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Correction to the Dirac field entropy of a Schwarzschild black hole by a modified dispersion relation

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Taking the WKB approximation to solve the Dirac field equation in a Schwarzschild black hole spacetime, we can get the classical momenta. Employing the classical momenta and state density equation corrected by a modified dispersion relation, we will obtain the number of quantum states with energy less than ω. Then, it is used to calculate the statistical-mechanical entropy of the Dirac field. Solving the integral for r exactly, we obtain the leading term of entropy that is proportional to the event horizon area, and correction terms take the form of A^{-1}, but the logarithmic correction term by this approach is not given.

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I. INTRODUCTION

About three decades ago, Bekenstein and Hawking [1,2] found that the black hole entropy is proportional to the area of the event horizon by comparing black hole physics with thermodynamics. This is one of the most profound discoveries in modern physics. Entropy is a statistical-mechanical concept, but the study of general relativity shows that a black hole has no hair. So, the statistical origin of black hole entropy becomes an important question in theoretical physics. Progress has been made by 't Hooft [3], whose brick wall model is extensively used to calculate the entropy in a variety of black holes. In this model, the Bekenstein-Hawking entropy is identified with the statistical-mechanical entropy arising from a thermal bath of quantum fields propagating outside the event horizon. Subsequently, the brick wall model was applied to study a variety of black hole entropies [4–13]. However, in order to remove the ultraviolet divergence near the horizon, one must introduce an artificial cutoff ε. There are other drawbacks in the brick wall model such as the little mass approximation, neglecting logarithm term, and taking the infrared term L^3 as a contribution of the long-distance vacuum. Solving these problems, an improved brick wall method has been introduced by taking the thin layer outside the event horizon of a black hole as the integral region [14]. As a result, this model has solved these drawbacks except for the artificial ultraviolet cutoff.

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The origin of ultraviolet divergence is that the number of quantum states is divergent at the horizon. Recently, in order to remove this divergence, people employ the idea that minimal length would correct the state density. The minimal length originates from the quantum effect of the gravitational field in which the Feynman propagator displays an exponential ultraviolet cutoff of the form $\exp(-\lambda p^2)$, where the parameter λ actually plays the role of a minimal length [15]. Moreover, quantum gravity phenomenology has been tackled with effective models based on generalized uncertainty principles and/or modified dispersion relations [16,17] containing a minimal length as a natural ultraviolet cutoff [18]. At the quantum mechanical level, the essence of the ultraviolet finiteness of the Feynman propagator can be captured by a modified dispersion relation $p = f(k)$, where p and k are the momentum and the wave vector of the particle, respectively. At the same time, the commutator between the operators x_i and p_j is generalized to $[x_i, p_j] = i\frac{\partial p_i}{\partial k_j}$; moreover, the usual momentum measure in $3+1$-dimensional spacetime d^3p is modified to

$$d^3p \det\left|\frac{\partial k_i}{\partial p_j}\right| = d^3p\prod_i\left|\frac{\partial k_i}{\partial p_i}\right|, \tag{1}$$

where $\frac{\partial p_i}{\partial k_j} = \delta_{ij}e^{\lambda p_i^2}$ [15,19–21]. From this momentum measure, we can easily obtain the number of quantum states in a volume element in phase cell space

$$dn = e^{-\lambda p^2}\frac{d^3xd^3p}{(2\pi)^3}, \tag{2}$$

where $p = p^ip_i$ is the square of momentum.

Up to now, a series of papers have studied black hole entropy by state density (2) [16,17,21]. All these works obtained the entropy to the leading-order term that is proportional to the horizon area without any artificial cut-off. In this paper, we employ the corrected state density (2) to study statistical entropy arising from the Dirac field in a Schwarzschild black hole background. In calculating, we attempt to calculate the integral for r exactly and introduce a correction to the leading-order term.

II. ENTROPY OF THE DIRAC FIELD IN A SCHWARZSCHILD BLACK HOLE

Taking the unit $G = c = 1$, a Schwarzschild black hole metric is given by

$$ds^2 = f(r)dt^2 - f^{-1}(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2), \quad (3)$$

where $f(r) = 1 - 2M/r$. Its horizon is located by $r_h = 2M$, and horizon area is $A = 4\pi r_h^2$.

In order to express the Dirac equation in the Newman-Penrose formalism, we take covariant components of the null tetrad vectors as

$$\begin{aligned} l_\mu &= \frac{1}{\sqrt{2}}\left(\sqrt{f}, \frac{1}{\sqrt{f}}, 0, 0\right), \\ n_\mu &= \frac{1}{\sqrt{2}}\left(\sqrt{f}, -\frac{1}{\sqrt{f}}, 0, 0\right), \\ m_\mu &= \frac{1}{\sqrt{2}}(0, 0, r, ir\sin\theta), \\ \bar{m}_\mu &= \frac{1}{\sqrt{2}}(0, 0, r, -ir\sin\theta). \end{aligned} \quad (4)$$

Then, we can find the nonzero spin coefficients

$$\begin{aligned} \mu = \rho &= \frac{1}{\sqrt{2}}\frac{\sqrt{f}}{r}, \\ \alpha = -\beta &= \frac{\cot\theta}{2\sqrt{2}r}, \\ \gamma = \varepsilon &= -\frac{f'}{4\sqrt{2}f}, \end{aligned} \quad (5)$$

and the differential operators read as

$$\begin{aligned} D = l^\mu\partial_\mu &= \frac{1}{\sqrt{2f}}\frac{\partial}{\partial t} - \sqrt{\frac{f}{2}}\frac{\partial}{\partial r}, \\ \Delta = n^\mu\partial_\mu &= \frac{1}{\sqrt{2f}}\frac{\partial}{\partial t} + \sqrt{\frac{f}{2}}\frac{\partial}{\partial r}, \\ \delta = m^\mu\partial_\mu &= -\frac{1}{\sqrt{2}r}\frac{\partial}{\partial\theta} - \frac{i}{\sqrt{2}r\sin\theta}\frac{\partial}{\partial\varphi}, \\ \bar{\delta} = \bar{m}^\mu\partial_\mu &= -\frac{1}{\sqrt{2}r}\frac{\partial}{\partial\theta} + \frac{i}{\sqrt{2}r\sin\theta}\frac{\partial}{\partial\varphi}. \end{aligned} \quad (6)$$

The Dirac equation in a curved spacetime is given by [22]

$$\begin{aligned} &(D + \varepsilon - \rho)F_1 + (\bar{\delta} + \pi - \alpha)F_2 = imG_1/\sqrt{2}\hbar, \\ &(\Delta + \mu - \gamma)F_2 + (\delta + \beta - \tau)F_1 = imG_2/\sqrt{2}\hbar, \\ &(D + \bar{\varepsilon} - \bar{\rho})G_2 - (\delta + \bar{\pi} - \bar{\alpha})G_1 = imF_2/\sqrt{2}\hbar, \\ &(\Delta + \bar{\mu} - \bar{\gamma})G_1 + (\bar{\delta} + \bar{\beta} - \bar{\tau})G_2 = imF_1/\sqrt{2}\hbar, \end{aligned} \quad (7)$$

where m is the mass of the particle. Substituting Eqs. (5) and (6) into this equation, we can obtain

$$\begin{aligned} &\left(\frac{1}{\sqrt{f}}\frac{\partial}{\partial t} - \sqrt{f}\frac{\partial}{\partial r} - \frac{f'}{4\sqrt{f}} - \frac{\sqrt{f}}{r}\right)F_1 + \left(-\frac{1}{r}\frac{\partial}{\partial\theta} + \frac{i}{r\sin\theta}\frac{\partial}{\partial\varphi} - \frac{\cot\theta}{2r}\right)F_2 - \frac{imG_1}{\hbar} = 0, \\ &\left(\frac{1}{\sqrt{f}}\frac{\partial}{\partial t} + \sqrt{f}\frac{\partial}{\partial r} + \frac{f'}{4\sqrt{f}} + \frac{\sqrt{f}}{r}\right)F_2 + \left(-\frac{1}{r}\frac{\partial}{\partial\theta} - \frac{i}{r\sin\theta}\frac{\partial}{\partial\varphi} - \frac{\cot\theta}{2r}\right)F_1 - \frac{imG_2}{\hbar} = 0, \\ &\left(\frac{1}{\sqrt{f}}\frac{\partial}{\partial t} - \sqrt{f}\frac{\partial}{\partial r} - \frac{f'}{4\sqrt{f}} - \frac{\sqrt{f}}{r}\right)G_2 - \left(-\frac{1}{r}\frac{\partial}{\partial\theta} - \frac{i}{r\sin\theta}\frac{\partial}{\partial\varphi} - \frac{\cot\theta}{2r}\right)G_1 - \frac{imF_2}{\hbar} = 0, \\ &\left(\frac{1}{\sqrt{f}}\frac{\partial}{\partial t} + \sqrt{f}\frac{\partial}{\partial r} + \frac{f'}{4\sqrt{f}} + \frac{\sqrt{f}}{r}\right)G_1 + \left(\frac{1}{r}\frac{\partial}{\partial\theta} - \frac{i}{r\sin\theta}\frac{\partial}{\partial\varphi} + \frac{\cot\theta}{2r}\right)G_2 - \frac{imF_1}{\hbar} = 0. \end{aligned} \quad (8)$$

These equations can be solved by the WKB approximation with

$$\begin{aligned} F_1 &= e^{-(i\omega t/\hbar)}e^{(i/\hbar)s_{11}(r,\theta,\varphi)}, \\ F_2 &= e^{-(i\omega t/\hbar)}e^{(i/\hbar)s_{12}(r,\theta,\varphi)}, \\ G_1 &= e^{-(i\omega t/\hbar)}e^{(i/\hbar)s_{21}(r,\theta,\varphi)}, \\ G_2 &= e^{-(i\omega t/\hbar)}e^{(i/\hbar)s_{22}(r,\theta,\varphi)}, \end{aligned} \quad (9)$$

where

$$s_{ij}(r,\theta,\varphi) = s_{ij}^{(0)}(r,\theta,\varphi) + \frac{\hbar}{i}s_{ij}^{(1)}(r,\theta,\varphi) + \cdots (i,j = 1,2). \quad (10)$$

Substituting this into Eq. (8) and equating the leading order of $\hbar$ on both sides, we can obtain

$$\begin{aligned} &A_1f_1 + B_1f_2 - mg_1 = 0, \qquad A_2f_2 + B_2f_1 - mg_2 = 0, \\ &C_1g_2 + D_1g_1 - mf_2 = 0, \qquad C_2g_1 + D_2g_2 - mf_1 = 0, \end{aligned} \quad (11)$$

where

$$f_1 = e^{(i/\hbar)s_{11}(r,\theta,\varphi)}, \qquad f_2 = e^{(i/\hbar)s_{12}(r,\theta,\varphi)},$$
$$g_1 = e^{(i/\hbar)s_{21}(r,\theta,\varphi)}, \qquad g_2 = e^{(i/\hbar)s_{22}(r,\theta,\varphi)},$$
$$A_1 = -\frac{\omega}{\sqrt{f}} - \sqrt{f}p_{11r}, \qquad B_1 = -\frac{1}{r}p_{12\theta} + \frac{i}{r\sin\theta}p_{12\varphi},$$
$$A_2 = -\frac{\omega}{\sqrt{f}} + \sqrt{f}p_{12r}, \qquad B_2 = -\frac{1}{r}p_{11\theta} + \frac{i}{r\sin\theta}p_{11\varphi},$$
$$C_1 = -\frac{\omega}{\sqrt{f}} + \sqrt{f}p_{22r}, \qquad D_1 = \frac{1}{r}p_{21\theta} + \frac{i}{r\sin\theta}p_{21\varphi},$$
$$C_2 = -\frac{\omega}{\sqrt{f}} + \sqrt{f}p_{21r}, \qquad D_2 = \frac{1}{r}p_{22\theta} - \frac{i}{r\sin\theta}p_{22\varphi}, \tag{12}$$

in which $p_{ijk} = \frac{\partial s_{ij}^{(0)}}{\partial x^k}(i, j = 1, 2, k = 1, 2, 3)$ are classical momenta. From the third and fourth identity of Eq. (11), one can solve out $f_1 = \frac{1}{m}(C_2 g_1 + D_2 g_2)$ and $f_2 = \frac{1}{m}(C_1 g_2 + D_1 g_1)$. Substituting this into the first and second identities of Eq. (11), we have

$$\begin{aligned}&\left(\frac{A_1C_2}{m} + \frac{B_1D_1}{m} - m\right)g_1 + \left(\frac{A_1D_2}{m} + \frac{B_1C_1}{m}\right)g_2 = 0,\\ &\left(\frac{A_2D_1}{m} + \frac{B_2C_2}{m}\right)g_1 + \left(\frac{A_2C_1}{m} + \frac{B_2D_2}{m} - m\right)g_2 = 0.\end{aligned} \tag{13}$$

Similarly, solving out $g_1 = \frac{1}{m}(A_1 f_1 + B_1 f_2)$ and $g_2 = \frac{1}{m}(A_2 f_2 + B_2 f_1)$ from the first and second identity of Eq. (11), and substituting this into the third identity of Eq. (11) we can obtain

$$\left(\frac{A_2C_1}{m} + \frac{B_1D_1}{m} - m\right)f_2 + \left(\frac{B_2C_1}{m} + \frac{A_1D_1}{m}\right)f_1 = 0. \tag{14}$$

From the third identity of Eq. (11), one can solve out $g_2 = \frac{m}{C_1}f_2 - \frac{D_1}{C_1}g_1$; substituting it and $f_1 = \frac{1}{m}(C_2g_1 + D_2g_2)$ into the first identity of Eq. (11), the relation between g_1 and f_2 can be found:

$$\left(\frac{A_1C_2}{m} - \frac{A_1D_1D_2}{mC_1} - m\right)g_1 + \left(\frac{A_1D_2}{C_1} + B_1\right)f_2 = 0. \tag{15}$$

Taking the derivative to r of Eqs. (13)–(15), and using the WKB approximation again, we can obtain

$$\begin{aligned}&\left(\frac{A_1C_2}{m} + \frac{B_1D_1}{m} - m\right)g_1p_{21r} + \left(\frac{A_1D_2}{m} + \frac{B_1C_1}{m}\right)g_2p_{22r} = 0,\\ &\left(\frac{A_2D_1}{m} + \frac{B_2C_2}{m}\right)g_1p_{21r} + \left(\frac{A_2C_1}{m} + \frac{B_2D_2}{m} - m\right)g_2p_{22r} = 0,\\ &\left(\frac{A_1C_2}{m} - \frac{A_1D_1D_2}{mC_1} - m\right)g_1p_{21r} + \left(\frac{A_1D_2}{C_1} + B_1\right)f_2p_{12r} = 0,\\ &\left(\frac{A_2C_1}{m} + \frac{B_1D_1}{m} - m\right)f_2p_{12r} + \left(\frac{B_2C_1}{m} + \frac{A_1D_1}{m}\right)f_1p_{11r} = 0.\end{aligned} \tag{16}$$

From Eqs. (13)–(16), we can easily find the relation

$$p_{11r} = p_{12r} = p_{21r} = p_{22r}, \tag{17}$$

which shows that the radial momenta are equal for different component wave functions. Taking the derivative to θ, φ of Eq. (13), and using the same method, we have the analogous identity

$$\begin{aligned}&p_{11\theta} = p_{12\theta} = p_{21\theta} = p_{22\theta},\\ &p_{11\varphi} = p_{12\varphi} = p_{21\varphi} = p_{22\varphi}.\end{aligned} \tag{18}$$

Solving Eq. (13), we can find g_1 and g_2. If we require that the solution is nontrivial, it is necessary for the determinant of the matrix of coefficients to become zero, that is,

$$\begin{vmatrix} \frac{A_1C_2}{m} + \frac{B_1D_1}{m} - m & \frac{A_1D_2}{m} + \frac{B_1C_1}{m} \\ \frac{A_2D_1}{m} + \frac{B_2C_2}{m} & \frac{A_2C_1}{m} + \frac{B_2D_2}{m} - m \end{vmatrix} = 0. \tag{19}$$

Substituting Eqs. (17) and (18) into Eq. (12), we have $A_1 = C_1$, $A_2 = C_2$, $B_1 = -D_2$, $B_2 = -D_1$. Considering this relation, Eq. (19) gives that

$$\begin{aligned}p_{11}^2 &= p_{11}^i p_{11i} = -fp_{11r}^2 - \frac{1}{r^2}p_{11\theta}^2 - \frac{1}{r^2\sin^2\theta}p_{11\varphi}^2\\ &= \frac{\omega^2}{f} - m^2.\end{aligned} \tag{20}$$

This is the square of classical momentum for the first component wave function. There is a similar equation for other component wave functions.

In order to simplify calculations, we let $m = 0$. Then, the number of quantum states with energy less than ω can be given out by substituting Eqs. (17), (18), and (20) into Eq. (2); it reads

$$\begin{aligned}g(\omega) &= g_{11}(\omega) + g_{12}(\omega) + g_{21}(\omega) + g_{22}(\omega)\\ &= \frac{4}{(2\pi)^3}\int e^{-\lambda p_{11}^2}drd\theta d\varphi dp_{11r}dp_{11\theta}dp_{11\varphi}\\ &= \frac{8}{3\pi}\int \frac{r^2}{\sqrt{f}}\left(\frac{\omega^2}{f}\right)^{3/2}e^{-\lambda(\omega^2/f)}dr,\end{aligned} \tag{21}$$

where the integral interval for r is $[r_h, r_h + \varepsilon]$ because we are only interested in the contribution from a thin layer at the vicinity near the horizon whose proper thickness is the minimal length $\sqrt{\frac{e\lambda}{2}}$. So, the minimal length parameters λ and ε satisfy the following relation due to Ref. [23]:

$$\sqrt{\frac{e\lambda}{2}} = \int_{r_h}^{r_h+\varepsilon}\frac{1}{\sqrt{f}}dr \approx \int_{r_h}^{r_h+\varepsilon}\frac{1}{\sqrt{2\kappa(r - r_h)}}dr = \sqrt{\frac{2\varepsilon}{\kappa}}, \tag{22}$$

where $\kappa = \frac{1}{2r_h}$ is the surface gravity at the event horizon. Because of this relation, ε can be written as a function of λ, that is, $\varepsilon(\lambda) = \frac{e\lambda}{8r_h}$. In order to obtain a more exact correction term, we must require a more exact function relationship between ε and λ. For this purpose, taking the derivative to λ in both sides of $\sqrt{\frac{e\lambda}{2}} = \int_{r_h}^{r_h+\varepsilon} \frac{1}{\sqrt{f}} dr$, one can obtain a differential equation

$$[r_h + \varepsilon(\lambda)]\left(\frac{d\varepsilon(\lambda)}{d\lambda}\right)^2 - \frac{e}{8\lambda}\varepsilon(\lambda) = 0, \tag{23}$$

with an initial condition $\varepsilon(0) = 0$. By the series method, its solution is given by

$$\varepsilon(\lambda) = \frac{e\lambda}{8r_h} - \frac{e^2\lambda^2}{192r_h^3} + \frac{11e^3\lambda^3}{23040r_h^5} - \frac{73e^4\lambda^4}{1290240r_h^7} + \frac{887e^5\lambda^5}{116121600r_h^9} + O(\lambda^6). \tag{24}$$

Its leading term is just as the function coming from Eq. (22), and other terms are corrections to higher orders of λ.

Now, we calculate the integral for r in Eq. (21). At first, rewriting it as

$$\int_{r_h}^{r_h+\varepsilon} \frac{r^2}{f^2} e^{-\lambda\omega^2/f} dr = -\int_{r_h}^{r_h+\varepsilon} \frac{r^2}{f'} e^{-\lambda\omega^2/f} d\left(\frac{1}{f}\right), \tag{25}$$

we noticed that $r = \frac{r_h}{1-f}$, and the integral in the right side of the above equation is reduced to

$$\begin{aligned}\int_{r_h}^{r_h+\varepsilon} &\frac{r^2}{f'} e^{-\lambda\omega^2/f} d\left(\frac{1}{f}\right) \\ &= \int_{\infty}^{f^{-1}(r_h+\varepsilon)} \frac{r_h^3 u^4}{(u-1)^4} e^{-\lambda\omega^2 u} du \\ &= r_h^3\left(\frac{1}{\lambda\omega^2} - \frac{6}{1-a} + \frac{2}{(1-a)^2} - \frac{1}{3(1-a)^3} + \frac{2\lambda\omega^2}{1-a} - \frac{\lambda\omega^2}{6(1-a)^2} - \frac{\lambda^2\omega^4}{6(1-a)}\right)\mathrm{e}^{-\lambda\omega^2 a} \\ &\quad - r_h^3(-1/6\lambda^3\omega^6 + 2\lambda^2\omega^4 - 6\lambda\omega^2 + 4)\mathrm{e}^{-\lambda\omega^2} \\ &\quad \times Ei(\lambda\omega^2(1-a)),\end{aligned} \tag{26}$$

where $Ei(x)$ is the exponential integral function defined by $Ei(x) = \int_{-x}^{\infty} \frac{e^{-t}}{t} dt (x < 0)$ [24]. The notation a is $f^{-1}(r_h + \varepsilon)$. Considering Eq. (24), a can be expressed by

$$\begin{aligned}a &= f^{-1}(r_h + \varepsilon) = 1 + \frac{r_h}{\varepsilon} \\ &= \frac{4}{3} + \frac{8r_h^2}{e\lambda} - \frac{e\lambda}{60r_h^2} + \frac{5e^2\lambda^2}{3024r_h^4} - \frac{71e^3\lambda^3}{345600r_h^6} - \frac{7453e^4\lambda^4}{174182400r_h^8} + O(r_h^{-9}) \\ &= \frac{2A}{\pi e\lambda} + \frac{4}{3} - \frac{e\lambda\pi}{15A} + \frac{5e^2\lambda^2\pi^2}{189A^2} - \frac{71e^3\lambda^3\pi^3}{5400A^3} - \frac{7453e^4\lambda^4\pi^4}{680400A^4} + O(A^{-5}).\end{aligned} \tag{27}$$

Substituting Eqs. (26) and (27) into Eq. (21), the number of quantum states less than energy ω can be found that would contain an exponential integral function $Ei(x)$. We will deal with it properly later.

For the Dirac field, the free energy is

$$F(\beta) = \frac{1}{\beta}\int dg(\omega)\ln(1 + e^{-\beta\omega}) = -\int_0^\infty \frac{g(\omega)}{e^{\beta\omega} + 1} d\omega. \tag{28}$$

Then, the entropy is given by

$$\begin{aligned}S &= \beta^2 \frac{\partial F}{\partial \beta} \\ &= \int_0^\infty \frac{8r_h^3\beta^{-3}x^4e^x}{3\pi(e^x+1)^2} \times \Bigg[\left(\frac{\beta^2}{\lambda x^2} - \frac{6}{1-a} + \frac{2}{(1-a)^2} - \frac{1}{3(1-a)^3} + \frac{2\lambda x^2\beta^{-2}}{1-a} - \frac{\lambda x^2\beta^{-2}}{6(1-a)^2} - \frac{\lambda^2x^4\beta^{-4}}{6(1-a)}\right)e^{-\lambda x^2\beta^{-2}a} - (-1/6\lambda^3x^6\beta^{-6} + 2\lambda^2x^4\beta^{-4} \\ &\quad - 6\lambda x^2\beta^{-2} + 4)e^{-\lambda x^2\beta^{-2}} Ei(\lambda x^2\beta^{-2}(1-a))\Bigg]dx,\end{aligned} \tag{29}$$

where we have let $x = \beta\omega$. It is not difficult to prove that the limit of $\sqrt{\lambda x^2\beta^{-2}(a-1)}$ at $\varepsilon = 0$ is

$$\sqrt{\frac{2x^2\beta^{-2}}{e}} \lim_{\varepsilon\to 0}\left(\int_{r_h}^{r_h+\varepsilon}\frac{dr}{\sqrt{f}}\right)\sqrt{\frac{1}{f(r_h+\varepsilon)} - 1} = 0. \tag{30}$$

So, the exponential integral function can be expanded to the series at $z = 0$ ($z = \lambda x^2\beta^{-2}(a-1)$) [24],

$$\begin{aligned}Ei(\lambda x^2\beta^{-2}(1-a)) &= \gamma + \ln(\lambda x^2\beta^{-2}(a-1)) + \sum_{k=1}^{\infty}\frac{1}{k\cdot k!}[\lambda x^2\beta^{-2}(1-a)]^k \\ &= Ei\left(-\frac{x^2}{2e\pi^2}\right) + \frac{k_1}{A} + \frac{k_2}{A^2} + O(A^{-3}),\end{aligned} \tag{31}$$

where

$$\begin{aligned}k_1 &= \frac{e\lambda\pi}{6} - \frac{\lambda x^2}{12\pi}\sum_{k=1}^{\infty}\frac{1}{k!}\left(-\frac{x^2}{2e\pi^2}\right)^{k-1}, \\ k_2 &= -\frac{17e^2\lambda^2\pi^2}{360} + \sum_{k=1}^{\infty}\left[\frac{1}{k!}\frac{e\lambda x^2}{60}\left(-\frac{x^2}{2e\pi^2}\right)^{k-1} + \frac{1}{(k+1)(k-1)!}\frac{\lambda^2x^4}{288\pi^2}\left(-\frac{x^2}{2e\pi^2}\right)^{k-1}\right].\end{aligned} \tag{32}$$

Addtionally, the exponential function has the following series expression:

$$e^{-\lambda x^2\beta^{-2}} = 1 - \frac{x^2\lambda}{4\pi A} + \frac{\lambda^2 x^4}{32\pi^2 A^2} + O(A^{-3}),$$
$$e^{-\lambda x^2\beta^{-2}a} = e^{-(x^2/2e\pi^2)}\left[1 - \frac{\lambda x^2}{3\pi A} + \left(\frac{e\lambda^2 x^2}{60} + \frac{\lambda^2 x^4}{18\pi^2}\right)\frac{1}{A^2}\right] + O(A^{-3}). \quad (33)$$

Substituting Eqs. (27) and (31)–(33) into Eq. (29), after some algebra the entropy is expressed by

$$S = C_1(\lambda)\frac{A}{4} + C_2(\lambda)A^{-1} + O(A^{-2}) + C_0, \quad (34)$$

where $C_i(\lambda)$ is given by the following integrals:

$$C_0 = -\int_0^\infty \frac{x^4 e^x}{18\pi^4(e^x+1)^2}\left[e^{-(x^2/2e\pi^2)} + 3Ei\left(-\frac{x^2}{2e\pi^2}\right)\right]dx,$$
$$C_1(\lambda) = \int_0^\infty \frac{2x^2 e^x}{3\pi^3\lambda(e^x+1)^2}e^{-(x^2/2e\pi^2)}dx,$$
$$C_2(\lambda) = \int_0^\infty \frac{x^4 e^x}{24\pi^3(e^x+1)^2}\left(\frac{46}{15}e\lambda + \frac{2\lambda x^2}{9\pi^2}\right)e^{-(x^2/2e\pi^2)} + \frac{x^4 e^x}{24\pi^4(e^x+1)^2}\left[4k_1 - \frac{5x^2\lambda}{2\pi}Ei\left(-\frac{x^2}{2e\pi^2}\right)\right]dx. \quad (35)$$

The integrals in the above equation contain a factor of $e^x(e^x+1)^{-2}$, and its asymptotic behavior is e^{-x}, so their value must be convergent. They have only one adjustable parameter λ. Supposing $C_1(\lambda_0) = 1$ and solving this equation, we can find

$$\lambda_0 = \int_0^\infty \frac{2x^2 e^x}{3\pi^3(e^x+1)^2}e^{-(x^2/2e\pi^2)}dx \approx 0.0282. \quad (36)$$

By the numerical method, the value of C_0 and $C_2(\lambda_0)$ is

$$C_0 \approx -0.0388, \qquad C_2(\lambda_0) \approx 0.0084. \quad (37)$$

So, the entropy is finally given by

$$S = \frac{A}{4} + 0.0084A^{-1} + O(A^{-2}) - 0.0388. \quad (38)$$

III. SUMMARY

We have studied the statistical-mechanical entropy arising from the Dirac field in a Schwarzschild black hole by carefully counting the number of quantum states in the vicinity near the horizon based on the state density corrected by a modified dispersion relation. Because the corrected factor in Eq. (2) shows negative exponential decay, the number of quantum states with energy less than ω is finite at the horizon. As a result, we have obtained the desired convergent entropy without any artificial cutoff.

We have also given out correction terms to entropy. The leading term of entropy for Eq. (38) is $A/4$ under the condition $C_1(\lambda_0) = 1$, which agrees with Bekenstein-Hawking entropy. The correction terms take the form of A^{-1}, A^{-2}, and so on. However, it does not give a logarithmic correction. In solving the Dirac field equation, we employ the WKB approximation only to the leading order. One may obtain the logarithmic correction term through the considerations of the higher-order WKB approximation. We will investigate this in future work.

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