5 kB	7	4	0	2	536 B	1
a_3	0.02	91	63 kB	26	15	0.003	7	5 kB	2
a_4	0.2	1058	949 kB	113	68	0.015	56	51 kB	5
a_5	3.6	13 972	15 MB	611	380	0.1	507	559 MB	14
a_6	76	199 264	254 MB	⋯	⋯	1.1	4988	6.3 MB	⋯
a_7	1489	2 987 366	4.4 GB	⋯	⋯	17	51 700	75 MB	⋯
a_8	⋯	⋯	⋯	⋯	⋯	254	554 715	910 MB	⋯
a_9	⋯	⋯	⋯	⋯	⋯	3373	6 098 069	10.9 GB	⋯

TABLE II. Calculation performance of the Avramidi method for computing the terms $V_{0(n)}$ of order n in the covariant series expansion of V_0 for both general (4 dimensional) and vacuum spacetimes. In each case, we list the computation time (in seconds), number of terms, and memory consumed.

| | General | | | Vacuum, massless | | |
Order	Time	Terms	Memory	Time	Terms	Memory
0	0.001	3	760 B	0.001	0	16 B
1	0.001	2	288 B	0.001	0	16 B
2	0.002	10	3.8 kB	0.001	2	432 B
3	0.003	15	6.1 kB	0.002	2	432 B
4	0.005	47	22.0 kB	0.003	5	2.5 kB
5	0.007	81	40.7 kB	0.004	7	3.6 kB
6	0.014	206	112 kB	0.009	22	12 kB
7	0.024	383	221 kB	0.015	39	23 kB
8	0.047	856	526 kB	0.019	94	59 kB
9	0.084	1641	1.03 MB	0.03	177	115 kB
10	0.16	3414	2.25 MB	0.05	384	260 kB
11	0.30	6547	4.51 MB	0.1	729	515 kB
12	0.58	13 064	9.34 MB	0.19	1480	1.1 MB
13	1.1	24 870	18.5 MB	0.33	2811	2.1 MB
14	2.1	48 167	37.1 MB	0.61	5485	4.2 MB
15	4.1	90 808	72.3 MB	1.1	10 320	8.3 MB
16	7.8	172 214	141 MB	2.1	19 637	16 MB
17	15	321 145	271 MB	3.7	36 556	30 MB
18	28	599 460	522 MB	6.8	68 295	58 MB
19	53	1 106 459	987 MB	12	125 852	110 MB
20	99	2 039 285	1.81 GB	23	231 837	208 MB

The relative compactness of our expressions after canonicalization means that they may be readily computed for a given choice of spacetime. For example, evaluating the expressions for the coincidence limits, $V_r(x, x) = V_{r(0)}(x)$, of the first five Hadamard coefficients given in the Appendix for Schwarzschild spacetime gives

$$
\begin{aligned}
V_{0(0)} &= 0, \\
V_{1(0)} &= \frac{M^2}{15r^6}, \\
V_{2(0)} &= \frac{M^2}{1008r^9}(194M - 81r), \\
V_{3(0)} &= \frac{M^2}{3150r^{12}}(210r^2 - 1125rM + 1454M^2), \\
V_{4(0)} &= \frac{-M^2}{1\,663\,200r^{15}}(78\,750r^3 - 1\,746\,182M^3 \\
&\quad + 1\,932\,801rM^2 - 689\,775Mr^2).
\end{aligned}
\tag{4.35}
$$

A similar calculation can be done for spacetimes with less symmetry (such as Kerr) without any additional difficulty other than the fact that the results are somewhat less compact.

V. NUMERICAL SOLUTION OF TRANSPORT EQUATIONS

In this section, we describe the implementation of the numerical solution of the transport equations of Sec. III. We use the analytic results for σ, $\Delta^{1/2}$, $g_a{}^{b'}$, and V_0 for a scalar field in Nariai spacetime from Refs. [47,48] as a check on our numerical code.

For the purposes of numerical calculations, the operator $\mathcal{D}'$ acting on a general bitensor $T^{a'\cdots}{}_{b'\ldots}$ can be written as

$$
\begin{aligned}
\mathcal{D}'T^{a'\cdots}{}_{b'\ldots} &= (s' - s)\Big(\frac{d}{ds'}T^{a'\cdots}{}_{b'\ldots} + T^{\alpha'\cdots}{}_{b'\ldots}\Gamma^{a'}_{\alpha'\beta'}u^{\beta'} \\
&\quad + \cdots - T^{a'\cdots}{}_{\alpha'\ldots}\Gamma^{\alpha'}_{b'\beta'}u^{\beta'} - \cdots\Big),
\end{aligned}
\tag{5.1}
$$

where s' is the affine parameter, $\Gamma^{a'}_{b'c'}$ are the Christoffel symbols at x', and $u^{a'}$ is the four velocity at x'. Additionally, we make use of the fact that

$$
\sigma^{a'} = (s' - s)u^{a'},
\tag{5.2}
$$

which allows us to write Eqs. (3.11), (3.13), (3.18), (3.19), (3.20), (3.21), (3.22), (3.24), (3.29), (3.34), (3.45), and (3.46) as a system of coupled, tensor ordinary differential equations. These equations all have the general form

$$
\begin{aligned}
\frac{d}{ds'}T^{a'\cdots}{}_{b'\ldots} &= (s')^{-1}A^{a'\cdots}{}_{b'\ldots} + B^{a'\cdots}{}_{b'\ldots} + s'C^{a'\cdots}{}_{b'\ldots} \\
&\quad - T^{\alpha'\cdots}{}_{b'\ldots}\Gamma^{a'}_{\alpha'\beta'}u^{\beta'} - \cdots \\
&\quad + T^{a'\cdots}{}_{\alpha'\ldots}\Gamma^{\alpha'}_{b'\beta'}u^{\beta'} + \cdots,
\end{aligned}
\tag{5.3}
$$

where we have set $s = 0$ without loss of generality and where $A^{a'\cdots}{}_{b'\ldots} = 0$ initially (i.e., at $s' = 0$). It is not necessarily true, however, that the derivative of $A^{\alpha'\cdots}{}_{b'\ldots}$ is zero initially. This fact is important when considering initial data for the numerical scheme.

Solving this system of equations along with the geodesic equations for the spacetime of interest will then yield a numerical value for V_0. Moreover, since $V = V_0$ along a null geodesic, the transport equation for V_0 will effectively give the full value of V on the light cone. We have implemented this numerical integration scheme for geodesics in Nariai and Schwarzschild spacetimes using the Runge-Kutta-Fehlberg method (with adaptive time stepping) provided by the GNU Scientific Library [49]. The source code of our implementation is available online [21].

A. Initial conditions

Numerical integration of the transport equations requires initial conditions for each of the bitensors involved. These initial conditions are readily obtained by considering

TABLE III. Initial conditions for tensors used in the numerical calculation of V_0 with $P = \xi R$.

Equation	Bitensor	Initial Condition	$(s')^{-1}$ Initial Condition
(3.11)	$\xi^{a'}{}_{b'}$	$\delta^a{}_b$	0
(3.13)	$\eta^a{}_{b'}$	$-\delta^a{}_b$	0
(3.18)	$\sigma^{a'}{}_{b'c'}$	0	$-\frac{2}{3}R^a{}_{(\alpha|b|c)}u^\alpha$
(3.19)	$\sigma^a{}_{b'c'}$	0	$\frac{1}{2}R^a{}_{b\alpha c}u^\alpha - \frac{1}{3}R^a{}_{(\alpha|b|c)}u^\alpha$
(3.20)	$\sigma^{a'}{}_{b'c'd'}$	$-\frac{2}{3}R^a{}_{(c|b|d)}$	$\frac{1}{2}R^a{}_{(c|b|d;\alpha)}u^\alpha - \frac{2}{3}R^a{}_{(\alpha|b|d);c}u^\alpha - \frac{2}{3}R^a{}_{(\alpha|b|c);d}u^\alpha$
(3.21)	$\sigma^a{}_{b'c'd'}$	$-\frac{1}{3}R^a{}_{(c|b|d)} - \frac{1}{2}R^a{}_{bcd}$	$-\frac{1}{2}R^a{}_{(c|b|d;\alpha)}u^\alpha + \frac{2}{3}R^a{}_{(\alpha|b|d);c}u^\alpha$
(3.22)	$g_{a'}{}^b$	$\delta_a{}^b$	0
(3.24)	$g_a{}^{b'}{}_{;c'}$	0	$\frac{1}{2}R^b{}_{a\alpha c}u^\alpha$
(3.29)	$g_a{}^{b'}{}_{;c'd'}$	$-\frac{1}{2}R^b{}_{acd}$	$-\frac{2}{3}R^b{}_{ac(d;\alpha)}u^\alpha$
(3.34)	$\Delta^{1/2}$	1	0
(3.46)	V_0	$\frac{1}{2}m^2 + \frac{1}{2}(\xi - \frac{1}{6})R$	$-\frac{1}{4}(\xi - \frac{1}{6})R_{;\alpha}u^\alpha$

the covariant series expansions of V_0, $\Delta^{1/2}$, $\xi^{a'}{}_{b'}$, $\eta^a{}_{b'}$, and $g_a{}^{b'}$ and their covariant derivatives at x'. Initial conditions for all bitensors used for calculating V_0 are given in Table III, where we list the transport equation for the bitensor, the bitensor itself, and its initial value.

Additionally, as is indicated in Eq. (5.3), many of the transport equations will contain terms involving $(s')^{-1}$. These terms must obviously be treated with care in any numerical implementation. Then, for the initial time step ($s' = 0$), we require analytic expressions for

$$\lim_{s' \to 0}(s')^{-1}A^{\alpha'\cdots}{}_{b'\ldots}, \qquad (5.4)$$

which may then be used to numerically compute an accurate initial value for the derivative. This limit can be computed from the first order term in the covariant series of $A^{\alpha'\cdots}{}_{b'\ldots}$, which is found most easily by considering the covariant series of its constituent bitensors. For this reason, we also list in Table III the limit as $s' \to 0$ of all required constituent bitensors multiplied by $(s')^{-1}$. In Table IV, we list the terms $(s')^{-1}A^{\alpha'\cdots}{}_{b'\ldots}$ for each transport equation involving $(s')^{-1}$, along with their limit as $s' \to 0$.

B. Results

The accuracy of our numerical code may be verified by comparing with the results of Refs. [47,48], which give analytic expressions for many of the bitensors used in this paper in Nariai spacetime. In Figs. 1 and 2, we compare analytic and numerical expressions for $\Delta^{1/2}$ and V_0,

TABLE IV. Initial conditions for transport equations required for the numerical calculation of V_0 with $P = \xi R$.

| Equation	Terms involving $(s')^{-1}$	Initial condition for $(s')^{-1}$ terms
(3.11)	$-(s')^{-1}(\xi^{a'}{}_{\alpha'}\xi^{\alpha'}{}_{b'} - \xi^{a'}{}_{b'})$	0
(3.13)	$-(s')^{-1}(\eta^a{}_{\alpha'}\xi^{\alpha'}{}_{b'} - \eta^a{}_{b'})$	0
(3.18)	$(s')^{-1}(\sigma^{a'}{}_{b'c'} - \xi^{a'}{}_{\alpha'}\sigma^{\alpha'}{}_{b'c'} - \xi^{\alpha'}{}_{b'}\sigma^{a'}{}_{\alpha'c'} - \xi^{\alpha'}{}_{c'}\sigma^{a'}{}_{\alpha'b'})$	$-\frac{2}{3}R^a{}_{(b|\alpha|c)}u^\alpha$
(3.19)	$(s')^{-1}(\sigma^a{}_{b'c'} - \eta^a{}_{\alpha'}\sigma^{\alpha'}{}_{b'c'} - \xi^{\alpha'}{}_{b'}\sigma^a{}_{\alpha'c'} - \xi^{\alpha'}{}_{c'}\sigma^a{}_{\alpha'b'})$	$-\frac{1}{2}R^a{}_{\alpha bc}u^\alpha - \frac{1}{3}R^a{}_{(b|\alpha|c)}u^\alpha$
(3.20)	$(s')^{-1}(\sigma^{a'}{}_{b'c'd'} - \sigma^{a'}{}_{\alpha'b'}\sigma^{\alpha'}{}_{c'd'} - \sigma^{a'}{}_{\alpha'c'}\sigma^{\alpha'}{}_{b'd'} - \sigma^{a'}{}_{\alpha'd'}\sigma^{\alpha'}{}_{b'c'}$ $-\sigma^{a'}{}_{\alpha'b'c'}\xi^{\alpha'}{}_{d'} - \sigma^{a'}{}_{\alpha'b'd'}\xi^{\alpha'}{}_{c'} - \sigma^{a'}{}_{\alpha'c'd'}\xi^{\alpha'}{}_{b'} - \sigma^{\alpha'}{}_{b'c'd'}\xi^{a'}{}_{\alpha'})$	$-\frac{3}{2}R^a{}_{(b|\alpha|c;d)}u^\alpha$
(3.21)	$(s')^{-1}(\sigma^a{}_{b'c'd'} - \sigma^a{}_{\alpha'b'}\sigma^{\alpha'}{}_{c'd'} - \sigma^a{}_{\alpha'c'}\sigma^{\alpha'}{}_{b'd'} - \sigma^a{}_{\alpha'd'}\sigma^{\alpha'}{}_{b'c'}$ $-\sigma^a{}_{\alpha'b'c'}\xi^{\alpha'}{}_{d'} - \sigma^a{}_{\alpha'b'd'}\xi^{\alpha'}{}_{c'} - \sigma^a{}_{\alpha'c'd'}\xi^{\alpha'}{}_{b'} - \sigma^{\alpha'}{}_{b'c'd'}\eta^a{}_{\alpha'})$	$\frac{1}{2}R^a{}_{(c|\alpha|d);b}u^\alpha - \frac{5}{6}R^a{}_{b\alpha(c;d)}u^\alpha + \frac{7}{6}R^a{}_{(d|\alpha b|;c)}u^\alpha$
(3.22)	0	0
(3.24)	$-(s')^{-1}g_a{}^{b'}{}_{;\alpha'}\xi^{\alpha'}{}_{c'}$	$-\frac{1}{2}R^b{}_{a\alpha c}u^\alpha$
(3.29)	$-(s')^{-1}(g_a{}^{b'}{}_{;\alpha'd'}\xi^{\alpha'}{}_{c'} + g_a{}^{b'}{}_{;\alpha'c'}\xi^{\alpha'}{}_{d'} + g_a{}^{b'}{}_{;\alpha'}\sigma^{\alpha'}{}_{c'd'})$	$-\frac{2}{3}R^b{}_{a\alpha(c;d)}u^\alpha$
(3.34)	$-(s')^{-1}\Delta^{1/2}(\xi^{a'}{}_{a'} - \delta^{a'}{}_{a'})$	0
(3.46)	$-(s')^{-1}[(\xi^{a'}{}_{a'} - \delta^{a'}{}_{a'})V_0 + 2V_0 + (\Box' - m^2 - \xi R)\Delta^{1/2}]$	$\frac{1}{4}(\xi - \frac{1}{6})R_{;\alpha}u^\alpha$
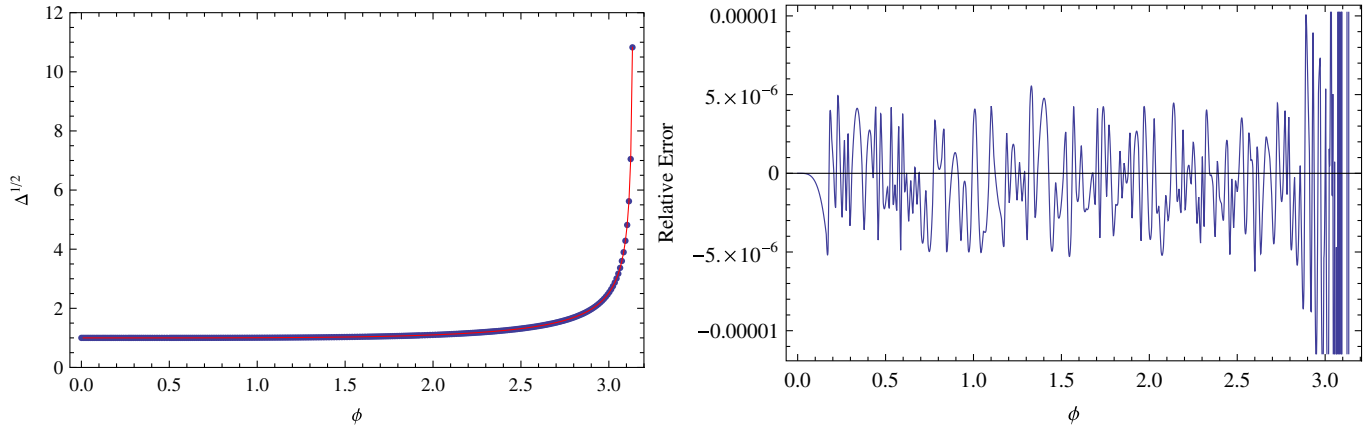
FIG. 1 (color online). Comparison of numerical and exact analytic calculations of $\Delta^{1/2}$ as a function of the angle, ϕ, along an orbiting null geodesic in Nariai spacetime. Left: The numerical calculation (blue dots) is a close match with the analytic expression (red line). Right: With parameters so that the code completes in 1 minute, the relative error is within 0.0001% up to the boundary of the normal neighborhood (at $\phi = \pi$).

respectively. We consider the null geodesic, which starts at $\rho = 0.5$ and moves inwards to $\rho = 0.25$ before turning around and going out to $\rho = 0.789$, where it reaches the edge of the normal neighborhood. The affine parameter, s', has been scaled so that it is equal to the angle coordinate, ϕ. We find that the numerical results faithfully match the analytic solution up to the boundary of the normal neighborhood. For the case of $\Delta^{1/2}$, Fig. 1, the error remains less than one part in 10^{-6} to within a short distance of the normal neighborhood boundary. The results for $V_0(x, x')$ are less accurate, but nonetheless the relative error remains less than 1%.

In Fig. 3, we use our numerical code to illustrate how $\Delta^{1/2}$ varies over the whole light cone in Schwarzschild spacetime. We find that it remains close to its initial value of 1 far away from the caustic. As geodesics get close to the

caustic, $\Delta^{1/2}$ grows and is eventually singular at the caustic. This is exactly as expected: $\Delta^{1/2}$ is a measure of the strength of focusing of geodesics, where values greater than 1 correspond to focusing and values less than 1 correspond to defocusing. At the caustic, geodesics are focused to a point and correspondingly $\Delta^{1/2}$ is singular there.

In Fig. 4, we give a similar plot (again calculated from our numerical code), which indicates how $V(x, x')$ varies over the light cone in Schwarzschild spacetime. In this case, there is considerably more structure than was previously the case with $\Delta^{1/2}$. There is the expected singularity at the caustic. However, travelling along a geodesic, $V(x, x')$ also becomes negative for a period before turning positive and eventually becoming singular at the caustic.
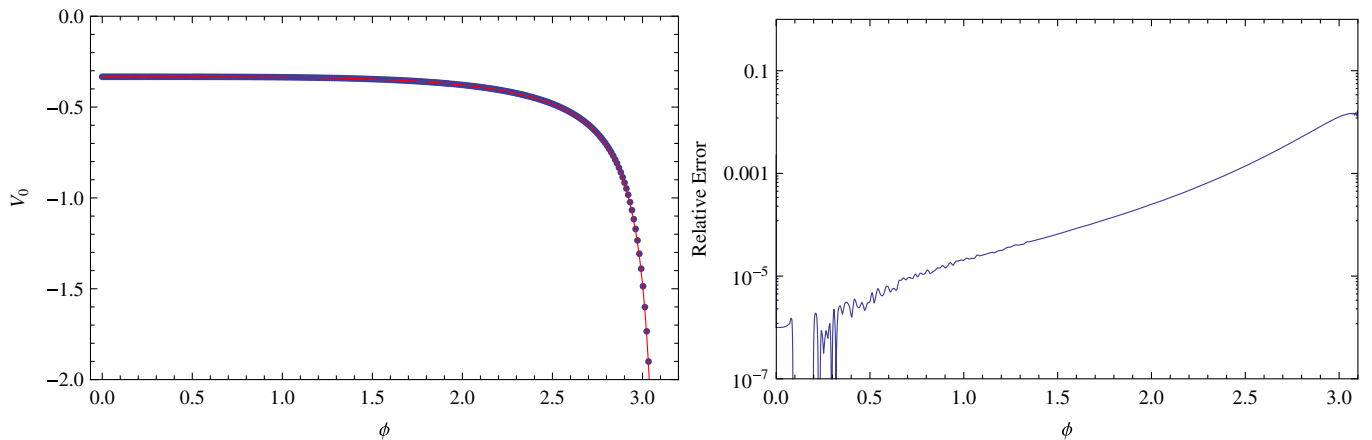

FIG. 2 (color online). Comparison of numerical and exact analytic calculations of V_0 for a massless, minimally coupled scalar field as a function of the angle, ϕ, along an orbiting null geodesic in Nariai spacetime. Left: The numerical calculation (blue dots) is a close match with the analytic expression (red line). The coincidence value is $V_0(x, x) = \frac{1}{2}(\xi - \frac{1}{6})R = -\frac{1}{3}$, as expected. Right: With parameters so that the code completes in 1 min, the relative error in the numerical calculation is within 1% up to the boundary of the normal neighborhood (at $\phi = \pi$).
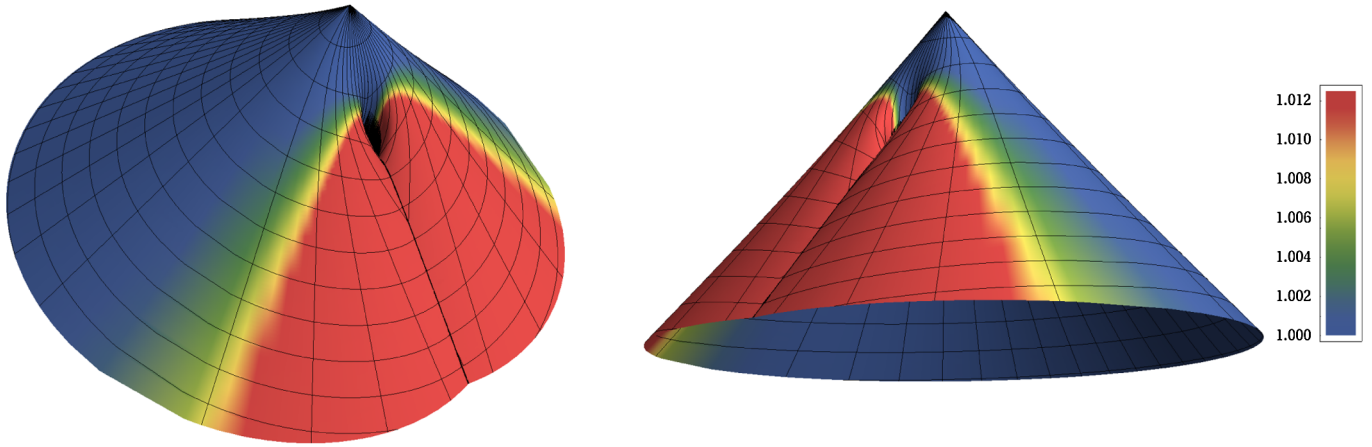
FIG. 3 (color online). $\Delta^{1/2}$ along the light cone in Schwarzschild spacetime. The point x at the vertex of the cone is fixed at $r = 10M$. $\Delta^{1/2}$ increases along a geodesic up to the caustic where it is singular.
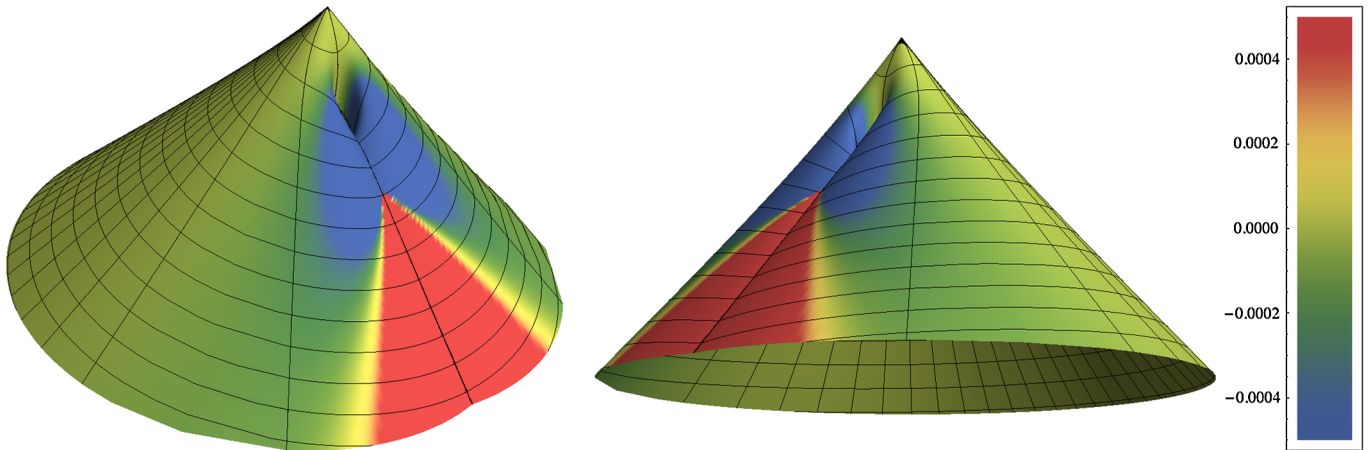

FIG. 4 (color online). $V(x, x')$ for a massless, scalar field along the light cone in Schwarzschild spacetime. The point x (the vertex of the cone) is fixed at $r = 6M$. $V(x, x')$ is 0 initially, then, travelling along a geodesic, it goes negative for a period before turning positive and eventually becoming singular at the caustic. (Note that V coincides with V_0 on the light cone.)
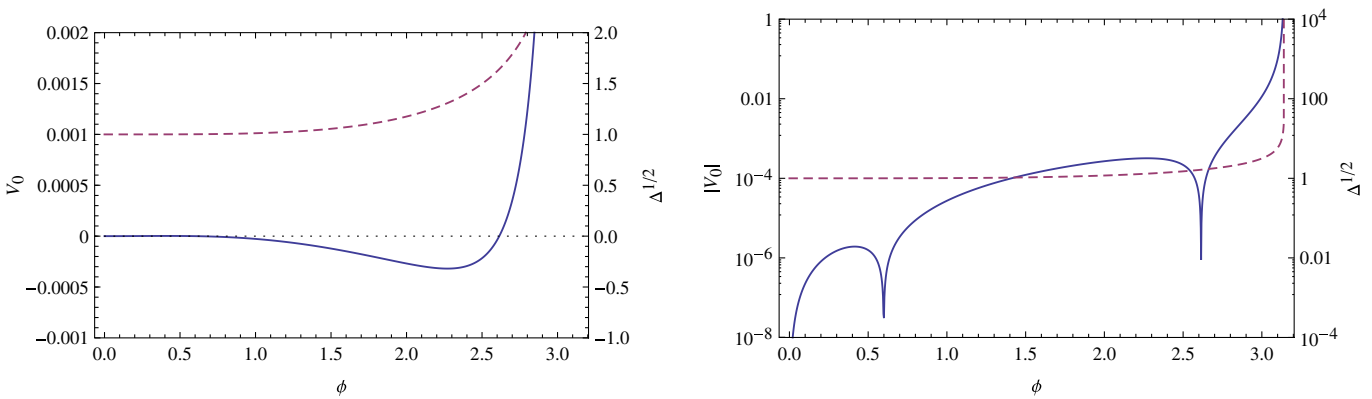

FIG. 5 (color online). $V_0(x, x')$ (solid blue line) and $\Delta^{1/2}$ (dashed purple line) for a massless, scalar field along the timelike circular orbit at $r = 10M$ in Schwarzschild spacetime as a function of the angle, ϕ through which the geodesic has passed. In the logarithmic plot, the absolute value of $V_0(x, x')$ is plotted to illustrate the divergence close to the caustic. [Since $V_0(x, x')$ passes through 0 at $\phi \approx 0.6$ and $\phi \approx 2.6$ there are corresponding features in the logarithmic plot.]

The transport equations may also be applied to calculate $V_r(x, x')$ along a timelike geodesic. In Fig. 5, we apply our numerical code to the calculation of $V_0(x, x')$ along the timelike circular orbit at $r = 10M$ in Schwarzschild.

VI. DISCUSSION

Several of the covariant expansion expressions computed by our code using the Avramidi method have been previously given in Ref. [19], albeit to considerably lower order [for example, in their paper, Décanini and Folacci give $V(x, x')$ to order $(\sigma^a)^4$ compared to order $(\sigma^a)^{20}$ here]. Comparison between the two results gives exact agreement, providing a reassuring confirmation of the accuracy of both our expressions and those of Ref. [19] (and confirming the error in Ref. [50] found by Décanini and Folacci). Furthermore, several of the expansions not given by Décanini and Folacci may be compared with those found by Christensen [13,14]. Again, we have found that our code is in exact agreement with Christensen's results.

Our MATHEMATICA implementation of the semirecursive approach (Sec. IV) is given as a practical tool for computing high order covariant expansions. While it already exhibits a high level of efficiency, we believe that further improvement could be achieved, particularly in the limiting area of memory requirements. The initial expressions for the DeWitt coefficients as computed by our code are very general. However, they are not necessarily given as a minimal set. For example, with $P = 0$, the DeWitt coefficient a_3 may be written as a sum of four terms, yet our code produces a sum of seven equivalent terms. It is possible, however, to use a set of transformation rules to reduce our expression to a canonical basis such as that of Ref. [16]. As our code is already written in MATHEMATICA [51] and has the ability to output into the XTENSOR [22] notation used by INVAR [45,46], we were able to quickly canonicalize the scalar invariants appearing in our coincidence limit expressions. An extension of the INVAR package to allow for the canonicalization of tensor invariants would allow our nondiagonal coefficients to also be immediately canonicalized with no further effort.

In Sec. V, we discussed a numerical implementation of the transport equation approach to the calculation of V_0. This implementation is capable of computing V_0 for any spacetime, although we have chosen Nariai and Schwarzschild spacetimes as examples. The choice of Nariai spacetime has the benefit that an expression for V_0 is known exactly [47]. This makes it possible to compare our numerical results with the analytic expressions to determine both the validity of the approach and the accuracy of the numerical calculation. Given parameters allowing the code to run in under a minute, we find that the numerical implementation is accurate to less than 1% out

as far as the location of the singularity of V_0 at the edge of the normal neighborhood.

In integrating the transport equations along a specific geodesic, we are not limited to the normal neighborhood. The only difficulty arises at *caustics*, where some bitensors such as $\Delta^{1/2}$ and V_0 become singular. However, this is not an insurmountable problem. The singular components may be separated out and methods of complex analysis employed to integrate through the caustics, beyond which the bitensors once more become regular (but not necessarily real valued) [2]. This is highlighted in Fig. 5, where our plot of $\Delta^{1/2}$ and V_0 extends outside the normal neighborhood, the boundary of which is at $\phi \approx 1.25$, where the first null geodesic reintersects the orbit. It does not necessarily follow, however, that the Green function outside the normal neighborhood is given by this value for $V(x, x')$. Instead, one might expect to obtain the Green function by considering the sum of the contributions obtained by integrating along *all* geodesics connecting x and x' (there will be a discrete number of such geodesics except at caustics).

ACKNOWLEDGMENTS

APPENDIX A: CANONICAL FORM OF HADAMARD AND DEWITT COEFFICIENTS

In this Appendix, we present expressions for the diagonal DeWitt coefficients $a_{0(0)}$, $a_{1(0)}$, $a_{2(0)}$, $a_{3(0)}$, $a_{4(0)}$, and $a_{5(0)}$ [where $a_r(x, x) = a_{r(0)}(x)$] in the canonical form produced by INVAR [45,46]. These have previously been given in various forms in the literature: a_1 and a_2 by DeWitt [7], a_3 by Sakai [52], and by Gilkey [53], a_4 by Amsterdamski, Berkin, and O'Connor [54] and by Avramidi [55,56] and a_5 by van den Ven [36]. However, to our knowledge, this is the first time that they have all been given in a simplified canonical form. We also note that our code is capable of producing expressions for a_6 and a_7 and for the off-diagonal coefficients (in noncanonical form) in a matter of minutes on a laptop computer.

During the canonicalization process, we have allowed INVAR to use identities which are valid only in four spacetime dimensions, as our primary motivation is to study

black hole spacetimes such as Schwarzschild and Kerr. This additional simplification is not essential, but does lead to more compact expressions.

Although our code is also capable of producing expressions for the off-diagonal coefficients, support for canonicalization of such expressions involving free indices is not yet available in INVAR. For this reason, we restrict ourselves here to only the diagonal coefficients. We have also made these expressions, along with the corresponding (noncanonical) off-diagonal coefficients available online as MATHEMATICA code [20].

We also note that Eq. (2.28) allows us to directly relate the Hadamard coefficients $V_{0(0)}$, $V_{1(0)}$, $V_{2(0)}$, $V_{3(0)}$, and $V_{4(0)}$ to these DeWitt coefficients:

$$
\begin{aligned}
V_{0(0)} &= \tfrac{1}{2}(m^2 a_{0(0)} - a_{1(0)}), \\
V_{1(0)} &= \tfrac{1}{8}(m^4 a_{0(0)} - 2m^2 a_{1(0)} + 2a_{2(0)}), \\
V_{2(0)} &= \tfrac{1}{96}(m^6 a_{0(0)} - 3m^4 a_{1(0)} + 6m^2 a_{2(0)} - 6a_{3(0)}), \\
V_{3(0)} &= \tfrac{1}{2304}(m^8 a_{0(0)} - 4m^6 a_{1(0)} + 12m^4 a_{2(0)} - 24m^2 a_{3(0)} + 24a_{4(0)}), \\
V_{4(0)} &= \tfrac{1}{92\,160}(m^{10} a_{0(0)} - 5m^8 a_{1(0)} + 20m^6 a_{2(0)} - 60m^4 a_{3(0)} + 120m^2 a_{4(0)} - 120a_{5(0)}).
\end{aligned}
\tag{A1}
$$

Finally, we note that our expressions for the coefficients $V_{0(0)}$, $V_{1(0)}$, and $V_{2(0)}$ are in agreement with Ref. [19], after canonicalization. In addition, our expressions for the coefficients $a_{3(0)}$ and $a_{4(0)}$ are in agreement with Ref. [44].

In the following, we group the expressions in powers of ξ and denote by $a_r^{(k)}$ the term involving the kth power of ξ in the diagonal DeWitt coefficient, $a_{r(0)}$, so that

$$
a_{r(0)} = \sum_{k=0}^{(r+1)} a_r^{(k)} \xi^k.
\tag{A2}
$$

In this notation, the diagonal DeWitt coefficients are

$$
a_0^{(0)} = 1,
\tag{A3}
$$

$$
a_1^{(0)} = \tfrac{1}{6}R, \qquad a_1^{(1)} = -R,
\tag{A4}
$$

$$
a_2^{(0)} = \tfrac{1}{360}(-2R_{\alpha\beta}R^{\alpha\beta} + 5R^2 + 2R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} + 12R^{;\alpha}{}_\alpha), \qquad a_2^{(1)} = \tfrac{1}{6}(-R^2 - R^{;\alpha}{}_\alpha), \qquad a_2^{(2)} = \tfrac{1}{2}R^2,
\tag{A5}
$$

$$
\begin{aligned}
a_3^{(0)} &= \tfrac{1}{15\,120}(584R_{\alpha}{}^{\gamma}R^{\alpha\beta}R_{\beta\gamma} - 654R_{\alpha\beta}R^{\alpha\beta}R + 99R^3 + 456R^{\alpha\beta}R^{\gamma\delta}R_{\alpha\gamma\beta\delta} + 72RR_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - 80R_{\alpha\beta}{}^{\epsilon\rho}R^{\alpha\beta\gamma\delta}R_{\gamma\delta\epsilon\rho} \\
&\quad + 51R_{;\alpha}R^{;\alpha} - 12R_{\alpha\gamma;\beta}R^{\alpha\beta;\gamma} - 6R_{\alpha\beta;\gamma}R^{\alpha\beta;\gamma} + 27R_{\alpha\beta\gamma\delta;\epsilon}R^{\alpha\beta\gamma\delta;\epsilon} + 84RR^{;\alpha}{}_\alpha + 36R_{\alpha\beta}R^{;\alpha\beta} \\
&\quad - 24R^{\alpha\beta}R_{\alpha\beta}{}^{;\gamma}{}_\gamma + 144R_{\alpha\gamma\beta\delta}R^{\alpha\beta;\gamma\delta} + 54R^{;\alpha}{}_\alpha{}^\beta{}_\beta),
\end{aligned}
\tag{A6a}
$$

$$
a_3^{(1)} = \tfrac{1}{360}(2R_{\alpha\beta}R^{\alpha\beta}R - 5R^3 - 2RR_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - 12R_{;\alpha}R^{;\alpha} - 22RR^{;\alpha}{}_\alpha - 4R_{\alpha\beta}R^{;\alpha\beta} - 6R^{;\alpha}{}_\alpha{}^\beta{}_\beta),
\tag{A6b}
$$

$$
a_3^{(2)} = \tfrac{1}{12}(R^3 + R_{;\alpha}R^{;\alpha} + 2RR^{;\alpha}{}_\alpha), \qquad a_3^{(3)} = \tfrac{-1}{6}R^3,
\tag{A6c}
$$

$$
\begin{aligned}
a_4^{(0)} = \frac{1}{1814400}\big(
&-32736 R_\alpha{}^\gamma R^{\alpha\beta} R_\beta{}^\delta R_{\gamma\delta} + 8436 R_{\alpha\beta} R^{\alpha\beta} R_{\gamma\delta} R^{\gamma\delta} + 59136 R_\alpha{}^\gamma R^{\alpha\beta} R_{\beta\gamma} R - 43518 R_{\alpha\beta} R^{\alpha\beta} R^2 + 5743 R^4 \\
&+ 13944 R^{\alpha\beta} R^{\gamma\delta} R R_{\alpha\gamma\beta\delta} + 3618 R^2 R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} + 168 R^{\alpha\beta} R^{\gamma\delta} R_{\alpha\gamma}{}^{\epsilon\rho} R_{\beta\delta\epsilon\rho} - 4480 R R_{\alpha\beta}{}^{\epsilon\rho} R^{\alpha\beta\gamma\delta} R_{\gamma\delta\epsilon\rho} \\
&+ 14832 R^{\alpha\beta} R^{\gamma\delta} R_\alpha{}^\epsilon{}_\beta{}^\rho R_{\gamma\epsilon\delta\rho} - 3282 R_{\alpha\beta} R^{\alpha\beta} R_{\gamma\delta\epsilon\rho} R^{\gamma\delta\epsilon\rho} - 2496 R_{\alpha\beta}{}^{\epsilon\rho} R^{\alpha\beta\gamma\delta} R_{\gamma\epsilon}{}^{\sigma\tau} R_{\delta\rho\sigma\tau} \\
&+ 1248 R_{\alpha\beta}{}^{\epsilon\rho} R^{\alpha\beta\gamma\delta} R_{\gamma\delta}{}^{\sigma\tau} R_{\epsilon\rho\sigma\tau} + 696 R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} R_{\epsilon\rho\sigma\tau} R^{\epsilon\rho\sigma\tau} - 65040 R^{\beta\gamma} R_{\beta\gamma;\alpha} R^{;\alpha} + 13740 R R_{;\alpha} R^{;\alpha} \\
&+ 6960 R^{\beta\gamma\delta\epsilon} R_{\beta\gamma\delta\epsilon;\alpha} R^{;\alpha} + 1440 R^{\alpha\beta} R^{\gamma\delta}{}_{;\alpha} R_{\gamma\delta;\beta} - 1560 R_{\alpha\beta} R^{;\alpha} R^{;\beta} + 2880 R^{\beta\gamma} R^{;\alpha} R_{\alpha\beta;\gamma} - 2160 R R_{\alpha\gamma;\beta} R^{\alpha\beta;\gamma} \\
&- 5760 R_\alpha{}^{\delta\epsilon\rho} R_{\gamma\delta\epsilon\rho;\beta} R^{\alpha\beta;\gamma} - 28920 R R_{\alpha\beta;\gamma} R^{\alpha\beta;\gamma} + 27840 R_{\alpha\delta\beta\epsilon} R^{\delta\epsilon}{}_{;\gamma} R^{\alpha\beta;\gamma} - 7680 R^{\alpha\beta} R_{\gamma\delta;\beta} R_\alpha{}^{\gamma;\delta} \\
&+ 4800 R^{\alpha\beta} R_{\beta\delta;\gamma} R_\alpha{}^{\gamma;\delta} + 85440 R^{\alpha\beta} R_{\beta\gamma;\delta} R_\alpha{}^{\gamma;\delta} - 1920 R_{\alpha\beta\gamma\delta} R^{;\alpha} R^{\beta\gamma;\delta} - 1920 R_{\beta\gamma\delta\epsilon} R^{\alpha\beta;\gamma} R_\alpha{}^{\delta;\epsilon} \\
&- 7680 R_{\alpha\delta\beta\epsilon} R^{\alpha\beta;\gamma} R_\gamma{}^{\delta;\epsilon} + 14400 R^{\alpha\beta} R_{\alpha\gamma\beta\epsilon;\delta} R^{\gamma\delta;\epsilon} + 34080 R^{\alpha\beta} R_{\alpha\gamma\beta\delta;\epsilon} R^{\gamma\delta;\epsilon} + 2700 R R_{\alpha\beta\gamma\delta;\epsilon} R^{\alpha\beta\gamma\delta;\epsilon} \\
&+ 12960 R^{\alpha\beta} R_{\beta\delta\gamma\rho;\epsilon} R_\alpha{}^{\gamma\delta\epsilon;\rho} - 10800 R^{\alpha\beta\gamma\delta} R_{\gamma\delta\epsilon\rho;\sigma} R_{\alpha\beta}{}^{\epsilon\rho;\sigma} - 9792 R_{\beta\gamma} R^{\beta\gamma} R^{;\alpha}{}_\alpha + 4608 R^2 R^{;\alpha}{}_\alpha \\
&+ 1296 R_{\beta\gamma\delta\epsilon} R^{\beta\gamma\delta\epsilon} R^{;\alpha}{}_\alpha + 432 R_{\alpha\beta} R R^{;\alpha\beta} + 1632 R^{\gamma\delta} R_{\alpha\gamma\beta\delta} R^{;\alpha\beta} + 936 R_{;\alpha\beta} R^{;\alpha\beta} - 11136 R^{\alpha\beta} R^{\gamma\delta} R_{\alpha\gamma;\beta\delta} \\
&+ 1008 R^{;\alpha}{}_\alpha R^{;\beta}{}_\beta + 10464 R^{\alpha\beta} R^{\gamma\delta} R_{\alpha\beta;\gamma\delta} - 18000 R^{\alpha\beta} R R_{\alpha\beta}{}^{;\gamma}{}_\gamma + 624 R^{;\alpha\beta} R_{\alpha\beta}{}^{;\gamma}{}_\gamma + 8352 R R_{\alpha\gamma\beta\delta} R^{\alpha\beta;\gamma\delta} \\
&- 14016 R_{\alpha\gamma}{}^{\epsilon\rho} R_{\beta\delta\epsilon\rho} R^{\alpha\beta;\gamma\delta} + 384 R_\alpha{}^\epsilon{}_\beta{}^\rho R_{\gamma\epsilon\delta\rho} R^{\alpha\beta;\gamma\delta} + 1872 R_{\gamma\delta;\alpha\beta} R^{\alpha\beta;\gamma\delta} - 4032 R_{\alpha\gamma;\beta\delta} R^{\alpha\beta;\gamma\delta} \\
&+ 1872 R_{\alpha\beta;\gamma\delta} R^{\alpha\beta;\gamma\delta} + 2304 R^{\alpha\beta} R^{\gamma\delta\epsilon\rho} R_{\alpha\gamma\beta\epsilon;\delta\rho} - 216 R^{\alpha\beta;\gamma}{}_\gamma R_{\alpha\beta}{}^{;\delta}{}_\delta + 23904 R_\alpha{}^\gamma R^{\alpha\beta} R_{\beta\gamma}{}^{;\delta}{}_\delta \\
&+ 8448 R^{\alpha\beta} R_{\beta\delta\gamma\epsilon} R_\alpha{}^{\gamma;\delta\epsilon} + 12288 R^{\alpha\beta} R_{\alpha\gamma\beta\delta} R^{\gamma\delta;\epsilon}{}_\epsilon + 576 R_{\alpha\beta\gamma\delta;\epsilon\rho} R^{\alpha\beta\gamma\delta;\epsilon\rho} + 2640 R^{;\alpha} R_{;\alpha}{}^\beta{}_\beta + 960 R_{\alpha\beta;\gamma} R^{;\alpha\beta\gamma} \\
&- 480 R^{\alpha\beta;\gamma} R_{\alpha\gamma;\beta}{}^\delta{}_\delta - 240 R^{\alpha\beta;\gamma} R_{\alpha\beta;\gamma}{}^\delta{}_\delta + 5760 R_{\alpha\gamma\beta\delta;\epsilon} R^{\alpha\beta;\gamma\delta\epsilon} + 1080 R R^{;\alpha}{}_\alpha{}^\beta{}_\beta + 960 R_{\alpha\beta} R^{;\alpha\beta\gamma}{}_\gamma \\
&- 240 R^{\alpha\beta} R_{\alpha\beta}{}^{;\gamma}{}_\gamma{}^\delta{}_\delta + 1920 R_{\alpha\gamma\beta\delta} R^{\alpha\beta;\gamma\delta\epsilon}{}_\epsilon + 480 R^{;\alpha}{}_\alpha{}^\beta{}_\beta{}^\gamma{}_\gamma \big),
\end{aligned} \tag{A7a}
$$

$$
\begin{aligned}
a_4^{(1)} = \frac{1}{15120}\big(
&-584 R_\alpha{}^\gamma R^{\alpha\beta} R_{\beta\gamma} R + 654 R_{\alpha\beta} R^{\alpha\beta} R^2 - 99 R^4 - 456 R^{\alpha\beta} R^{\gamma\delta} R R_{\alpha\gamma\beta\delta} - 72 R^2 R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} \\
&+ 80 R R_{\alpha\beta}{}^{\epsilon\rho} R^{\alpha\beta\gamma\delta} R_{\gamma\delta\epsilon\rho} + 12 R^{\beta\gamma} R_{\beta\gamma;\alpha} R^{;\alpha} - 135 R R_{;\alpha} R^{;\alpha} - 36 R^{\beta\gamma\delta\epsilon} R_{\beta\gamma\delta\epsilon;\alpha} R^{;\alpha} + 102 R_{\alpha\beta} R^{;\alpha} R^{;\beta} \\
&+ 24 R^{\beta\gamma} R^{;\alpha} R_{\alpha\beta;\gamma} + 12 R R_{\alpha\gamma;\beta} R^{\alpha\beta;\gamma} + 6 R R_{\alpha\beta;\gamma} R^{\alpha\beta;\gamma} + 24 R_{\alpha\beta\gamma\delta} R^{;\alpha} R^{\beta\gamma;\delta} - 27 R R_{\alpha\beta\gamma\delta;\epsilon} R^{\alpha\beta\gamma\delta;\epsilon} \\
&+ 30 R_{\beta\gamma} R^{\beta\gamma} R^{;\alpha}{}_\alpha - 123 R^2 R^{;\alpha}{}_\alpha - 18 R_{\beta\gamma\delta\epsilon} R^{\beta\gamma\delta\epsilon} R^{;\alpha}{}_\alpha - 48 R_{\alpha\beta} R R^{;\alpha\beta} - 72 R^{\gamma\delta} R_{\alpha\gamma\beta\delta} R^{;\alpha\beta} - 72 R_{;\alpha\beta} R^{;\alpha\beta} \\
&- 84 R^{;\alpha}{}_\alpha R^{;\beta}{}_\beta + 24 R^{\alpha\beta} R R_{\alpha\beta}{}^{;\gamma}{}_\gamma - 24 R^{;\alpha\beta} R_{\alpha\beta}{}^{;\gamma}{}_\gamma - 144 R R_{\alpha\gamma\beta\delta} R^{\alpha\beta;\gamma\delta} - 210 R^{;\alpha} R_{;\alpha}{}^\beta{}_\beta - 36 R_{\alpha\beta;\gamma} R^{;\alpha\beta\gamma} \\
&- 96 R R^{;\alpha}{}_\alpha{}^\beta{}_\beta - 36 R_{\alpha\beta} R^{;\alpha\beta\gamma}{}_\gamma - 18 R^{;\alpha}{}_\alpha{}^\beta{}_\beta{}^\gamma{}_\gamma \big),
\end{aligned} \tag{A7b}
$$

$$
\begin{aligned}
a_4^{(2)} = \frac{1}{720}\big(
&-2 R_{\alpha\beta} R^{\alpha\beta} R^2 + 5 R^4 + 2 R^2 R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} + 34 R R_{;\alpha} R^{;\alpha} - 12 R_{\alpha\beta} R^{;\alpha} R^{;\beta} + 32 R^2 R^{;\alpha}{}_\alpha + 8 R_{\alpha\beta} R R^{;\alpha\beta} \\
&+ 8 R_{;\alpha\beta} R^{;\alpha\beta} + 10 R^{;\alpha}{}_\alpha R^{;\beta}{}_\beta + 24 R^{;\alpha} R_{;\alpha}{}^\beta{}_\beta + 12 R R^{;\alpha}{}_\alpha{}^\beta{}_\beta \big),
\end{aligned} \tag{A7c}
$$

$$
a_4^{(3)} = \frac{1}{36}\big(-R^4 - 3 R R_{;\alpha} R^{;\alpha} - 3 R^2 R^{;\alpha}{}_\alpha\big), \qquad a_4^{(4)} = \frac{1}{24} R^4, \tag{A7d}
$$

$$a_5^{(0)} = \frac{1}{119\,750\,400}(-900736 R_{\alpha\beta}R^{\alpha\beta}R_\gamma{}^\epsilon R^{\gamma\delta}R_{\delta\epsilon} - 1297920 R_\alpha{}^\gamma R^{\alpha\beta}R_\beta{}^\delta R_{\gamma\delta}R + 1367004 R_{\alpha\beta}R^{\alpha\beta}R_{\gamma\delta}R^{\gamma\delta}R$$

$$+ 1858640 R_\alpha{}^\gamma R^{\alpha\beta}R_{\beta\gamma}R^2 - 1332402 R_{\alpha\beta}R^{\alpha\beta}R^3 + 152693 R^5 + 219000 R^{\alpha\beta}R^{\gamma\delta}R^2 R_{\alpha\gamma\beta\delta} + 89530 R^3 R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$$

$$+ 156312 R^{\alpha\beta}R^{\gamma\delta}R R_{\alpha\gamma}{}^{\epsilon\rho}R_{\beta\delta\epsilon\rho} - 107800 R^2 R_{\alpha\beta}{}^{\epsilon\rho}R^{\alpha\beta\gamma\delta}R_{\gamma\delta\epsilon\rho} - 1021632 R_\alpha{}^\gamma R^{\alpha\beta}R_\beta{}^\delta R^{\epsilon\rho}R_{\gamma\epsilon\delta\rho}$$

$$- 471360 R_{\alpha\beta}R^{\alpha\beta}R^{\gamma\delta}R^{\epsilon\rho}R_{\gamma\epsilon\delta\rho} + 438768 R^{\alpha\beta}R^{\gamma\delta}R R_\alpha{}^\epsilon{}_\beta{}^\rho R_{\gamma\epsilon\delta\rho} - 332262 R_{\alpha\beta}R^{\alpha\beta}R R_{\gamma\delta\epsilon\rho}R^{\gamma\delta\epsilon\rho}$$

$$- 142656 R_\alpha{}^\gamma R^{\alpha\beta}R^{\delta\epsilon}R_\beta{}^\rho{}_\gamma{}^\sigma R_{\delta\rho\epsilon\sigma} + 165312 R^{\alpha\beta}R^{\gamma\delta}R_{\alpha\gamma}{}^{\epsilon\rho}R_{\beta\epsilon}{}^{\sigma\tau}R_{\delta\rho\sigma\tau} + 16512 R^{\alpha\beta}R^{\gamma\delta}R_\alpha{}^\epsilon{}_\beta{}^\rho R_{\gamma\epsilon}{}^{\sigma\tau}R_{\delta\rho\sigma\tau}$$

$$- 122400 R R_{\alpha\beta}{}^{\epsilon\rho}R^{\alpha\beta\gamma\delta}R_{\gamma\epsilon}{}^{\sigma\tau}R_{\delta\rho\sigma\tau} + 188288 R_\alpha{}^\gamma R^{\alpha\beta}R_{\beta\gamma}R_{\delta\epsilon\rho\sigma}R^{\delta\epsilon\rho\sigma} - 82656 R^{\alpha\beta}R^{\gamma\delta}R_{\alpha\gamma}{}^{\epsilon\rho}R_{\beta\delta}{}^{\sigma\tau}R_{\epsilon\rho\sigma\tau}$$

$$+ 61200 R R_{\alpha\beta}{}^{\epsilon\rho}R^{\alpha\beta\gamma\delta}R_{\gamma\delta}{}^{\sigma\tau}R_{\epsilon\rho\sigma\tau} + 159136 R_{\alpha\beta}R^{\alpha\beta}R_{\gamma\delta}{}^{\sigma\tau}R^{\gamma\delta\epsilon\rho}R_{\epsilon\rho\sigma\tau} + 77376 R^{\alpha\beta}R^{\gamma\delta}R_{\alpha\gamma\beta\delta}R_{\epsilon\rho\sigma\tau}R^{\epsilon\rho\sigma\tau}$$

$$+ 39984 R R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}R_{\epsilon\rho\sigma\tau}R^{\epsilon\rho\sigma\tau} + 91392 R_{\alpha\beta}{}^{\epsilon\rho}R^{\alpha\beta\gamma\delta}R_{\gamma\delta}{}^{\sigma\tau}R_{\epsilon\sigma}{}^{\kappa\lambda}R_{\rho\tau\kappa\lambda} - 45696 R_{\alpha\beta}{}^{\epsilon\rho}R^{\alpha\beta\gamma\delta}R_{\gamma\delta}{}^{\sigma\tau}R_{\epsilon\rho}{}^{\kappa\lambda}R_{\sigma\tau\kappa\lambda}$$

$$- 34368 R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}R_{\epsilon\rho}{}^{\kappa\lambda}R^{\epsilon\rho\sigma\tau}R_{\sigma\tau\kappa\lambda} + 19558344 R^{\beta\gamma}R R_{\beta\gamma;\alpha}R^{:\alpha} - 21437712 R_\beta{}^\delta R^{\beta\gamma}R_{\gamma\delta;\alpha}R^{:\alpha}$$

$$- 4124208 R^{\beta\gamma}R_{\beta\delta\gamma\epsilon}R^{\delta\epsilon}{}_{;\alpha}R^{:\alpha} + 5396424 R_{\beta\gamma}R^{\beta\gamma}R_{:\alpha}R^{:\alpha} - 4045236 R^2 R_{:\alpha}R^{:\alpha} + 241398 R_{\beta\gamma\delta\epsilon}R^{\beta\gamma\delta\epsilon}R_{:\alpha}R^{:\alpha}$$

$$- 836028 R R^{\beta\gamma\delta\epsilon}R_{\beta\gamma\delta\epsilon;\alpha}R^{:\alpha} - 3149568 R^{\beta\gamma}R^{\delta\epsilon}R_{\beta\delta\gamma\epsilon;\alpha}R^{:\alpha} - 326592 R_{\beta\gamma}{}^{\rho\sigma}R^{\beta\gamma\delta\epsilon}R_{\delta\epsilon\rho\sigma;\alpha}R^{:\alpha}$$

$$+ 807696 R^{\alpha\beta}R R^{\gamma\delta}{}_{;\alpha}R_{\gamma\delta;\beta} - 1932624 R_\alpha{}^\beta R^{\gamma\delta}R^{:\alpha}R_{\gamma\delta;\beta} - 5280 R^{\beta\gamma}R_{\alpha\delta\gamma\epsilon}R^{:\alpha}R^{\delta\epsilon}{}_{;\beta} + 807072 R^{\alpha\beta}R^{\gamma\delta}R^{\epsilon\rho}{}_{;\alpha}R_{\gamma\epsilon\delta\rho;\beta}$$

$$- 526200 R_\alpha{}^\gamma R_{\beta\gamma}R^{:\alpha}R^{:\beta} + 191772 R_{\alpha\beta}R R^{:\alpha}R^{:\beta} - 736536 R^{\gamma\delta}R_{\alpha\gamma\beta\delta}R^{:\alpha}R^{:\beta} + 6856176 R^{\beta\gamma}R R^{:\alpha}R_{\alpha\beta;\gamma}$$

$$+ 241992 R_\alpha{}^\gamma R^{\alpha\beta}R^{\delta\epsilon}{}_{;\beta}R_{\delta\epsilon;\gamma} + 4805376 R^{\alpha\beta}R_{\beta\epsilon\delta\rho}R^{\gamma\delta}{}_{;\alpha}R^{\epsilon\rho}{}_{;\gamma} + 4684080 R^2 R_{\alpha\gamma;\beta}R^{\alpha\beta;\gamma}$$

$$+ 155208 R_{\delta\epsilon\rho\sigma}R^{\delta\epsilon\rho\sigma}R_{\alpha\gamma;\beta}R^{\alpha\beta;\gamma} - 806304 R R_\alpha{}^{\delta\epsilon\rho}R_{\gamma\delta\epsilon\rho;\beta}R^{\alpha\beta;\gamma} - 763296 R_\alpha{}^{\delta\epsilon\rho}R_{\epsilon\rho}{}^{\sigma\tau}R_{\gamma\delta\sigma\tau;\beta}R^{\alpha\beta;\gamma}$$

$$+ 2580132 R^2 R_{\alpha\beta;\gamma}R^{\alpha\beta;\gamma} - 487944 R_{\delta\epsilon\rho\sigma}R^{\delta\epsilon\rho\sigma}R_{\alpha\beta;\gamma}R^{\alpha\beta;\gamma} + 1798176 R R_{\alpha\delta\beta\epsilon}R^{\delta\epsilon}{}_{;\gamma}R^{\alpha\beta;\gamma}$$

$$- 1507920 R_{\alpha\delta}{}^{\rho\sigma}R_{\beta\epsilon\rho\sigma}R^{\delta\epsilon}{}_{;\gamma}R^{\alpha\beta;\gamma} + 2898720 R_\alpha{}^\rho{}_\beta{}^\sigma R_{\delta\rho\epsilon\sigma}R^{\delta\epsilon}{}_{;\gamma}R^{\alpha\beta;\gamma} - 6241920 R_\beta{}^\delta R^{\beta\gamma}R^{:\alpha}R_{\alpha\gamma;\delta}$$

$$+ 1822080 R_\alpha{}^\beta R^{\gamma\delta}R^{:\alpha}R_{\beta\gamma;\delta} + 2472096 R^{\alpha\beta}R^{\gamma\delta}R_\alpha{}^\epsilon{}_{;\gamma}R_{\beta\epsilon;\delta} - 1651104 R^{\alpha\beta}R^{\gamma\delta}R_\alpha{}^\epsilon{}_{;\beta}R_{\gamma\epsilon;\delta}$$

$$- 10000416 R^{\alpha\beta}R R_{\gamma\delta;\beta}R_\alpha{}^{\gamma;\delta} - 9098544 R^{\alpha\beta}R R_{\beta\delta;\gamma}R_\alpha{}^{\gamma;\delta} - 10746912 R^{\alpha\beta}R R_{\beta\gamma;\delta}R_\alpha{}^{\gamma;\delta}$$

$$- 859920 R R_{\alpha\beta\gamma\delta}R^{:\alpha}R^{\beta\gamma;\delta} - 349824 R_\alpha{}^\epsilon{}_\delta{}^\rho R_{\beta\epsilon\gamma\rho}R^{:\alpha}R^{\beta\gamma;\delta} - 470544 R_{\alpha\beta}{}^{\epsilon\rho}R_{\gamma\delta\epsilon\rho}R^{:\alpha}R^{\beta\gamma;\delta}$$

$$+ 356832 R_{\alpha\beta}{}^{\epsilon\rho}R^{\alpha\beta\gamma\delta}R_{\rho\sigma\tau\kappa;\delta}R_\gamma{}^{\sigma\tau\kappa}{}_{;\epsilon} - 5862720 R^{\alpha\beta}R^{\gamma\delta}R_{\gamma\epsilon;\delta}R_{\alpha\beta}{}^{;\epsilon} - 8640336 R^{\alpha\beta}R^{\gamma\delta}R_{\gamma\delta;\epsilon}R_{\alpha\beta}{}^{;\epsilon}$$

$$+ 5012928 R^{\alpha\beta}R^{\gamma\delta}R_{\beta\epsilon;\delta}R_{\alpha\gamma}{}^{;\epsilon} + 8810064 R^{\alpha\beta}R^{\gamma\delta}R_{\beta\delta;\epsilon}R_{\alpha\gamma}{}^{;\epsilon} - 2592384 R^{\beta\gamma}R_{\beta\delta\gamma\epsilon}R^{:\alpha}R_\alpha{}^{\delta;\epsilon}$$

$$- 3291264 R R_{\beta\gamma\delta\epsilon}R^{\alpha\beta;\gamma}R_\alpha{}^{\delta;\epsilon} + 33120 R_{\beta\epsilon}{}^{\rho\sigma}R_{\gamma\delta\rho\sigma}R^{\alpha\beta;\gamma}R_\alpha{}^{\delta;\epsilon} - 33120 R_{\beta\delta}{}^{\rho\sigma}R_{\gamma\epsilon\rho\sigma}R^{\alpha\beta;\gamma}R_\alpha{}^{\delta;\epsilon}$$

$$+ 134880 R_{\beta\gamma}{}^{\rho\sigma}R_{\delta\epsilon\rho\sigma}R^{\alpha\beta;\gamma}R_\alpha{}^{\delta;\epsilon} - 89472 R_\beta{}^\rho{}_\gamma{}^\sigma R_{\delta\rho\epsilon\sigma}R^{\alpha\beta;\gamma}R_\alpha{}^{\delta;\epsilon} - 3247104 R^{\beta\gamma}R_{\alpha\delta\gamma\epsilon}R^{:\alpha}R_\beta{}^{\delta;\epsilon}$$

$$+ 2323680 R^{\beta\gamma}R_{\alpha\epsilon\gamma\delta}R^{:\alpha}R_\beta{}^{\delta;\epsilon} + 8320800 R_\alpha{}^\gamma R^{\alpha\beta}R_{\delta\epsilon;\gamma}R_\beta{}^{\delta;\epsilon} + 9750384 R_\alpha{}^\gamma R^{\alpha\beta}R_{\gamma\epsilon;\delta}R_\beta{}^{\delta;\epsilon}$$

$$+ 11572176 R_\alpha{}^\gamma R^{\alpha\beta}R_{\gamma\delta;\epsilon}R_\beta{}^{\delta;\epsilon} - 7426464 R R_{\alpha\delta\beta\epsilon}R^{\alpha\beta;\gamma}R_\gamma{}^{\delta;\epsilon} + 493920 R_{\alpha\delta}{}^{\rho\sigma}R_{\beta\epsilon\rho\sigma}R^{\alpha\beta;\gamma}R_\gamma{}^{\delta;\epsilon}$$

$$+ 912000 R_\alpha{}^\rho{}_\beta{}^\sigma R_{\delta\rho\epsilon\sigma}R^{\alpha\beta;\gamma}R_\gamma{}^{\delta;\epsilon} + 5089056 R^{\alpha\beta}R_{\alpha\delta\beta\rho}R_\epsilon{}^\rho{}_{;\gamma}R^{\gamma\delta;\epsilon} - 7634784 R_{\alpha\beta}R^{\alpha\beta}R_{\gamma\epsilon;\delta}R^{\gamma\delta;\epsilon}$$

$$- 4831872 R^{\alpha\beta}R R_{\alpha\gamma\beta\epsilon;\delta}R^{\gamma\delta;\epsilon} + 89856 R^{\alpha\beta}R_\gamma{}^\rho{}_\epsilon{}^\sigma R_{\alpha\rho\beta\sigma;\delta}R^{\gamma\delta;\epsilon} + 3264960 R^{\alpha\beta}R_\alpha{}^\rho{}_\gamma{}^\sigma R_{\beta\rho\epsilon\sigma;\delta}R^{\gamma\delta;\epsilon}$$

$$+ 1777152 R^{\alpha\beta}R_\alpha{}^\rho{}_\epsilon{}^\sigma R_{\beta\sigma\gamma\rho;\delta}R^{\gamma\delta;\epsilon} - 4651392 R^{\alpha\beta}R_\alpha{}^\rho{}_\gamma{}^\sigma R_{\beta\sigma\epsilon\rho;\delta}R^{\gamma\delta;\epsilon} + 3936192 R^{\alpha\beta}R_\alpha{}^\rho{}_\beta{}^\sigma R_{\gamma\rho\epsilon\sigma;\delta}R^{\gamma\delta;\epsilon}$$

$$- 1655496 R_{\alpha\beta}R^{\alpha\beta}R_{\gamma\delta;\epsilon}R^{\gamma\delta;\epsilon} - 6984576 R^{\alpha\beta}R_{\alpha\delta\beta\rho}R_\gamma{}^\rho{}_{;\epsilon}R^{\gamma\delta;\epsilon} - 3366432 R^{\alpha\beta}R R_{\alpha\gamma\beta\delta;\epsilon}R^{\gamma\delta;\epsilon}$$

$$- 6791232 R^{\alpha\beta}R_\gamma{}^\rho{}_\delta{}^\sigma R_{\alpha\rho\beta\sigma;\epsilon}R^{\gamma\delta;\epsilon} + 23185728 R^{\alpha\beta}R_\alpha{}^\rho{}_\gamma{}^\sigma R_{\beta\rho\delta\sigma;\epsilon}R^{\gamma\delta;\epsilon} - 10610304 R^{\alpha\beta}R_\alpha{}^\rho{}_\gamma{}^\sigma R_{\beta\sigma\delta\rho;\epsilon}R^{\gamma\delta;\epsilon}$$

$$- 2938176 R^{\alpha\beta}R_\alpha{}^\rho{}_\beta{}^\sigma R_{\gamma\rho\delta\sigma;\epsilon}R^{\gamma\delta;\epsilon} + 171594 R^2 R_{\alpha\beta\gamma\delta;\epsilon}R^{\alpha\beta\gamma\delta;\epsilon} + 314160 R^{\alpha\beta\gamma\delta}R^{\epsilon\rho\sigma\tau}R_{\epsilon\rho\sigma\tau;\kappa}R_{\alpha\beta\gamma\delta}{}^{;\kappa}$$

$$- 2461824 R_\alpha{}^\epsilon{}_\gamma{}^\rho R^{\alpha\beta\gamma\delta}R_{\epsilon\sigma\rho\tau;\kappa}R_\beta{}^\sigma{}_\delta{}^{\tau;\kappa} + 35856 R_{\alpha\beta}{}^{\epsilon\rho}R^{\alpha\beta\gamma\delta}R_{\epsilon\rho\sigma\tau;\kappa}R_{\gamma\delta}{}^{\sigma\tau;\kappa} + 202356 R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}R_{\epsilon\rho\sigma\tau;\kappa}R^{\epsilon\rho\sigma\tau;\kappa}$$

$$- 1522176 R^{\beta\gamma}R_\beta{}^{\delta\epsilon\rho}R^{:\alpha}R_{\alpha\delta\gamma\epsilon;\rho} - 39168 R_\gamma{}^{\delta\epsilon\rho}R_\delta{}^\sigma{}_\epsilon{}^\tau R^{\alpha\beta;\gamma}R_{\alpha\sigma\beta\tau;\rho} + 341952 R_\alpha{}^\beta R^{\gamma\delta\epsilon\rho}R^{:\alpha}R_{\beta\gamma\delta\epsilon;\rho}$$

$$- 356832 R_{\alpha\beta}{}^{\epsilon\rho}R^{\alpha\beta\gamma\delta}R_\gamma{}^{\sigma\tau\kappa}{}_{;\epsilon}R_{\delta\sigma\tau\kappa;\rho} + 356832 R_{\alpha\beta}{}^{\epsilon\rho}R^{\alpha\beta\gamma\delta}R_\gamma{}^{\sigma\tau\kappa}{}_{;\delta}R_{\epsilon\sigma\tau\kappa;\rho}$$

$$+ 116160 R^{\alpha\beta} R^{\gamma\delta} R_{\gamma\epsilon\delta\rho;\beta} R_\alpha{}^{\epsilon;\rho} - 791424 R^{\alpha\beta} R^{\gamma\beta} R_{\beta\gamma\delta\rho;\epsilon} R_\alpha{}^{\epsilon;\rho} + 1156512 R^{\alpha\beta} R_{\gamma\epsilon\delta\rho} R_\alpha{}^{\gamma;\delta} R_\beta{}^{\epsilon;\rho}$$

$$+ 1855296 R^{\alpha\beta} R_{\alpha\epsilon\beta\rho} R^{\gamma\delta;\epsilon} R_{\gamma\delta}{}^{;\rho} + 1174080 R^{\alpha\beta} R_{\alpha\delta\beta\rho} R^{\gamma\delta;\epsilon} R_{\gamma\epsilon}{}^{;\rho} - 5450112 R^{\alpha\beta} R_{\beta\delta\epsilon\rho} R_\alpha{}^{\gamma;\delta} R_\gamma{}^{\epsilon;\rho}$$

$$+ 5065152 R^{\alpha\beta} R_{\beta\epsilon\delta\rho} R_\alpha{}^{\gamma;\delta} R_\gamma{}^{\epsilon;\rho} - 1258752 R^{\alpha\beta} R_{\beta\gamma\epsilon\rho} R_\alpha{}^{\gamma;\delta} R_\delta{}^{\epsilon;\rho} + 5230464 R^{\alpha\beta} R_{\beta\epsilon\gamma\rho} R_\alpha{}^{\gamma;\delta} R_\delta{}^{\epsilon;\rho}$$

$$+ 1191168 R^{\alpha\beta} R_{\gamma\delta\epsilon\rho} R_\alpha{}^\gamma{}_{;\beta} R^{\delta\epsilon;\rho} - 1375296 R^{\alpha\beta} R_{\gamma\delta\epsilon\rho} R_{\alpha\beta}{}^{;\gamma} R^{\delta\epsilon;\rho} + 7328064 R_\alpha{}^\gamma R^{\alpha\beta} R_{\beta\delta\gamma\rho;\epsilon} R^{\delta\epsilon;\rho}$$

$$- 1192224 R_\alpha{}^\gamma R^{\alpha\beta} R_{\beta\delta\gamma\epsilon;\rho} R^{\delta\epsilon;\rho} - 1588704 R^{\alpha\beta} R R_{\beta\delta\gamma\rho;\epsilon} R_\alpha{}^{\gamma\delta\epsilon;\rho} + 3409920 R^{\alpha\beta} R_{\gamma\epsilon}{}^{\rho\sigma} R^{\gamma\delta;\epsilon} R_{\alpha\delta\beta\rho;\sigma}$$

$$+ 1821696 R^{\alpha\beta} R_\alpha{}^\rho{}_\gamma{}^\sigma R^{\gamma\delta;\epsilon} R_{\beta\delta\epsilon\rho;\sigma} - 474864 R R^{\alpha\beta\gamma\delta} R_{\gamma\delta\epsilon\rho;\sigma} R_{\alpha\beta}{}^{\epsilon\rho;\sigma} + 361008 R^{\alpha\beta} R^{\gamma\delta} R_{\beta\delta\epsilon\rho;\sigma} R_{\alpha\gamma}{}^{\epsilon\rho;\sigma}$$

$$- 838176 R^{\alpha\beta} R^{\gamma\delta} R_{\gamma\epsilon\delta\rho;\sigma} R_\alpha{}^\epsilon{}_\beta{}^{\rho;\sigma} + 3496512 R_\alpha{}^\gamma R^{\alpha\beta} R_{\gamma\epsilon\delta\sigma;\rho} R_\beta{}^{\delta\epsilon\rho;\sigma} - 293028 R_{\alpha\beta} R^{\alpha\beta} R_{\gamma\delta\epsilon\rho;\sigma} R^{\gamma\delta\epsilon\rho;\sigma}$$

$$+ 1459200 R_{\alpha\gamma}{}^{\delta\epsilon} R_\beta{}^{\rho\sigma\tau} R^{\alpha\beta;\gamma} R_{\delta\epsilon\rho\sigma;\tau} + 4167936 R^{\alpha\beta} R^{\gamma\delta\epsilon\rho} R_{\delta\epsilon\rho\tau;\sigma} R_{\alpha\gamma\beta}{}^{\sigma;\tau} - 2542848 R^{\alpha\beta} R_\alpha{}^{\gamma\delta\epsilon} R_{\gamma\rho\epsilon\tau;\sigma} R_{\beta\delta}{}^{\rho\sigma;\tau}$$

$$+ 1998912 R^{\alpha\beta} R_\alpha{}^\gamma{}_\beta{}^\delta R_{\delta\rho\epsilon\tau;\sigma} R_\gamma{}^{\epsilon\rho\sigma;\tau} + 193920 R^{\alpha\beta;\gamma} R_\gamma{}^{\delta;\epsilon} R_{\delta\epsilon;\alpha\beta} - 96192 R^{\alpha\beta} R^{\gamma\delta} R_{\gamma\epsilon\delta\rho} R^{\epsilon\rho}{}_{;\alpha\beta}$$

$$+ 357672 R^{;\alpha} R^{;\beta} R_{;\alpha\beta} + 74016 R^{;\alpha} R^{\beta\gamma;\delta} R_{\beta\delta;\alpha\gamma} - 1763424 R^{;\alpha} R^{\beta\gamma;\delta} R_{\beta\gamma;\alpha\delta} + 1029888 R^{\alpha\beta;\gamma} R_{\beta\gamma\delta\rho;\epsilon} R^{\delta\epsilon}{}_{;\alpha}{}^\rho$$

$$- 103488 R_\beta{}^\delta R^{\beta\gamma} R_{\gamma\delta} R^{;\alpha}{}_\alpha + 193776 R_{\beta\gamma} R^{\beta\gamma} R R^{;\alpha}{}_\alpha - 15620 R^3 R^{;\alpha}{}_\alpha - 129696 R^{\beta\gamma} R^{\delta\epsilon} R_{\beta\delta\gamma\epsilon} R^{;\alpha}{}_\alpha$$

$$+ 29580 R R_{\beta\gamma\delta\epsilon} R^{\beta\gamma\delta\epsilon} R^{;\alpha}{}_\alpha - 46216 R_{\beta\gamma}{}^{\rho\sigma} R^{\beta\gamma\delta\epsilon} R_{\delta\epsilon\rho\sigma} R^{;\alpha}{}_\alpha + 211152 R_{\beta\delta;\gamma} R^{\beta\gamma;\delta} R^{;\alpha}{}_\alpha - 124176 R_{\beta\gamma;\delta} R^{\beta\gamma;\delta} R^{;\alpha}{}_\alpha$$

$$+ 25560 R_{\beta\gamma\delta\epsilon;\rho} R^{\beta\gamma\delta\epsilon;\rho} R^{;\alpha}{}_\alpha + 667392 R_\alpha{}^\gamma R_\beta{}^\delta R_{\gamma\delta} R^{;\alpha\beta} - 310128 R_{\alpha\beta} R_{\gamma\delta} R^{\gamma\delta} R^{;\alpha\beta} - 651168 R_\alpha{}^\gamma R_{\beta\gamma} R R^{;\alpha\beta}$$

$$+ 274008 R_{\alpha\beta} R^2 R^{;\alpha\beta} - 179904 R^{\gamma\delta} R R_{\alpha\gamma\beta\delta} R^{;\alpha\beta} + 359040 R_\beta{}^\gamma R^{\delta\epsilon} R_{\alpha\delta\gamma\epsilon} R^{;\alpha\beta} + 142656 R^{\gamma\delta} R_\alpha{}^\epsilon{}_\gamma{}^\rho R_{\beta\epsilon\delta\rho} R^{;\alpha\beta}$$

$$- 167424 R^{\gamma\delta} R_\alpha{}^\epsilon{}_\beta{}^\rho R_{\gamma\epsilon\delta\rho} R^{;\alpha\beta} + 27408 R_{\alpha\beta} R_{\gamma\delta\epsilon\rho} R^{\gamma\delta\epsilon\rho} R^{;\alpha\beta} - 908832 R^{\gamma\delta}{}_{;\alpha} R_{\gamma\delta;\beta} R^{;\alpha\beta} + 125064 R^{\gamma\delta\epsilon\rho}{}_{;\alpha} R_{\gamma\delta\epsilon\rho;\beta} R^{;\alpha\beta}$$

$$+ 4992 R_{\gamma\delta;\beta} R_\alpha{}^{\gamma;\delta} R^{;\alpha\beta} - 21120 R_{\beta\delta;\gamma} R_\alpha{}^{\gamma;\delta} R^{;\alpha\beta} + 6528 R_{\beta\gamma;\delta} R_\alpha{}^{\gamma;\delta} R^{;\alpha\beta} + 43776 R_{\alpha\gamma\delta\epsilon;\beta} R^{\gamma\delta;\epsilon} R^{;\alpha\beta}$$

$$+ 23616 R_{\alpha\gamma\beta\delta;\epsilon} R^{\gamma\delta;\epsilon} R^{;\alpha\beta} - 1028256 R^{\gamma\delta} R_{\gamma\delta;\alpha\beta} R^{;\alpha\beta} + 210696 R R_{;\alpha\beta} R^{;\alpha\beta} + 119040 R^{\gamma\delta\epsilon\rho} R_{\gamma\delta\epsilon\rho;\alpha\beta} R^{;\alpha\beta}$$

$$- 382080 R^{;\alpha} R^{\beta\gamma;\delta} R_{\alpha\delta;\beta\gamma} - 221376 R^{\alpha\beta} R^{\gamma\delta} R R_{\alpha\gamma;\beta\delta} - 197568 R^{\alpha\beta;\gamma} R_\gamma{}^{\delta;\epsilon} R_{\alpha\epsilon;\beta\delta} + 792000 R^{\alpha\beta;\gamma} R_\alpha{}^{\delta;\epsilon} R_{\gamma\epsilon;\beta\delta}$$

$$- 194112 R^{\alpha\beta;\gamma} R_\gamma{}^{\delta;\epsilon} R_{\alpha\delta;\beta\epsilon} + 32832 R^{\alpha\beta} R^{\gamma\delta}{}_{;\alpha}{}^\epsilon R_{\gamma\delta;\beta\epsilon} + 2748 R_{;\alpha} R^{;\alpha} R^{;\beta}{}_\beta + 44568 R R^{;\alpha}{}_\alpha R^{;\beta}{}_\beta + 25920 R_{\beta\gamma;\alpha} R^{;\alpha} R^{;\beta\gamma}$$

$$- 29088 R^{;\alpha} R_{\alpha\beta;\gamma} R^{;\beta\gamma} + 7200 R_{\beta\gamma} R^{;\alpha}{}_\alpha R^{;\beta\gamma} + 173280 R^{\alpha\beta} R^{\gamma\delta} R R_{\alpha\beta;\gamma\delta} + 145056 R^{;\alpha} R^{\beta\gamma;\delta} R_{\alpha\beta;\gamma\delta}$$

$$+ 24480 R^{\gamma\delta} R^{;\alpha\beta} R_{\alpha\beta;\gamma\delta} - 153792 R^{\alpha\beta;\gamma} R_\alpha{}^{\delta;\epsilon} R_{\beta\epsilon;\gamma\delta} + 86016 R_\alpha{}^\gamma R^{\alpha\beta} R^{\delta\epsilon} R_{\beta\delta;\gamma\epsilon} + 2651328 R^{\alpha\beta;\gamma} R_\alpha{}^{\delta;\epsilon} R_{\beta\delta;\gamma\epsilon}$$

$$+ 643680 R^{\alpha\beta;\gamma} R^{\delta\epsilon;\rho} R_{\alpha\delta\beta\epsilon;\gamma\rho} + 1569024 R^{\alpha\beta;\gamma} R_{\alpha\delta\beta\epsilon;\rho} R^{\delta\epsilon}{}_{;\gamma}{}^\rho - 31104 R_{\alpha\gamma} R^{;\alpha\beta} R^{;\gamma}{}_\beta - 208080 R^{\alpha\beta} R^2 R_{\alpha\beta}{}^{;\gamma}{}_\gamma$$

$$- 39456 R^{\alpha\beta} R_{\delta\epsilon\rho\sigma} R^{\delta\epsilon\rho\sigma} R_{\alpha\beta}{}^{;\gamma}{}_\gamma + 84960 R^{;\alpha} R^{;\beta} R_{\alpha\beta}{}^{;\gamma}{}_\gamma + 1680 R R^{;\alpha\beta} R_{\alpha\beta}{}^{;\gamma}{}_\gamma - 56664 R^{\delta\epsilon\rho\sigma}{}_{;\alpha} R_{\delta\epsilon\rho\sigma;\beta} R^{\alpha\beta;\gamma}{}_\gamma$$

$$+ 6528 R^{\delta\epsilon\rho\sigma} R_{\delta\epsilon\rho\sigma;\alpha\beta} R^{\alpha\beta;\gamma}{}_\gamma + 215712 R^2 R_{\alpha\gamma\beta\delta} R^{\alpha\beta;\gamma\delta} - 426240 R R_{\alpha\gamma}{}^{\epsilon\rho} R_{\beta\delta\epsilon\rho} R^{\alpha\beta;\gamma\delta} - 19584 R R_\alpha{}^\epsilon{}_\beta{}^\rho R_{\gamma\epsilon\delta\rho} R^{\alpha\beta;\gamma\delta}$$

$$- 43008 R_\alpha{}^{\epsilon\rho\sigma} R_\beta{}^\tau{}_{\rho\sigma} R_{\gamma\epsilon\delta\tau} R^{\alpha\beta;\gamma\delta} - 246528 R_{\alpha\gamma}{}^{\epsilon\rho} R_{\beta\epsilon}{}^{\sigma\tau} R_{\delta\rho\sigma\tau} R^{\alpha\beta;\gamma\delta} + 123264 R_{\alpha\gamma}{}^{\epsilon\rho} R_{\beta\delta}{}^{\sigma\tau} R_{\epsilon\rho\sigma\tau} R^{\alpha\beta;\gamma\delta}$$

$$+ 81216 R_{\alpha\gamma\beta\delta} R_{\epsilon\rho\sigma\tau} R^{\epsilon\rho\sigma\tau} R^{\alpha\beta;\gamma\delta} - 482112 R_\alpha{}^{\epsilon\rho\sigma}{}_{;\beta} R_{\gamma\epsilon\rho\sigma;\delta} R^{\alpha\beta;\gamma\delta} + 6912 R_{\gamma\epsilon\delta\rho;\sigma} R_\alpha{}^\epsilon{}_\beta{}^{\rho;\sigma} R^{\alpha\beta;\gamma\delta}$$

$$+ 31056 R R_{\gamma\delta;\alpha\beta} R^{\alpha\beta;\gamma\delta} - 172224 R R_{\alpha\gamma;\beta\delta} R^{\alpha\beta;\gamma\delta} - 251904 R_\alpha{}^{\epsilon\rho\sigma} R_{\gamma\epsilon\rho\sigma;\beta\delta} R^{\alpha\beta;\gamma\delta} - 358128 R R_{\alpha\beta;\gamma\delta} R^{\alpha\beta;\gamma\delta}$$

$$+ 494208 R_{\alpha\epsilon\beta\rho} R^{\epsilon\rho}{}_{;\gamma\delta} R^{\alpha\beta;\gamma\delta} + 42336 R_{\alpha\gamma\beta\delta} R^{;\alpha\beta} R^{;\gamma\delta} + 5376 R^{\gamma\delta} R^{;\alpha\beta} R_{\alpha\gamma;\delta\beta} - 15648 R^{\delta\epsilon}{}_{;\gamma} R^{\alpha\beta;\gamma} R_{\alpha\beta;\delta\epsilon}$$

$$+ 95424 R^{\alpha\beta;\gamma} R_\gamma{}^{\delta;\epsilon} R_{\alpha\beta;\delta\epsilon} - 39552 R_\alpha{}^\gamma R^{\alpha\beta} R^{\delta\epsilon} R_{\beta\gamma;\delta\epsilon} - 341952 R^{\alpha\beta;\gamma} R_\alpha{}^{\delta;\epsilon} R_{\beta\gamma;\delta\epsilon} + 960 R^{\alpha\beta;\gamma} R^{\delta\epsilon;\rho} R_{\alpha\gamma\beta\rho;\delta\epsilon}$$

$$+ 16128 R^{\alpha\beta} R R^{\gamma\delta\epsilon\rho} R_{\alpha\gamma\beta\epsilon;\delta\rho} + 17280 R^{\alpha\beta;\gamma} R_\gamma{}^{\delta\epsilon\rho;\sigma} R_{\alpha\epsilon\beta\sigma;\delta\rho} + 345600 R^{\alpha\beta;\gamma} R_\alpha{}^{\delta\epsilon\rho;\sigma} R_{\beta\epsilon\gamma\sigma;\delta\rho}$$

$$- 69384 R R^{\alpha\beta;\gamma}{}_\gamma R_{\alpha\beta}{}^{;\delta}{}_\delta + 301824 R_\alpha{}^\gamma R^{\alpha\beta} R R_{\beta\gamma}{}^{;\delta}{}_\delta + 48288 R^{\beta\gamma}{}_{;\alpha} R^{;\alpha} R_{\beta\gamma}{}^{;\delta}{}_\delta - 108672 R^{;\alpha} R_\alpha{}^{\beta;\gamma} R_{\beta\gamma}{}^{;\delta}{}_\delta$$

$$- 178848 R^{\beta\gamma} R^{;\alpha}{}_\alpha R_{\beta\gamma}{}^{;\delta}{}_\delta + 6528 R_\alpha{}^\gamma R^{;\alpha\beta} R_{\beta\gamma}{}^{;\delta}{}_\delta - 10368 R^{\alpha\beta} R R_{\beta\delta\gamma\epsilon} R_\alpha{}^{\gamma;\delta\epsilon} + 63360 R^{\alpha\beta} R_{\delta\epsilon;\beta\gamma} R_\alpha{}^{\gamma;\delta\epsilon}$$

$$- 195840 R^{\alpha\beta} R_{\gamma\delta;\beta\epsilon} R_\alpha{}^{\gamma;\delta\epsilon} + 161664 R^{\alpha\beta} R_{\beta\delta;\gamma\epsilon} R_\alpha{}^{\gamma;\delta\epsilon} + 1341120 R^{\alpha\beta} R_{\beta\gamma;\delta\epsilon} R_\alpha{}^{\gamma;\delta\epsilon} + 146592 R^{;\alpha} R_{\alpha\delta\beta\epsilon;\gamma} R^{\beta\gamma;\delta\epsilon}$$

$$- 101664 R^{;\alpha} R_{\alpha\beta\gamma\delta;\epsilon} R^{\beta\gamma;\delta\epsilon} + 119808 R_{\beta\delta\gamma\epsilon} R^{;\alpha}{}_\alpha R^{\beta\gamma;\delta\epsilon} + 653760 R^{;\alpha} R^{\beta\gamma\delta\epsilon;\rho} R_{\alpha\beta\gamma\delta;\epsilon\rho}$$

$$+ 26496 R^{\alpha\beta} R^{\gamma\delta} R^{\epsilon\rho} R_{\alpha\gamma\beta\delta;\epsilon\rho} - 139200 R^{\alpha\beta;\gamma} R^{\delta\epsilon;\rho} R_{\alpha\gamma\beta\delta;\epsilon\rho} + 253440 R_\alpha{}^\gamma R^{\alpha\beta} R^{\delta\epsilon\rho\sigma} R_{\beta\delta\gamma\rho;\epsilon\sigma}$$

$$- 36864 R_{\alpha\gamma\delta\epsilon} R^{;\alpha\beta} R^{\gamma\delta;\epsilon}{}_\beta + 211776 R^{\alpha\beta} R_\alpha{}^{\gamma;\delta}{}_\delta R_{\beta\gamma}{}^{;\epsilon}{}_\epsilon - 418560 R_\gamma{}^\delta{}_{;\alpha} R^{\alpha\beta;\gamma} R_{\beta\delta}{}^{;\epsilon}{}_\epsilon + 229152 R_\alpha{}^\delta{}_{;\gamma} R^{\alpha\beta;\gamma} R_{\beta\delta}{}^{;\epsilon}{}_\epsilon$$

$$+ 198336 R^{\alpha\beta;\gamma} R_{\alpha\gamma}{}^{;\delta} R_{\beta\delta}{}^{;\epsilon}{}_\epsilon + 960 R_\alpha{}^\gamma R^{\alpha\beta} R_\beta{}^\delta R_{\gamma\delta}{}^{;\epsilon}{}_\epsilon - 206976 R_{\alpha\beta} R^{\alpha\beta} R^{\gamma\delta} R_{\gamma\delta}{}^{;\epsilon}{}_\epsilon - 388368 R^{\alpha\beta;\gamma} R_{\alpha\beta}{}^{;\delta} R_{\gamma\delta}{}^{;\epsilon}{}_\epsilon$$

$$+ 81696 R^{\alpha\beta} R^{\gamma\delta}{}_{;\alpha\beta} R_{\gamma\delta}{}^{;\epsilon}{}_\epsilon - 192000 R^{\alpha\beta} R_\alpha{}^\gamma{}_{;\beta}{}^\delta R_{\gamma\delta}{}^{;\epsilon}{}_\epsilon + 93408 R^{\alpha\beta} R_{\alpha\beta}{}^{;\gamma\delta} R_{\gamma\delta}{}^{;\epsilon}{}_\epsilon + 12960 R^{\alpha\beta} R R_{\alpha\gamma\beta\delta} R^{\gamma\delta;\epsilon}{}_\epsilon$$

$$+ 378048 R^{\alpha\beta} R_\alpha{}^\rho{}_\gamma{}^\sigma R_{\beta\rho\delta\sigma} R^{\gamma\delta;\epsilon}{}_\epsilon + 36096 R^{\alpha\beta} R_\alpha{}^\rho{}_\beta{}^\sigma R_{\gamma\rho\delta\sigma} R^{\gamma\delta;\epsilon}{}_\epsilon + 14400 R_{\alpha\gamma\beta\delta} R^{;\alpha\beta} R^{\gamma\delta;\epsilon}{}_\epsilon$$

$$+ 86976 R^{\alpha\beta} R^{\gamma\delta} R_{\gamma\epsilon\delta\rho} R_{\alpha\beta}{}^{;\epsilon\rho} + 126720 R^{\alpha\beta;\gamma} R_{\beta\epsilon\gamma\rho;\delta} R_\alpha{}^{\delta;\epsilon\rho} - 668160 R^{\alpha\beta;\gamma} R_{\beta\gamma\delta\epsilon;\rho} R_\alpha{}^{\delta;\epsilon\rho}$$

$$- 260352 R^{\alpha\beta;\gamma} R_{\beta\delta\gamma\epsilon;\rho} R_\alpha{}^{\delta;\epsilon\rho} + 540672 R_\alpha{}^\gamma R^{\alpha\beta} R_{\gamma\epsilon\delta\rho} R_\beta{}^{\delta;\epsilon\rho} - 9216 R_{\alpha\epsilon\beta\rho} R^{\alpha\beta;\gamma\delta} R_{\gamma\delta}{}^{;\epsilon\rho}$$

$$+ 590400 R^{\alpha\beta;\gamma} R_{\alpha\epsilon\beta\rho;\delta} R_\gamma{}^{\delta;\epsilon\rho} - 140544 R^{\alpha\beta;\gamma} R_{\alpha\delta\beta\epsilon;\rho} R_\gamma{}^{\delta;\epsilon\rho} + 388608 R^{\alpha\beta} R_{\alpha\gamma\delta}{}^\sigma R_{\beta\epsilon\rho\sigma} R^{\gamma\delta;\epsilon\rho}$$

$$+ 99840 R^{\alpha\beta} R_{\alpha\gamma\epsilon}{}^\sigma R_{\beta\sigma\delta\rho} R^{\gamma\delta;\epsilon\rho} - 278976 R_{\alpha\beta} R^{\alpha\beta} R_{\gamma\epsilon\delta\rho} R^{\gamma\delta;\epsilon\rho} - 367872 R^{\alpha\beta} R_{\alpha\gamma\beta}{}^\sigma R_{\delta\rho\epsilon\sigma} R^{\gamma\delta;\epsilon\rho}$$

$$+ 32256 R^{\alpha\beta} R_{\alpha\epsilon\beta\rho;\gamma\delta} R^{\gamma\delta;\epsilon\rho} + 486912 R^{\alpha\beta} R_{\alpha\gamma\beta\epsilon;\delta\rho} R^{\gamma\delta;\epsilon\rho} + 360960 R^{\alpha\beta} R_{\alpha\gamma\beta\delta;\epsilon\rho} R^{\gamma\delta;\epsilon\rho}$$

$$+ 27264 R R_{\alpha\beta\gamma\delta;\epsilon\rho} R^{\alpha\beta\gamma\delta;\epsilon\rho} + 293760 R^{\alpha\beta;\gamma} R_\alpha{}^{\delta\epsilon\rho;\sigma} R_{\beta\gamma\delta\epsilon;\rho\sigma} + 116736 R^{\alpha\beta} R_\alpha{}^{\gamma\delta\epsilon} R_\gamma{}^{\rho\sigma\tau} R_{\beta\delta\epsilon\sigma;\rho\tau}$$

$$- 98304 R^{\alpha\beta} R_\alpha{}^{\gamma\delta\epsilon} R_\beta{}^{\rho\sigma\tau} R_{\gamma\delta\epsilon\sigma;\rho\tau} - 12288 R^{\alpha\beta} R_\alpha{}^\gamma{}_\beta{}^\delta R^{\epsilon\rho\sigma\tau} R_{\gamma\epsilon\delta\sigma;\rho\tau} + 176256 R_{\beta\gamma\epsilon\rho} R^{\alpha\beta;\gamma\delta} R_\delta{}^{\epsilon;\rho}{}_\alpha$$

$$- 28800 R_{\beta\epsilon\gamma\rho} R^{\alpha\beta;\gamma\delta} R_\delta{}^{\epsilon;\rho}{}_\alpha + 81024 R_{\beta\delta\epsilon\rho} R^{\alpha\beta;\gamma}{}_\gamma R^{\delta\epsilon;\rho}{}_\alpha - 120576 R^{\alpha\beta} R^{\gamma\delta} R_{\gamma\epsilon\delta\rho} R_\alpha{}^{\epsilon;\rho}{}_\beta$$

$$+ 188928 R^{\alpha\beta} R^{\gamma\delta} R_{\beta\delta\epsilon\rho} R_\alpha{}^{\epsilon;\rho}{}_\gamma + 768 R^{\alpha\beta} R^{\gamma\delta} R_{\beta\epsilon\delta\rho} R_\alpha{}^{\epsilon;\rho}{}_\gamma - 205056 R_{\beta\gamma\epsilon\rho} R^{\alpha\beta;\gamma\delta} R_\alpha{}^{\epsilon;\rho}{}_\delta$$

$$- 96000 R_{\alpha\epsilon\beta\rho} R^{\alpha\beta;\gamma\delta} R_\gamma{}^{\epsilon;\rho}{}_\delta - 768 R^{\alpha\beta} R^{\gamma\delta} R_{\beta\gamma\delta\epsilon} R_\alpha{}^{\epsilon;\rho}{}_\rho - 302400 R_{\alpha\delta\beta\epsilon;\gamma} R^{\alpha\beta;\gamma} R^{\delta\epsilon;\rho}{}_\rho$$

$$+ 1066560 R^{\alpha\beta;\gamma} R_{\alpha\gamma\beta\delta;\epsilon} R^{\delta\epsilon;\rho}{}_\rho + 75744 R_{\alpha\delta\beta\epsilon} R^{\alpha\beta;\gamma}{}_\gamma R^{\delta\epsilon;\rho}{}_\rho + 396288 R^{\alpha\beta} R_{\beta\delta\gamma\rho;\epsilon\sigma} R_\alpha{}^{\gamma\delta\epsilon;\rho\sigma}$$

$$+ 453600 R^{\alpha\beta\gamma\delta;\epsilon} R_{\alpha\beta}{}^{\rho\sigma;\tau} R_{\gamma\delta\epsilon\rho;\sigma\tau} - 156672 R^{\alpha\beta\gamma\delta} R_{\gamma\delta\epsilon\rho;\sigma\tau} R_{\alpha\beta}{}^{\epsilon\rho;\sigma\tau} - 36864 R^{\alpha\beta\gamma\delta} R_{\beta\sigma\delta\tau;\epsilon\rho} R_\alpha{}^\epsilon{}_\gamma{}^{\rho;\sigma\tau}$$

$$+ 9600 R^{\delta\epsilon\rho\sigma} R^{\alpha\beta;\gamma} R_{\delta\epsilon\rho\sigma;\alpha\beta\gamma} + 40320 R^{\alpha\beta} R^{\gamma\delta;\epsilon} R_{\gamma\epsilon;\alpha\beta\delta} + 442176 R^{\alpha\beta} R^{\gamma\delta;\epsilon} R_{\gamma\delta;\alpha\beta\epsilon} + 281568 R R^{;\alpha} R_{;\alpha}{}^\beta{}_\beta$$

$$- 634560 R^{\beta\gamma} R^{;\alpha} R_{\beta\gamma;\alpha}{}^\delta{}_\delta + 171840 R^{\alpha\beta} R_{\beta\epsilon\gamma\rho;\delta} R^{\gamma\delta}{}_{;\alpha}{}^{\epsilon\rho} - 88608 R_{\beta\delta\gamma\epsilon} R^{\alpha\beta;\gamma} R^{\delta\epsilon}{}_{;\alpha}{}^\rho{}_\rho - 642144 R^{\gamma\delta} R_{\gamma\delta;\alpha} R^{;\alpha\beta}{}_\beta$$

$$+ 68760 R^{\gamma\delta\epsilon\rho} R_{\gamma\delta\epsilon\rho;\alpha} R^{;\alpha\beta}{}_\beta + 33888 R^{\gamma\delta} R_{\alpha\gamma;\delta} R^{;\alpha\beta}{}_\beta - 18912 R_{\alpha\gamma\delta\epsilon} R^{\gamma\delta;\epsilon} R^{;\alpha\beta}{}_\beta + 7872 R R_{\alpha\beta;\gamma} R^{;\alpha\beta\gamma}$$

$$- 11712 R_\alpha{}^\delta R_{\beta\delta;\gamma} R^{;\alpha\beta\gamma} + 30144 R_{\alpha\delta\beta\epsilon} R^{\delta\epsilon}{}_{;\gamma} R^{;\alpha\beta\gamma} + 26880 R^{\delta\epsilon} R_{\alpha\delta\beta\epsilon;\gamma} R^{;\alpha\beta\gamma} + 1056 R_\alpha{}^\delta R_{\beta\gamma;\delta} R^{;\alpha\beta\gamma}$$

$$+ 61824 R_{\alpha\delta\beta\epsilon} R_\gamma{}^{\delta;\epsilon} R^{;\alpha\beta\gamma} + 7104 R_{;\alpha\beta\gamma} R^{;\alpha\beta\gamma} - 42240 R^{\alpha\beta} R^{\gamma\delta;\epsilon} R_{\alpha\epsilon;\beta\gamma\delta} + 18864 R^{;\alpha}{}_\alpha{}^\beta R_{;\beta}{}^\gamma{}_\gamma$$

$$- 931584 R^{\alpha\beta} R^{\gamma\delta;\epsilon} R_{\alpha\gamma;\beta\delta\epsilon} - 10560 R R^{\alpha\beta;\gamma} R_{\alpha\gamma;\beta}{}^\delta{}_\delta + 23976 R^{\alpha\beta} R^{\gamma\delta}{}_{;\alpha} R_{\gamma\delta;\beta}{}^\epsilon{}_\epsilon - 42768 R^{\alpha\beta} R_\alpha{}^{\gamma;\delta} R_{\gamma\delta;\beta}{}^\epsilon{}_\epsilon$$

$$+ 6240 R_{\beta\gamma} R^{;\alpha} R^{;\beta\gamma}{}_\alpha - 21120 R_{\alpha\beta} R^{;\alpha} R^{;\beta\gamma}{}_\gamma + 467328 R^{\alpha\beta} R^{\gamma\delta;\epsilon} R_{\alpha\beta;\gamma\delta\epsilon} + 10560 R^{\beta\gamma} R^{;\alpha} R_{\alpha\beta;\gamma}{}^\delta{}_\delta$$

$$- 607680 R R^{\alpha\beta;\gamma} R_{\alpha\beta;\gamma}{}^\delta{}_\delta + 10656 R^{;\alpha\beta\gamma} R_{\alpha\beta;\gamma}{}^\delta{}_\delta + 26352 R^{\alpha\beta} R_\alpha{}^{\gamma;\delta} R_{\beta\delta;\gamma}{}^\epsilon{}_\epsilon + 522048 R_{\alpha\delta\beta\epsilon} R^{\alpha\beta;\gamma} R^{\delta\epsilon}{}_{;\gamma}{}^\rho{}_\rho$$

$$- 2736 R_{\alpha\delta;\beta}{}^\epsilon{}_\epsilon R^{\alpha\beta;\gamma}{}_\gamma{}^\delta - 47616 R_\alpha{}^{\epsilon\rho\sigma} R_{\gamma\epsilon\rho\sigma;\beta} R^{\alpha\beta;\gamma\delta}{}_\delta - 1536 R_\alpha{}^\gamma{}_\gamma{}^\sigma R_{\delta\rho\epsilon\sigma;\beta} R^{\alpha\beta;\gamma\delta\epsilon} + 273984 R R_{\alpha\gamma\beta\delta;\epsilon} R^{\alpha\beta;\gamma\delta\epsilon}$$

$$- 747648 R_{\alpha\gamma}{}^{\rho\sigma} R_{\beta\delta\rho\sigma;\epsilon} R^{\alpha\beta;\gamma\delta\epsilon} + 16896 R_\alpha{}^\rho{}_\beta{}^\sigma R_{\gamma\rho\delta\sigma;\epsilon} R^{\alpha\beta;\gamma\delta\epsilon} + 42624 R_{\gamma\delta;\alpha\beta\epsilon} R^{\alpha\beta;\gamma\delta\epsilon}$$

$$- 87552 R_{\alpha\gamma;\beta\delta\epsilon} R^{\alpha\beta;\gamma\delta\epsilon} + 43008 R_{\alpha\beta;\gamma\delta\epsilon} R^{\alpha\beta;\gamma\delta\epsilon} - 1368 R^{\alpha\beta;\gamma}{}_\gamma{}^\delta R_{\alpha\beta;\delta}{}^\epsilon{}_\epsilon - 91632 R^{\alpha\beta} R^{\gamma\delta}{}_{;\alpha} R_{\beta\gamma;\delta}{}^\epsilon{}_\epsilon$$

$$+ 1728624 R^{\alpha\beta} R_\alpha{}^{\gamma;\delta} R_{\beta\gamma;\delta}{}^\epsilon{}_\epsilon + 48000 R^{\alpha\beta} R^{\gamma\delta\epsilon\rho;\sigma} R_{\alpha\beta\epsilon;\delta\rho\sigma} + 301248 R_{\beta\delta\gamma\epsilon} R^{;\alpha} R^{\beta\gamma;\delta\epsilon}{}_\alpha$$

$$- 16416 R_{\alpha\beta\gamma\delta} R^{;\alpha} R^{\beta\gamma;\delta\epsilon}{}_\epsilon + 32448 R^{\alpha\beta} R_{\beta\delta\gamma\epsilon;\rho} R_\alpha{}^{\gamma;\delta\epsilon\rho} + 2304 R_{\beta\epsilon\delta\rho} R^{\alpha\beta;\gamma} R_\gamma{}^{\delta;\epsilon\rho}{}_\alpha$$

$$- 27936 R_{\beta\gamma\delta\epsilon} R^{\alpha\beta;\gamma} R_\alpha{}^{\delta;\epsilon\rho}{}_\rho - 43008 R_{\alpha\delta\beta\epsilon} R^{\alpha\beta;\gamma} R_\gamma{}^{\delta;\epsilon\rho}{}_\rho$$

$$
\begin{aligned}
&+ 93\,888 R^{\alpha\beta} R_{\alpha\gamma\beta\epsilon;\delta} R^{\gamma\delta;\epsilon\rho}{}_{\rho} + 379\,392 R^{\alpha\beta} R_{\alpha\gamma\beta\delta;\epsilon} R^{\gamma\delta;\epsilon\rho}{}_{\rho} + 4800 R_{\alpha\beta\gamma\delta;\epsilon\rho\sigma} R^{\alpha\beta\gamma\delta;\epsilon\rho\sigma} \\
&- 2304 R_{\gamma\delta\epsilon\rho} R^{\alpha\beta;\gamma} R^{\delta\epsilon;\rho}{}_{\alpha\beta} + 172\,800 R_{\beta\delta\epsilon\rho} R^{\alpha\beta;\gamma} R^{\delta\epsilon;\rho}{}_{\alpha\gamma} + 27\,648 R^{;\alpha\beta} R_{;\alpha\beta}{}^{\gamma}{}_{\gamma} + 27\,648 R^{\alpha\beta;\gamma\delta} R_{\gamma\delta;\alpha\beta}{}^{\epsilon}{}_{\epsilon} \\
&+ 34\,560 R^{\alpha\beta} R_{\gamma\epsilon\delta\rho} R^{\gamma\delta}{}_{;\alpha\beta}{}^{\epsilon\rho} - 58\,824 R_{\gamma\delta} R^{\gamma\delta} R^{;\alpha}{}_{\alpha}{}^{\beta}{}_{\beta} + 27\,900 R^2 R^{;\alpha}{}_{\alpha}{}^{\beta}{}_{\beta} + 8640 R_{\gamma\delta\epsilon\rho} R^{\gamma\delta\epsilon\rho} R^{;\alpha}{}_{\alpha}{}^{\beta}{}_{\beta} \\
&+ 6336 R_{\alpha\beta} R R^{;\alpha\beta\gamma}{}_{\gamma} + 24\,192 R^{\delta\epsilon} R_{\alpha\delta\beta\epsilon} R^{;\alpha\beta\gamma}{}_{\gamma} + 9216 R_{\alpha\beta}{}^{;\delta}{}_{\delta} R^{;\alpha\beta\gamma}{}_{\gamma} + 2400 R_{\alpha\beta} R_{\gamma\delta} R^{;\alpha\beta\gamma\delta} \\
&+ 1920 R_{\alpha}{}^{\epsilon}{}_{\beta}{}^{\rho} R_{\gamma\epsilon\delta\rho} R^{;\alpha\beta\gamma\delta} + 8640 R_{\alpha\beta;\gamma\delta} R^{;\alpha\beta\gamma\delta} - 60\,672 R^{\alpha\beta} R^{\gamma\delta} R_{\alpha\gamma;\beta\delta}{}^{\epsilon}{}_{\epsilon} - 62\,208 R^{\alpha\beta;\gamma\delta} R_{\alpha\gamma;\beta\delta}{}^{\epsilon}{}_{\epsilon} \\
&+ 14\,256 R^{;\alpha}{}_{\alpha} R^{;\beta}{}_{\beta}{}^{\gamma}{}_{\gamma} + 51\,072 R^{\alpha\beta} R^{\gamma\delta} R_{\alpha\beta;\gamma\delta}{}^{\epsilon}{}_{\epsilon} + 27\,648 R^{\alpha\beta;\gamma\delta} R_{\alpha\beta;\gamma\delta}{}^{\epsilon}{}_{\epsilon} - 99\,984 R^{\alpha\beta} R R_{\alpha\beta}{}^{;\gamma}{}_{\gamma}{}^{\delta}{}_{\delta} \\
&+ 3456 R^{;\alpha\beta} R_{\alpha\beta}{}^{;\gamma}{}_{\gamma}{}^{\delta}{}_{\delta} + 48\,576 R R_{\alpha\gamma\beta\delta} R^{\alpha\beta;\gamma\delta\epsilon}{}_{\epsilon} - 76\,032 R_{\alpha\gamma}{}^{\rho\sigma} R_{\beta\delta\rho\sigma} R^{\alpha\beta;\gamma\delta\epsilon}{}_{\epsilon} + 6144 R_{\alpha}{}^{\rho}{}_{\beta}{}^{\sigma} R_{\gamma\rho\delta\sigma} R^{\alpha\beta;\gamma\delta\epsilon}{}_{\epsilon} \\
&+ 69\,120 R_{\alpha\gamma\beta\delta;\epsilon\rho} R^{\alpha\beta;\gamma\delta\epsilon\rho} - 2448 R^{\alpha\beta;\gamma}{}_{\gamma} R_{\alpha\beta}{}^{;\delta}{}_{\delta}{}^{\epsilon}{}_{\epsilon} + 132\,288 R_{\alpha}{}^{\gamma} R^{\alpha\beta} R_{\beta\gamma}{}^{;\delta}{}_{\delta}{}^{\epsilon}{}_{\epsilon} + 69\,888 R^{\alpha\beta} R_{\beta\delta\gamma\epsilon} R_{\alpha}{}^{\gamma;\delta\epsilon\rho}{}_{\rho} \\
&+ 67\,488 R^{\alpha\beta} R_{\alpha\gamma\beta\delta} R^{\gamma\delta;\epsilon}{}_{\epsilon}{}^{\rho}{}_{\rho} + 19\,440 R^{;\alpha} R_{;\alpha}{}^{\beta}{}_{\beta}{}^{\gamma}{}_{\gamma} + 14\,400 R_{\alpha\beta;\gamma} R^{;\alpha\beta\gamma\delta}{}_{\delta} - 2880 R^{\alpha\beta;\gamma} R_{\alpha\gamma;\beta}{}^{\delta}{}_{\delta}{}^{\epsilon}{}_{\epsilon} \\
&- 1440 R^{\alpha\beta;\gamma} R_{\alpha\beta;\gamma}{}^{\delta}{}_{\delta}{}^{\epsilon}{}_{\epsilon} + 43\,200 R_{\alpha\gamma\beta\delta;\epsilon} R^{\alpha\beta;\gamma\delta\epsilon\rho}{}_{\rho} + 5280 R R^{;\alpha}{}_{\alpha}{}^{\beta}{}_{\beta}{}^{\gamma}{}_{\gamma} + 7200 R_{\alpha\beta} R^{;\alpha\beta\gamma}{}_{\gamma}{}^{\delta}{}_{\delta} \\
&- 960 R^{\alpha\beta} R_{\alpha\beta}{}^{;\gamma}{}_{\gamma}{}^{\delta}{}_{\delta}{}^{\epsilon}{}_{\epsilon} + 9600 R_{\alpha\gamma\beta\delta} R^{\alpha\beta;\gamma\delta\epsilon}{}_{\epsilon}{}^{\rho}{}_{\rho} + 1800 R^{;\alpha}{}_{\alpha}{}^{\beta}{}_{\beta}{}^{\gamma}{}_{\gamma}{}^{\delta}{}_{\delta}), \tag{A8a}
\end{aligned}
$$

$$
\begin{aligned}
a_5^{(1)} = \frac{1}{1\,814\,400}(&32\,736 R_{\alpha}{}^{\gamma} R^{\alpha\beta} R_{\beta}{}^{\delta} R_{\gamma\delta} R - 8436 R_{\alpha\beta} R^{\alpha\beta} R_{\gamma\delta} R^{\gamma\delta} R - 59\,136 R_{\alpha}{}^{\gamma} R^{\alpha\beta} R_{\beta\gamma} R^2 + 43\,518 R_{\alpha\beta} R^{\alpha\beta} R^3 \\
&- 5743 R^5 - 13\,944 R^{\alpha\beta} R^{\gamma\delta} R^2 R_{\alpha\gamma\beta\delta} - 3618 R^3 R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} - 168 R^{\alpha\beta} R^{\gamma\delta} R R_{\alpha\gamma}{}^{\epsilon\rho} R_{\beta\delta\epsilon\rho} \\
&+ 4480 R^2 R_{\alpha\beta}{}^{\epsilon\rho} R^{\alpha\beta\gamma\delta} R_{\gamma\delta\epsilon\rho} - 14\,832 R^{\alpha\beta} R^{\gamma\delta} R R_{\alpha}{}^{\epsilon}{}_{\beta}{}^{\rho} R_{\gamma\epsilon\delta\rho} + 3282 R_{\alpha\beta} R^{\alpha\beta} R R_{\gamma\delta\epsilon\rho} R^{\gamma\delta\epsilon\rho} \\
&+ 2496 R R_{\alpha\beta}{}^{\epsilon\rho} R^{\alpha\beta\gamma\delta} R_{\gamma\epsilon}{}^{\sigma\tau} R_{\delta\rho\sigma\tau} - 1248 R R_{\alpha\beta}{}^{\epsilon\rho} R^{\alpha\beta\gamma\delta} R_{\gamma\delta}{}^{\sigma\tau} R_{\epsilon\rho\sigma\tau} - 696 R R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} R_{\epsilon\rho\sigma\tau} R^{\epsilon\rho\sigma\tau} \\
&+ 100\,256 R^{\beta\gamma} R R_{\beta\gamma;\alpha} R^{;\alpha} - 47\,648 R_{\beta}{}^{\delta} R^{\beta\gamma} R_{\gamma\delta;\alpha} R^{;\alpha} - 25\,664 R^{\beta\gamma} R_{\beta\delta\gamma\epsilon} R^{\delta\epsilon}{}_{;\alpha} R^{;\alpha} + 19\,152 R_{\beta\gamma} R^{\beta\gamma} R_{;\alpha} R^{;\alpha} \\
&- 22\,052 R^2 R_{;\alpha} R^{;\alpha} - 2272 R_{\beta\gamma\delta\epsilon} R^{\beta\gamma\delta\epsilon} R_{;\alpha} R^{;\alpha} - 11\,240 R R^{\beta\gamma\delta\epsilon} R_{\beta\gamma\delta\epsilon;\alpha} R^{;\alpha} - 14\,560 R^{\beta\gamma} R^{\delta\epsilon} R_{\beta\delta\gamma\epsilon;\alpha} R^{;\alpha} \\
&+ 7824 R_{\beta\gamma}{}^{\rho\sigma} R^{\beta\gamma\delta\epsilon} R_{\delta\epsilon\rho\sigma;\alpha} R^{;\alpha} - 1440 R^{\alpha\beta} R R^{\gamma\delta}{}_{;\alpha} R_{\gamma\delta;\beta} + 7520 R_{\alpha}{}^{\beta} R^{\gamma\delta} R^{;\alpha} R_{\gamma\delta;\beta} - 2640 R^{\beta\gamma} R_{\alpha\delta\gamma\epsilon} R^{;\alpha} R^{\delta\epsilon}{}_{;\beta} \\
&- 360 R_{\alpha}{}^{\gamma} R_{\beta\gamma} R^{;\alpha} R^{;\beta} + 1240 R_{\alpha\beta} R R^{;\alpha} R^{;\beta} + 2640 R^{\gamma\delta} R_{\alpha\gamma\beta\delta} R^{;\alpha} R^{;\beta} + 2864 R^{\beta\gamma} R R^{;\alpha} R_{\alpha\beta;\gamma} + 2160 R^2 R_{\alpha\gamma;\beta} R^{\alpha\beta;\gamma} \\
&+ 5760 R R_{\alpha}{}^{\delta\epsilon\rho} R_{\gamma\delta\epsilon\rho;\beta} R^{\alpha\beta;\gamma} + 28\,920 R^2 R_{\alpha\beta;\gamma} R^{\alpha\beta;\gamma} - 27\,840 R R_{\alpha\delta\beta\epsilon} R^{\delta\epsilon}{}_{;\gamma} R^{\alpha\beta;\gamma} - 3840 R_{\beta}{}^{\delta} R^{\beta\gamma} R^{;\alpha} R_{\alpha\gamma;\delta} \\
&- 8480 R_{\alpha}{}^{\beta} R^{\gamma\delta} R^{;\alpha} R_{\beta\gamma;\delta} + 7680 R^{\alpha\beta} R R_{\gamma\delta;\beta} R_{\alpha}{}^{\gamma;\delta} - 4800 R^{\alpha\beta} R R_{\beta\delta;\gamma} R_{\alpha}{}^{\gamma;\delta} - 85\,440 R^{\alpha\beta} R R_{\beta\gamma;\delta} R_{\alpha}{}^{\gamma;\delta} \\
&- 1600 R R_{\alpha\beta\gamma\delta} R^{;\alpha} R^{\beta\gamma;\delta} - 4800 R_{\alpha}{}^{\epsilon}{}_{\delta}{}^{\rho} R_{\beta\epsilon\gamma\rho} R^{;\alpha} R^{\beta\gamma;\delta} + 1440 R_{\alpha\beta}{}^{\epsilon\rho} R_{\gamma\delta\epsilon\rho} R^{;\alpha} R^{\beta\gamma;\delta} - 6112 R^{\beta\gamma} R_{\beta\delta\gamma\epsilon} R^{;\alpha} R_{\alpha}{}^{\delta;\epsilon} \\
&+ 1920 R R_{\beta\gamma\delta\epsilon} R^{\alpha\beta;\gamma} R_{\alpha}{}^{\delta;\epsilon} + 11\,840 R^{\beta\gamma} R_{\alpha\delta\gamma\epsilon} R^{;\alpha} R_{\beta}{}^{\delta;\epsilon} + 1600 R^{\beta\gamma} R_{\alpha\epsilon\gamma\delta} R^{;\alpha} R_{\beta}{}^{\delta;\epsilon} + 7680 R R_{\alpha\delta\beta\epsilon} R^{\alpha\beta;\gamma} R_{\gamma}{}^{\delta;\epsilon} \\
&- 14\,400 R^{\alpha\beta} R R_{\alpha\gamma\beta\epsilon;\delta} R^{\gamma\delta;\epsilon} - 34\,080 R^{\alpha\beta} R R_{\alpha\gamma\beta\delta;\epsilon} R^{\gamma\delta;\epsilon} - 2700 R^2 R_{\alpha\beta\gamma\delta;\epsilon} R^{\alpha\beta\gamma\delta;\epsilon} + 3136 R^{\beta\gamma} R_{\beta}{}^{\delta\epsilon\rho} R^{;\alpha} R_{\alpha\delta\gamma\epsilon;\rho} \\
&- 960 R_{\alpha}{}^{\beta} R^{\gamma\delta\epsilon\rho} R^{;\alpha} R_{\beta\gamma\delta\epsilon;\rho} - 12\,960 R^{\alpha\beta} R R_{\beta\delta\gamma\rho;\epsilon} R_{\alpha}{}^{\gamma\delta\epsilon;\rho} + 10\,800 R R^{\alpha\beta\gamma\delta} R_{\gamma\delta\epsilon\rho;\sigma} R_{\alpha\beta}{}^{\epsilon\rho;\sigma} - 3880 R^{;\alpha} R^{;\beta} R_{;\alpha\beta} \\
&+ 960 R^{;\alpha} R^{\beta\gamma;\delta} R_{\beta\delta;\alpha\gamma} - 480 R^{;\alpha} R^{\beta\gamma;\delta} R_{\beta\gamma;\alpha\delta} - 8608 R_{\beta}{}^{\delta} R^{\beta\gamma} R_{\gamma\delta} R^{;\alpha}{}_{\alpha} + 16\,440 R_{\beta\gamma} R^{\beta\gamma} R R^{;\alpha}{}_{\alpha} - 5064 R^3 R^{;\alpha}{}_{\alpha} \\
&- 7680 R^{\beta\gamma} R^{\delta\epsilon} R_{\beta\delta\gamma\epsilon} R^{;\alpha}{}_{\alpha} - 2484 R R_{\beta\gamma\delta\epsilon} R^{\beta\gamma\delta\epsilon} R^{;\alpha}{}_{\alpha} + 1912 R_{\beta\gamma}{}^{\rho\sigma} R^{\beta\gamma\delta\epsilon} R_{\delta\epsilon\rho\sigma} R^{;\alpha}{}_{\alpha} + 240 R_{\beta\delta;\gamma} R^{\beta\gamma;\delta} R^{;\alpha}{}_{\alpha} \\
&+ 600 R_{\beta\gamma;\delta} R^{\beta\gamma;\delta} R^{;\alpha}{}_{\alpha} - 660 R_{\beta\gamma\delta\epsilon;\rho} R^{\beta\gamma\delta\epsilon;\rho} R^{;\alpha}{}_{\alpha} - 12\,352 R_{\alpha}{}^{\gamma} R_{\beta}{}^{\delta} R_{\gamma\delta} R^{;\alpha\beta} + 7056 R_{\alpha\beta} R_{\gamma\delta} R^{\gamma\delta} R^{;\alpha\beta} \\
&+ 13\,888 R_{\alpha}{}^{\gamma} R_{\beta\gamma} R R^{;\alpha\beta} - 6696 R_{\alpha\beta} R^2 R^{;\alpha\beta} + 1184 R^{\gamma\delta} R R_{\alpha\gamma\beta\delta} R^{;\alpha\beta} - 2240 R_{\beta}{}^{\gamma} R^{\delta\epsilon} R_{\alpha\delta\gamma\epsilon} R^{;\alpha\beta} \\
&- 6912 R^{\gamma\delta} R_{\alpha}{}^{\epsilon}{}_{\gamma}{}^{\rho} R_{\beta\epsilon\delta\rho} R^{;\alpha\beta} + 4544 R^{\gamma\delta} R_{\alpha}{}^{\epsilon}{}_{\beta}{}^{\rho} R_{\gamma\epsilon\delta\rho} R^{;\alpha\beta} - 816 R_{\alpha\beta} R_{\gamma\delta\epsilon\rho} R^{\gamma\delta\epsilon\rho} R^{;\alpha\beta} - 80 R^{\gamma\delta}{}_{;\alpha} R_{\gamma\delta;\beta} R^{;\alpha\beta} \\
&- 600 R^{\gamma\delta\epsilon\rho}{}_{;\alpha} R_{\gamma\delta\epsilon\rho;\beta} R^{;\alpha\beta} - 640 R_{\gamma\delta;\beta} R_{\alpha}{}^{\gamma;\delta} R^{;\alpha\beta} + 1440 R_{\beta\delta;\gamma} R_{\alpha}{}^{\gamma;\delta} R^{;\alpha\beta} - 480 R_{\beta\gamma;\delta} R_{\alpha}{}^{\gamma;\delta} R^{;\alpha\beta} \\
&+ 640 R_{\alpha\gamma\delta\epsilon;\beta} R^{\gamma\delta;\epsilon} R^{;\alpha\beta} - 1920 R_{\alpha\gamma\beta\delta;\epsilon} R^{\gamma\delta;\epsilon} R^{;\alpha\beta} - 1728 R^{\gamma\delta} R_{\gamma\delta;\alpha\beta} R^{;\alpha\beta} - 2376 R R_{;\alpha\beta} R^{;\alpha\beta} \\
&- 640 R^{\gamma\delta\epsilon\rho} R_{\gamma\delta\epsilon\rho;\alpha\beta} R^{;\alpha\beta} - 480 R^{;\alpha} R^{\beta\gamma;\delta} R_{\alpha\delta;\beta\gamma} + 11\,136 R^{\alpha\beta} R^{\gamma\delta} R R_{\alpha\gamma;\beta\delta} - 3156 R_{;\alpha} R^{;\alpha} R^{;\beta}{}_{\beta} \\
&- 2880 R R^{;\alpha}{}_{\alpha} R^{;\beta}{}_{\beta} - 3840 R_{\beta\gamma;\alpha} R^{;\alpha} R^{;\beta\gamma} + 7952 R^{;\alpha} R_{\alpha\beta;\gamma} R^{;\beta\gamma} - 1008 R_{\beta\gamma} R^{;\alpha}{}_{\alpha} R^{;\beta\gamma} - 10\,464 R^{\alpha\beta} R^{\gamma\delta} R R_{\alpha\beta;\gamma\delta} \\
&+ 960 R^{;\alpha} R^{\beta\gamma;\delta} R_{\alpha\beta;\gamma\delta} - 1728 R^{\gamma\delta} R^{;\alpha\beta} R_{\alpha\beta;\gamma\delta} + 3328 R_{\alpha\gamma} R^{;\alpha\beta} R^{;\gamma}{}_{\beta} + 18\,000 R^{\alpha\beta} R^2 R_{\alpha\beta}{}^{;\gamma}{}_{\gamma} + 1800 R^{;\alpha} R^{;\beta} R_{\alpha\beta}{}^{;\gamma}{}_{\gamma}
\end{aligned}
$$

$$
\begin{aligned}
&- 720RR^{;\alpha\beta}R_{\alpha\beta}{}^{;\gamma}{}_{\gamma} - 8352R^2R_{\alpha\gamma\beta\delta}R^{\alpha\beta;\gamma\delta} + 14\,016RR_{\alpha\gamma}{}^{\epsilon\rho}R_{\beta\delta\epsilon\rho}R^{\alpha\beta;\gamma\delta} - 384RR_{\alpha}{}^{\epsilon}{}_{\beta}{}^{\rho}R_{\gamma\epsilon\delta\rho}R^{\alpha\beta;\gamma\delta} \\
&- 1872RR_{\gamma\delta;\alpha\beta}R^{\alpha\beta;\gamma\delta} + 4032RR_{\alpha\gamma;\beta\delta}R^{\alpha\beta;\gamma\delta} - 1872RR_{\alpha\beta;\gamma\delta}R^{\alpha\beta;\gamma\delta} - 5856R_{\alpha\gamma\beta\delta}R^{;\alpha\beta}R^{;\gamma\delta} \\
&+ 4096R^{\gamma\delta}R^{;\alpha\beta}R_{\alpha\gamma;\delta\beta} - 2304R^{\alpha\beta}RR^{\gamma\delta\epsilon\rho}R_{\alpha\gamma\beta\epsilon;\delta\rho} + 216RR^{\alpha\beta;\gamma}{}_{\gamma}R_{\alpha\beta}{}^{;\delta}{}_{\delta} - 23\,904R_{\alpha}{}^{\gamma}R^{\alpha\beta}RR_{\beta\gamma}{}^{;\delta}{}_{\delta} \\
&+ 240R^{\beta\gamma}{}_{;\alpha}R^{;\alpha}R_{\beta\gamma}{}^{;\delta}{}_{\delta} + 384R^{;\alpha}R_{\alpha}{}^{\beta;\gamma}R_{\beta\gamma}{}^{;\delta}{}_{\delta} + 1248R^{\beta\gamma}R^{;\alpha}{}_{\alpha}R_{\beta\gamma}{}^{;\delta}{}_{\delta} - 480R_{\alpha}{}^{\gamma}R^{;\alpha\beta}R_{\beta\gamma}{}^{;\delta}{}_{\delta} \\
&- 8448R^{\alpha\beta}RR_{\beta\delta\gamma\epsilon}R_{\alpha}{}^{\gamma;\delta\epsilon} - 3744R^{;\alpha}R_{\alpha\delta\beta\epsilon;\gamma}R^{\beta\gamma;\delta\epsilon} + 4320R^{;\alpha}R_{\alpha\beta\gamma\delta;\epsilon}R^{\beta\gamma;\delta\epsilon} - 3648R_{\beta\delta\gamma\epsilon}R^{;\alpha}{}_{\alpha}R^{\beta\gamma;\delta\epsilon} \\
&- 5760R^{;\alpha}R^{\beta\gamma\delta\epsilon;\rho}R_{\alpha\beta\gamma\delta;\epsilon\rho} + 512R_{\alpha\gamma\delta\epsilon}R^{;\alpha\beta}R^{\gamma\delta;\epsilon}{}_{\beta} - 12\,888R^{\alpha\beta}RR_{\alpha\gamma\beta\delta}R^{\gamma\delta;\epsilon}{}_{\epsilon} - 1440R_{\alpha\gamma\beta\delta}R^{;\alpha\beta}R^{\gamma\delta;\epsilon}{}_{\epsilon} \\
&- 576RR_{\alpha\beta\gamma\delta;\epsilon\rho}R^{\alpha\beta\gamma\delta;\epsilon\rho} - 7080RR^{;\alpha}R_{;\alpha}{}^{\beta}{}_{\beta} + 240R^{\beta\gamma}R^{;\alpha}R_{\beta\gamma;\alpha}{}^{\delta}{}_{\delta} + 1200R^{\gamma\delta}R_{\gamma\delta;\alpha}R^{;\alpha\beta}{}_{\beta} \\
&- 1120R^{\gamma\delta\epsilon\rho}R_{\gamma\delta\epsilon\rho;\alpha}R^{;\alpha\beta}{}_{\beta} + 640R^{\gamma\delta}R_{\alpha\gamma;\delta}R^{;\alpha\beta}{}_{\beta} + 640R_{\alpha\gamma\delta\epsilon}R^{\gamma\delta;\epsilon}R^{;\alpha\beta}{}_{\beta} - 1200RR_{\alpha\beta;\gamma}R^{;\alpha\beta\gamma} \\
&+ 800R_{\alpha}{}^{\delta}R_{\beta\delta;\gamma}R^{;\alpha\beta\gamma} - 2080R_{\alpha\delta\beta\epsilon}R^{\delta\epsilon}{}_{;\gamma}R^{;\alpha\beta\gamma} - 2400R^{\delta\epsilon}R_{\alpha\delta\beta\epsilon;\gamma}R^{;\alpha\beta\gamma} - 80R_{\alpha}{}^{\delta}R_{\beta\gamma;\delta}R^{;\alpha\beta\gamma} \\
&- 4160R_{\alpha\delta\beta\epsilon}R_{\gamma}{}^{\delta;\epsilon}R^{;\alpha\beta\gamma} - 960R_{;\alpha\beta\gamma}R^{;\alpha\beta\gamma} - 2640R^{;\alpha}{}_{\alpha}R_{;\beta}{}^{\gamma}{}_{\gamma} + 480RR^{\alpha\beta;\gamma}R_{\alpha\gamma;\beta}{}^{\delta}{}_{\delta} - 1360R_{\beta\gamma}R^{;\alpha}R^{;\beta\gamma}{}_{\alpha} \\
&+ 2480R_{\alpha\beta}R^{;\alpha}R^{;\beta\gamma}{}_{\gamma} + 480R^{\beta\gamma}R^{;\alpha}R_{\alpha\beta;\gamma}{}^{\delta}{}_{\delta} + 240RR^{\alpha\beta;\gamma}R_{\alpha\beta;\gamma}{}^{\delta}{}_{\delta} - 720R^{;\alpha\beta\gamma}R_{\alpha\beta;\gamma}{}^{\delta}{}_{\delta} - 5760RR_{\alpha\gamma\beta\delta;\epsilon}R^{\alpha\beta;\gamma\delta\epsilon} \\
&- 3840R_{\beta\delta\gamma\epsilon}R^{;\alpha}R^{\beta\gamma;\delta\epsilon}{}_{\alpha} + 480R_{\alpha\beta\gamma\delta}R^{;\alpha}R^{\beta\gamma;\delta\epsilon}{}_{\epsilon} - 3792R^{;\alpha\beta}R_{;\alpha\beta}{}^{\gamma}{}_{\gamma} + 584R_{\gamma\delta}R^{\gamma\delta}R^{;\alpha}{}_{\alpha}{}^{\beta}{}_{\beta} - 1604R^2R^{;\alpha}{}_{\alpha}{}^{\beta}{}_{\beta} \\
&- 272R_{\gamma\delta\epsilon\rho}R^{\gamma\delta\epsilon\rho}R^{;\alpha}{}_{\alpha}{}^{\beta}{}_{\beta} - 1264R_{\alpha\beta}RR^{;\alpha\beta\gamma}{}_{\gamma} - 1888R^{\delta\epsilon}R_{\alpha\delta\beta\epsilon}R^{;\alpha\beta\gamma}{}_{\gamma} - 624R_{\alpha\beta}{}^{;\delta}{}_{\delta}R^{;\alpha\beta\gamma}{}_{\gamma} - 160R_{\alpha\beta}R_{\gamma\delta}R^{;\alpha\beta\gamma\delta} \\
&- 128R_{\alpha}{}^{\epsilon}{}_{\beta}{}^{\rho}R_{\gamma\epsilon\delta\rho}R^{;\alpha\beta\gamma\delta} - 576R_{\alpha\beta;\gamma\delta}R^{;\alpha\beta\gamma\delta} - 2088R^{;\alpha}{}_{\alpha}R^{;\beta}{}_{\beta}{}^{\gamma}{}_{\gamma} + 240R^{\alpha\beta}RR_{\alpha\beta}{}^{;\gamma}{}_{\gamma}{}^{\delta}{}_{\delta} - 240R^{;\alpha\beta}R_{\alpha\beta}{}^{;\gamma}{}_{\gamma}{}^{\delta}{}_{\delta} \\
&- 1920RR_{\alpha\gamma\beta\delta}R^{\alpha\beta;\gamma\delta\epsilon}{}_{\epsilon} - 2760R^{;\alpha}R_{;\alpha}{}^{\beta}{}_{\beta}{}^{\gamma}{}_{\gamma} - 960R_{\alpha\beta;\gamma}R^{;\alpha\beta\gamma\delta}{}_{\delta} - 840RR^{;\alpha}{}_{\alpha}{}^{\beta}{}_{\beta}{}^{\gamma}{}_{\gamma} - 480R_{\alpha\beta}R^{;\alpha\beta\gamma}{}_{\gamma}{}^{\delta}{}_{\delta} \\
&- 120R^{;\alpha}{}_{\alpha}{}^{\beta}{}_{\beta}{}^{\gamma}{}_{\gamma}{}^{\delta}{}_{\delta}),
\end{aligned}
\tag{A8b}
$$

$$
\begin{aligned}
a_5^{(2)} = \frac{1}{30\,240}\big(&584R_{\alpha}{}^{\gamma}R^{\alpha\beta}R_{\beta\gamma}R^2 - 654R_{\alpha\beta}R^{\alpha\beta}R^3 + 99R^5 + 456R^{\alpha\beta}R^{\gamma\delta}R^2R_{\alpha\gamma\beta\delta} + 72R^3R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} \\
&- 80R^2R_{\alpha\beta}{}^{\epsilon\rho}R^{\alpha\beta\gamma\delta}R_{\gamma\delta\epsilon\rho} - 24R^{\beta\gamma}RR_{\beta\gamma;\alpha}R^{;\alpha} - 26R_{\beta\gamma}R^{\beta\gamma}R_{;\alpha}R^{;\alpha} + 257R^2R_{;\alpha}R^{;\alpha} + 17R_{\beta\gamma\delta\epsilon}R^{\beta\gamma\delta\epsilon}R_{;\alpha}R^{;\alpha} \\
&+ 72RR^{\beta\gamma\delta\epsilon}R_{\beta\gamma\delta\epsilon;\alpha}R^{;\alpha} + 90R_{\alpha}{}^{\gamma}R_{\beta\gamma}R^{;\alpha}R^{;\beta} - 300R_{\alpha\beta}RR^{;\alpha}R^{;\beta} + 48R^{\gamma\delta}R_{\alpha\gamma\beta\delta}R^{;\alpha}R^{;\beta} - 48R^{\beta\gamma}RR^{;\alpha}R_{\alpha\beta;\gamma} \\
&- 12R^2R_{\alpha\gamma;\beta}R^{\alpha\beta;\gamma} - 6R^2R_{\alpha\beta;\gamma}R^{\alpha\beta;\gamma} - 48RR_{\alpha\beta\gamma\delta}R^{;\alpha}R^{\beta\gamma;\delta} + 27R^2R_{\alpha\beta\gamma\delta;\epsilon}R^{\alpha\beta\gamma\delta;\epsilon} + 312R^{;\alpha}R^{;\beta}R_{;\alpha\beta} \\
&- 60R_{\beta\gamma}R^{\beta\gamma}RR^{;\alpha}{}_{\alpha} + 162R^3R^{;\alpha}{}_{\alpha} + 36RR_{\beta\gamma\delta\epsilon}R^{\beta\gamma\delta\epsilon}R^{;\alpha}{}_{\alpha} + 60R_{\alpha\beta}R^2R^{;\alpha\beta} + 144R^{\gamma\delta}RR_{\alpha\gamma\beta\delta}R^{;\alpha\beta} \\
&+ 200RR_{;\alpha\beta}R^{;\alpha\beta} + 252R_{;\alpha}R^{;\alpha}R^{;\beta}{}_{\beta} + 238RR^{;\alpha}{}_{\alpha}R^{;\beta}{}_{\beta} + 168R_{\beta\gamma;\alpha}R^{;\alpha}R^{;\beta\gamma} - 312R^{;\alpha}R_{\alpha\beta;\gamma}R^{;\beta\gamma} \\
&+ 56R_{\beta\gamma}R^{;\alpha}{}_{\alpha}R^{;\beta\gamma} - 128R_{\alpha\gamma}R^{;\alpha\beta}R^{;\gamma}{}_{\beta} - 24R^{\alpha\beta}R^2R_{\alpha\beta}{}^{;\gamma}{}_{\gamma} - 72R^{;\alpha}R^{;\beta}R_{\alpha\beta}{}^{;\gamma}{}_{\gamma} + 48RR^{;\alpha\beta}R_{\alpha\beta}{}^{;\gamma}{}_{\gamma} \\
&+ 144R^2R_{\alpha\gamma\beta\delta}R^{\alpha\beta;\gamma\delta} + 224R_{\alpha\gamma\beta\delta}R^{;\alpha\beta}R^{;\gamma\delta} + 588RR^{;\alpha}R_{;\alpha}{}^{\beta}{}_{\beta} + 72RR_{\alpha\beta;\gamma}R^{;\alpha\beta\gamma} + 36R_{;\alpha\beta\gamma}R^{;\alpha\beta\gamma} \\
&+ 102R^{;\alpha}{}_{\alpha}{}^{\beta}R_{;\beta}{}^{\gamma}{}_{\gamma} + 72R_{\beta\gamma}R^{;\alpha}R^{;\beta\gamma}{}_{\alpha} - 102R_{\alpha\beta}R^{;\alpha}R^{;\beta\gamma}{}_{\gamma} + 144R^{;\alpha\beta}R_{;\alpha\beta}{}^{\gamma}{}_{\gamma} + 138R^2R^{;\alpha}{}_{\alpha}{}^{\beta}{}_{\beta} \\
&+ 72R_{\alpha\beta}RR^{;\alpha\beta\gamma}{}_{\gamma} + 84R^{;\alpha}{}_{\alpha}R^{;\beta}{}_{\beta}{}^{\gamma}{}_{\gamma} + 108R^{;\alpha}R_{;\alpha}{}^{\beta}{}_{\beta}{}^{\gamma}{}_{\gamma} + 36RR^{;\alpha}{}_{\alpha}{}^{\beta}{}_{\beta}{}^{\gamma}{}_{\gamma}),
\end{aligned}
\tag{A8c}
$$

$$
\begin{aligned}
a_5^{(3)} = \frac{1}{2160}\big(&2R_{\alpha\beta}R^{\alpha\beta}R^3 - 5R^5 - 2R^3R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - 66R^2R_{;\alpha}R^{;\alpha} + 36R_{\alpha\beta}RR^{;\alpha}R^{;\beta} - 36R^{;\alpha}R^{;\beta}R_{;\alpha\beta} - 42R^3R^{;\alpha}{}_{\alpha} \\
&- 12R_{\alpha\beta}R^2R^{;\alpha\beta} - 24RR_{;\alpha\beta}R^{;\alpha\beta} - 30R_{;\alpha}R^{;\alpha}R^{;\beta}{}_{\beta} - 30R^{;\alpha}{}_{\alpha}R^{;\beta}{}_{\beta} - 72RR^{;\alpha}R_{;\alpha}{}^{\beta}{}_{\beta} - 18R^2R^{;\alpha}{}_{\alpha}{}^{\beta}{}_{\beta}),
\end{aligned}
\tag{A8d}
$$

$$
a_5^{(4)} = \frac{1}{144}(R^5 + 6R^2R_{;\alpha}R^{;\alpha} + 4R^3R^{;\alpha}{}_{\alpha}), \qquad a_5^{(5)} = -\frac{1}{120}R^5,
\tag{A8e}
$$

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