

General maximum entropy principle for self-gravitating perfect fluid

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We consider a self-gravitating system consisting of perfect fluid with spherical symmetry. Using the general expression of entropy density, we extremize the total entropy S under the constraint that the total number of particles is fixed. We show that extrema of S coincides precisely with the relativistic Tolman-Oppenheimer-Volkoff equation of hydrostatic equilibrium. Furthermore, we apply the maximum entropy principle to a charged perfect fluid and derive the generalized Tolman-Oppenheimer-Volkoff equation. Our work provides strong evidence for the fundamental relationship between general relativity and ordinary thermodynamics.

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I. INTRODUCTION

In the past few decades, research in general relativity has suggested a very deep connection between gravitation and thermodynamics. The four laws of black hole mechanics were originally derived from the Einstein equation at the purely classical level [1,2]. The discovery of the Hawking radiation [3] allows a consistent interpretation of the laws of black hole mechanics as the ordinary laws of thermodynamics. By turning the logic around, Jacobson [4] showed that the Einstein equation may be derived from the first law of local Rindler horizons. Inspired by Jacobson's work, a lot of efforts have been made to derive the dynamical equations from black hole thermodynamics [5–9]. In fact, this idea can be traced back even before the establishment of black hole mechanics. In 1965, Cocks [10] proposed a maximum entropy principle for self-gravitating fluid spheres. Let S be the total entropy of spherically symmetric perfect fluid. Cocks showed that the requirement that S be an extremum yields the equation of hydrostatic equilibrium which was originally derived from the Einstein equation. However, a critical assumption in Cocks's derivation is that the fluid is in adiabatic motion so that the total entropy is invariant. By imposing the adiabatic condition, the entropy density s is expressed as the function of the energy density ρ , while for a general fluid, s is a function of at least two thermodynamic variables. In my opinion, the variation of S is performed on a spacelike hypersurface and the dynamic evolution of the fluid is irrelevant. Furthermore, variation of S is not consistent with the adiabatic condition. If the entropy is required to be invariant, as indicated by the adiabatic condition, the variation of entropy would be meaningless. Thus, it is not appropriate and consistent to impose the adiabatic condition. In relation to Cocks's work, Sorkin, Wald and Zhang (SWZ) [11] developed a different entropy principle for radiation. The major difference is that the adiabatic condition was not needed in SWZ's derivation.

Moreover, only the Einstein constraint equation was used in the proof while Cocks used both the constraint equation and the radial-radial component of Einstein's equation. However, SWZ's discussion was restricted to radiation for which the thermodynamic relations can be expressed explicitly and the entropy density only depends on one thermodynamic variable. It is important to know whether SWZ's treatment can be generalized to an arbitrary perfect fluid. In this paper, we prove a maximum entropy principle for a general self-gravitating perfect fluid. The new arguments used in our proof are as follows. First, we use the Gibbs-Duhem relation as the expression of entropy density for a general fluid. Second, our maximum entropy principle is under the constraint that the total number of particles is invariant. Consequently, the method of Lagrange multipliers plays an important role in our derivation. Third, in addition to the Einstein constraint equation, we only make use of the ordinary thermodynamic relations to derive the Tolman-Oppenheimer-Volkoff (TOV) equation. No other assumptions are needed. Finally, we extend our treatment to a general charged fluid. With modified arguments, we derive the generalized TOV equation for the charged fluid.

II. REVIEW OF SWZ'S DERIVATION ON SELF-GRAVITATING RADIATION

Since our work is closely related to SWZ's prescription, we shall give a brief review on the derivation in [11]. Consider a spherical box of radiation having total energy M and confined within a radius R . For thermal radiation, the pressure p and energy density ρ satisfy the equation of state

$$p = \frac{1}{3}\rho, \quad (1)$$

and then the stress energy tensor is given by

$$T_{ab} = \rho u_a u_b + \frac{1}{3}\rho(g_{ab} + u_a u_b), \quad (2)$$

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where u^a is the 4-velocity of the local rest frame of the radiation. In terms of the locally measured temperature T , the energy density ρ and entropy density s are given by

$$\rho = bT^4, \quad (3)$$

$$s = \frac{4}{3}bT^3, \quad (4)$$

where b is a constant. So s can also be expressed as

$$s = \alpha\rho^{3/4} \quad (5)$$

with $\alpha = \frac{4}{3}b^{1/4}$. As shown by SWZ, the extrema of the total entropy S corresponds to a static spacetime metric

$$ds^2 = g_{tt}(r)dt^2 + \left[1 - \frac{2m(r)}{r}\right]^{-1} dr^2 + r^2 d\Omega^2. \quad (6)$$

The constraint equation, which is obtained from the time-time component of the Einstein equation, yields

$$\rho = \frac{m'(r)}{4\pi r^2}. \quad (7)$$

Thus, $m(r)$ is a mass function.

Let the gas be confined in the region $r \leq R$. Then the total entropy is given by

$$\begin{aligned} S &= 4\pi \int_0^R s(r) \left[1 - \frac{2m(r)}{r}\right]^{-1/2} r^2 dr \\ &= 4\pi\alpha \int_0^R \rho^{3/4} \left[1 - \frac{2m(r)}{r}\right]^{-1/2} r^2 dr \\ &= (4\pi)^{1/4} \alpha \int_0^R \left[\frac{1}{r^2} m'(r)\right]^{3/4} \left[1 - \frac{2m(r)}{r}\right]^{-1/2} r^2 dr. \end{aligned} \quad (8)$$

Our task is to find a function $m(r)$ such that the total entropy S is extremized. Note that the total mass M within R is

$$M = 4\pi \int_0^R \rho(r) r^2 dr = m(R), \quad (9)$$

and obviously

$$m(0) = 0. \quad (10)$$

Hence, all the variations must satisfy

$$\delta m(0) = \delta m(R) = 0. \quad (11)$$

By using this condition, the extrema of S is equivalent to the Euler-Lagrange equation

$$\frac{d}{dr} \left(\frac{\partial L}{\partial m'} \right) - \frac{\partial L}{\partial m} = 0 \quad (12)$$

for the Lagrangian

$$L = (m')^{3/4} \left[1 - \frac{2m(r)}{r}\right]^{-1/2} r^{1/2}. \quad (13)$$

By straightforward calculation, Eq. (12) yields

$$-\frac{3}{16} m'' r^2 + \frac{3}{8} m'' m r + \frac{3}{8} m' r - \frac{1}{4} m'^2 r - \frac{3}{2} m' m = 0. \quad (14)$$

By substituting Eq. (7), one can show that Eq. (14) is equivalent to

$$\frac{d}{dr} (\rho/3) = -\frac{(\rho + \rho/3)[m(r) + 4\pi r^3(\rho/3)]}{r[r - 2m(r)]}. \quad (15)$$

Since $p = \rho/3$ for radiation, we see immediately that Eq. (15) is just the relativistic Tolman-Oppenheimer-Volkoff equation.

III. MAXIMUM ENTROPY PRINCIPLE FOR PERFECT FLUID

To generalize SWZ's prescription to an arbitrary perfect fluid, we first need to find a formula for entropy density s . Because radiation has a vanishing chemical potential, its entropy density depends on only one thermodynamic variable, e.g. T or ρ . For fluids consisting of particles, there are at least two independent variables. We start with the familiar first law

$$dS = \frac{1}{T} dE + \frac{p}{T} dV - \frac{\mu}{T} dN, \quad (16)$$

where S , E , N represent the total entropy, energy, and particle number within the volume V . Write Eq. (16) in terms of densities

$$d(sV) = \frac{1}{T} d(\rho V) + \frac{p}{T} dV - \frac{\mu}{T} d(nV). \quad (17)$$

By expansion, we have

$$sdV + Vds = \frac{1}{T} \rho dV + Vd\rho + \frac{p}{T} dV - \frac{\mu}{T} ndV - \frac{\mu}{T} Vdn. \quad (18)$$

Applying Eq. (16) to a unit volume, we find

$$ds = \frac{1}{T} d\rho - \frac{\mu}{T} dn. \quad (19)$$

Combining Eqs. (18) and (19), we arrive at the integrated form of the Gibbs-Duhem relation [12]

$$s = \frac{1}{T} (\rho + p - \mu n). \quad (20)$$

To derive this formula, we only used the first law of the ordinary thermodynamics. So it is a general expression for perfect fluid. We treat (ρ, n) as two independent variables, e.g.,

$$s = s(\rho, n), \quad \mu = \mu(\rho, n). \quad (21)$$

The equation of state is assumed to be

$$p = p(\rho). \tag{22}$$

For example, the thermodynamic quantities for a monatomic ideal gas are given by [13]

$$\rho = \frac{3}{2}nkT, \tag{23}$$

$$p = nkT, \tag{24}$$

$$s = \frac{3}{2}nk \ln T - nk \ln n + \frac{3}{2}nk \left[\frac{5}{3} + \ln \left(\frac{2\pi mk}{h^2} \right) \right]. \tag{25}$$

Our task is to extremize the total entropy

$$S = 4\pi \int_0^R s(r) \left[1 - \frac{2m(r)}{r} \right]^{-1/2} r^2 dr. \tag{26}$$

In addition to the constraint Eq. (11), it is natural to require the total number of particles

$$N = 4\pi \int_0^R n(r) \left[1 - \frac{2m(r)}{r} \right]^{-1/2} r^2 dr \tag{27}$$

to be invariant, i.e.,

$$\delta N = 0. \tag{28}$$

Following the standard method of Lagrange multipliers, the equation of variation becomes

$$\delta S + \lambda \delta N = 0. \tag{29}$$

Define the “total Lagrangian” by

$$L(m, m', n) = s(\rho(m'), n) \left[1 - \frac{2m(r)}{r} \right]^{-1/2} r^2 + \lambda n(r) \left[1 - \frac{2m(r)}{r} \right]^{-1/2} r^2. \tag{30}$$

Now the constrained Euler-Lagrange equation is given by

$$\frac{\partial L}{\partial n} = 0, \tag{31}$$

$$\frac{d}{dr} \frac{\partial L}{\partial m'} + \frac{\partial L}{\partial m} = 0. \tag{32}$$

Thus, Eq. (31) yields

$$\frac{\partial s}{\partial n} + \lambda = 0. \tag{33}$$

Using Eq. (19), we have

$$-\frac{\mu}{T} + \lambda = 0, \tag{34}$$

which shows that $\frac{\mu}{T}$ must be a constant for self-gravitating fluid.

From Eq. (30), we have

$$\frac{\partial L}{\partial m} = r \left(1 - \frac{2m}{r} \right)^{-3/2} (n\lambda + s), \tag{35}$$

and

$$\frac{\partial L}{\partial m'} = \frac{\partial s}{\partial m'} r^2 \left(1 - \frac{2m}{r} \right)^{-1/2}. \tag{36}$$

Here

$$\frac{\partial s}{\partial m'} = \frac{\partial s}{\partial \rho} \frac{\partial \rho}{\partial m'} = \frac{1}{T} \frac{1}{4\pi r^2}, \tag{37}$$

where Eqs. (7) and (19) have been used. Hence,

$$\frac{\partial L}{\partial m'} = \frac{1}{4\pi T} \left(1 - \frac{2m}{r} \right)^{-1/2}, \tag{38}$$

and

$$\frac{d}{dr} \frac{\partial L}{\partial m'} = \frac{T(m'r - m) - r(r - 2m)T'}{4\pi T^2 (r - 2m)^{3/2} r^2}. \tag{39}$$

Using Eqs. (34), (20), and (35) becomes

$$\frac{\partial L}{\partial m} = r \left(1 - \frac{2m}{r} \right)^{-3/2} \left(\frac{\rho + p}{T} \right). \tag{40}$$

So the Euler-Lagrange Eq. (32) yields

$$(4\pi p r^3 + m)T + (r - 2m)rT' = 0. \tag{41}$$

Since there are two independent variables, it is convenient to assume $T = T(\rho, \mu)$. Then

$$T'(r) = \frac{\partial T}{\partial \rho} \Big|_{\mu} \rho' + \frac{\partial T}{\partial \mu} \Big|_{\rho} \mu'. \tag{42}$$

The constraint Eq. (34) yields

$$\mu' = \lambda T'. \tag{43}$$

Substitution in Eq. (42) gives

$$T' = \left(1 - \lambda \frac{\partial T}{\partial \mu} \Big|_{\rho} \right)^{-1} \frac{\partial T}{\partial \rho} \Big|_{\mu} \rho'. \tag{44}$$

Rewrite Eq. (20) as

$$p = Ts + \mu n - \rho. \tag{45}$$

The differential of p is

$$dp = Tds + sdT + \mu dn + nd\mu - d\rho. \tag{46}$$

By substituting Eq. (19), we have

$$dp = sdT + nd\mu. \tag{47}$$

Since

$$p = p(\rho), \tag{48}$$

Eq. (47) yields

$$\left. \frac{\partial T}{\partial \mu} \right|_{\rho} = -\frac{n}{s}, \quad (49)$$

$$\left. \frac{\partial T}{\partial \rho} \right|_{\mu} = \frac{p'(\rho)}{s}. \quad (50)$$

Hence, Eq. (44) becomes

$$T' = \left(1 + \frac{\mu}{T} \frac{n}{s}\right)^{-1} \frac{p'(\rho)}{s} \rho'(r) = \frac{T}{p + \rho} p'(r). \quad (51)$$

Substituting Eq. (51) into Eq. (41), we obtain the desired TOV equation

$$p' = -\frac{(p + \rho)(4\pi r^3 p + m)}{r(r - 2m)}. \quad (52)$$

IV. MAXIMUM ENTROPY PRINCIPLE FOR CHARGED FLUID

For a charged fluid, the local thermodynamic relations remain unchanged. For example, we can still use Eq. (20) as the expression of entropy density. But the presence of charge will change the distribution of the fluid in spacetime.

In coordinates (t, r, θ, ϕ) , assume that a spherically symmetric charged fluid has the line element

$$ds^2 = g_{tt}(r)dt^2 + \left[1 - \frac{2m(r)}{r} + \frac{Q^2(r)}{r^2}\right]^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2. \quad (53)$$

Here $Q(r)$ is defined as the total charge up to the radius r and $m(r)$ will be determined later. The matter field consists of a charged fluid. Let

$$u^a = \frac{1}{\sqrt{-g_{tt}}}\left(\frac{\partial}{\partial t}\right)^a \quad (54)$$

be the four-velocity of the fluid. Then the total stress-energy tensor can be written as

$$T_{ab} = \tilde{T}_{ab} + T_{ab}^{\text{EM}}, \quad (55)$$

where

$$\tilde{T}_{ab} = \rho u^a u^b + p(g_{ab} + u_a u_b), \quad (56)$$

$$T_{ab}^{\text{EM}} = \frac{1}{4\pi} \left(F_a{}^c F_{bc} - \frac{1}{4} g_{ab} F^{cd} F_{cd} \right). \quad (57)$$

The electromagnetic field F_{ab} satisfies the Maxwell's equations

$$\nabla_b F^{ab} = 4\pi j^a = 4\pi \rho_e u^a, \quad (58)$$

and

$$\nabla_{[a} F_{bc]} = 0, \quad (59)$$

where ρ_e is the charge density measured by the comoving observers. By using the identity $\Gamma_{a\mu}^a = \frac{\partial}{\partial x^\mu} \ln \sqrt{-g}$ (see [14]) Eq. (58) becomes

$$\frac{\partial}{\partial x^\nu} [\sqrt{-g} F^{\mu\nu}] = 4\pi j^\mu \sqrt{-g}. \quad (60)$$

Because of spherical symmetry, the only nonvanishing components of F^{ab} are $F^{tr}(r) = -F^{rt}(r)$. Thus, Eq. (60) yields

$$\partial_r (r^2 \sqrt{-g_{tt} g_{rr}} F^{tr}) = 4\pi j^t r^2 \sqrt{-g_{tt} g_{rr}} = 4\pi \rho_e r^2 \sqrt{g_{rr}}, \quad (61)$$

where Eqs. (54) and (58) have been used in the last step. So by definition, the function $Q(r)$ in Eq. (53) can be written as

$$Q(r) = \int_0^r 4\pi r'^2 \sqrt{g_{rr}} \rho_e dr'. \quad (62)$$

By comparing Eqs. (62) and (61), one finds immediately that

$$F^{tr} = \frac{1}{r^2 \sqrt{-g_{tt} g_{rr}}} Q(r) \quad (63)$$

is a solution of Eq. (61).

Then the time-time component of Einstein's equation gives

$$m'(r) = 4\pi r^2 \rho + \frac{QQ'}{r}. \quad (64)$$

This formula is consistent with the result in [15]. Now we derive the hydroelectrostatic equation from the maximum entropy principle. The total entropy of matter takes the form

$$S = \int_0^R s(r) \left[1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right]^{-1/2} r^2 dr. \quad (65)$$

For simplicity, we assume all the particles have the same charge q . Thus, the charge density is proportional to the particle number density n

$$\rho_e = qn. \quad (66)$$

Together with Eq. (62), we have

$$n = \frac{Q'}{4\pi r^2 q} \left[1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right]^{1/2}. \quad (67)$$

Now we treat $Q(r)$, $Q'(r)$ as independent variables in the Lagrangian formalism. So the Lagrangian is written as

$$L(m, m', Q, Q') = s \left[1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right]^{-1/2} r^2. \quad (68)$$

The conservation of particle number N is equivalent to the conservation of charge with the radius R . Now the constraints are

$$m(0) = Q(0) = 0, \quad m(R) = \text{constant}, \quad (69)$$

$$Q(R) = \text{constant}.$$

With these constraints, the extrema of S leads to the following Euler-Lagrange equations

$$\frac{d}{dr} \frac{\partial L}{\partial Q'} + \frac{\partial L}{\partial Q} = 0 \quad (70)$$

$$\frac{d}{dr} \frac{\partial L}{\partial m'} + \frac{\partial L}{\partial m} = 0. \quad (71)$$

To calculate the Euler-Lagrange equations, we first note that

$$s = s(\rho, n) = s(\rho(m', Q, Q'), n(Q, m, Q')). \quad (72)$$

With the help of Eqs. (64) and (67), we have

$$\frac{\partial s}{\partial Q'} = \frac{\partial s}{\partial \rho} \frac{\partial \rho}{\partial Q'} + \frac{\partial s}{\partial n} \frac{\partial n}{\partial Q'} \quad (73)$$

$$= -\frac{1}{T} \frac{Q}{4\pi r^3} - \frac{\mu}{T} \frac{1}{q} \frac{1}{4\pi r^2} \left[1 - \frac{2m}{r} + \frac{Q^2}{r^2} \right]^{1/2}. \quad (74)$$

Thus,

$$\frac{\partial L}{\partial Q'} = -\frac{1}{T} \frac{Q}{4\pi r} \left[1 - \frac{2m}{r} + \frac{Q^2}{r^2} \right]^{-1/2} - \frac{\mu}{T} \frac{1}{q} \frac{1}{4\pi}. \quad (75)$$

To calculate $\frac{\partial L}{\partial Q}$, first note that

$$\frac{\partial s}{\partial Q} = \frac{\partial s}{\partial \rho} \frac{\partial \rho}{\partial Q} + \frac{\partial s}{\partial n} \frac{\partial n}{\partial Q}$$

$$= -\frac{1}{T} \frac{Q'}{4\pi r^3} - \frac{\mu}{T} \frac{Q Q'}{4\pi q r^4} \left[1 - \frac{2m}{r} + \frac{Q^2}{r^2} \right]^{-1/2}. \quad (76)$$

Then

$$\frac{\partial L}{\partial Q} = -\frac{4\pi r^2 q Q s T + (f q r + \sqrt{f} Q \mu) Q'}{4\pi r^2 q T f^{3/2}}, \quad (77)$$

where

$$f = 1 - \frac{2m}{r} + \frac{Q^2}{r^2}. \quad (78)$$

By substituting Eqs. (75) and (77), Eq. (70) becomes

$$0 = qQ^3 T' + Q[-mqT + qrT - qrTm' + 4\pi q r^3 s T^2$$

$$+ \sqrt{f} r T \mu Q' - 2mqrT' + qr^2 T'] + \sqrt{f} r^2 (r - 2m)$$

$$\times (\mu T' - T \mu') + Q^2 [qTQ' + \sqrt{f} r (\mu T' - T \mu')]. \quad (79)$$

Using the thermodynamic relation (49) and (50), we find

$$T'(r) = \frac{\partial T}{\partial \mu} \Big|_{\rho} \mu' + \frac{\partial T}{\partial \rho} \Big|_{\mu} \rho' = -\frac{n}{s} \mu' + \frac{p'(\rho)}{s} \rho'$$

$$= -\frac{n}{s} \mu' + \frac{p'(r)}{s}. \quad (80)$$

Using Eq. (80) to eliminate μ' in Eq. (79), we have

$$0 = qQ^3 T' + \frac{\sqrt{f} r^2 (r - 2m)(s T T' + n \mu T' - T p')}{n}$$

$$+ \frac{Q^2 \sqrt{f} r (s T T' + n \mu T' - T p')}{n} + qTQ^2 Q'$$

$$+ Q[-mqT + qrT - qrTm' + 4\pi q r^3 s T^2$$

$$+ \sqrt{f} r T \mu Q' - 2mqrT' + qr^2 T']. \quad (81)$$

Eliminating s , μ , and n via Eqs. (20) and (67), we rewrite Eq. (81) as

$$0 = 4\pi r^3 (r^2 - 2mr + Q^2)(p + \rho) T'$$

$$- 4\pi r^3 (r^2 - 2mr + Q^2) T p' + T Q^2 Q'^2$$

$$+ Q Q' (rT + 4\pi r^3 (p + \rho) T + Q^2 T' + r^2 T'$$

$$- mT - 2rmT' - rm'T). \quad (82)$$

Now we begin to calculate Eq. (71). Note that

$$\frac{\partial s}{\partial m'} = \frac{\partial s}{\partial \rho} \frac{\partial \rho}{\partial m'} = \frac{1}{4\pi r^2 T}. \quad (83)$$

Then

$$\frac{\partial L}{\partial m'} = \frac{1}{4\pi r^2 T} \left[1 - \frac{2m}{r} + \frac{Q^2}{r^2} \right]^{-1/2} r^2. \quad (84)$$

Equation (72) yields

$$\frac{\partial s}{\partial m} = \frac{\partial s}{\partial n} \frac{\partial n}{\partial m} = \frac{\mu}{T} \frac{Q'}{4\pi r^3 q \sqrt{1 - \frac{2m}{r} + \frac{Q^2}{r^2}}}. \quad (85)$$

Here we have used Eqs. (19) and (67). From Eq. (68), we find

$$\frac{\partial L}{\partial m} = \frac{\partial s}{\partial m} \left[1 - \frac{2m}{r} + \frac{Q^2}{r^2} \right]^{-1/2} r^2 + sr \left[1 - \frac{2m}{r} + \frac{Q^2}{r^2} \right]^{-3/2}$$

$$= \frac{\mu}{T} \frac{Q'}{4\pi r^2 q} \left[1 - \frac{2m}{r} + \frac{Q^2}{r^2} \right]^{-1} r + sr \left[1 - \frac{2m}{r} + \frac{Q^2}{r^2} \right]^{-3/2}$$

$$= r \left[1 - \frac{2m}{r} + \frac{Q^2}{r^2} \right]^{-3/2}$$

$$\times \left[\frac{\mu}{T} \frac{Q'}{4\pi r^2 q} \left[1 - \frac{2m}{r} + \frac{Q^2}{r^2} \right]^{1/2} + s \right]$$

$$= r \left[1 - \frac{2m}{r} + \frac{Q^2}{r^2} \right]^{-3/2} \left[\frac{\mu n}{T} + s \right]$$

$$= r \left[1 - \frac{2m}{r} + \frac{Q^2}{r^2} \right]^{-3/2} \frac{\rho + p}{T}. \quad (86)$$

Thus, Eq. (71) becomes

$$Q^2T - 4\pi r^4 T(p + \rho) + m'Tr^2 - TrQQ' - rT'Q^2 - r^3T' - mrT + 2mr^2T' = 0. \quad (87)$$

Combining Eq. (82) and (87), one can eliminate T' . Then by substituting Eq. (64) for m' , we finally find

$$p' = \frac{QQ'}{4\pi r^4} - (\rho + p)\left(4\pi r p + \frac{m}{r^2} - \frac{Q^2}{r^3}\right) \times \left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right)^{-1}. \quad (88)$$

This is exactly the generalized Oppenheimer-Volkoff equation for charged fluid [15].

V. CONCLUSIONS

By applying the maximum entropy principle to a general self-gravitating fluid, we have derived the TOV equation of

hydrostatic equilibrium. We only used the Einstein's constraint equation and ordinary thermodynamic relations. By similar assumptions but more complicated arguments, we have shown that the generalized TOV equation for a charged fluid can also be derived by extremizing the total entropy. The TOV equation is an important equation for self-gravitating system which was originally derived from the Einstein equation. Our results show that the Einstein equation can be derived from ordinary thermodynamic laws. This is direct evidence for the fundamental relationship between gravitation and thermodynamics.

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