Residue theorem and summing over Kaluza-Klein excitations

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(Received 17 March 2011; revised manuscript received 8 September 2011; published 18 November 2011)

Applying the equations of motion together with corresponding boundary conditions of bulk profiles at infrared and ultraviolet branes, we verify some lemmas on the eigenvalues of Kaluza-Klein modes in extension of the standard model with a warped extra dimension and the custodial symmetry $SU(3)_c \times SU(2)_L \times SU(2)_R \times U(1)_X \times P_{LR}$. Using the lemmas and performing properly analytic extensions of bulk profiles, we present the sufficient condition for a convergent series of Kaluza-Klein excitations and sum over the series through the residue theorem. The method can also be applied to sum over the infinite series of Kaluza-Klein excitations in a universal extra dimension. Furthermore, we analyze the possible connection between the propagators in five-dimensional full theory and the product of bulk profiles with corresponding propagators of exciting Kaluza-Klein modes in four-dimensional effective theory, and recover some relations presented in the literature for warped and universal extra dimensions, respectively. As an example, we present the correction from new physics to the branching ratio of $\bar{B} \to X_s \gamma$ to the order $\mathcal{O}(\mu_{\rm EW}^2/\Lambda_{\rm KK}^2)$ in extension of the standard model with a warped extra dimension and the custodial symmetry, where $\Lambda_{\rm KK}$ denotes the energy scale of low-lying Kaluza-Klein excitations and $\mu_{\rm EW}$ denotes the electroweak energy scale.

DOI: 10.1103/PhysRevD.84.096012 PACS numbers: 11.10.Kk, 04.50.−h

I. INTRODUCTION

Extensions of the standard model (SM) with a warped dimension [1–4], where all SM fields are propagating in the bulk, provide a naturally geometrical solution to the hierarchy problem regarding the huge difference between the Planck scale and the electroweak one. The small mixing between zero modes and heavy Kaluza-Klein (KK) excitations can induce the observed fermion masses and corresponding weak mixing angles [5,6], and suppress flavor-changing neutral current (FCNC) couplings [7,8]. In addition, realistic models of electroweak symmetry breaking in the framework with a warped extra dimension are constructed in Refs. [9–14], and the gauge coupling unification with a warped extra dimension is also analyzed in Refs. [15,16].

If the SM gauge group $SU(2)_L \times U(1)_Y$ is chosen in the bulk for extensions of the SM with a warped extra dimension, the electroweak precision observables, for example, the experimental data on S, T parameters and the well-measured $Z\bar{b}_L b_L$ coupling [17–20], generally require that the exciting KK modes are heavier than 10 TeV and exceed the reach of colliders running now. In order to accommodate low-lying KK excitations with $\mathcal{O}(1\ \text{TeV})$ masses in the framework with a warped extra dimension, Refs. [21,22] enlarge the gauge group in the bulk to $SU(3)_c \times SU(2)_L \times SU(2)_R \times U(1)_X \times P_{LR}$. With an appropriate choice of quark bulk masses, one indeed obtains the agreement with the electroweak precision data in the presence of light KK excitations [23,24]. Actually, the electroweak precision observables are consistent with the light fermion KK modes with masses even below 1 TeV while the masses of KK gauge bosons are forced to be at least 2–3 TeV to be consistent with experimental data on the parameter S.

However, the large FCNC transitions are aroused by light exciting KK modes if we assume that the hierarchy of fermion masses together with corresponding weak mixings solely originate from geometry and the fundamental Yukawa couplings in five dimensions are anarchic [25–30]. To suppress the large FCNC processes mediated by light exciting KK modes, Ref. [31] introduces the bulk and brane flavor symmetries where the naturally geometric explanation of fermion hierarchies is abandoned. The author of Ref. [32] proposes the minimal flavor protection where a global $U(3)$ bulk flavor symmetry is imposed on the triplet representations of the local gauge symmetry $SU(2)_L \times SU(2)_R$ in the quark sector. In a similar way, Ref. [33] chooses the global flavor symmetry $U(3)_L \times U(3)_R$ on the lepton sector to control relevant FCNC transitions. Another approach to suppress FCNC processes in a warped extra dimension introduces two horizontal $U(1)$ symmetries which guarantee bulk masses in alignment with Yukawa couplings for charge $-1/3$ quarks and charge -1 leptons, respectively [34]. An analogous strategy to solve the FCNC problems in warped geometry is based on A_4 flavor symmetry [35–37]. A very different approach has been presented in Refs. [38,39], where the bulk mass matrices are expressed in terms of five-dimensional Yukawa couplings, thus flavor violation at low energy can be suppressed rationally. Comparing with the choices mentioned

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above, the warped extra dimension with a soft-wall [40–45] perhaps provides a natural solution to accommodate FCNC transitions at an acceptable level and the lightest KK excitations around $\mathcal{O}(1\ \text{TeV})$ masses simultaneously [46].

In the framework with a warped extra dimension and the custodial symmetry $SU(3)_c \times SU(2)_L \times SU(2)_R \times U(1)_X \times P_{LR}$, a meticulous analysis on the electroweak and flavor structure is provided in Ref. [47], a complete study of rare K and B meson decays is presented in Ref. [48], and the impact from KK excitations of fermions on the couplings among the SM particles is given in Ref. [49]. Assuming all fields are propagating in the bulk, the authors of Ref. [50] analyze the FCNC processes mediated by a light Higgs, the authors of Ref. [51] present an analysis on production and decay of KK gravitons at the LHC, and the authors of Ref. [52] study signals for KK excitations of electroweak and strong gauge bosons in the LHC. Additionally, an analysis of loop-induced rare lepton FCNC transition $\mu \to e\gamma$ in a warped extra dimension with anarchic Yukawa couplings and IR brane localized Higgs boson is presented in Ref. [53] through the five-dimensional mixed position/momentum space formalism [54] and mass insertion approach.

It is well known that all virtual KK excitations contribute their corrections to theoretical predictions on the physical quantities at electroweak scale, and those theoretical corrections should be summed over infinite KK modes in principle [55–57]. In this work, we verify some lemmas on the eigenvalues of exciting KK modes in extension of the SM with a warped extra dimension and the custodial symmetry $SU(3)_c \times SU(2)_L \times SU(2)_R \times U(1)_X \times P_{LR}$ [8,58]. Performing properly analytic extensions of the bulk profiles, we sum over the infinite series of KK modes through the residue theorem. In addition, we also present the sufficient condition for a convergent series of infinite KK modes in extensions of the SM with a warped extra dimension. The method can also be applied to sum over the infinite series of KK modes in a universal extra dimension. The emphasized point here is that the authors of Ref. [56] also propose to sum over infinite series of KK modes applying the equations of motion and corresponding completeness relations of bulk profiles. Nevertheless, the method proposed there can only be applied to sum over the infinite series of KK modes for the five-dimensional fields with zero bulk mass and zero modes in extensions of the SM with a warped extra dimension.

The radiative decay $\bar{B} \to X_s\gamma$ definitely provides a very strong constraint on extensions of the SM. The SM prediction on the branching ratio of $\bar{B} \to X_s\gamma$ at next-to-next to-leading order [59–61] (NNLO) reads

$$\mathrm{Br}\,(\bar{B} \to X_s\gamma) = (3.15 \pm 0.23) \times 10^{-4}, \tag{1}$$

which is calculated for a photon-energy cutoff $E_\gamma >$ 1.6 GeV in the $\bar{B}$-meson rest frame. This result certainly coincides with the current experimental observation [62],

$$\mathrm{Br}\,(\bar{B} \to X_s\gamma) = (3.55 \pm 0.24 \pm 0.09) \times 10^{-4}, \tag{2}$$

with the same energy cutoff E_γ. Using the residue technique mentioned above, we also analyze the corrections from exciting KK modes to the branching ratio of $\bar{B} \to X_s\gamma$ in the scenario with a warped extra dimension and the custodial symmetry.

Our presentation is organized as follows. The main ingredients of the SM extension with a warped extra dimension and the custodial symmetry are summarized briefly in Sec. II; the KK decompositions of all five-dimensional fields and relevant bulk profiles are given here also. In Sec. III, we present the verification of relevant lemmas on the eigenvalues of exciting KK modes in detail. In order to sum over the infinite series of KK modes properly, we also discuss how to extend bulk profiles analytically. We discuss the possible relation between the perturbative expansions in four-dimensional effective theory and five-dimensional full theory in Sec. IV. Furthermore, we also recover the equations presented in Ref. [18] through the residue theorem. In Sec. V, we show how to sum over the infinite series of KK modes in a universal extra dimension by the residue theorem through recovering an important relation applied extensively in the literature. In Sec. VI, we present the theoretical prediction on the effective Hamiltonian of $\bar{B} \to X_s\gamma$ to the order $\mathcal{O}(\mu_{\mathrm{EW}}^2/\Lambda_{\mathrm{KK}}^2)$ through residue technique in the framework with a warped extra dimension and the custodial symmetry, where Λ_{KK} denotes the energy scale of low-lying KK excitations and μ_{EW} denotes the electroweak energy scale. Additionally, we also present the possible constraint from experimental data of $\bar{B} \to X_s\gamma$ on the parameter space of the SM extension with a warped extra dimension and the custodial symmetry. Our conclusions are summarized in Sec. VII.

II. A WARPED EXTRA DIMENSION WITH THE CUSTODIAL SYMMETRY

In the Randall-Sundrum (RS) scenario, four-dimensional Minkowskian space-time is embedded into a slice of five-dimensional anti de–Sitter (ADS_5) space with curvature k. The fifth dimension is a $S^1/Z_2 \times Z_2'$ orbifold of size r labeled by a coordinate $\phi \in [-\pi, \pi]$, thus the points (x^μ, ϕ), $(x^\mu, \pi - \phi)$, $(x^\mu, \pi + \phi)$, and $(x^\mu, -\phi)$ are identified all. The corresponding metric of nonfactorizable RS geometry is written as

$$ds^2 = e^{-2\sigma(\phi)}\eta_{\mu\nu}dx^\mu dx^\nu - r^2 d\phi^2, \qquad \sigma(\phi) = kr|\phi|, \tag{3}$$

where x^μ ($\mu = 0, 1, 2, 3$) are the coordinates on the four-dimensional hypersurfaces of constant ϕ with metric $\eta_{\mu\nu} = (1, -1, -1, -1)$, and e^σ is called the warp factor. Two branes are located on the orbifold fixed points $\phi = 0$ and $\phi = \pi/2$, respectively. The brane on $\phi = 0$ is called the Planck or ultraviolet (UV) brane, and the brane on $\phi = \pi/2$ is called the TeV or infrared (IR) brane. Assuming the parameters k and $1/r$ to be of order the fundamental Planck scale $M_{\rm PI}$ and choosing the product $kr \simeq 24$, one gets the inverse warp factor

$$\epsilon = \frac{\Lambda_{\rm IR}}{\Lambda_{\rm UV}} \equiv e^{-kr\pi/2} \simeq 10^{-16}, \tag{4}$$

which explains the hierarchy between the electroweak and Planck scale naturally. Meanwhile, the mass scale of low-lying KK excitations is set as

$$\Lambda_{\rm KK} \equiv k\epsilon = ke^{-kr\pi/2} = \mathcal{O}(1\ {\rm TeV}). \tag{5}$$

In the gauge symmetry $SU(2)_L \times SU(2)_R \times U(1)_X \times P_{LR}$, the discrete symmetry P_{LR} interchanging the local groups $SU(2)_L$ and $SU(2)_R$ implies that the five-dimensional gauge couplings satisfy the relation $g_{5L} = g_{5R} = g_5$. The local gauge group $SU(2)_L \times SU(2)_R \times U(1)_X$ is broken to the SM gauge group by the boundary conditions (BCs) on the UV brane,

$$\begin{aligned} &W^{1,2,3}_{L,\mu}(++), \qquad B_\mu(++), \qquad W^{1,2}_{R,\mu}(-+), \\ &Z_{X,\mu}(-+), \qquad (\mu = 0, 1, 2, 3), \\ &W^{1,2,3}_{L,5}(--), \qquad B_5(--), \qquad W^{1,2}_{R,5}(+-), \qquad Z_{X,5}(+-). \end{aligned} \tag{6}$$

Here, the first (second) sign is the BC on the UV (IR) brane, $+$ denotes a Neumann BC, and $-$ denotes a Dirichlet BC. The zero modes of fields with $(++)$ BCs are massless before the electroweak symmetry is spontaneously broken, and are identified as the SM gauge bosons. The fields with other BCs only contain massive KK modes. The third component of $SU(2)_R$ gauge fields $W^3_{R,M}$ and the $U(1)_X$ gauge field $\tilde{B}_M$ are expressed in terms of the neutral gauge fields $Z_{X,M}$ and B_M as

$$\begin{aligned} W^3_{R,M} &= \frac{g_5 Z_{X,M} + g_{5X} B_M}{\sqrt{g_5^2 + g_{5X}^2}}, \\ \tilde{B}_M &= -\frac{g_{5X} Z_{X,M} - g_5 B_M}{\sqrt{g_5^2 + g_{5X}^2}}, \qquad (M = 0, 1, 2, 3, 5), \end{aligned} \tag{7}$$

where g_{5X} is the five-dimensional gauge coupling of $U(1)_X$. The Lagrangian for gauge sector is written as

$$\begin{aligned} \mathcal{L}_{\rm gauge} = \frac{\sqrt{\mathcal{G}}}{r} \mathcal{G}^{KM} \mathcal{G}^{LN} \Big(&-\frac{1}{4} W^i_{KL} W^i_{MN} - \frac{1}{4} \tilde{W}^i_{KL} \tilde{W}^i_{MN} \\ &- \frac{1}{4} \tilde{B}_{KL} \tilde{B}_{MN} - \frac{1}{4} G^a_{KL} G^a_{MN} \Big). \end{aligned} \tag{8}$$

For convenience in our discussion further, we define the following five-dimensional fields [52]:

$$\begin{aligned} A_M &= \frac{\sqrt{g_5^2 + g_{5X}^2} B_M + g_{5X} W^3_{L,M}}{\sqrt{g_5^2 + 2g_{5X}^2}}, \\ Z_M &= \frac{-g_{5X} B_M + \sqrt{g_5^2 + g_{5X}^2} W^3_{L,M}}{\sqrt{g_5^2 + 2g_{5X}^2}}, \\ W^\pm_{L,M} &= \frac{1}{\sqrt{2}} (W^1_{L,M} \mp i W^2_{L,M}), \\ W^\pm_{R,M} &= \frac{1}{\sqrt{2}} (W^1_{R,M} \mp i W^2_{R,M}). \end{aligned} \tag{9}$$

As regards the matter fields, the quarks of one generation are embedded into the multiplets [63],

$$\begin{aligned} Q^i_L &= \begin{pmatrix} \chi^i_{u_L}(-+)_{5/3} & q^i_{u_L}(++)_{2/3} \\ \chi^i_{d_L}(-+)_{2/3} & q^i_{d_L}(++)_{-1/3} \end{pmatrix}, \\ Q^i_{u_R} &= u^i_R(++)_{2/3} \\ \tilde{Q}^i_{d_R} &= \begin{pmatrix} X^i_R(-+)_{5/3} \\ U^i_R(-+)_{2/3} \\ D^i_R(-+)_{-1/3} \end{pmatrix}, \\ Q^i_{d_R} &= \begin{pmatrix} \tilde{X}^i_R(-+)_{5/3} \\ \tilde{U}^i_R(-+)_{2/3} \\ d^i_R(++)_{-1/3} \end{pmatrix}, \end{aligned} \tag{10}$$

and the states with opposite chirality are written as

$$\begin{aligned} Q^i_R &= \begin{pmatrix} \chi^i_{u_R}(+-)_{5/3} & q^i_{u_R}(--)_{2/3} \\ \chi^i_{d_R}(+-)_{2/3} & q^i_{d_R}(--)_{-1/3} \end{pmatrix}, \\ Q^i_{u_L} &= u^i_L(--)_{2/3} \\ \tilde{Q}^i_{d_L} &= \begin{pmatrix} X^i_L(+-)_{5/3} \\ U^i_L(+-)_{2/3} \\ D^i_L(+-)_{-1/3} \end{pmatrix}, \\ Q^i_{d_L} &= \begin{pmatrix} \tilde{X}^i_L(+-)_{5/3} \\ \tilde{U}^i_L(+-)_{2/3} \\ d^i_L(--)_{-1/3} \end{pmatrix}. \end{aligned} \tag{11}$$

Here, $i = 1, 2, 3$ denotes the index of generation and the $U(1)_X$ charges are all assigned as

$$Y_{Q^i} = Y_{u^i} = Y_{Q^i_d} = \frac{2}{3}. \tag{12}$$

In order to give the kinetic terms of triplets, we redefine the quarks in triplet as

$$\tilde{T}^i_{Q_R} = \begin{pmatrix} \frac{1}{\sqrt{2}}(X^i_R + D^i_R) \\ \frac{i}{\sqrt{2}}(X^i_R - D^i_R) \\ U^i_R \end{pmatrix}, \qquad T^i_{Q_R} = \begin{pmatrix} \frac{1}{\sqrt{2}}(\tilde{X}^i_R + d^i_R) \\ \frac{i}{\sqrt{2}}(\tilde{X}^i_R - d^i_R) \\ \tilde{U}^i_R \end{pmatrix},$$
$$\tilde{T}^i_{Q_L} = \begin{pmatrix} \frac{1}{\sqrt{2}}(X^i_L + D^i_L) \\ \frac{i}{\sqrt{2}}(X^i_L - D^i_L) \\ U^i_L \end{pmatrix}, \qquad T^i_{Q_L} = \begin{pmatrix} \frac{1}{\sqrt{2}}(\tilde{X}^i_L + d^i_L) \\ \frac{i}{\sqrt{2}}(\tilde{X}^i_L - d^i_L) \\ \tilde{U}^i_L \end{pmatrix}. \tag{13}$$

Correspondingly, the Lagrangian for kinetic terms of quarks can be written as

$$\begin{aligned}\mathcal{L}_Q = \frac{\sqrt{\mathcal{G}}}{2r}\sum_{i=1}^{3}\Big\{&(\bar{Q}^i)_{a_1a_2} iE^M_A\gamma^A\Big[\Big(\frac{1}{2}(\partial_M - \overleftarrow{\partial}_M) + ig_{5s}T^aG^a_M + ig_{5X}Y_{Q^i}\tilde{B}_M\Big)\delta_{a_1b_1}\delta_{a_2b_2} + ig_5\Big(\frac{\sigma^{c_1}}{2}\Big)_{a_1b_1}W^{c_1}_{L,M}\delta_{a_2b_2}\\
&+ ig_5\Big(\frac{\sigma^{c_2}}{2}\Big)_{a_2b_2}W^{c_2}_{R,M}\delta_{a_1b_1}\Big](Q^i)_{b_1b_2} + (\bar{Q}^i)_{a_1a_2}[iE^M_A\gamma^A\omega_M - \mathrm{sgn}(\phi)k(c_B)_{ij}](Q^j)_{a_1a_2}\\
&+ \bar{u}^i\Big[iE^M_A\gamma^A\Big(\frac{1}{2}(\partial_M - \overleftarrow{\partial}_M) + ig_{5s}T^aG^a_M + ig_{5X}Y_{u^i}\tilde{B}_M\Big)\delta_{ij} + iE^M_A\gamma^A\omega_M - \mathrm{sgn}(\phi)k(c_S)_{ij}\Big]u^j\\
&+ (\bar{\tilde{T}}^i_Q)_{a1} iE^M_A\gamma^A\Big[\Big(\frac{1}{2}(\partial_M - \overleftarrow{\partial}_M) + ig_{5s}T^aG^a_M + ig_{5X}Y_{Q^i_d}\tilde{B}_M\Big)\delta_{a_1b_1} + g_5\varepsilon_{a_1b_1c_1}W^{c_1}_{L,M}\Big](\tilde{T}^i_Q)_{b_1}\\
&+ (\bar{\tilde{T}}^i_Q)_{a_1}[iE^M_A\gamma^A\omega_M - \mathrm{sgn}(\phi)(\eta_3)_{ij}](\tilde{T}^j_Q)_{a_1} + (\bar{T}^i_Q)_{a_1} iE^M_A\gamma^A\Big[\Big(\frac{1}{2}(\partial_M - \overleftarrow{\partial}_M) + ig_{5s}T^aG^a_M + ig_{5X}Y_{Q^i_d}\tilde{B}_M\Big)\delta_{a_1b_1}\\
&+ g_5\varepsilon_{a_1b_1c_1}W^{c_1}_{R,M}\Big](T^i_Q)_{b_1} + (\bar{T}^i_Q)_{a_1}[iE^M_A\gamma^A\omega_M - \mathrm{sgn}(\phi)k(c_T)_{ij}](T^j_Q)_{a_1} + \mathrm{H.c.}\Big\},\end{aligned} \tag{14}$$

with $\gamma^A = (\gamma^\mu, -i\gamma^5)$, the inverse vielbein $E^A_B = \mathrm{diag}(e^{\sigma(\phi)}, e^{\sigma(\phi)}, e^{\sigma(\phi)}, e^{\sigma(\phi)}, \frac{1}{r})$, and the spin connection $\omega_A = (\mathrm{sgn}(\phi)\frac{i}{2}ke^{-\sigma(\phi)}\gamma_\mu\gamma^5, 0)$. Generally, three bulk mass matrices c_B, c_S, c_T are arbitrary Hermitian 3×3 matrices.

To break down the electroweak symmetry, we introduce an IR brane located Higgs which transforms as a self-dual bidoublet under the gauge group $SU(2)_L \times SU(2)_R$, and transforms as a singlet with charge $Y_H = 0$ under the gauge group $U(1)_X$,

$$H = \begin{pmatrix} -i\pi^+/\sqrt{2} & -(h^0 - i\pi^0)/2 \\ (h^0 + i\pi^0)/2 & i\pi^-/\sqrt{2} \end{pmatrix}. \tag{15}$$

After normalizing the kinetic term of Higgs in five dimensions, we write the corresponding Lagrangian as

$$\begin{aligned}\mathcal{L}_H = &\mathrm{Tr}[(D_\mu\Phi(x))^\dagger(D^\mu\Phi(x))] - \mu^2\mathrm{Tr}(\Phi^\dagger(x)\Phi(x))\\ &+ \frac{\lambda}{2}[\mathrm{Tr}(\Phi^\dagger(x)\Phi(x))]^2,\end{aligned} \tag{16}$$

with $D_MH = \partial_MH + \frac{i}{2}g_5(\sum_{a=1}^3 W^a_{L,M}\sigma^a)H + \frac{i}{2}g_5H(\sum_{a=1}^3 W^a_{R,M}\sigma^a)^T$. Accordingly, the Yukawa couplings between quarks and Higgs field can be formulated as

$$\begin{aligned}\mathcal{L}^Q_Y = e^{kr\pi/2}\sqrt{-\mathcal{G}_{\mathrm{IR}}}\sum_{i,j=1}^{3}\{&\sqrt{2}\lambda^u_{ij}\bar{Q}^i_{a\alpha}H_{a\alpha}u^j\\ &- 2\lambda^d_{ij}[\bar{Q}^i_{a\alpha}(\tau^c)_{ab}(\tilde{T}^j_d)_cH_{b\alpha}\\ &+ \bar{Q}^i_{a\alpha}(\tau^c)_{\alpha\beta}(T^j_d)_cH_{a\beta}] + \mathrm{H.c.}\},\end{aligned} \tag{17}$$

here the metric on IR brane $\mathcal{G}^{\mu\nu}_{\mathrm{IR}} = e^{kr\pi/2}\eta^{\mu\nu}$. In the following we choose to work in the background gauge [64]. Furthermore, we also assume that three bulk mass matrices c_B, c_S, c_T are real and diagonal, i.e. each of them is described by three real parameters. This can always be obtained through some appropriate field redefinitions.

For convenience in our analysis, we define the gauge couplings in four dimensions which are related to the five-dimensional gauge couplings via

$$g = \frac{g_5}{\sqrt{2\pi r}}, \qquad g_X = \frac{g_{5X}}{\sqrt{2\pi r}}. \tag{18}$$

Correspondingly, the constant of electromagnetic coupling and Weinberg angle in four dimensions are given through

$$e = \frac{gg_X}{\sqrt{g^2 + 2g_X^2}}, \qquad \sin\theta_W = \frac{g_X}{\sqrt{g^2 + 2g_X^2}}. \tag{19}$$

In terms of the Weinberg angle θ_W and the constant of electromagnetic coupling e, the gauge couplings in Eq. (18) are written as

$$g = \frac{e}{\sin\theta_W}, \qquad g_X = \frac{e}{\sqrt{1 - 2\sin^2\theta_W}}. \tag{20}$$

The KK decompositions of five-dimensional gauge fields are extensively studied in the literature, and can be written in our notations as

$$\begin{aligned}
A_\mu(x,\phi) &= \frac{1}{\sqrt{r}} \sum_{n=0}^{\infty} A_\mu^{(n)}(x) \chi_{(++)}^G (y_{(++)}^{G(n)}, t),\\
Z_\mu(x,\phi) &= \frac{1}{\sqrt{r}} \sum_{n=0}^{\infty} Z_\mu^{(n)}(x) \chi_{(++)}^G (y_{(++)}^{G(n)}, t),\\
Z_{X,\mu}(x,\phi) &= \frac{1}{\sqrt{r}} \sum_{n=1}^{\infty} Z_{X,\mu}^{(n)}(x) \chi_{(-+)}^G (y_{(-+)}^{G(n)}, t),\\
W_{L,\mu}^{\pm}(x,\phi) &= \frac{1}{\sqrt{r}} \sum_{n=0}^{\infty} W_{L,\mu}^{\pm(n)}(x) \chi_{(++)}^G (y_{(++)}^{G(n)}, t),\\
W_{R,\mu}^{\pm}(x,\phi) &= \frac{1}{\sqrt{r}} \sum_{n=1}^{\infty} W_{R,\mu}^{\pm(n)}(x) \chi_{(-+)}^G (y_{(-+)}^{G(n)}, t),\\
G_\mu^a(x,\phi) &= \frac{1}{\sqrt{r}} \sum_{n=0}^{\infty} G_\mu^{a,(n)}(x) \chi_{(++)}^G (y_{(++)}^{G(n)}, t).
\end{aligned} \tag{21}$$

As $G = A, Z, W_L^{\pm}, g$, $y_{(++)}^{G(n)}$ $(n = 0, 1, \cdots, \infty)$ denote the roots of equation $z^2 R_{(++)}^{G,\epsilon}(z) \equiv 0$ with

$$R_{(++)}^{G,\epsilon}(z) = Y_0(z)J_0(z\epsilon) - J_0(z)Y_0(z\epsilon). \tag{22}$$

When $G = Z_X, W_R^{\pm}$, $y_{(-+)}^{G(n)}$ $(n = 1, 2, \cdots, \infty)$ denote the roots of equation $R_{(-+)}^{G,\epsilon}(z) \equiv 0$ with

$$R_{(-+)}^{G,\epsilon}(z) = Y_0(z)J_1(z\epsilon) - J_0(z)Y_1(z\epsilon). \tag{23}$$

In order to give the bulk profiles of those five-dimensional fields conveniently, we introduce a coordinate $t = \epsilon\exp(\sigma(\phi))$ which takes values between $t = \epsilon$ (UV brane) and $t = 1$ (IR brane). In terms of the variable t, we write the bulk files for those gauge fields as

$$\begin{aligned}
\chi_{(++)}^G(y_{(++)}^{G(n)}, t) &= \frac{t\Phi_{(uv)}^{G(+)}(y_{(++)}^{G(n)}, t)}{N_{(uv)}^{G(+)}(y_{(++)}^{G(n)})} = \frac{t\Phi_{(ir)}^{G(+)}(y_{(++)}^{G(n)}, t)}{N_{(ir)}^{G(+)}(y_{(++)}^{G(n)})},\\
\chi_{(-+)}^G(y_{(-+)}^{G(n)}, t) &= \frac{t\Phi_{(uv)}^{G(-)}(y_{(-+)}^{G(n)}, t)}{N_{(uv)}^{G(-)}(y_{(-+)}^{G(n)})} = \frac{t\Phi_{(ir)}^{G(+)}(y_{(-+)}^{G(n)}, t)}{N_{(ir)}^{G(+)}(y_{(-+)}^{G(n)})},
\end{aligned} \tag{24}$$

with

$$\begin{aligned}
\Phi_{(uv)}^{G(+)}(y,t) &= Y_0(y\epsilon)J_1(yt) - J_0(y\epsilon)Y_1(yt),\\
\Phi_{(ir)}^{G(+)}(y,t) &= Y_0(y)J_1(yt) - J_0(y)Y_1(yt),\\
\Phi_{(uv)}^{G(-)}(y,t) &= Y_1(y\epsilon)J_1(yt) - J_1(y\epsilon)Y_1(yt),\\
\Psi_{(uv)}^{G(-)}(y,t) &= Y_0(y\epsilon)J_0(yt) - J_0(y\epsilon)Y_0(yt),\\
\Psi_{(ir)}^{G(-)}(y,t) &= Y_0(y)J_0(yt) - J_0(y)Y_0(yt),\\
\Psi_{(uv)}^{G(+)}(y,t) &= Y_1(y\epsilon)J_0(yt) - J_1(y\epsilon)Y_0(yt).
\end{aligned} \tag{25}$$

It is easy to check Eq. (25) satisfying the equations of motion,

$$\begin{aligned}
t\frac{\partial\Phi_{(uv)}^{G(+)}}{\partial t}(y,t) + \Phi_{(uv)}^{G(+)}(y,t) &= yt\Psi_{(uv)}^{G(-)}(y,t),\\
t\frac{\partial\Phi_{(ir)}^{G(+)}}{\partial t}(y,t) + \Phi_{(ir)}^{G(+)}(y,t) &= yt\Psi_{(ir)}^{G(-)}(y,t),\\
t\frac{\partial\Phi_{(uv)}^{G(-)}}{\partial t}(y,t) + \Phi_{(uv)}^{G(-)}(y,t) &= yt\Psi_{(uv)}^{G(+)}(y,t),\\
t\frac{\partial\Psi_{(uv)}^{G(-)}}{\partial t}(y,t) &= -yt\Phi_{(uv)}^{G(+)}(y,t),\\
t\frac{\partial\Psi_{(ir)}^{G(-)}}{\partial t}(y,t) &= -yt\Phi_{(ir)}^{G(+)}(y,t),\\
t\frac{\partial\Psi_{(uv)}^{G(+)}}{\partial t}(y,t) &= -yt\Phi_{(uv)}^{G(-)}(y,t).
\end{aligned} \tag{26}$$

The corresponding normalization constants are formulated as

$$\begin{aligned}
[N_{(uv)}^{G(+)}(y)]^2 &= \frac{2}{kr}\Big\{(\Phi_{(uv)}^{G(+)}(y,1))^2 - \epsilon^2(\Phi_{(uv)}^{G(+)}(y,\epsilon))^2 + (\Psi_{(uv)}^{G(-)}(y,1))^2 - \frac{2}{y}\Phi_{(uv)}^{G(+)}(y,1)\Psi_{(uv)}^{G(-)}(y,1)\Big\},\\
[N_{(ir)}^{G(+)}(y)]^2 &= \frac{2}{kr}\Big\{(\Phi_{(ir)}^{G(+)}(y,1))^2 - \epsilon^2(\Phi_{(ir)}^{G(+)}(y,\epsilon))^2 - \epsilon^2(\Psi_{(ir)}^{G(-)}(y,\epsilon))^2 + \frac{2\epsilon}{y}\Phi_{(ir)}^{G(+)}(y,\epsilon)\Psi_{(ir)}^{G(-)}(y,\epsilon)\Big\},\\
[N_{(uv)}^{G(-)}(y)]^2 &= \frac{2}{kr}\Big\{(\Psi_{(uv)}^{G(+)}(y,1))^2 - \epsilon^2(\Psi_{(uv)}^{G(+)}(y,\epsilon))^2 + (\Phi_{(uv)}^{G(-)}(y,1))^2 - \frac{2}{y}\Psi_{(uv)}^{G(+)}(y,1)\Phi_{(uv)}^{G(-)}(y,1)\Big\}.
\end{aligned} \tag{27}$$

Similarly, the KK decompositions of five-dimensional quark fields are written as

$$\begin{aligned}
\chi^i_{u_L}(x,\phi) &= \frac{e^{2\sigma(\phi)}}{\sqrt{r}}\sum_n \chi^{i,(n)}_{u_L}(x) f^{L,c^i_B}_{(-+)}(y^{c^i_B(n)}_{(\mp\pm)},t), & \chi^i_{d_L}(x,\phi) &= \frac{e^{2\sigma(\phi)}}{\sqrt{r}}\sum_n \chi^{i,(n)}_{d_L}(x) f^{L,c^i_B}_{(-+)}(y^{c^i_B(n)}_{(\mp\pm)},t),\\
q^i_{u_L}(x,\phi) &= \frac{e^{2\sigma(\phi)}}{\sqrt{r}}\sum_n q^{i,(n)}_{u_L}(x) f^{L,c^i_B}_{(++)}(y^{c^i_B(n)}_{(\pm\pm)},t), & q^i_{d_L}(x,\phi) &= \frac{e^{2\sigma(\phi)}}{\sqrt{r}}\sum_n q^{i,(n)}_{d_L}(x) f^{L,c^i_B}_{(++)}(y^{c^i_B(n)}_{(\pm\pm)},t),\\
u^i_R(x,\phi) &= \frac{e^{2\sigma(\phi)}}{\sqrt{r}}\sum_n u^{i,(n)}_R(x) f^{R,c^i_S}_{(++)}(y^{c^i_S(n)}_{(\mp\mp)},t), & X^i_R(x,\phi) &= \frac{e^{2\sigma(\phi)}}{\sqrt{r}}\sum_n X^{i,(n)}_R(x) f^{R,c^i_T}_{(-+)}(y^{c^i_T(n)}_{(\pm\mp)},t),\\
U^i_R(x,\phi) &= \frac{e^{2\sigma(\phi)}}{\sqrt{r}}\sum_n U^{i,(n)}_R(x) f^{R,c^i_T}_{(-+)}(y^{c^i_T(n)}_{(\pm\mp)},t), & D^i_R(x,\phi) &= \frac{e^{2\sigma(\phi)}}{\sqrt{r}}\sum_n D^{i,(n)}_R(x) f^{R,c^i_T}_{(-+)}(y^{c^i_T(n)}_{(\pm\mp)},t),\\
\tilde{X}^i_R(x,\phi) &= \frac{e^{2\sigma(\phi)}}{\sqrt{r}}\sum_n \tilde{X}^{i,(n)}_R(x) f^{R,c^i_T}_{(-+)}(y^{c^i_T(n)}_{(\pm\mp)},t), & \tilde{U}^i_R(x,\phi) &= \frac{e^{2\sigma(\phi)}}{\sqrt{r}}\sum_n \tilde{U}^{i,(n)}_R(x) f^{R,c^i_T}_{(-+)}(y^{c^i_T(n)}_{(\pm\mp)},t),\\
d^i_R(x,\phi) &= \frac{e^{2\sigma(\phi)}}{\sqrt{r}}\sum_n d^{i,(n)}_R(x) f^{R,c^i_T}_{(++)}(y^{c^i_T(n)}_{(\mp\mp)},t), & X^i_L(x,\phi) &= \frac{e^{2\sigma(\phi)}}{\sqrt{r}}\sum_n X^{i,(n)}_L(x) f^{L,c^i_T}_{(+-)}(y^{c^i_T(n)}_{(\pm\mp)},t),\\
U^i_L(x,\phi) &= \frac{e^{2\sigma(\phi)}}{\sqrt{r}}\sum_n U^{i,(n)}_L(x) f^{L,c^i_T}_{(+-)}(y^{c^i_T(n)}_{(\pm\mp)},t), & D^i_L(x,\phi) &= \frac{e^{2\sigma(\phi)}}{\sqrt{r}}\sum_n D^{i,(n)}_L(x) f^{L,c^i_T}_{(+-)}(y^{c^i_T(n)}_{(\pm\mp)},t),\\
\tilde{X}^i_L(x,\phi) &= \frac{e^{2\sigma(\phi)}}{\sqrt{r}}\sum_n \tilde{X}^{i,(n)}_L(x) f^{L,c^i_T}_{(+-)}(y^{c^i_T(n)}_{(\pm\mp)},t), & \tilde{U}^i_L(x,\phi) &= \frac{e^{2\sigma(\phi)}}{\sqrt{r}}\sum_n \tilde{U}^{i,(n)}_L(x) f^{L,c^i_T}_{(+-)}(y^{c^i_T(n)}_{(\pm\mp)},t),\\
d^i_L(x,\phi) &= \frac{e^{2\sigma(\phi)}}{\sqrt{r}}\sum_n d^{i,(n)}_L(x) f^{L,c^i_T}_{(--)}(y^{c^i_T(n)}_{(\mp\mp)},t), & u^i_L(x,\phi) &= \frac{e^{2\sigma(\phi)}}{\sqrt{r}}\sum_n u^{i,(n)}_L(x) f^{L,c^i_S}_{(--)}(y^{c^i_S(n)}_{(\mp\mp)},t),\\
\chi^i_{u_R}(x,\phi) &= \frac{e^{2\sigma(\phi)}}{\sqrt{r}}\sum_n \chi^{i,(n)}_{u_R}(x) f^{R,c^i_B}_{(+-)}(y^{c^i_B(n)}_{(\mp\pm)},t), & \chi^i_{d_R}(x,\phi) &= \frac{e^{2\sigma(\phi)}}{\sqrt{r}}\sum_n \chi^{i,(n)}_{d_R}(x) f^{R,c^i_B}_{(+-)}(y^{c^i_B(n)}_{(\mp\pm)},t),\\
q^i_{u_R}(x,\phi) &= \frac{e^{2\sigma(\phi)}}{\sqrt{r}}\sum_n q^{i,(n)}_{u_R}(x) f^{R,c^i_B}_{(--)}(y^{c^i_B(n)}_{(\pm\pm)},t), & q^i_{d_R}(x,\phi) &= \frac{e^{2\sigma(\phi)}}{\sqrt{r}}\sum_n q^{i,(n)}_{d_R}(x) f^{R,c^i_B}_{(--)}(y^{c^i_B(n)}_{(\pm\pm)},t).
\end{aligned} \tag{28}$$

In Eq. (28), the eigenvalues $y^{c(n)}_{(\pm\pm)}(n\geq 1)$ satisfy the equation $R^{c,\epsilon}_{(\pm\pm)}(z)\equiv 0$, $y^{c(n)}_{(\mp\mp)}(n\geq 1)$ satisfy the equation $R^{c,\epsilon}_{(\mp\mp)}(z)\equiv 0$, $y^{c(n)}_{(\pm\mp)}(n\geq 1)$ satisfy the equation $R^{c,\epsilon}_{(\pm\mp)}(z)\equiv 0$, and the eigenvalues $y^{c(n)}_{(\mp\pm)}(n\geq 1)$ satisfy the equation $R^{c,\epsilon}_{(\mp\pm)}(z)\equiv 0$, respectively. Here, the concrete expressions of $R^{c,\epsilon}_{(\pm\pm)}(z)$, $R^{c,\epsilon}_{(\pm\mp)}(z)$, $R^{c,\epsilon}_{(\mp\pm)}(z)$, $R^{c,\epsilon}_{(\mp\mp)}(z)$ are

$$\begin{aligned}
R^{c,\epsilon}_{(\pm\pm)}(z) &= \begin{cases} Y_N(z)J_N(z\epsilon)-J_N(z)Y_N(z\epsilon), & c=N+\frac{1}{2}\\ J_{-c+(1/2)}(z)J_{c-(1/2)}(z\epsilon)-J_{c-(1/2)}(z)J_{-c+(1/2)}(z\epsilon), & c\neq N+\frac{1}{2}\end{cases},\\
R^{c,\epsilon}_{(\pm\mp)}(z) &= \begin{cases} J_{N+1}(z)Y_N(z\epsilon)-Y_{N+1}(z)J_N(z\epsilon), & c=N+\frac{1}{2}\\ J_{c+(1/2)}(z)J_{-c+(1/2)}(z\epsilon)+J_{-c-(1/2)}(z)J_{c-(1/2)}(z\epsilon), & c\neq N+\frac{1}{2}\end{cases},\\
R^{c,\epsilon}_{(\mp\pm)}(z) &= \begin{cases} Y_N(z)J_{N+1}(z\epsilon)-J_N(z)Y_{N+1}(z\epsilon), & c=N+\frac{1}{2}\\ J_{-c+(1/2)}(z)J_{c+(1/2)}(z\epsilon)+J_{c-(1/2)}(z)J_{-c-(1/2)}(z\epsilon), & c\neq N+\frac{1}{2}\end{cases},\\
R^{c,\epsilon}_{(\mp\mp)}(z) &= \begin{cases} J_{N+1}(z)Y_{N+1}(z\epsilon)-Y_{N+1}(z)J_{N+1}(z\epsilon), & c=N+\frac{1}{2}\\ J_{c+(1/2)}(z)J_{-c-(1/2)}(z\epsilon)-J_{-c-(1/2)}(z)J_{c+(1/2)}(z\epsilon), & c\neq N+\frac{1}{2}\end{cases}.
\end{aligned} \tag{29}$$

In terms of the variable t, those bulk profiles for five-dimensional fermions are formulated as

$$
\begin{aligned}
f^{L,c}_{(++)}(y^{c(n)}_{(\pm\pm)}, t) &= \frac{\sqrt{t}\varphi^{c(+)}_{L_{(uv)}}(y^{c(n)}_{(\pm\pm)}, t)}{N^{c(+)}_{L_{(uv)}}(y^{c(n)}_{(\pm\pm)})} = \frac{\sqrt{t}\varphi^{c(+)}_{L_{(ir)}}(y^{c(n)}_{(\pm\pm)}, t)}{N^{c(+)}_{L_{(ir)}}(y^{c(n)}_{(\pm\pm)})}, &\quad f^{L,c}_{(+-)}(y^{c(n)}_{(\pm\mp)}, t) &= \frac{\sqrt{t}\varphi^{c(+)}_{L_{(uv)}}(y^{c(n)}_{(\pm\mp)}, t)}{N^{c(+)}_{L_{(uv)}}(y^{c(n)}_{(\pm\mp)})} = \frac{\sqrt{t}\varphi^{c(-)}_{L_{(ir)}}(y^{c(n)}_{(\pm\mp)}, t)}{N^{c(-)}_{L_{(ir)}}(y^{c(n)}_{(\pm\mp)})},\\
f^{L,c}_{(-+)}(y^{c(n)}_{(\mp\pm)}, t) &= \frac{\sqrt{t}\varphi^{c(-)}_{L_{(uv)}}(y^{c(n)}_{(\mp\pm)}, t)}{N^{c(-)}_{L_{(uv)}}(y^{c(n)}_{(\mp\pm)})} = \frac{\sqrt{t}\varphi^{c(+)}_{L_{(ir)}}(y^{c(n)}_{(\mp\pm)}, t)}{N^{c(+)}_{L_{(ir)}}(y^{c(n)}_{(\mp\pm)})}, &\quad f^{L,c}_{(--)}(y^{c(n)}_{(\mp\mp)}, t) &= \frac{\sqrt{t}\varphi^{c(-)}_{L_{(uv)}}(y^{c(n)}_{(\mp\mp)}, t)}{N^{c(-)}_{L_{(uv)}}(y^{c(n)}_{(\mp\mp)})} = \frac{\sqrt{t}\varphi^{c(-)}_{L_{(ir)}}(y^{c(n)}_{(\mp\mp)}, t)}{N^{c(-)}_{L_{(ir)}}(y^{c(n)}_{(\mp\mp)})},\\
f^{R,c}_{(++)}(y^{c(n)}_{(\mp\mp)}, t) &= \frac{\sqrt{t}\varphi^{c(+)}_{R_{(uv)}}(y^{c(n)}_{(\mp\mp)}, t)}{N^{c(+)}_{R_{(uv)}}(y^{c(n)}_{(\mp\mp)})} = \frac{\sqrt{t}\varphi^{c(+)}_{R_{(ir)}}(y^{c(n)}_{(\mp\mp)}, t)}{N^{c(+)}_{R_{(ir)}}(y^{c(n)}_{(\mp\mp)})}, &\quad f^{R,c}_{(+-)}(y^{c(n)}_{(\mp\pm)}, t) &= \frac{\sqrt{t}\varphi^{c(+)}_{R_{(uv)}}(y^{c(n)}_{(\mp\pm)}, t)}{N^{c(+)}_{R_{(uv)}}(y^{c(n)}_{(\mp\pm)})} = \frac{\sqrt{t}\varphi^{c(-)}_{R_{(ir)}}(y^{c(n)}_{(\mp\pm)}, t)}{N^{c(-)}_{R_{(ir)}}(y^{c(n)}_{(\mp\pm)})},\\
f^{R,c}_{(-+)}(y^{c(n)}_{(\pm\mp)}, t) &= \frac{\sqrt{t}\varphi^{c(-)}_{R_{(uv)}}(y^{c(n)}_{(\pm\mp)}, t)}{N^{c(-)}_{R_{(uv)}}(y^{c(n)}_{(\pm\mp)})} = \frac{\sqrt{t}\varphi^{c(+)}_{R_{(ir)}}(y^{c(n)}_{(\pm\mp)}, t)}{N^{c(+)}_{R_{(ir)}}(y^{c(n)}_{(\pm\mp)})}, &\quad f^{R,c}_{(--)}(y^{c(n)}_{(\pm\pm)}, t) &= \frac{\sqrt{t}\varphi^{c(-)}_{R_{(uv)}}(y^{c(n)}_{(\pm\pm)}, t)}{N^{c(-)}_{R_{(uv)}}(y^{c(n)}_{(\pm\pm)})} = \frac{\sqrt{t}\varphi^{c(-)}_{R_{(ir)}}(y^{c(n)}_{(\pm\pm)}, t)}{N^{c(-)}_{R_{(ir)}}(y^{c(n)}_{(\pm\pm)})}
\end{aligned}
\tag{30}
$$

with

$$
\begin{aligned}
\varphi^{c(+)}_{L_{(ir)}}(y, t) &= \begin{cases} Y_N(y)J_{N+1}(yt) - J_N(y)Y_{N+1}(yt), & c = N + \frac{1}{2}\\ J_{-c+(1/2)}(y)J_{c+(1/2)}(yt) + J_{c-(1/2)}(y)J_{-c-(1/2)}(yt), & c \neq N + \frac{1}{2}\end{cases},\\
\varphi^{c(+)}_{L_{(uv)}}(y, t) &= \begin{cases} Y_N(y\epsilon)J_{N+1}(yt) - J_N(y\epsilon)Y_{N+1}(yt), & c = N + \frac{1}{2}\\ J_{-c+(1/2)}(y\epsilon)J_{c+(1/2)}(yt) + J_{c-(1/2)}(y\epsilon)J_{-c-(1/2)}(yt), & c \neq N + \frac{1}{2}\end{cases},\\
\varphi^{c(-)}_{L_{(ir)}}(y, t) &= \begin{cases} Y_{N+1}(y)J_{N+1}(yt) - J_{N+1}(y)Y_{N+1}(yt), & c = N + \frac{1}{2}\\ J_{-c-(1/2)}(y)J_{c+(1/2)}(yt) - J_{c+(1/2)}(y)J_{-c-(1/2)}(yt), & c \neq N + \frac{1}{2}\end{cases},\\
\varphi^{c(-)}_{L_{(uv)}}(y, t) &= \begin{cases} Y_{N+1}(y\epsilon)J_{N+1}(yt) - J_{N+1}(y\epsilon)Y_{N+1}(yt), & c = N + \frac{1}{2}\\ J_{-c-(1/2)}(y\epsilon)J_{c+(1/2)}(yt) - J_{c+(1/2)}(y\epsilon)J_{-c-(1/2)}(yt), & c \neq N + \frac{1}{2}\end{cases},\\
\varphi^{c(-)}_{R_{(ir)}}(y, t) &= \begin{cases} Y_N(y)J_N(yt) - J_N(y)Y_N(yt), & c = N + \frac{1}{2}\\ J_{-c+(1/2)}(y)J_{c-(1/2)}(yt) - J_{c-(1/2)}(y)J_{-c+(1/2)}(yt), & c \neq N + \frac{1}{2}\end{cases},\\
\varphi^{c(-)}_{R_{(uv)}}(y, t) &= \begin{cases} Y_N(y\epsilon)J_N(yt) - J_N(y\epsilon)Y_N(yt), & c = N + \frac{1}{2}\\ J_{-c+(1/2)}(y\epsilon)J_{c-(1/2)}(yt) - J_{c-(1/2)}(y\epsilon)J_{-c+(1/2)}(yt), & c \neq N + \frac{1}{2}\end{cases},\\
\varphi^{c(+)}_{R_{(ir)}}(y, t) &= \begin{cases} Y_{N+1}(y)J_N(yt) - J_{N+1}(y)Y_N(yt), & c = N + \frac{1}{2}\\ J_{-c-(1/2)}(y)J_{c-(1/2)}(yt) + J_{c+(1/2)}(y)J_{-c+(1/2)}(yt), & c \neq N + \frac{1}{2}\end{cases},\\
\varphi^{c(+)}_{R_{(uv)}}(y, t) &= \begin{cases} Y_{N+1}(y\epsilon)J_N(yt) - J_{N+1}(y\epsilon)Y_N(yt), & c = N + \frac{1}{2}\\ J_{-c-(1/2)}(y\epsilon)J_{c-(1/2)}(yt) + J_{c+(1/2)}(y\epsilon)J_{-c+(1/2)}(yt), & c \neq N + \frac{1}{2}\end{cases}.
\end{aligned}
\tag{31}
$$

It should be pointed out that those bulk profiles satisfy the following equations of motion:

$$
\begin{aligned}
&t\frac{\partial\varphi^{c(\pm)}_{L_{(ir)}}}{\partial t}(y, t) + \left(c + \frac{1}{2}\right)\varphi^{c(\pm)}_{L_{(ir)}}(y, t) = yt\varphi^{c(\mp)}_{R_{(ir)}}(y, t), \qquad t\frac{\partial\varphi^{c(\pm)}_{R_{(ir)}}}{\partial t}(z, t) - \left(c - \frac{1}{2}\right)\varphi^{c(\pm)}_{R_{(ir)}}(y, t) = -yt\varphi^{c(\mp)}_{L_{(ir)}}(y, t),\\
&t\frac{\partial\varphi^{c(\pm)}_{L_{(uv)}}}{\partial t}(y, t) + \left(c + \frac{1}{2}\right)\varphi^{c(\pm)}_{L_{(uv)}}(y, t) = yt\varphi^{c(\mp)}_{R_{(uv)}}(y, t), \qquad t\frac{\partial\varphi^{c(\pm)}_{R_{(uv)}}}{\partial t}(y, t) - \left(c - \frac{1}{2}\right)\varphi^{c(\pm)}_{R_{(uv)}}(y, t) = -yt\varphi^{c(\mp)}_{L_{(uv)}}(y, t).
\end{aligned}
\tag{32}
$$

Meanwhile, the normalization factors can be written as

$$
\begin{aligned}
[N^{c(+)}_{L_{(ir)}}(y)]^2 &= \frac{2}{kr\epsilon}\Big\{(\varphi^{c(+)}_{L_{(ir)}}(y,1))^2 - \epsilon^2(\varphi^{c(+)}_{L_{(ir)}}(y,\epsilon))^2 - \epsilon^2(\varphi^{c(-)}_{R_{(ir)}}(y,\epsilon))^2 + (2c+1)\frac{\epsilon}{y}\varphi^{c(+)}_{L_{(ir)}}(y,\epsilon)\varphi^{c(-)}_{R_{(ir)}}(y,\epsilon)\Big\},\\
[N^{c(-)}_{R_{(ir)}}(y)]^2 &= \frac{2}{kr\epsilon}\Big\{(\varphi^{c(+)}_{L_{(ir)}}(y,1))^2 - \epsilon^2(\varphi^{c(+)}_{L_{(ir)}}(y,\epsilon))^2 - \epsilon^2(\varphi^{c(-)}_{R_{(ir)}}(y,\epsilon))^2 + (2c-1)\frac{\epsilon}{y}\varphi^{c(+)}_{L_{(ir)}}(y,\epsilon)\varphi^{c(-)}_{R_{(ir)}}(y,\epsilon)\Big\},\\
[N^{c(-)}_{L_{(ir)}}(y)]^2 &= \frac{2}{kr\epsilon}\Big\{-\epsilon^2(\varphi^{c(-)}_{L_{(ir)}}(y,\epsilon))^2 + (\varphi^{c(+)}_{R_{(ir)}}(y,1))^2 - \epsilon^2(\varphi^{c(+)}_{R_{(ir)}}(y,\epsilon))^2 + (2c+1)\frac{\epsilon}{y}\varphi^{c(-)}_{L_{(ir)}}(y,\epsilon)\varphi^{c(+)}_{R_{(ir)}}(y,\epsilon)\Big\},\\
[N^{c(+)}_{R_{(ir)}}(y)]^2 &= \frac{2}{kr\epsilon}\Big\{-\epsilon^2(\varphi^{c(-)}_{L_{(ir)}}(y,\epsilon))^2 + (\varphi^{c(+)}_{R_{(ir)}}(y,1))^2 - \epsilon^2(\varphi^{c(+)}_{R_{(ir)}}(y,\epsilon))^2 + (2c-1)\frac{\epsilon}{y}\varphi^{c(-)}_{L_{(ir)}}(y,\epsilon)\varphi^{c(+)}_{R_{(ir)}}(y,\epsilon)\Big\},\\
[N^{c(+)}_{L_{(uv)}}(y)]^2 &= \frac{2}{kr\epsilon}\Big\{(\varphi^{c(+)}_{L_{(uv)}}(y,1))^2 - \epsilon^2(\varphi^{c(+)}_{L_{(uv)}}(y,\epsilon))^2 + (\varphi^{c(-)}_{R_{(uv)}}(y,1))^2 - \frac{2c+1}{y}\varphi^{c(+)}_{L_{(uv)}}(y,1)\varphi^{c(-)}_{R_{(uv)}}(y,1)\Big\},\\
[N^{c(-)}_{R_{(uv)}}(y)]^2 &= \frac{2}{kr\epsilon}\Big\{(\varphi^{c(+)}_{L_{(uv)}}(y,1))^2 - \epsilon^2(\varphi^{c(+)}_{L_{(uv)}}(y,\epsilon))^2 + (\varphi^{c(-)}_{R_{(uv)}}(y,1))^2 - \frac{2c-1}{y}\varphi^{c(+)}_{L_{(uv)}}(y,1)\varphi^{c(-)}_{R_{(uv)}}(y,1)\Big\},\\
[N^{c(-)}_{L_{(uv)}}(y)]^2 &= \frac{2}{kr\epsilon}\Big\{(\varphi^{c(-)}_{L_{(uv)}}(y,1))^2 + (\varphi^{c(+)}_{R_{(uv)}}(y,1))^2 - \epsilon^2(\varphi^{c(+)}_{R_{(uv)}}(y,\epsilon))^2 - \frac{2c+1}{y}\varphi^{c(-)}_{L_{(uv)}}(y,1)\varphi^{c(+)}_{R_{(uv)}}(y,1)\Big\},\\
[N^{c(+)}_{R_{(uv)}}(y)]^2 &= \frac{2}{kr\epsilon}\Big\{(\varphi^{c(-)}_{L_{(uv)}}(y,1))^2 + (\varphi^{c(+)}_{R_{(uv)}}(y,1))^2 - \epsilon^2(\varphi^{c(+)}_{R_{(uv)}}(y,\epsilon))^2 - \frac{2c-1}{y}\varphi^{c(-)}_{L_{(uv)}}(y,1)\varphi^{c(+)}_{R_{(uv)}}(y,1)\Big\}.
\end{aligned}
\tag{33}
$$

With the preparations above, we verify some lemmas on the eigenvalues of KK modes in extensions of the SM with a warped extra dimension, then show how to sum over the infinite series of KK modes using the residue theorem.

III. SUMMING OVER INFINITE SERIES OF KK MODES

As mentioned above, the radiative corrections from all virtual KK modes to the physics quantities at electroweak scale should be summed over in principle in order to obtain the theoretical predictions in extensions of the SM with a warped or universal extra dimension. When $f(z)$ is an analytic function except some limited-in-number nonzero poles $z_1, z_2, \cdots, z_{n_0}$ and the possible pole $z = 0$, we construct another function $G(z)$ in order to sum over the infinite series $\sum_{n=1}^{\infty}(f(y^{c(n)}_{(\mathrm{BCs})}) + f(y^{c(-n)}_{(\mathrm{BCs})}))$. Here the function $G(z)$ is analytic except its poles of order one $y^{c(\pm1)}_{(\mathrm{BCs})}, \cdots, y^{c(\pm n)}_{(\mathrm{BCs})}, \cdots$, and the possible pole of order m: $z = 0 (m \geq 1)$. Furthermore, the residues of $G(z)$ are uniform ones at nonzero poles $z = y^{c(\pm1)}_{(\mathrm{BCs})}, \cdots, y^{c(\pm n)}_{(\mathrm{BCs})}, \cdots$. If the closed rectifiable contour $C_{r_{(n)}}$ does not pass through any of the points $y^{c(\pm k)}_{(\mathrm{BCs})}(k = 1, 2, \cdots, n)$, and the region with the boundary $C_{r_{(n)}}$ contains the points $0, y^{c(\pm1)}_{(\mathrm{BCs})}, \cdots, y^{c(\pm n)}_{(\mathrm{BCs})}$ and $z_1, z_2, \cdots, z_{n_0}$, we obtain

$$
\begin{aligned}
\oint_{C_{r_{(n)}}} G(z)f(z)dz = i2\pi\Big\{&\sum_{i=1}^{n}[f(y^{c(i)}_{(\mathrm{BCs})}) + f(y^{c(-i)}_{(\mathrm{BCs})})]\\
&+ \mathrm{Res}(G(z)f(z), z=0)\\
&+ \sum_{i=1}^{n_0}\mathrm{Res}(G(z)f(z), z=z_i)\Big\}
\end{aligned}
\tag{34}
$$

according the residue theorem [65]. If the limit

$$\lim_{n\to\infty}\oint_{C_{r_{(n)}}} G(z)f(z)dz = 0,$$

then we can sum over the infinite series

$$
\begin{aligned}
\sum_{i=1}^{\infty}[f(y^{c(i)}_{(\mathrm{BCs})}) + f(y^{c(-i)}_{(\mathrm{BCs})})] = &-\mathrm{Res}(G(z)f(z), z=0)\\
&- \sum_{i=1}^{n_0}\mathrm{Res}(G(z)f(z), z=z_i).
\end{aligned}
\tag{35}
$$

To construct the function $G(z)$ and find the suitable contour $C_{r_{(n)}}$, we verify some lemmas on the eigenvalues of KK modes first.

Lemma 1.—If $y^{c(n)}_{(\mathrm{BCs})}(n = 1, 2, \cdots, \infty)$ satisfy the equation $R^{c,\epsilon}_{(\mathrm{BCs})}(y^{c(n)}_{(\mathrm{BCs})}) = 0$, where c denotes the bulk mass of five-dimensional fermions, then $y^{c(n)}_{(\mathrm{BCs})}$ is real.

Proof.—Taking (BCs) = $(\pm\pm)$ as an example, we show how to demonstrate that the roots of $R^{c,\epsilon}_{\pm\pm}(y^{c(n)}_{\pm\pm}) = 0$ are real. If the left-handed five-dimensional fermion satisfies the BCs $(++)$ and the dual right-handed five-dimensional fermion satisfies the BCs $(--)$, the corresponding bulk files are written as

$$
\begin{aligned}
f^{L,c}_{(++)}(y^{c(n)}_{(\pm\pm)}, t) &= \frac{\sqrt{t}\varphi^{c(+)}_{L_{(ir)}}(y^{c(n)}_{(\pm\pm)}, t)}{N^{c(+)}_{L_{(ir)}}(y^{c(n)}_{(\pm\pm)})} = \frac{\sqrt{t}\varphi^{c(+)}_{L_{(uv)}}(y^{c(n)}_{(\pm\pm)}, t)}{N^{c(+)}_{L_{(uv)}}(y^{c(n)}_{(\pm\pm)})},\\
f^{R,c}_{(--)}(y^{c(n)}_{(\pm\pm)}, t) &= \frac{\sqrt{t}\varphi^{c(-)}_{R_{(ir)}}(y^{c(n)}_{(\pm\pm)}, t)}{N^{c(-)}_{R_{(ir)}}(y^{c(n)}_{(\pm\pm)})} = \frac{\sqrt{t}\varphi^{c(-)}_{R_{(uv)}}(y^{c(n)}_{(\pm\pm)}, t)}{N^{c(-)}_{R_{(uv)}}(y^{c(n)}_{(\pm\pm)})},
\end{aligned}
\tag{36}
$$

where $\varphi^{c(+)}_{L_{(ir)}}(y^{c(n)}_{(\pm\pm)},t)$, $\varphi^{c(+)}_{L_{(uv)}}(y^{c(n)}_{(\pm\pm)},t)$, $\varphi^{c(-)}_{R_{(ir)}}(y^{c(n)}_{(\pm\pm)},t)$, $\varphi^{c(-)}_{R_{(uv)}}(y^{c(n)}_{(\pm\pm)},t)$ satisfy the equations of motion in Eq. (32). Using Eq. (32), we can easily derive the following equations:

$$\begin{aligned}
&\frac{1}{t}\frac{\partial}{\partial t}\Big(t\frac{\partial\varphi^{c(+)}_{L_{(ir)}}}{\partial t}\Big)(y^{c(n)}_{(\pm\pm)},t)\\
&\quad+\Big([y^{c(n)}_{(\pm\pm)}]^2-\frac{(c+\frac{1}{2})^2}{t^2}\Big)\varphi^{c(+)}_{L_{(ir)}}(y^{c(n)}_{(\pm\pm)},t)=0,\\
&\frac{1}{t}\frac{\partial}{\partial t}\Big(t\frac{\partial\varphi^{c(-)}_{R_{(ir)}}}{\partial t}\Big)(y^{c(n)}_{(\pm\pm)},t)\\
&\quad+\Big([y^{c(n)}_{(\pm\pm)}]^2-\frac{(c-\frac{1}{2})^2}{t^2}\Big)\varphi^{c(-)}_{R_{(ir)}}(y^{c(n)}_{(\pm\pm)},t)=0.
\end{aligned}\tag{37}$$

If $y^{c(n)}_{(\pm\pm)}=r+is$ (r, s are real and $s\neq 0$) satisfies $R^{c,\epsilon}_{(\pm\pm)}(y^{c(n)}_{(\pm\pm)})=0$, then its conjugate $\bar{y}^{c(n)}_{(\pm\pm)}=r-is$ also satisfies $R^{c,\epsilon}_{(\pm\pm)}(\bar{y}^{c(n)}_{(\pm\pm)})=0$. This implies

$$\begin{aligned}
&\frac{1}{t}\frac{\partial}{\partial t}\Big(t\frac{\partial\varphi^{c(+)}_{L_{(ir)}}}{\partial t}\Big)(\bar{y}^{c(n)}_{(\pm\pm)},t)\\
&\quad+\Big([\bar{y}^{c(n)}_{(\pm\pm)}]^2-\frac{(c+\frac{1}{2})^2}{t^2}\Big)\varphi^{c(+)}_{L_{(ir)}}(\bar{y}^{c(n)}_{(\pm\pm)},t)=0.
\end{aligned}\tag{38}$$

Using the first equation in Eq. (37) and that in Eq. (38), we find

$$\begin{aligned}
&([y^{c(n)}_{(\pm\pm)}]^2-[\bar{y}^{c(n)}_{(\pm\pm)}]^2)\int_\epsilon^1 dtt\varphi^{c(+)}_{L_{(ir)}}(y^{c(n)}_{(\pm\pm)},t)\varphi^{c(+)}_{L_{(ir)}}(\bar{y}^{c(n)}_{(\pm\pm)},t)\\
&=\Big\{t\varphi^{c(+)}_{L_{(ir)}}(y^{c(n)}_{(\pm\pm)},t)\frac{\partial\varphi^{c(+)}_{L_{(ir)}}}{\partial t}(\bar{y}^{c(n)}_{(\pm\pm)},t)\\
&\quad-t\varphi^{c(+)}_{L_{(ir)}}(\bar{y}^{c(n)}_{(\pm\pm)},t)\frac{\partial\varphi^{c(+)}_{L_{(ir)}}}{\partial t}(y^{c(n)}_{(\pm\pm)},t)\Big\}_\epsilon^1\\
&=\{t\bar{y}^{c(n)}_{(\pm\pm)}\varphi^{c(+)}_{L_{(ir)}}(y^{c(n)}_{(\pm\pm)},t)\varphi^{c(-)}_{R_{(ir)}}(\bar{y}^{c(n)}_{(\pm\pm)},t)\\
&\quad-t\bar{y}^{c(n)}_{(\pm\pm)}\varphi^{c(+)}_{L_{(ir)}}(\bar{y}^{c(n)}_{(\pm\pm)},t)\varphi^{c(-)}_{R_{(ir)}}(y^{c(n)}_{(\pm\pm)},t)\}_\epsilon^1.
\end{aligned}\tag{39}$$

Similarly, we also obtain

$$\begin{aligned}
&([y^{c(n)}_{(\pm\pm)}]^2-[\bar{y}^{c(n)}_{(\pm\pm)}]^2)\int_\epsilon^1 dtt\varphi^{c(-)}_{R_{(ir)}}(y^{c(n)}_{(\pm\pm)},t)\varphi^{c(-)}_{R_{(ir)}}(\bar{y}^{c(n)}_{(\pm\pm)},t)\\
&\quad=\{-t\bar{y}^{c(n)}_{(\pm\pm)}\varphi^{c(-)}_{R_{(ir)}}(y^{c(n)}_{(\pm\pm)},t)\varphi^{c(+)}_{L_{(ir)}}(\bar{y}^{c(n)}_{(\pm\pm)},t)\\
&\qquad+t\bar{y}^{c(n)}_{(\pm\pm)}\varphi^{c(-)}_{R_{(ir)}}(\bar{y}^{c(n)}_{(\pm\pm)},t)\varphi^{c(+)}_{L_{(ir)}}(y^{c(n)}_{(\pm\pm)},t)\}_\epsilon^1.
\end{aligned}\tag{40}$$

Since $t\in(\epsilon,1)$ is real, $\varphi^{c(+)}_{L_{(ir)}}(\bar{y}^{c(n)}_{(\pm\pm)},t)=[\varphi^{c(+)}_{L_{(ir)}}(y^{c(n)}_{(\pm\pm)},t)]^*$, $\varphi^{c(-)}_{R_{(ir)}}(\bar{y}^{c(n)}_{(\pm\pm)},t)=[\varphi^{c(-)}_{R_{(ir)}}(y^{c(n)}_{(\pm\pm)},t)]^*$, then

$$\begin{aligned}
&\int_\epsilon^1 dtt\varphi^{c(+)}_{L_{(ir)}}(y^{c(n)}_{(\pm\pm)},t)\varphi^{c(+)}_{L_{(ir)}}(\bar{y}^{c(n)}_{(\pm\pm)},t)\\
&\quad=\int_\epsilon^1 dtt|\varphi^{c(+)}_{L_{(ir)}}(y^{c(n)}_{(\pm\pm)},t)|^2>0,\\
&\int_\epsilon^1 dtt\varphi^{c(-)}_{R_{(ir)}}(y^{c(n)}_{(\pm\pm)},t)\varphi^{c(-)}_{R_{(ir)}}(\bar{y}^{c(n)}_{(\pm\pm)},t)\\
&\quad=\int_\epsilon^1 dtt|\varphi^{c(-)}_{R_{(ir)}}(y^{c(n)}_{(\pm\pm)},t)|^2>0
\end{aligned}\tag{41}$$

for the nontrivial functions $\varphi^{c(+)}_{L_{(ir)}}(y^{c(n)}_{(\pm\pm)},t)$, $\varphi^{c(-)}_{R_{(ir)}}(y^{c(n)}_{(\pm\pm)},t)$. If $r\neq 0$, then $[y^{c(n)}_{(\pm\pm)}]^2-[\bar{y}^{c(n)}_{(\pm\pm)}]^2=i4rs\neq 0$. Applying $(--)$ BCs satisfied by the bulk profiles of right-handed fermions, we derive the following equations from Eqs. (39) and (40):

$$\begin{aligned}
&\int_\epsilon^1 dtt|\varphi^{c(+)}_{L_{(ir)}}(y^{c(n)}_{(\pm\pm)},t)|^2=0,\\
&\int_\epsilon^1 dtt|\varphi^{c(-)}_{R_{(ir)}}(y^{c(n)}_{(\pm\pm)},t)|^2=0,
\end{aligned}\tag{42}$$

which are contrary to the inequalities in Eq. (41). For $r=0$,

$$\begin{aligned}
\int_\epsilon^1 dtt|\varphi^{c(+)}_{L_{(ir)}}(y^{c(n)}_{(\pm\pm)},t)|^2&=\lim_{r\to 0}\frac{1}{i4rs}\{t\bar{y}^{c(n)}_{(\pm\pm)}\varphi^{c(+)}_{L_{(ir)}}(y^{c(n)}_{(\pm\pm)},t)\varphi^{c(-)}_{R_{(ir)}}(\bar{y}^{c(n)}_{(\pm\pm)},t)-t\bar{y}^{c(n)}_{(\pm\pm)}\varphi^{c(+)}_{L_{(ir)}}(\bar{y}^{c(n)}_{(\pm\pm)},t)\varphi^{c(-)}_{R_{(ir)}}(y^{c(n)}_{(\pm\pm)},t)\}_\epsilon^1\\
&=\frac{1}{i4s}\Big\{t\varphi^{c(+)}_{L_{(ir)}}(y^{c(n)}_{(\pm\pm)},t)\varphi^{c(-)}_{R_{(ir)}}(\bar{y}^{c(n)}_{(\pm\pm)},t)+\frac{\bar{y}^{c(n)}_{(\pm\pm)}}{y^{c(n)}_{(\pm\pm)}}t^2\frac{\partial\varphi^{c(+)}_{L_{(ir)}}}{\partial t}(y^{c(n)}_{(\pm\pm)},t)\varphi^{c(-)}_{R_{(ir)}}(\bar{y}^{c(n)}_{(\pm\pm)},t)\\
&\quad+t^2\varphi^{c(+)}_{L_{(ir)}}(y^{c(n)}_{(\pm\pm)},t)\frac{\partial\varphi^{c(-)}_{R_{(ir)}}}{\partial t}(\bar{y}^{c(n)}_{(\pm\pm)},t)-t\varphi^{c(+)}_{L_{(ir)}}(\bar{y}^{c(n)}_{(\pm\pm)},t)\varphi^{c(-)}_{R_{(ir)}}(y^{c(n)}_{(\pm\pm)},t)\\
&\quad-\frac{y^{c(n)}_{(\pm\pm)}}{\bar{y}^{c(n)}_{(\pm\pm)}}t^2\frac{\partial\varphi^{c(+)}_{L_{(ir)}}}{\partial t}(\bar{y}^{c(n)}_{(\pm\pm)},t)\varphi^{c(-)}_{R_{(ir)}}(y^{c(n)}_{(\pm\pm)},t)-t^2\varphi^{c(+)}_{L_{(ir)}}(\bar{y}^{c(n)}_{(\pm\pm)},t)\frac{\partial\varphi^{c(-)}_{R_{(ir)}}}{\partial t}(y^{c(n)}_{(\pm\pm)},t)\Big\}_\epsilon^1\\
&=\frac{1}{2}\{|\varphi^{c(+)}_{L_{(ir)}}(y^{c(n)}_{(\pm\pm)},1)|^2-\epsilon^2|\varphi^{c(+)}_{L_{(ir)}}(y^{c(n)}_{(\pm\pm)},\epsilon)|^2\}.
\end{aligned}\tag{43}$$

In the last step, we apply the equation of motion for the bulk profiles of fermions in Eq. (32) and $(--)$ BCs satisfied by the bulk files of right-handed fermions. Similarly, we can derive

$$\int_\epsilon^1 dtt|\varphi_{R_{(ir)}}^{c(-)}(y_{(\pm\pm)}^{c(n)}, t)|^2 = -\frac{1}{2}\{|\varphi_{L_{(ir)}}^{c(+)}(y_{(\pm\pm)}^{c(n)}, 1)|^2 - \epsilon^2|\varphi_{L_{(ir)}}^{c(+)}(y_{(\pm\pm)}^{c(n)}, \epsilon)|^2\}. \quad (44)$$

Equations (43) and (44) are also contrary to the inequalities in Eq. (41). In other words, $R_{(\pm\pm)}^{c,\epsilon}(z) = 0$ only has the real roots. Analogously, we can verify that the equations $R_{(\pm\mp)}^{c,\epsilon}(z) = 0$, $R_{(\mp\pm)}^{c,\epsilon}(z) = 0$, $R_{(\mp\mp)}^{c,\epsilon}(z) = 0$, $R_{(++)}^{G,\epsilon}(z) = 0$, and $R_{(-+)}^{G,\epsilon}(z) = 0$ only have real roots.

Lemma 2.—If $y_{(\mathrm{BCs})}^{c(n)}(n = 1, 2, \cdots, \infty)$ satisfy $R_{(\mathrm{BCs})}^{c,\epsilon}(y_{(\mathrm{BCs})}^{c(n)}) = 0$, then $y_{(\mathrm{BCs})}^{c(-n)} = -y_{(\mathrm{BCs})}^{c(n)}$ and $\pm y_{(\mathrm{BCs})}^{c(1)}, \cdots, \pm y_{(\mathrm{BCs})}^{c(n)}, \cdots$ are the zeros of order one of the function $R_{(\mathrm{BCs})}^{c,\epsilon}(z)$.

Proof.—Assuming (BCs) $= (\pm\pm)$, we first demonstrate that $y_{(\pm\pm)}^{c(1)}, \cdots, y_{(\pm\pm)}^{c(n)}, \cdots$ are the zeros of order one of the function $R_{(\pm\pm)}^{c,\epsilon}(z)$. For $c \neq N + \frac{1}{2}$

$$\begin{aligned}\frac{dR_{(\pm\pm)}^{c,\epsilon}}{dz}(z) &= [J_{-c-(1/2)}(z)J_{c-(1/2)}(z\epsilon) + J_{c+(1/2)}(z)J_{-c+(1/2)}(z\epsilon)] \\ &\quad - \epsilon[J_{-c+(1/2)}(z)J_{c+(1/2)}(z\epsilon) + J_{c-(1/2)}(z)J_{-c-(1/2)}(z\epsilon)] \\ &\quad + \frac{2c-1}{z}[J_{-c+(1/2)}(z)J_{c-(1/2)}(z\epsilon) - J_{c-(1/2)}(z)J_{-c+(1/2)}(z\epsilon)] \\ &= \varphi_{R_{(ir)}}^{c(+)}(z, \epsilon) - \epsilon\varphi_{L_{(ir)}}^{c(+)}(z, \epsilon) + \frac{2c-1}{z}\varphi_{R_{(ir)}}^{c(-)}(z, \epsilon). \end{aligned} \quad (45)$$

As $c = N + 1/2$,

$$\begin{aligned}\frac{dR_{(\pm\pm)}^{c,\epsilon}}{dz}(z) &= -[Y_{N+1}(z)J_N(z\epsilon) - J_{N+1}(z)Y_N(z\epsilon)] \\ &\quad - \epsilon[Y_N(z)J_{N+1}(z\epsilon) - J_N(z)Y_{N+1}(z\epsilon)] \\ &\quad + \frac{2N}{z}[Y_N(z)J_N(z\epsilon) - J_N(z)Y_N(z\epsilon)] \\ &= \varphi_{R_{(ir)}}^{c(+)}(z, \epsilon) - \epsilon\varphi_{L_{(ir)}}^{c(+)}(z, \epsilon) + \frac{2c-1}{z}\varphi_{R_{(ir)}}^{c(-)}(z, \epsilon). \end{aligned} \quad (46)$$

When $y_{(\pm\pm)}^{c(n)}(n = 1, 2, \cdots, \infty)$ satisfy the equation $R_{(\pm\pm)}^{c,\epsilon}(y_{(\pm\pm)}^{c(n)}) = 0$, then

$$\frac{dR_{(\pm\pm)}^{c,\epsilon}}{dz}(z)|_{z=y_{(\pm\pm)}^{c(n)}} = \varphi_{R_{(ir)}}^{c(+)}(y_{(\pm\pm)}^{c(n)}, \epsilon) - \epsilon\varphi_{L_{(ir)}}^{c(+)}(y_{(\pm\pm)}^{c(n)}, \epsilon) \neq 0. \quad (47)$$

In other words, $y_{(\pm\pm)}^{c(n)}(n = 1, 2, \cdots, \infty)$ are the zeros of order one for the function $R_{(\pm\pm)}^{c,\epsilon}(z)$. Using concrete expressions of the Bessel functions J_ν and Y_ν, we can verify directly $R_{(\pm\pm)}^{c,\epsilon}(-y_{(\pm\pm)}^{c(n)}) = 0$ if $R_{(\pm\pm)}^{c,\epsilon}(y_{(\pm\pm)}^{c(n)}) = 0$. Furthermore, we can obtain those similar results on zeros of the functions $R_{(\pm\mp)}^{c,\epsilon}(z)$, $R_{(\mp\pm)}^{c,\epsilon}(z)$, $R_{(\mp\mp)}^{c,\epsilon}(z)$, $R_{(++)}^{G,\epsilon}(z)$ and $R_{(-+)}^{G,\epsilon}(z)$.

When $z \to 0$ and $c \neq N + \frac{1}{2}$, the function $R_{(\pm\pm)}^{c,\epsilon}(z)$ is approximated as

$$R_{(\pm\pm)}^{c,\epsilon}(z) = \frac{2(\epsilon^{c-(1/2)} - \epsilon^{(1/2)-c})}{(1-2c)\Gamma(\frac{1}{2}-c)\Gamma(\frac{1}{2}+c)}\{1 + \mathcal{O}(z^2)\}. \quad (48)$$

When $z \to 0$ and $c = N + \frac{1}{2}$, the function $R_{(\pm\pm)}^{c,\epsilon}(z)$ is approximated as

$$R_{(\pm\pm)}^{c,\epsilon}(z) = \begin{cases} \frac{1-\epsilon^{2N}}{N\pi\epsilon^N}\{1 + \mathcal{O}(z^2)\}, & N \neq 0 \\ -\frac{2\ln\epsilon}{\pi}\{1 + \mathcal{O}(z^2)\}, & N = 0 \end{cases}. \quad (49)$$

In other words, $z = 0$ is not the zero of $R_{(\pm\pm)}^{c,\epsilon}(z)$. Similarly, we find that $z = 0$ is not the zero of the functions $R_{(\mp\mp)}^{c,\epsilon}(z)$ as well as $R_{(++)}^{G,\epsilon}(z)$ also, and is the pole of order one of the functions $R_{(\pm\mp)}^{c,\epsilon}(z)$, $R_{(\mp\pm)}^{c,\epsilon}(z)$ together with $R_{(-+)}^{G,\epsilon}(z)$.

Lemma 3.—Let function $f(z)$ be analytic except for limited-in-number isolated singularities on the complex plane. If there are two constants $\mathcal{M} > 0$ and $\mathcal{R} > 0$, we have $|zf(z)| \leq \mathcal{M}$ when $|z| > \mathcal{R}$. Then

$$\lim_{n\to\infty}\oint_{C_{r_{(n)}}}\left\{\frac{2}{z} + \frac{1}{R_{(\mathrm{BCs})}^{c,\epsilon}(z)}\frac{dR_{(\mathrm{BCs})}^{c,\epsilon}}{dz}(z)\right\}f(z)dz = 0, \quad (50)$$

where the path $C_{r_{(n)}}$ is the rectangular contour with four vertices $(1 \pm i)r_{(n)}$ and $(-1 \pm i)r_{(n)}$ with $y_{(\mathrm{BCs})}^{c(n)} < r_{(n)} < y_{(\mathrm{BCs})}^{c(n+1)}$.

Proof.—First, we illustrate how to demonstrate the lemma for the case (BCs) $= (\pm\pm)$. Since $|zf(z)| \leq \mathcal{M}$ when $|z| > \mathcal{R}$, all singularities of the function $f(z)$ all distribute within the region $|z| \leq \mathcal{R}$. This implies that $zf(z)$ is analytic at $z = \infty$,

$$zf(z) = a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots, \qquad |z| > \mathcal{R}, \quad (51)$$

or

$$f(z) = \frac{a_0}{z} + \frac{a_1}{z^2} + \frac{a_2}{z^3} + \cdots, \qquad |z| > \mathcal{R}. \quad (52)$$

Applying the residue theorem, we have

$$\oint_{C_{r_{(n)}}}\left\{\frac{2}{z^2}+\frac{1}{zR^{c,\epsilon}_{(\pm\pm)}(z)}\frac{dR^{c,\epsilon}_{(\pm\pm)}}{dz}(z)\right\}dz = i2\pi\left\{\mathrm{Res}\left(\frac{2}{z^2}+\frac{1}{zR^{c,\epsilon}_{(\pm\pm)}(z)}\frac{dR^{c,\epsilon}_{(\pm\pm)}}{dz}(z), z=0\right) + \sum_{i=1}^{n}\left[\mathrm{Res}\left(\frac{1}{zR^{c,\epsilon}_{(\pm\pm)}(z)}\frac{dR^{c,\epsilon}_{(\pm\pm)}}{dz}(z), z = y^{c(i)}_{(\pm\pm)}\right) + \mathrm{Res}\left(\frac{1}{zR^{c,\epsilon}_{(\pm\pm)}(z)}\frac{dR^{c,\epsilon}_{(\pm\pm)}}{dz}(z), z=-y^{c(i)}_{(\pm\pm)}\right)\right]\right\}. \tag{53}$$

Because $2/z^2 + (dR^{c,\epsilon}_{(\pm\pm)}(z)/dz)/(zR^{c,\epsilon}_{(\pm\pm)}(z))$ is an even function of z, its Laurent series at the point $z = 0$ does not contain the term which is proportional to $1/z$. One directly has

$$\oint_{C_{r_{(n)}}}\left\{\frac{2}{z^2}+\frac{1}{zR^{c,\epsilon}_{(\pm\pm)}(z)}\frac{dR^{c,\epsilon}_{(\pm\pm)}}{dz}(z)\right\}dz = i2\pi\sum_{i=1}^{n}\left[\mathrm{Res}\left(\frac{1}{zR^{c,\epsilon}_{(\pm\pm)}(z)}\frac{dR^{c,\epsilon}_{(\pm\pm)}}{dz}(z), z = y^{c(i)}_{(\pm\pm)}\right) + \mathrm{Res}\left(\frac{1}{zR^{c,\epsilon}_{(\pm\pm)}(z)}\frac{dR^{c,\epsilon}_{(\pm\pm)}}{dz}(z), z = -y^{c(i)}_{(\pm\pm)}\right)\right] = i2\pi\sum_{i=1}^{n}\left[\frac{1}{y^{c(i)}_{(\pm\pm)}}+\frac{1}{-y^{c(i)}_{(\pm\pm)}}\right] = 0, \tag{54}$$

then gets

$$\oint_{C_{r_{(n)}}}\left\{\frac{2}{z}+\frac{1}{R^{c,\epsilon}_{(\pm\pm)}(z)}\frac{dR^{c,\epsilon}_{(\pm\pm)}}{dz}(z)\right\}f(z)dz = \oint_{C_{r_{(n)}}}\left\{\frac{2}{z}+\frac{1}{R^{c,\epsilon}_{(\pm\pm)}(z)}\frac{dR^{c,\epsilon}_{(\pm\pm)}}{dz}(z)\right\}\left\{f(z)-\frac{a_0}{z}\right\}dz. \tag{55}$$

For $f(z) - a_0/z = 1/z^2\{a_1 + a_2/z + \cdots + a_{n+1}/z^n + \cdots\}$, $a_1 + a_2/z + \cdots + a_{n+1}/z^n + \cdots$ is the analytic function and its absolute value has an upper limit in the region $|z| \geq \mathcal{R}'$, $\mathcal{R}' > \mathcal{R}$. Assuming

$$\left| a_1 + \frac{a_2}{z} + \cdots + \frac{a_{n+1}}{z^n} + \cdots \right| \leq \mathcal{M}', \quad \text{as } |z| \geq \mathcal{R}', \tag{56}$$

then one gets

$$\left| f(z) - \frac{a_0}{z} \right| \leq \frac{\mathcal{M}'}{|z|^2}, \quad \text{as } |z| \geq \mathcal{R}'. \tag{57}$$

As n is sufficiently large, we have $|z| \geq \mathcal{R}'$ for $z \in C_{r_{(n)}}$, then get

$$\left|\oint_{C_{r_{(n)}}}\left\{\frac{2}{z}+\frac{1}{R^{c,\epsilon}_{(\pm\pm)}(z)}\frac{dR^{c,\epsilon}_{(\pm\pm)}}{dz}(z)\right\}\left\{f(z)-\frac{a_0}{z}\right\}dz\right| \leq \frac{4r_{(n)}\mathcal{M}'}{r^2_{(n)}}\times\left\{\text{the upper bound of }\left|\frac{1}{R^{c,\epsilon}_{(\pm\pm)}(z)}\frac{dR^{c,\epsilon}_{(\pm\pm)}}{dz}(z)\right| \text{ for } z\in C_{r_{(n)}}\right\}. \tag{58}$$

In order to obtain the upper bound of $|(dR^{c,\epsilon}_{(\pm\pm)}(z)/dz)/R^{c,\epsilon}_{(\pm\pm)}(z)|$ for $z \in C_{r_{(n)}}$, we express the Bessel functions as [66]

$$\begin{aligned} J_\nu(z) = \frac{1}{\sqrt{2\pi z}}\Big\{&[e^{i(z-(\nu\pi/2)-(\pi/4))} + e^{-i(z-(\nu\pi/2)-(\pi/4))}]\left(1+\mathcal{O}\left(\frac{1}{z^2}\right)\right) \\ &+ \frac{i}{2\nu}\left(\nu^2-\frac{1}{4}\right)[e^{i(z-(\nu\pi/2)-(\pi/4))} - e^{-i(z-(\nu\pi/2)-(\pi/4))}]\left(1+\mathcal{O}\left(\frac{1}{z^2}\right)\right)\Big\}, \\ Y_\nu(z) = \frac{1}{\sqrt{2\pi z}}\Big\{&-i[e^{i(z-(\nu\pi/2)-(\pi/4))} - e^{-i(z-(\nu\pi/2)-(\pi/4))}]\left(1+\mathcal{O}\left(\frac{1}{z^2}\right)\right) \\ &+ \frac{1}{2\nu}\left(\nu^2-\frac{1}{4}\right)[e^{i(z-(\nu\pi/2)-(\pi/4))} + e^{-i(z-(\nu\pi/2)-(\pi/4))}]\left(1+\mathcal{O}\left(\frac{1}{z^2}\right)\right)\Big\}, \end{aligned} \tag{59}$$

for $|z| \to \infty$. Using the above equations, we approximate $R^{c,\epsilon}_{(\pm\pm)}(z)$ and $dR^{c,\epsilon}_{(\pm\pm)}(z)/dz$ as

$$\begin{aligned} R^{c,\epsilon}_{(\pm\pm)}(z) = -\frac{i\cos c\pi}{\pi\sqrt{\epsilon}z}\Big\{&[e^{i(1-\epsilon)z} - e^{-i(1-\epsilon)z}]\left(1+\mathcal{O}\left(\frac{1}{z^2}\right)\right) \\ &+ \frac{ic(c-1)}{2z}\left(\frac{1}{\epsilon}-1\right)[e^{i(1-\epsilon)z} + e^{-i(1-\epsilon)z}]\left(1+\mathcal{O}\left(\frac{1}{z^2}\right)\right)\Big\}, \\ \frac{dR^{c,\epsilon}_{(\pm\pm)}}{dz}(z) = \frac{\cos c\pi}{\pi\sqrt{\epsilon}z}\Big\{&(1-\epsilon)[e^{i(1-\epsilon)z} + e^{-i(1-\epsilon)z}]\left(1+\mathcal{O}\left(\frac{1}{z^2}\right)\right) \\ &+ \frac{i}{z}\left[1+\left(1-\frac{(1+\epsilon^2)}{2\epsilon}\right)c(c-1)\right][e^{i(1-\epsilon)z} - e^{-i(1-\epsilon)z}]\left(1+\mathcal{O}\left(\frac{1}{z^2}\right)\right)\Big\}, \end{aligned} \tag{60}$$

for $c \neq N + 1/2$ and $|z| \to \infty$. When $c = N + 1/2$, the functions $R^{c,\epsilon}_{(\pm\pm)}(z)$ and $dR^{c,\epsilon}_{(\pm\pm)}(z)/dz$ at $|z| \to \infty$ can be similarly approximated as

$$R^{c,\epsilon}_{(\pm\pm)}(z) = -\frac{i}{\pi\sqrt{\epsilon}z}\left\{[e^{i(1-\epsilon)z} - e^{-i(1-\epsilon)z}]\left(1+\mathcal{O}\left(\frac{1}{z^2}\right)\right) + \frac{i}{2z}\left(N^2-\frac{1}{4}\right)\left(\frac{1}{\epsilon}-1\right)[e^{i(1-\epsilon)z} + e^{-i(1-\epsilon)z}]\left(1+\mathcal{O}\left(\frac{1}{z^2}\right)\right)\right\},$$

$$\frac{dR^{c,\epsilon}_{(\pm\pm)}}{dz}(z) = \frac{1}{\pi\sqrt{\epsilon}z}\left\{(1-\epsilon)[e^{i(1-\epsilon)z} + e^{-i(1-\epsilon)z}]\left(1+\mathcal{O}\left(\frac{1}{z^2}\right)\right) + \frac{i}{z}\left[1+\left(1-\frac{(1+\epsilon^2)}{2\epsilon}\right)\left(N^2-\frac{1}{4}\right)\right] \times [e^{i(1-\epsilon)z} - e^{-i(1-\epsilon)z}]\left(1+\mathcal{O}\left(\frac{1}{z^2}\right)\right)\right\}. \quad (61)$$

Using Eq. (60) and (61), one obtains

$$\frac{1}{R^{c,\epsilon}_{(\pm\pm)}(z)}\frac{dR^{c,\epsilon}_{(\pm\pm)}}{dz}(z) = i(1-\epsilon)\frac{e^{i(1-\epsilon)z} + e^{-i(1-\epsilon)z} + \frac{i}{z}[\frac{1}{1-\epsilon} - \frac{(1-\epsilon)c(c-1)}{2\epsilon}][e^{i(1-\epsilon)z} - e^{-i(1-\epsilon)z}]}{e^{i(1-\epsilon)z} - e^{-i(1-\epsilon)z} + \frac{ic(c-1)}{2z}(\frac{1}{\epsilon}-1)[e^{i(1-\epsilon)z} + e^{-i(1-\epsilon)z}]}\left[1+\mathcal{O}\left(\frac{1}{z^2}\right)\right]. \quad (62)$$

If n is sufficiently large, $y^{c(n)}_{(\pm\pm)}$ is approximately given by [63]

$$y^{c(n)}_{(\pm\pm)} \simeq \left[n + \frac{1}{2}\left(\left|c+\frac{1}{2}\right| - 1\right) - \frac{1}{4}\right]\pi. \quad (63)$$

The fact implies that the interval between $y^{c(n)}_{(\pm\pm)}$ and $y^{c(n+1)}_{(\pm\pm)}$ is about π as $n \gg 1$. When

$$y^{c(n)}_{(\pm\pm)} < \frac{\pi}{1-\epsilon}\left(N_0+\frac{1}{4}\right) < \frac{\pi}{1-\epsilon}\left(N_0+\frac{3}{4}\right) \le y^{c(n+1)}_{(\pm\pm)} \quad (64)$$

or

$$y^{c(n)}_{(\pm\pm)} < \frac{\pi}{1-\epsilon}\left(N_0+\frac{1}{4}\right) < y^{c(n+1)}_{(\pm\pm)} \le \frac{\pi}{1-\epsilon}\left(N_0+\frac{3}{4}\right), \quad (65)$$

where the positive integer N_0 obviously turns large along with increasing of the number n, one can choose

$$r_{(n)} = \frac{\pi}{1-\epsilon}\left(N_0+\frac{1}{4}\right). \quad (66)$$

When the point z belongs to the left and right borders of $C_{r_{(n)}}$, i.e. $z = \mp(N_0+1/4)\pi/(1-\epsilon) + iy$ with $-(N_0+1/4)\pi/(1-\epsilon) \le y \le (N_0+1/4)\pi/(1-\epsilon)$, we have

$$\left|\frac{1}{R^{c,\epsilon}_{(\pm\pm)}(z)}\frac{dR^{c,\epsilon}_{(\pm\pm)}}{dz}(z)\right| \le (1-\epsilon)\frac{|e^{i(1-\epsilon)z} + e^{-i(1-\epsilon)z}| + \frac{1}{|z|}|\frac{1}{1-\epsilon} - \frac{(1-\epsilon)c(c-1)}{2\epsilon}||e^{i(1-\epsilon)z} - e^{-i(1-\epsilon)z}|}{|e^{i(1-\epsilon)z} - e^{-i(1-\epsilon)z}| - \frac{|c(c-1)|}{2|z|}(\frac{1}{\epsilon}-1)|e^{i(1-\epsilon)z} + e^{-i(1-\epsilon)z}|}\left[1+\mathcal{O}\left(\frac{1}{|z|^2}\right)\right|$$
$$\le (1-\epsilon)\left\{1+\frac{1}{r_{(n)}}\left[\frac{1}{1-\epsilon} + |c(c-1)|\left(\frac{1}{\epsilon}-1\right)\right]\right\}\left[1+\mathcal{O}\left(\frac{1}{r^2_{(n)}}\right)\right]. \quad (67)$$

When the point z belongs to the upper and lower borders of $C_{r_{(n)}}$, i.e. $z = x \pm i(N_0+1/4)\pi/(1-\epsilon)$ with $-(N_0+1/4)\pi/(1-\epsilon) \le x \le (N_0+1/4)\pi/(1-\epsilon)$, we similarly obtain

$$\left|\frac{1}{R^{c,\epsilon}_{(\pm\pm)}(z)}\frac{dR^{c,\epsilon}_{(\pm\pm)}}{dz}(z)\right| \le (1-\epsilon)\frac{|e^{i2(1-\epsilon)x} + e^{(2N_0+(1/2))\pi}| + \frac{1}{|z|}|\frac{1}{1-\epsilon} - \frac{(1-\epsilon)c(c-1)}{2\epsilon}||e^{i2(1-\epsilon)x} - e^{(2N_0+(1/2))\pi}|}{|e^{i2(1-\epsilon)x} - e^{(2N_0+(1/2))\pi}| - \frac{|c(c-1)|}{2|z|}(\frac{1}{\epsilon}-1)|e^{i2(1-\epsilon)x} + e^{(2N_0+(1/2))\pi}|}\left[1+\mathcal{O}\left(\frac{1}{|z|^2}\right)\right|$$
$$\le (1-\epsilon)\left\{1+\frac{1}{|z|}\left|\frac{1}{1-\epsilon} - \frac{(1-\epsilon)c(c-1)}{2\epsilon}\right| + \frac{|c(c-1)|}{2|z|}\left(\frac{1}{\epsilon}-1\right)\right\}$$
$$\times \frac{e^{(2N_0+(1/2))\pi}+1}{e^{(2N_0+(1/2))\pi} - 1 - \frac{|c(c-1)|}{|z|}(\frac{1}{\epsilon}-1)}\left[1+\mathcal{O}\left(\frac{1}{|z|^2}\right)\right]$$
$$\le 2(1-\epsilon)\left\{1+\frac{1}{r_{(n)}}\left[\frac{1}{1-\epsilon} + |c(c-1)|\left(\frac{1}{\epsilon}-1\right)\right]\right\}\left[1+\mathcal{O}\left(\frac{1}{r^2_{(n)}}\right)\right]. \quad (68)$$

As

$$\frac{\pi}{1-\epsilon}\left(N_0+\frac{1}{4}\right) \le y^{c(n)}_{(\pm\pm)} < \frac{\pi}{1-\epsilon}\left(N_0+\frac{3}{4}\right) < y^{c(n+1)}_{(\pm\pm)}, \quad (69)$$

we choose

$$r_{(n)} = \frac{\pi}{1-\epsilon}\left(N_0 + \frac{3}{4}\right), \tag{70}$$

and similarly get the upper bounds in Eqs. (67) and (68). Applying Eq. (58), we have

$$\lim_{r_{(n)}\to\infty}\left|\oint_{C_{r_{(n)}}}\left\{\frac{2}{z} + \frac{1}{R^{c,\epsilon}_{(\pm\pm)}(z)}\frac{dR^{c,\epsilon}_{(\pm\pm)}}{dz}(z)\right\}\left\{f(z) - \frac{a_0}{z}\right\}dz\right| \leq \lim_{r_{(n)}\to\infty}\frac{4(1-\epsilon)\mathcal{M}'}{r_{(n)}}\left\{1 + \frac{1}{r_{(n)}}\left[\frac{1}{1-\epsilon} + |c(c-1)|\left(\frac{1}{\epsilon} - 1\right)\right]\right\}\left[1 + \mathcal{O}\left(\frac{1}{r^2_{(n)}}\right)\right] = 0. \tag{71}$$

In other words, the integral

$$\lim_{r_{(n)}\to\infty}\oint_{C_{r_{(n)}}}\left\{\frac{2}{z} + \frac{1}{R^{c,\epsilon}_{(\pm\pm)}(z)}\frac{dR^{c,\epsilon}_{(\pm\pm)}}{dz}(z)\right\}f(z) = 0. \tag{72}$$

Using Eqs. (29) and (59), we have

$$\frac{1}{R^{c,\epsilon}_{(\pm\mp)}(z)}\frac{dR^{c,\epsilon}_{(\pm\mp)}}{dz}(z) = i(1-\epsilon)\frac{e^{i(1-\epsilon)z} - e^{-i(1-\epsilon)z}}{e^{i(1-\epsilon)z} + e^{-i(1-\epsilon)z}}\left\{1 - \frac{i}{2z}\left[\frac{c(c-1)}{\epsilon} - c - 1 + \frac{(c^2-1)\epsilon^2}{1-\epsilon}\right]\frac{e^{i(1-\epsilon)z} + e^{-i(1-\epsilon)z}}{e^{i(1-\epsilon)z} - e^{-i(1-\epsilon)z}} + \frac{ic}{2z}\left(\frac{c-1}{\epsilon} - c - 1\right)\frac{e^{i(1-\epsilon)z} - e^{-i(1-\epsilon)z}}{e^{i(1-\epsilon)z} + e^{-i(1-\epsilon)z}}\right\} \times\left[1 + \mathcal{O}\left(\frac{1}{z^2}\right)\right], \tag{73}$$

for the case BCs $= (\pm\mp)$. If n is sufficiently large, $y^{c(n)}_{(\pm\mp)}$ is approximately given by [63]

$$y^{c(n)}_{(\pm\mp)} \simeq \left[n + \frac{1}{2}\left(\left|c + \frac{1}{2}\right| - 1\right) + \frac{1}{4}\right]\pi. \tag{74}$$

The fact also implies that the interval between $y^{c(n)}_{(\pm\mp)}$ and $y^{c(n+1)}_{(\pm\mp)}$ is about π as $n \gg 1$. When

$$y^{c(n)}_{(\pm\pm)} < \frac{\pi}{1-\epsilon}\left(N_0 - \frac{1}{4}\right) < \frac{\pi}{1-\epsilon}\left(N_0 + \frac{1}{4}\right) \leq y^{c(n+1)}_{(\pm\pm)} \tag{75}$$

or

$$y^{c(n)}_{(\pm\pm)} < \frac{\pi}{1-\epsilon}\left(N_0 - \frac{1}{4}\right) < y^{c(n+1)}_{(\pm\pm)} \leq \frac{\pi}{1-\epsilon}\left(N_0 + \frac{1}{4}\right), \tag{76}$$

one can choose

$$r_{(n)} = \frac{\pi}{1-\epsilon}\left(N_0 - \frac{1}{4}\right), \tag{77}$$

where the positive integer N_0 obviously turns large along with increasing of the number n. As

$$\frac{\pi}{1-\epsilon}\left(N_0 - \frac{1}{4}\right) \leq y^{c(n)}_{(\pm\pm)} < \frac{\pi}{1-\epsilon}\left(N_0 + \frac{1}{4}\right) < y^{c(n+1)}_{(\pm\pm)}, \tag{78}$$

one can choose $r_{(n)} = (N_0 + 1/4)\pi/(1-\epsilon)$. Then performing the similar analysis above, we finally get

$$\lim_{r_{(n)}\to\infty}\oint_{C_{r_{(n)}}}\left\{\frac{2}{z} + \frac{1}{R^{c,\epsilon}_{(\pm\mp)}(z)}\frac{dR^{c,\epsilon}_{(\pm\mp)}}{dz}(z)\right\}f(z) = 0. \tag{79}$$

As for the functions $R^{c,\epsilon}_{(\mp\pm)}(z)$, $R^{c,\epsilon}_{(\mp\mp)}(z)$, $R^{G,\epsilon}_{(++)}(z)$, and $R^{G,\epsilon}_{(-+)}(z)$, we derive the similar equations.

Using the lemmas verified above and Eq. (34), we summarize the summing over infinite series of KK modes as

$$\sum_{i=1}^{\infty}[f(y^{c(i)}_{(\mathrm{BCs})}) + f(-y^{c(i)}_{(\mathrm{BCs})})] = -\mathrm{Res}\left[\left(\frac{2}{z} + \frac{1}{R^{c,\epsilon}_{(\mathrm{BCs})}(z)}\frac{dR^{c,\epsilon}_{(\mathrm{BCs})}}{dz}(z)\right)f(z), z = 0\right] - \sum_{i=1}^{n_0}\mathrm{Res}\left[\left(\frac{2}{z} + \frac{1}{R^{c,\epsilon}_{(\mathrm{BCs})}(z)}\frac{dR^{c,\epsilon}_{(\mathrm{BCs})}}{dz}(z)\right)f(z), z = z_i\right],$$

$$\sum_{i=1}^{\infty}[f(y^{G(i)}_{(\mathrm{BCs})}) + f(-y^{G(i)}_{(\mathrm{BCs})})] = -\mathrm{Res}\left[\left(\frac{2}{z} + \frac{1}{R^{G,\epsilon}_{(\mathrm{BCs})}(z)}\frac{dR^{G,\epsilon}_{(\mathrm{BCs})}}{dz}(z)\right)f(z), z = 0\right] - \sum_{i=1}^{n_0}\mathrm{Res}\left[\left(\frac{2}{z} + \frac{1}{R^{G,\epsilon}_{(\mathrm{BCs})}(z)}\frac{dR^{G,\epsilon}_{(\mathrm{BCs})}}{dz}(z)\right)f(z), z = z_i\right], \tag{80}$$

where

$$\lim_{|z|\to\infty}|zf(z)| \leq \mathcal{M}, \qquad 0 < \mathcal{M} < \infty. \tag{81}$$

Actually, Eq. (81) is the sufficient condition to judge if the infinite series $\sum_{i=1}^{\infty}[f(y^{c(i)}_{(\mathrm{BCs})}) + f(-y^{c(i)}_{(\mathrm{BCs})})]$ is convergent.

In extensions of the SM with a warped extra dimension, the bulk profiles in Eqs. (24) and (30) affect amplitudes for relevant processes in terms of $[\chi^G_{(\mathrm{BCs})}(y^{G(n)}_{(\mathrm{BCs})}, t)] \times [\chi^G_{(\mathrm{BCs})}(y^{G(n)}_{(\mathrm{BCs})}, t')]$, $[f^{L,c}_{(\mathrm{BCs})}(y^{c(n)}_{(\mathrm{BCs})}, t)][f^{L,c}_{(\mathrm{BCs})}(y^{c(n)}_{(\mathrm{BCs})}, t')]$, and $[f^{R,c}_{(\mathrm{BCs})}(y^{c(n)}_{(\mathrm{BCs})}, t)][f^{R,c}_{(\mathrm{BCs})}(y^{c(n)}_{(\mathrm{BCs})}, t')]$. In order to sum over the infinite series of KK modes properly, one should analytically extend the above combinations of bulk profiles to the complex plane. Here, we illustrate how to extend analytically the combinations of bulk profiles for gauge fields satisfying $(++)$ $[(-+)]$ BCs in the complex plane. When $y = y^{G(n)}_{(++)}$ satisfies the equation $R^{G,\epsilon}_{(++)}(y^{G(n)}_{(++)}) = 0$, the combination of bulk profiles for gauge fields with $(++)$ BCs can be formulated as

$$[\chi^G_{(++)}(y,t)][\chi^G_{(++)}(y,t')] = \frac{tt'\Phi^{G(+)}_{(uv)}(y,t)\Phi^{G(+)}_{(uv)}(y,t')}{[N^{G(+)}_{(uv)}(y)]^2} = \frac{tt'\Phi^{G(+)}_{(ir)}(y,t)\Phi^{G(+)}_{(ir)}(y,t')}{[N^{G(+)}_{(ir)}(y)]^2} = \frac{tt'\Phi^{G(+)}_{(uv)}(y,t)\Phi^{G(+)}_{(ir)}(y,t')}{[N^{G(+)}_{(uv)}(y)][N^{G(+)}_{(ir)}(y)]} = \frac{tt'\Phi^{G(+)}_{(ir)}(y,t)\Phi^{G(+)}_{(uv)}(y,t')}{[N^{G(+)}_{(uv)}(y)][N^{G(+)}_{(ir)}(y)]}. \quad (82)$$

When $y = y^{G(n)}_{(-+)}$ satisfies the equation $R^{G,\epsilon}_{(-+)}(y^{G(n)}_{(-+)}) = 0$, the combination of bulk profiles for gauge fields with the BCs $(-+)$ can be written as

$$[\chi^G_{(-+)}(y,t)][\chi^G_{(-+)}(y,t')] = \frac{tt'\Phi^{G(-)}_{(uv)}(y,t)\Phi^{G(-)}_{(uv)}(y,t')}{[N^{G(-)}_{(uv)}(y)]^2} = \frac{tt'\Phi^{G(+)}_{(ir)}(y,t)\Phi^{G(+)}_{(ir)}(y,t')}{[N^{G(+)}_{(ir)}(y)]^2} = \frac{tt'\Phi^{G(-)}_{(uv)}(y,t)\Phi^{G(+)}_{(ir)}(y,t')}{[N^{G(-)}_{(uv)}(y)][N^{G(+)}_{(ir)}(y)]} = \frac{tt'\Phi^{G(+)}_{(ir)}(y,t)\Phi^{G(-)}_{(uv)}(y,t')}{[N^{G(-)}_{(uv)}(y)][N^{G(+)}_{(ir)}(y)]}. \quad (83)$$

The combinations of bulk profiles for gauge fields certainly satisfy the corresponding BCs,

$$\begin{aligned} &\frac{\partial}{\partial t_{(uv)}}[\chi^G_{(++)}(y,t)][\chi^G_{(++)}(y,t')]|_{t_{(uv)}=\epsilon} = 0,\\ &\frac{\partial}{\partial t_{(ir)}}[\chi^G_{(++)}(y,t)][\chi^G_{(++)}(y,t')]|_{t_{(ir)}=1} = 0,\\ &[\chi^G_{(-+)}(y,t)][\chi^G_{(-+)}(y,t')]|_{t_{(uv)}=\epsilon} = 0,\\ &\frac{\partial}{\partial t_{(ir)}}[\chi^G_{(-+)}(y,t)][\chi^G_{(-+)}(y,t')]|_{t_{(ir)}=1} = 0, \end{aligned} \quad (84)$$

with $t_{(uv)} = \min(t,t')$, $t_{(ir)} = \max(t,t')$. Considering Eq. (84), we analytically extend the combinations of bulk profiles from Eqs. (82) and (83) in the complex plane as

$$\begin{aligned} &[\chi^G_{(++)}(z,t)][\chi^G_{(++)}(z,t')]\\ &= \frac{tt'}{[N^{G(+)}_{(uv)}(z)][N^{G(+)}_{(ir)}(z)]}\{\theta(t-t')\Phi^{G(+)}_{(uv)}(z,t')\Phi^{G(+)}_{(ir)}(z,t)\\ &\quad + \theta(t'-t)\Phi^{G(+)}_{(uv)}(z,t)\Phi^{G(+)}_{(ir)}(z,t')\},\\ &[\chi^G_{(-+)}(z,t)][\chi^G_{(-+)}(z,t')]\\ &= \frac{tt'}{[N^{G(-)}_{(uv)}(z)][N^{G(+)}_{(ir)}(z)]}\{\theta(t-t')\Phi^{G(-)}_{(uv)}(z,t')\Phi^{G(+)}_{(ir)}(z,t)\\ &\quad + \theta(t'-t)\Phi^{G(-)}_{(uv)}(z,t)\Phi^{G(+)}_{(ir)}(z,t')\}. \end{aligned} \quad (85)$$

Here, the step function $\theta(x)$ is defined as

$$\theta(x) = \begin{cases} 1, & x > 0;\\ \frac{1}{2}, & x = 0;\\ 0, & x < 0. \end{cases} \quad (86)$$

To guarantee that the combinations of bulk profiles are uniformly bounded in the complex plane, we analytically extend the corresponding normalization factors in Eq. (85) as

$$\begin{aligned} |N^{G(+)}_{(uv)}(z)|^2 &= \frac{2}{kr(z^2-\bar{z}^2)}\{\bar{z}\Phi^{G(+)}_{(uv)}(z,1)\Psi^{G(-)}_{(uv)}(\bar{z},1)\\ &\quad - z\Phi^{G(+)}_{(uv)}(\bar{z},1)\Psi^{G(-)}_{(uv)}(z,1)\} + \frac{2}{kr}\Upsilon(z),\\ |N^{G(-)}_{(uv)}(z)|^2 &= \frac{2}{kr(z^2-\bar{z}^2)}\{\bar{z}\Phi^{G(-)}_{(uv)}(z,1)\Psi^{G(+)}_{(uv)}(\bar{z},1)\\ &\quad - z\Phi^{G(-)}_{(uv)}(\bar{z},1)\Psi^{G(+)}_{(uv)}(z,1)\} + \frac{2}{kr}\Upsilon(z),\\ |N^{G(+)}_{(ir)}(z)|^2 &= \frac{2\epsilon}{kr(z^2-\bar{z}^2)}\{z\Phi^{G(+)}_{(ir)}(\bar{z},\epsilon)\Psi^{G(-)}_{(ir)}(z,\epsilon)\\ &\quad - \bar{z}\Phi^{G(+)}_{(ir)}(z,\epsilon)\Psi^{G(-)}_{(ir)}(\bar{z},\epsilon)\} + \frac{2}{kr}\Upsilon(z). \end{aligned} \quad (87)$$

Here, $\bar{z}$ represents the conjugate of z, and the non-negative function $\Upsilon(z)$ is defined as

$$\Upsilon(z) = \frac{1}{\pi^2|z|^2}\bigg\{(1-\epsilon)(e^{-i(1-\epsilon)(z-\bar{z})} + e^{i(1-\epsilon)(z-\bar{z})}) + \frac{e^{-i(1-\epsilon)(z-\bar{z})} - e^{i(1-\epsilon)(z-\bar{z})}}{i(z-\bar{z})}\bigg\}. \quad (88)$$

In the limit of $\bar{z} = z$ (i.e. z is real), one easily gets

$$\lim_{\bar{z}\to z}\Upsilon(z) = 0 \quad (89)$$

and the normalization factors in Eq. (87) recover the corresponding expressions in Eq. (27). Similarly, we can analytically generalize the normalization constants of bulk profiles for fermions as

$$
\begin{aligned}
|N^{c(+)}_{L_{(ir)}}(z)|^2 &= \frac{2}{kr(z^2-\bar{z}^2)}\{z\varphi^{c(+)}_{L_{(ir)}}(\bar{z},\epsilon)\varphi^{c(-)}_{R_{(ir)}}(z,\epsilon) - \bar{z}\varphi^{c(+)}_{L_{(ir)}}(z,\epsilon)\varphi^{c(-)}_{R_{(ir)}}(\bar{z},\epsilon)\} + \frac{2\cos^2 c\pi}{kr\epsilon}\Upsilon(z),\\
|N^{c(+)}_{L_{(uv)}}(z)|^2 &= \frac{2}{kr\epsilon(z^2-\bar{z}^2)}\{\bar{z}\varphi^{c(+)}_{L_{(uv)}}(z,1)\varphi^{c(-)}_{R_{(uv)}}(\bar{z},1) - z\varphi^{c(+)}_{L_{(uv)}}(\bar{z},1)\varphi^{c(-)}_{R_{(uv)}}(z,1)\} + \frac{2\cos^2 c\pi}{kr\epsilon}\Upsilon(z),\\
|N^{c(-)}_{R_{(ir)}}(z)|^2 &= \frac{2}{kr(z^2-\bar{z}^2)}\{\bar{z}\varphi^{c(+)}_{L_{(ir)}}(\bar{z},\epsilon)\varphi^{c(-)}_{R_{(ir)}}(z,\epsilon) - z\varphi^{c(+)}_{L_{(ir)}}(z,\epsilon)\varphi^{c(-)}_{R_{(ir)}}(\bar{z},\epsilon)\} + \frac{2\cos^2 c\pi}{kr\epsilon}\Upsilon(z),\\
|N^{c(-)}_{R_{(uv)}}(z)|^2 &= \frac{2}{kr\epsilon(z^2-\bar{z}^2)}\{z\varphi^{c(+)}_{L_{(uv)}}(z,1)\varphi^{c(-)}_{R_{(uv)}}(\bar{z},1) - \bar{z}\varphi^{c(+)}_{L_{(uv)}}(\bar{z},1)\varphi^{c(-)}_{R_{(uv)}}(z,1)\} + \frac{2\cos^2 c\pi}{kr\epsilon}\Upsilon(z),\\
|N^{c(-)}_{L_{(ir)}}(z)|^2 &= \frac{2}{kr(z^2-\bar{z}^2)}\{z\varphi^{c(-)}_{L_{(ir)}}(\bar{z},\epsilon)\varphi^{c(+)}_{R_{(ir)}}(z,\epsilon) - \bar{z}\varphi^{c(-)}_{L_{(ir)}}(z,\epsilon)\varphi^{c(+)}_{R_{(ir)}}(\bar{z},\epsilon)\} + \frac{2\cos^2 c\pi}{kr\epsilon}\Upsilon(z),\\
|N^{c(-)}_{L_{(uv)}}(z)|^2 &= \frac{2}{kr\epsilon(z^2-\bar{z}^2)}\{\bar{z}\varphi^{c(-)}_{L_{(uv)}}(z,1)\varphi^{c(+)}_{R_{(uv)}}(\bar{z},1) - z\varphi^{c(-)}_{L_{(uv)}}(\bar{z},1)\varphi^{c(+)}_{R_{(uv)}}(z,1)\} + \frac{2\cos^2 c\pi}{kr\epsilon}\Upsilon(z),\\
|N^{c(+)}_{R_{(ir)}}(z)|^2 &= \frac{2}{kr(z^2-\bar{z}^2)}\{\bar{z}\varphi^{c(-)}_{L_{(ir)}}(\bar{z},\epsilon)\varphi^{c(+)}_{R_{(ir)}}(z,\epsilon) - z\varphi^{c(-)}_{L_{(ir)}}(z,\epsilon)\varphi^{c(+)}_{R_{(ir)}}(\bar{z},\epsilon)\} + \frac{2\cos^2 c\pi}{kr\epsilon}\Upsilon(z),\\
|N^{c(+)}_{R_{(uv)}}(z)|^2 &= \frac{2}{kr\epsilon(z^2-\bar{z}^2)}\{z\varphi^{c(-)}_{L_{(uv)}}(z,1)\varphi^{c(+)}_{R_{(uv)}}(\bar{z},1) - \bar{z}\varphi^{c(-)}_{L_{(uv)}}(\bar{z},1)\varphi^{c(+)}_{R_{(uv)}}(z,1)\} + \frac{2\cos^2 c\pi}{kr\epsilon}\Upsilon(z).
\end{aligned}
\tag{90}
$$

Using the above normalization constants, we write the uniformly bounded combinations of bulk profiles for fermion fields in the complex plane as

$$
\begin{aligned}
[f^{L,c}_{(++)}(z,t)][f^{L,c}_{(++)}(z,t')] &= \frac{\sqrt{tt'}}{N^{c(+)}_{L_{(uv)}}(z)N^{c(+)}_{L_{(ir)}}(z)}\{\theta(t-t')\varphi^{c(+)}_{L_{(uv)}}(z,t')\varphi^{c(+)}_{L_{(ir)}}(z,t) + \theta(t'-t)\varphi^{c(+)}_{L_{(uv)}}(z,t)\varphi^{c(+)}_{L_{(ir)}}(z,t')\},\\
[f^{R,c}_{(--)}(z,t)][f^{R,c}_{(--)}(z,t')] &= \frac{\sqrt{tt'}}{N^{c(-)}_{R_{(uv)}}(z)N^{c(-)}_{R_{(ir)}}(z)}\{\theta(t-t')\varphi^{c(-)}_{R_{(uv)}}(z,t')\varphi^{c(-)}_{R_{(ir)}}(z,t) + \theta(t'-t)\varphi^{c(-)}_{R_{(uv)}}(z,t)\varphi^{c(-)}_{R_{(ir)}}(z,t')\}.\\
[f^{L,c}_{(+-)}(z,t)][f^{L,c}_{(+-)}(z,t')] &= \frac{\sqrt{tt'}}{N^{c(+)}_{L_{(uv)}}(z)N^{c(-)}_{L_{(ir)}}(z)}\{\theta(t-t')\varphi^{c(+)}_{L_{(uv)}}(z,t')\varphi^{c(-)}_{L_{(ir)}}(z,t) + \theta(t'-t)\varphi^{c(+)}_{L_{(uv)}}(z,t)\varphi^{c(-)}_{L_{(ir)}}(z,t')\},\\
[f^{R,c}_{(-+)}(z,t)][f^{R,c}_{(-+)}(z,t')] &= \frac{\sqrt{tt'}}{N^{c(-)}_{R_{(uv)}}(z)N^{c(+)}_{R_{(ir)}}(z)}\{\theta(t-t')\varphi^{c(-)}_{R_{(uv)}}(z,t')\varphi^{c(+)}_{R_{(ir)}}(z,t) + \theta(t'-t)\varphi^{c(-)}_{R_{(uv)}}(z,t)\varphi^{c(+)}_{R_{(ir)}}(z,t')\},\\
[f^{L,c}_{(-+)}(z,t)][f^{L,c}_{(-+)}(z,t')] &= \frac{\sqrt{tt'}}{N^{c(-)}_{L_{(uv)}}(z)N^{c(+)}_{L_{(ir)}}(z)}\{\theta(t-t')\varphi^{c(-)}_{L_{(uv)}}(z,t')\varphi^{c(+)}_{L_{(ir)}}(z,t) + \theta(t'-t)\varphi^{c(-)}_{L_{(uv)}}(z,t)\varphi^{c(+)}_{L_{(ir)}}(z,t')\},\\
[f^{R,c}_{(+-)}(z,t)][f^{R,c}_{(+-)}(z,t')] &= \frac{\sqrt{tt'}}{N^{c(+)}_{R_{(uv)}}(z)N^{c(-)}_{R_{(ir)}}(z)}\{\theta(t-t')\varphi^{c(+)}_{R_{(uv)}}(z,t')\varphi^{c(-)}_{R_{(ir)}}(z,t) + \theta(t'-t)\varphi^{c(+)}_{R_{(uv)}}(z,t)\varphi^{c(-)}_{R_{(ir)}}(z,t')\},\\
[f^{L,c}_{(--)}(z,t)][f^{L,c}_{(--)}(z,t')] &= \frac{\sqrt{tt'}}{N^{c(-)}_{L_{(uv)}}(z)N^{c(-)}_{L_{(ir)}}(z)}\{\theta(t-t')\varphi^{c(-)}_{L_{(uv)}}(z,t')\varphi^{c(-)}_{L_{(ir)}}(z,t) + \theta(t'-t)\varphi^{c(-)}_{L_{(uv)}}(z,t)\varphi^{c(-)}_{L_{(ir)}}(z,t')\},\\
[f^{R,c}_{(++)}(z,t)][f^{R,c}_{(++)}(z,t')] &= \frac{\sqrt{tt'}}{N^{c(+)}_{R_{(uv)}}(z)N^{c(+)}_{R_{(ir)}}(z)}\{\theta(t-t')\varphi^{c(+)}_{R_{(uv)}}(z,t')\varphi^{c(+)}_{R_{(ir)}}(z,t) + \theta(t'-t)\varphi^{c(+)}_{R_{(uv)}}(z,t)\varphi^{c(+)}_{R_{(ir)}}(z,t')\}.
\end{aligned}
\tag{91}
$$

The above expressions in Eq. (91) are valid for $c \neq N + 1/2$ and one gets the corresponding expressions for $c = N + 1/2$ after replacing $\cos^2 c\pi$ with 1 in the Eq. (90).

IV. THE FOUR- AND FIVE-DIMENSIONAL PERTURBATIVE EXPANSIONS

The KK excitations affect the theoretical predictions of electroweak scale although it is very difficult to produce them directly in the colliders running now. When KK excitations of gauge fields with $(++)$ BCs are virtual intermediate particles of one-loop Feynman diagrams in four-dimensional effective theory, the amplitudes possibly contain the factor

$$\frac{-i[\chi^G_{(++)}(y^{G(n)}_{(++)}, t)][\chi^G_{(++)}(y^{G(n)}_{(++)}, t')]}{p^2 - \Lambda^2_{\rm KK}[y^{G(n)}_{(++)}]^2}, \tag{92}$$

when we expand them according $\mu^2_{\rm EW}/\Lambda^2_{\rm KK}$. The denominator $p^2 - \Lambda^2_{\rm KK}[y^{G(n)}_{(++)}]^2$ originates from four-dimensional propagators of KK excitations for gauge fields in momentum space and $[\chi^G_{(++)}(y^{G(n)}_{(++)}, t)]$ and $[\chi^G_{(++)}(y^{G(n)}_{(++)}, t')]$ originate from the neighbor vertices in four-dimensional effective theory. Note that $\pm y^{G(1)}_{(++)}, \cdots, \pm y^{G(n)}_{(++)}, \cdots$ are zeros of the function $R^{G,\epsilon}_{(++)}(z)$, and the limit

$$\lim_{|z|\to\infty}\left|\frac{-iz[\chi^G_{(++)}(z, t)][\chi^G_{(++)}(z, t')]}{p^2 - \Lambda^2_{\rm KK} z^2}\right| = 0, \tag{93}$$

when we adopt the analytical extension for the combination of bulk profiles with $(++)$ BCs gauge fields in Eq. (85). Applying Eq. (80), we have

$$\begin{aligned} iD^G_{(++)}(p; \phi, \phi') &= \sum_{n=1}^{\infty}\frac{-i[\chi^G_{(++)}(y^{G(n)}_{(++)}, t)][\chi^G_{(++)}(y^{G(n)}_{(++)}, t')]}{p^2 - \Lambda^2_{\rm KK}[y^{G(n)}_{(++)}]^2} \\ &= \frac{i}{p^2}\left\{[\chi^G_{(++)}(0, t)][\chi^G_{(++)}(0, t')] - \left[\chi^G_{(++)}\left(\frac{p}{\Lambda_{\rm KK}}, t\right)\right]\left[\chi^G_{(++)}\left(\frac{p}{\Lambda_{\rm KK}}, t'\right)\right]\right\} \\ &\quad - \frac{i}{2\Lambda_{\rm KK} p}\frac{[\chi^G_{(++)}(\frac{p}{\Lambda_{\rm KK}}, t)][\chi^G_{(++)}(\frac{p}{\Lambda_{\rm KK}}, t')]}{R^{G,\epsilon}_{(++)}(\frac{p}{\Lambda_{\rm KK}})}\frac{\partial R^{G,\epsilon}_{(++)}}{\partial z}(z)|_{z=(p/\Lambda_{\rm KK})} \\ &= -\frac{i\Omega^G_{(++)}(p/\Lambda_{\rm KK})}{\pi\Lambda^2_{\rm KK}}\left\{\frac{\pi t t'}{2R^{G,\epsilon}_{(++)}(p/\Lambda_{\rm KK})}[\theta(t - t')\Phi^{G(+)}_{(\rm uv)}(p/\Lambda_{\rm KK}, t')\Phi^{G(+)}_{(ir)}(p/\Lambda_{\rm KK}, t)\right. \\ &\quad \left. + \theta(t' - t)\Phi^{G(+)}_{(\rm uv)}(p/\Lambda_{\rm KK}, t)\Phi^{G(+)}_{(ir)}(p/\Lambda_{\rm KK}, t')]\right\} + \frac{i}{4\pi p^2}, \end{aligned} \tag{94}$$

with

$$\Omega^G_{(++)}(x) = \frac{2R^{G,\epsilon}_{(++)}(x) + x[\Phi^{G(+)}_{(\rm uv)}(x, 1) - \epsilon\Phi^{G(+)}_{(ir)}(x, \epsilon)]}{x^2[N^{G(+)}_{(\rm uv)}(x)][N^{G(+)}_{(ir)}(x)]}. \tag{95}$$

The factor in the braces of the first term behind the third equality sign in Eq. (94) is the five-dimensional mixed position/momentum-space propagator derived in [64]. The factor $\Omega^G_{(++)}/\pi$ originates from the normalization constants of bulk profiles for gauge fields with $(++)$ BCs in four-dimensional effective theory, the different definitions of couplings involving gauge and fermion fields in five-dimensional full theory and corresponding four-dimensional effective theory, and the difference between normalization of kinetic terms of relevant fields in five-dimensional full theory and corresponding four-dimensional effective theory, respectively. Additionally, $D^G_{(++)}(p; \phi, \phi')$ satisfies the following equation:

$$\left\{\frac{1}{r^2}\frac{\partial}{\partial\phi}\left[e^{-2\sigma(\phi)}\frac{\partial}{\partial\phi}\right] + p^2\right\}D^G_{(++)}(p; \phi, \phi') = -\frac{i\Omega^G_{(++)}(p/\Lambda_{\rm KK})}{\pi\Lambda^2_{\rm KK} r}\delta(\phi - \phi') \tag{96}$$

and the corresponding BCs

$$\frac{\partial D^G_{(++)}}{\partial\phi_{(\rm uv)}}(p; \phi, \phi')|_{\phi_{(\rm uv)}=0} = 0, \qquad \frac{\partial D^G_{(++)}}{\partial\phi_{(ir)}}(p; \phi, \phi')|_{\phi_{(ir)}=\pi/2} = 0. \tag{97}$$

Here $\phi_{(\rm uv)} = \min(\phi, \phi')$, $\phi_{(ir)} = \max(\phi, \phi')$. For small z,

$$R^{G,\epsilon}_{(++)}(z) = -\frac{2\ln\epsilon}{\pi}\left\{1 - \frac{z^2}{4}\left[1 + \epsilon^2 + \frac{1-\epsilon^2}{\ln\epsilon}\right] + \mathcal{O}(z^4)\right\},$$

$$\frac{\partial R^{G,\epsilon}_{(++)}}{\partial z}(z) = \frac{z}{\pi}[1 - \epsilon^2 + (1+\epsilon^2)\ln\epsilon]\left\{1 - \frac{z^2}{16}\frac{2(1+4\epsilon^2+\epsilon^4)\ln\epsilon + 3(1-\epsilon^4)}{1-\epsilon^2+(1+\epsilon^2)\ln\epsilon} + \mathcal{O}(z^4)\right\},$$

$$\frac{t\Phi^{G(+)}_{(uv)}(z,t)}{[N^{G(+)}_{(\mathrm{uv})}(z)]} = \frac{1}{\sqrt{2\pi}}\left\{1 + \frac{z^2}{4}\left[t^2(1 - 2\ln t + 2\ln\epsilon) + 1 + \frac{1-\epsilon^2}{\ln\epsilon}\right] + \frac{z^4}{16}\left[-\frac{5}{4}t^4 + \epsilon^2 t^2 + (t^4 + 2\epsilon^2 t^2)(\ln t - \ln\epsilon)\right.\right.$$
$$\left.\left. + \left(\frac{1-\epsilon^2}{\ln\epsilon} + 1 + \epsilon^2\right)(t^2 - 2t^2\ln t + 2t^2\ln\epsilon) + 2 + \frac{\ln\epsilon}{2} + \frac{47 - 32\epsilon^2 - 15\epsilon^4}{16\ln\epsilon} + \frac{3(1-\epsilon^2)^2}{2\ln^2\epsilon}\right] + \mathcal{O}(z^6)\right\},$$

$$\frac{t\Phi^{G(+)}_{(ir)}(z,t)}{[N^{G(+)}_{(ir)}(z)]} = \frac{1}{\sqrt{2\pi}}\left\{1 + \frac{z^2}{4}\left[t^2(1 - 2\ln t) + \epsilon^2 + \frac{1-\epsilon^2}{\ln\epsilon}\right] + \frac{z^4}{16}\left[-\frac{5}{4}t^4 + t^2 + (t^4 + 2t^2)\ln t\right.\right.$$
$$\left.\left. + \left(\frac{1-\epsilon^2}{\ln\epsilon} + 1 + \epsilon^2\right)(t^2 - 2t^2\ln t) + 2\epsilon^4 - \frac{\epsilon^4\ln\epsilon}{2} + \frac{15 + 32\epsilon^2 - 47\epsilon^4}{16\ln\epsilon} + \frac{3(1-\epsilon^2)^2}{2\ln^2\epsilon}\right] + \mathcal{O}(z^6)\right\}. \qquad (98)$$

Inserting the above equations into Eq. (94) and assuming $p \to 0$, one obtains obviously

$$\Sigma^G_{(++)}(t,t') = \sum_{n=1}^{\infty}\frac{[\chi^G_{(++)}(y^{G(n)}_{(++)},t)][\chi^G_{(++)}(y^{G(n)}_{(++)},t')]}{[y^{G(n)}_{(++)}]^2}$$
$$= \frac{1}{8\pi}\left\{t^2(2\ln t - 1) + t'^2(2\ln t' - 1) - 2\ln\epsilon[t^2\theta(t'-t) + t'^2\theta(t-t')] - \frac{1-\epsilon^2}{\ln\epsilon}\right\}, \qquad (99)$$

which is coincided with Eq. (34) exactly in Ref. [18]. Actually, this result can also be gotten directly by the residue theorem. Applying Eqs. (80) and (98), we have

$$\Sigma^G_{(++)}(t,t') = \sum_{n=1}^{\infty}\frac{[\chi^G_{(++)}(y^{G(n)}_{(++)},t)][\chi^G_{(++)}(y^{G(n)}_{(++)},t')]}{[y^{G(n)}_{(++)}]^2}$$
$$= -\frac{1}{2}\,\mathrm{Res}\left\{\left[\frac{2}{z^3} + \frac{1}{z^2 R^{G,\epsilon}_{(++)}(z)}\frac{\partial R^{G,\epsilon}_{(++)}}{\partial z}(z)\right][\chi^G_{(++)}(z,t)][\chi^G_{(++)}(z,t')], z=0\right\}$$
$$= \frac{1}{8\pi}\left\{t^2(2\ln t - 1) + t'^2(2\ln t' - 1) - 2\ln\epsilon[t^2\theta(t'-t) + t'^2\theta(t-t')] - \frac{1-\epsilon^2}{\ln\epsilon}\right\},$$

$$\sum_{n=1}^{\infty}\frac{[\chi^G_{(++)}(y^{G(n)}_{(++)},t)][\chi^G_{(++)}(y^{G(n)}_{(++)},t')]}{[y^{G(n)}_{(++)}]^4} = -\frac{1}{2}\,\mathrm{Res}\left\{\left[\frac{2}{z^5} + \frac{1}{z^4 R^{G,\epsilon}_{(++)}(z)}\frac{\partial R^{G,\epsilon}_{(++)}}{\partial z}(z)\right][\chi^G_{(++)}(z,t)][\chi^G_{(++)}(z,t')], z=0\right\}$$
$$= -\frac{1}{32\pi}\left\{t^4\left[\ln t - \frac{5}{4}\right] + t^2\left[2 + \frac{1-\epsilon^2}{\ln\epsilon} - \frac{2(1-\epsilon^2)}{\ln\epsilon}\ln t\right]\right.$$
$$+ t'^4\left[\ln t' - \frac{5}{4}\right] + t'^2\left[2 + \frac{1-\epsilon^2}{\ln\epsilon} - \frac{2(1-\epsilon^2)}{\ln\epsilon}\ln t'\right]$$
$$+ 4t^2t'^2\left[\ln t\ln t' - \left(\ln\epsilon + \frac{1}{2}\right)\ln(tt') + \frac{1}{4} + \frac{1}{2}\ln\epsilon\right]$$
$$- \ln\epsilon[(t^4 - 4t^2t'^2\ln t)\theta(t'-t) + (t'^4 - 4t^2t'^2\ln t')\theta(t-t')]$$
$$\left. + \frac{5(1-\epsilon^4)}{8\ln\epsilon} + \frac{(1-\epsilon^2)^2}{\ln^2\epsilon}\right\}, \qquad (100)$$

where the second equation is coincided with the Eq. (36) exactly in Ref. [18]. For the bulk gauge bosons with $(-+)$ BCs, the Laurent series of $R^{G,\epsilon}_{-+}(z)$ and the profiles at $z=0$ can be written, respectively, as

$$R^{G,\epsilon}_{(-+)}(z) = \frac{2}{\pi\epsilon z}\Big\{1 - \frac{z^2}{4}[1-\epsilon^2+2\epsilon^2\ln\epsilon] + \mathcal{O}(z^4)\Big\},$$
$$\frac{\partial R^{G,\epsilon}_{(-+)}}{\partial z}(z) = -\frac{2}{\pi\epsilon z^2}\Big\{1 + \frac{z^2}{4}[1-\epsilon^2+2\epsilon^2\ln\epsilon] + \mathcal{O}(z^4)\Big\},$$
$$\frac{t\Phi^{G(-)}_{(uv)}(z,t)}{[N^{G(-)}_{(\mathrm{uv})}(z)]} = \Big(\frac{-2\epsilon\ln\epsilon}{\pi(1-4\epsilon^2+3\epsilon^4-4\epsilon^4\ln\epsilon)}\Big)^{1/2} t^{1/2}(t^2-\epsilon^2) \times\Big\{1 - \frac{z^2}{2}\Big[-\frac{1-9\epsilon^4+8\epsilon^6+6\epsilon^2\ln\epsilon-12\epsilon^4\ln\epsilon-6\epsilon^6\ln\epsilon}{6(1-4\epsilon^2+3\epsilon^4-4\epsilon^4\ln\epsilon)} - \frac{\epsilon^2t^2(\ln t-\ln\epsilon)}{t^2-\epsilon^2} + \frac{t^2+\epsilon^2}{4}\Big] + \mathcal{O}(z^4)\Big\}. \tag{101}$$

Using the residue theorem, we correspondingly derive

$$\begin{aligned}\Sigma^G_{(-+)}(t,t') &= \sum_{n=1}^{\infty}\frac{[\chi^G_{(-+)}(y^{G(n)}_{(-+)},t)][\chi^G_{(-+)}(y^{G(n)}_{(-+)},t')]}{[y^{G(n)}_{(-+)}]^2}\\ &= -\frac{1}{2}\,\mathrm{Res}\Big\{\Big[\frac{2}{z^3}+\frac{1}{z^2R^{G,\epsilon}_{(-+)}(z)}\frac{\partial R^{G,\epsilon}_{(-+)}}{\partial z}(z)\Big][\chi^G_{(-+)}(z,t)][\chi^G_{(-+)}(z,t')],\, z=0\Big\}\\ &= \frac{1}{4\pi}\Big(\frac{-\epsilon\ln\epsilon}{1-4\epsilon^2+3\epsilon^4-4\epsilon^4\ln\epsilon}\Big)^{1/2}\Big\{\theta(t-t')\sqrt{t'}(t'^2-\epsilon^2)\Big[1-\frac{\epsilon^2}{4}+2\epsilon^2\ln\epsilon\\ &\quad - \frac{1-9\epsilon^4+8\epsilon^6+6\epsilon^2\ln\epsilon-12\epsilon^4\ln\epsilon-6\epsilon^6\ln\epsilon}{6(1-4\epsilon^2+3\epsilon^4-4\epsilon^4\ln\epsilon)}\\ &\quad - \frac{\epsilon^2t'^2(\ln t'-\ln\epsilon)}{t'^2-\epsilon^2}+\frac{t'^2}{4}-\frac{t^2}{2}(1-2\ln t)+\frac{1-\epsilon^2}{2\ln\epsilon}\Big]+(t\leftrightarrow t')\Big\}.\end{aligned} \tag{102}$$

As $t \neq t'$, $\Sigma^G_{(-+)}(t,t')$ satisfies the corresponding BCs on UV and IR branes separately,

$$t_{ir}\frac{\partial\Sigma^G_{(-+)}}{\partial t_{ir}}(t,t')|_{t_{ir}=1} = 0, \qquad \Sigma^G_{(-+)}(t,t')|_{t_{\mathrm{uv}}=\epsilon} = 0. \tag{103}$$

When $t = t' \in [\epsilon, 1]$, the function $\Sigma^G_{(-+)}(t,t)$ is always non-negative.

When exciting KK modes of the left-handed fields with $(++)$ BCs are virtual intermediate particles of one-loop Feynman diagrams in four-dimensional effective theory, the amplitudes certainly contain the factor

$$\frac{i\{\not p[f^{L,c}_{(++)}(y^{c(n)}_{(\pm\pm)},t)][f^{L,c}_{(++)}(y^{c(n)}_{(\pm\pm)},t')] + \Lambda_{\mathrm{KK}}[y^{c(n)}_{(\pm\pm)}][f^{R,c}_{(--)}(y^{c(n)}_{(\pm\pm)},t)][f^{L,c}_{(++)}(y^{c(n)}_{(\pm\pm)},t')]\}}{p^2-\Lambda^2_{\mathrm{KK}}[y^{c(n)}_{(\pm\pm)}]^2}, \tag{104}$$

when we expand them according $\mu^2_{\mathrm{EW}}/\Lambda^2_{\mathrm{KK}}$. The limit

$$\lim_{|z|\to\infty}\Big|\frac{iz\not p[f^{L,c}_{(++)}(z,t)][f^{L,c}_{(++)}(z,t')]}{p^2-\Lambda^2_{\mathrm{KK}}z^2}\Big| = 0 \tag{105}$$

and the function

$$\Big|\frac{z^2\Lambda_{\mathrm{KK}}[f^{R,c}_{(--)}(z,t)][f^{L,c}_{(++)}(z,t')]}{p^2-\Lambda^2_{\mathrm{KK}}z^2}\Big| \tag{106}$$

is uniformly bounded. Since $\pm y^{c(1)}_{(\pm\pm)}, \cdots, \pm y^{c(n)}_{(\pm\pm)}, \cdots$ are zeros of the function $R^{c,\epsilon}_{(\pm\pm)}(z)$, we have the following equation using Eq. (80):

$$\begin{aligned}
iD^c_{L(++)}(p;\phi,\phi') &= \sum_{n=1}^{\infty}\frac{i}{p^2-\Lambda_{\rm KK}^2[y^{c(n)}_{(\pm\pm)}]^2}\{\not p[f^{L,c}_{(++)}(y^{c(n)}_{(\pm\pm)},t)][f^{L,c}_{(++)}(y^{c(n)}_{(\pm\pm)},t')] \\
&\quad + \Lambda_{\rm KK}[y^{c(n)}_{(\pm\pm)}][f^{R,c}_{(--)}(y^{c(n)}_{(\pm\pm)},t)][f^{L,c}_{(++)}(y^{c(n)}_{(\pm\pm)},t')]\} \\
&= \sum_{n=1}^{\infty}\frac{i\not p[f^{L,c}_{(++)}(y^{c(n)}_{(\pm\pm)},t)][f^{L,c}_{(++)}(y^{c(n)}_{(\pm\pm)},t')]}{p^2-\Lambda_{\rm KK}^2[y^{c(n)}_{(\pm\pm)}]^2} \\
&= \frac{i\not p}{p^2}\bigg\{[f^{L,c}_{(++)}(0,t)][f^{L,c}_{(++)}(0,t')] - \bigg[f^{L,c}_{(++)}\bigg(\frac{p}{\Lambda_{\rm KK}},t\bigg)\bigg]\bigg[f^{L,c}_{(++)}\bigg(\frac{p}{\Lambda_{\rm KK}},t'\bigg)\bigg]\bigg\} \\
&\quad - \frac{i\not p}{2\Lambda_{\rm KK}p}\frac{[f^{L,c}_{(++)}(\frac{p}{\Lambda_{\rm KK}},t)][f^{L,c}_{(++)}(\frac{p}{\Lambda_{\rm KK}},t')]}{R^{c,\epsilon}_{(\pm\pm)}(\frac{p}{\Lambda_{\rm KK}})}\frac{\partial R^{c,\epsilon}_{(\pm\pm)}}{\partial z}(z)|_{z=(p/\Lambda_{\rm KK})} \\
&= -\frac{i\not p\Omega^{L,c}_{(++)}(p/\Lambda_{\rm KK})}{\pi\Lambda_{\rm KK}^2}\bigg\{\frac{\pi t t'}{2R^{c,\epsilon}_{(\pm\pm)}(p/\Lambda_{\rm KK})}[\theta(t-t')\varphi^{c(+)}_{L_{(\rm uv)}}(p/\Lambda_{\rm KK},t')\varphi^{c(+)}_{L_{(ir)}}(p/\Lambda_{\rm KK},t) \\
&\quad + \theta(t'-t)\varphi^{c(+)}_{L_{(\rm uv)}}(p/\Lambda_{\rm KK},t)\varphi^{c(+)}_{L_{(ir)}}(p/\Lambda_{\rm KK},t')]\bigg\} + \frac{i\not p}{p^2}\bigg[\frac{-2(1-2c)\epsilon\ln\epsilon}{\pi(1-\epsilon^{1-2c})}\bigg]\frac{(tt')^{-c}}{4},
\end{aligned} \tag{107}$$

with

$$\Omega^{L,c}_{(++)}(x) = \frac{2R^{c,\epsilon}_{(\pm\pm)}(x) + x[\varphi^{c(+)}_{L_{(\rm uv)}}(x,1) - \epsilon\varphi^{c(+)}_{L_{(ir)}}(x,\epsilon)]}{x^2[N^{c(+)}_{L_{(\rm uv)}}(x)][N^{c(+)}_{L_{(ir)}}(x)]}. \tag{108}$$

For small z,

$$\begin{aligned}
R^{c,\epsilon}_{(\pm\pm)}(z) &= \frac{2\epsilon^{c-(1/2)}(1-\epsilon^{1-2c})}{(1-2c)\Gamma(\frac12-c)\Gamma(\frac12+c)}\bigg\{1-\frac{z^2}{2(1-\epsilon^{1-2c})}\bigg[\frac{1-\epsilon^{3-2c}}{3-2c}+\frac{\epsilon^2-\epsilon^{1-2c}}{1+2c}\bigg]+\mathcal{O}(z^4)\bigg\}, \\
\frac{\partial R^{c,\epsilon}_{(\pm\pm)}}{\partial z}(z) &= -\frac{2z\epsilon^{c-(1/2)}}{(1-2c)\Gamma(\frac12-c)\Gamma(\frac12+c)}\bigg\{\bigg[\frac{1-\epsilon^{3-2c}}{3-2c}+\frac{\epsilon^2-\epsilon^{1-2c}}{1+2c}\bigg]-\frac{z^2}{2}\bigg[\frac{1-\epsilon^{5-2c}}{(5-2c)(3-2c)} \\
&\quad + \frac{2(\epsilon^2-\epsilon^{3-2c})}{(3-2c)(1+2c)}+\frac{(\epsilon^4-\epsilon^{1-2c})}{(1+2c)(3+2c)}\bigg]+\mathcal{O}(z^4)\bigg\}, \\
\frac{\sqrt{t}\varphi^{c(+)}_{L_{(\rm uv)}}(z,t)}{[N^{c(+)}_{L_{(\rm uv)}}(z)]} &= \frac{t^{-c}}{2}\sqrt{\frac{-2(1-2c)\epsilon\ln\epsilon}{\pi(1-\epsilon^{1-2c})}}\bigg\{1-\frac{z^2}{2}\bigg[\frac{\epsilon^2}{1+2c}+\frac{t^2}{1-2c}-\frac{2\epsilon^{1-2c}t^{1+2c}}{(1-2c)(1+2c)} \\
&\quad - \frac{(1+2c)(1-\epsilon^{3-2c})+(3-2c)(\epsilon^2-\epsilon^{1-2c})}{(3-2c)(1+2c)(1-\epsilon^{1-2c})}\bigg]+\mathcal{O}(z^4)\bigg\}, \\
\frac{\sqrt{t}\varphi^{c(+)}_{L_{(ir)}}(z,t)}{[N^{c(+)}_{L_{(ir)}}(z)]} &= \frac{t^{-c}}{2}\sqrt{\frac{-2(1-2c)\epsilon\ln\epsilon}{\pi(1-\epsilon^{1-2c})}}\bigg\{1-\frac{z^2}{2}\bigg[\frac{1}{1+2c}+\frac{t^2}{1-2c}-\frac{2t^{1+2c}}{(1-2c)(1+2c)} \\
&\quad - \frac{(1+2c)(1-\epsilon^{3-2c})+(3-2c)(\epsilon^2-\epsilon^{1-2c})}{(3-2c)(1+2c)(1-\epsilon^{1-2c})}\bigg]+\mathcal{O}(z^4)\bigg\}.
\end{aligned} \tag{109}$$

Using Eq. (109), we derive the following equation:

$$\Sigma^{L,c}_{(\pm\pm)}(t,t') = \sum_{n=1}^{\infty} \frac{[f^{L,c}_{(++)}(y^{c(n)}_{(\pm\pm)},t)][f^{L,c}_{(++)}(y^{c(n)}_{(\pm\pm)},t')]}{[y^{c(n)}_{(\pm\pm)}]^2}$$
$$= -\frac{1}{2}\,\mathrm{Res}\left\{\left[\frac{2}{z^3} + \frac{1}{z^2 R^{c,\epsilon}_{(\pm\pm)}(z)} \frac{\partial R^{c,\epsilon}_{(\pm\pm)}}{\partial z}(z)\right][f^L_{(++)}(z,t)][f^L_{(++)}(z,t')],\, z=0\right\}$$
$$= -\frac{(1-2c)\epsilon \ln\epsilon}{(1-\epsilon^{1-2c})}\frac{(tt')^{-c}}{4\pi}\left\{\frac{2(1-2c)(1-\epsilon^{3-2c})}{(3-2c)(1+2c)(1-\epsilon^{1-2c})} + \frac{t^2+t'^2}{1-2c} - \frac{2}{(1-2c)(1+2c)}\left[\theta(t'-t)(t'^{1+2c}+\epsilon^{1-2c}t^{1+2c}) + \theta(t-t')(t^{1+2c}+\epsilon^{1-2c}t'^{1+2c})\right]\right\}. \quad (110)$$

As $t \neq t'$, $\Sigma^{L,c}_{(\pm\pm)}(t,t')$ satisfies the corresponding BCs on UV and IR branes respectively,

$$\left[t_{ir}\frac{\partial \Sigma^{L,c}_{(\pm\pm)}}{\partial t_{ir}} + c\Sigma^{L,c}_{(\pm\pm)}\right](t,t')|_{t_{ir}=1} = 0, \qquad \left[t_{\mathrm{uv}}\frac{\partial \Sigma^{L,c}_{(\pm\pm)}}{\partial t_{\mathrm{uv}}} + c\Sigma^{L,c}_{(\pm\pm)}\right](t,t')|_{t_{\mathrm{uv}}=\epsilon} = 0. \quad (111)$$

When $t = t' \in [\epsilon, 1]$, the function $\Sigma^{L,c}_{(\pm\pm)}(t,t)$ is always non-negative for any real bulk mass c. For the left-handed fermions with $(--)$ BCs, the Laurent series of $R^{c,\epsilon}_{\mp\mp}(z)$ and relevant profiles at $z=0$ can be written, respectively, as

$$R^{c,\epsilon}_{(\mp\mp)}(z) = \frac{2\epsilon^{-c-(1/2)}(1-\epsilon^{1+2c})}{(1+2c)\Gamma(\frac{1}{2}-c)\Gamma(\frac{1}{2}+c)}\left\{1 - \frac{z^2}{2(1-\epsilon^{1+2c})}\left[\frac{1-\epsilon^{3+2c}}{3+2c} + \frac{\epsilon^2-\epsilon^{1+2c}}{1-2c}\right] + \mathcal{O}(z^4)\right\},$$
$$\frac{\partial R^{c,\epsilon}_{(\mp\mp)}}{\partial z}(z) = -\frac{2z\epsilon^{-c-(1/2)}}{(1+2c)\Gamma(\frac{1}{2}-c)\Gamma(\frac{1}{2}+c)}\left\{\left[\frac{1-\epsilon^{3+2c}}{3+2c} + \frac{\epsilon^2-\epsilon^{1+2c}}{1-2c}\right] - \frac{z^2}{2}\left[\frac{1-\epsilon^{5+2c}}{(5+2c)(3+2c)} + \frac{2(\epsilon^2-\epsilon^{3+2c})}{(3+2c)(1-2c)} + \frac{(\epsilon^4-\epsilon^{1+2c})}{(1-2c)(3-2c)}\right] + \mathcal{O}(z^4)\right\},$$
$$\frac{\sqrt{t}\varphi^{c(-)}_{L_{(\mathrm{uv})}}(z,t)}{[N^{c(-)}_{L_{(\mathrm{uv})}}(z)]} = \left(\frac{-(1-2c)(3+2c)\ln\epsilon}{2\pi[\zeta^{(-)}_{(\mathrm{uv})}(c,\epsilon)]}\right)^{1/2} t^{1+c}\epsilon^{1+c}(t^{-1-2c}-\epsilon^{-1-2c})\left\{1 - \frac{z^2}{2}\left[\frac{t^{1-2c}-\epsilon^{1-2c}}{(1-2c)(t^{-1-2c}-\epsilon^{-1-2c})} + \frac{\epsilon^2 t^{-1-2c}-\epsilon^{-1-2c}t^2}{(3+2c)(t^{-1-2c}-\epsilon^{-1-2c})} - \frac{1+2\epsilon^2-\epsilon^{1-2c}-\epsilon^{3+2c}}{[\zeta^{(-)}_{(\mathrm{uv})}(c,\epsilon)]} - \frac{3(1+2c)^2\epsilon^4}{(3-2c)(5+2c)[\zeta^{(-)}_{(\mathrm{uv})}(c,\epsilon)]} + \frac{(1-2c)\epsilon^{-1-2c}}{(5+2c)[\zeta^{(-)}_{(\mathrm{uv})}(c,\epsilon)]} + \frac{(3+2c)\epsilon^{1+2c}}{(3-2c)[\zeta^{(-)}_{(\mathrm{uv})}(c,\epsilon)]}\right] + \mathcal{O}(z^4)\right\},$$
$$\frac{\sqrt{t}\varphi^{c(-)}_{L_{(ir)}}(z,t)}{[N^{c(-)}_{L_{(ir)}}(z)]} = \left(\frac{-(1-2c)(3+2c)\epsilon\ln\epsilon}{2\pi[\zeta^{(-)}_{(ir)}(c,\epsilon)]}\right)^{1/2} t^{1+c}(1-t^{-1-2c})\left\{1 - \frac{z^2}{2}\left[\frac{1-t^{1-2c}}{(1-2c)(1-t^{-1-2c})} + \frac{t^2-t^{-1-2c}}{(3+2c)(1-t^{-1-2c})} + \frac{\epsilon^{3+2c}+\epsilon^{1-2c}-2\epsilon^2-\epsilon^4}{[\zeta^{(-)}_{(ir)}(c,\epsilon)]} - \frac{3(1+2c)^2}{(3-2c)(5+2c)[\zeta^{(-)}_{(ir)}(c,\epsilon)]} + \frac{(1-2c)\epsilon^{5+2c}}{(5+2c)[\zeta^{(-)}_{(ir)}(c,\epsilon)]} + \frac{(3+2c)\epsilon^{3-2c}}{(3-2c)[\zeta^{(-)}_{(ir)}(c,\epsilon)]}\right] + \mathcal{O}(z^4)\right\}, \quad (112)$$

with

$$\zeta^{(-)}_{(\mathrm{uv})}(c,\epsilon) = (3+2c)\epsilon^{1+2c} + (1-2c)\epsilon^{-1-2c} - (1+2c)^2\epsilon^2 - (1-2c)(3+2c),$$
$$\zeta^{(-)}_{(ir)}(c,\epsilon) = -(3+2c)\epsilon^{1-2c} - (1-2c)\epsilon^{3+2c} + (1+2c)^2 + (1-2c)(3+2c)\epsilon^2. \quad (113)$$

From Eq. (112), one similarly obtains the following summation:

$$\Sigma^{L,c}_{(\mp\mp)}(t,t') = \sum_{n=1}^{\infty} \frac{[f^{L,c}_{(--)}(y^{c(n)}_{(\mp\mp)},t)][f^{L,c}_{(--)}(y^{c(n)}_{(\mp\mp)},t')]}{[y^{c(n)}_{(\mp\mp)}]^2}$$
$$= -\frac{1}{2}\,\mathrm{Res}\bigg\{\bigg[\frac{2}{z^3} + \frac{1}{z^2 R^{c,\epsilon}_{(\mp\mp)}(z)}\frac{\partial R^{c,\epsilon}_{(\mp\mp)}}{\partial z}(z)\bigg][f^L_{(--)}(z,t)][f^L_{(--)}(z,t')],\ z=0\bigg\}$$
$$= -\frac{(1-2c)(3+2c)\epsilon^{3/2+c}\ln\epsilon}{[\zeta^{(-)}_{(\mathrm{uv})}(c,\epsilon)\zeta^{(-)}_{(ir)}(c,\epsilon)]^{1/2}}\frac{(tt')^{1+c}}{4\pi}$$
$$\times\bigg\{\theta(t-t')(1-t^{-1-2c})(t'^{-1-2c}-\epsilon^{-1-2c})\bigg[\frac{1}{1-\epsilon^{1+2c}}\bigg(\frac{1-\epsilon^{3+2c}}{3+2c}+\frac{\epsilon^2-\epsilon^{1+2c}}{1-2c}\bigg)$$
$$+\frac{\epsilon^{3+2c}+\epsilon^{1-2c}-2\epsilon^2-\epsilon^4}{[\zeta^{(-)}_{(ir)}(c,\epsilon)]} - \frac{3(1+2c)^2}{(3-2c)(5+2c)[\zeta^{(-)}_{(ir)}(c,\epsilon)]} + \frac{(1-2c)\epsilon^{5+2c}}{(5+2c)[\zeta^{(-)}_{(ir)}(c,\epsilon)]}$$
$$+\frac{(3+2c)\epsilon^{3-2c}}{(3-2c)[\zeta^{(-)}_{(ir)}(c,\epsilon)]} - \frac{1+2\epsilon^2-\epsilon^{1-2c}-\epsilon^{3+2c}}{[\zeta^{(-)}_{(\mathrm{uv})}(c,\epsilon)]} - \frac{3(1+2c)^2\epsilon^4}{(3-2c)(5+2c)[\zeta^{(-)}_{(\mathrm{uv})}(c,\epsilon)]} + \frac{(1-2c)\epsilon^{-1-2c}}{(5+2c)[\zeta^{(-)}_{(\mathrm{uv})}(c,\epsilon)]}$$
$$+\frac{(3+2c)\epsilon^{1+2c}}{(3-2c)[\zeta^{(-)}_{(\mathrm{uv})}(c,\epsilon)]} + \frac{1-t^{1-2c}}{(1-2c)(1-t^{-1-2c})} + \frac{t^2-t^{-1-2c}}{(3+2c)(1-t^{-1-2c})}$$
$$+\frac{t'^{1-2c}-\epsilon^{1-2c}}{(1-2c)(t'^{-1-2c}-\epsilon^{-1-2c})} + \frac{\epsilon^2 t'^{-1-2c}-\epsilon^{-1-2c}t'^2}{(3+2c)(t'^{-1-2c}-\epsilon^{-1-2c})}\bigg] + (t\leftrightarrow t')\bigg\}. \tag{114}$$

Assuming $t \neq t'$, one easily finds that $\Sigma^{L,c}_{(\mp\mp)}(t,t')$ satisfies the corresponding BCs on UV and IR branes, respectively,

$$\Sigma^{L,c}_{(\mp\mp)}(t,t')|_{t_{ir}=1} = 0, \qquad \Sigma^{L,c}_{(\mp\mp)}(t,t')|_{t_{\mathrm{uv}}=\epsilon} = 0, \tag{115}$$

and the function $\Sigma^{L,c}_{(\mp\mp)}(t,t)$ is always non-negative for any real bulk mass c and $t\in[\epsilon,1]$. For the left-handed fermions satisfying $(+-)$ BCs, the Laurent series of the profiles at $z=0$ can be read from Eqs. (109) and (112). The Laurent series of $R^{c,\epsilon}_{\pm\mp}(z)$ at $z=0$ is given by

$$R^{c,\epsilon}_{(\pm\mp)}(z) = \frac{2\epsilon^{c-(1/2)}}{z\Gamma(\frac{1}{2}-c)\Gamma(\frac{1}{2}+c)}\bigg\{1-\frac{z^2}{2}\bigg[\frac{1}{1-2c}+\frac{\epsilon^2}{1+2c}-\frac{2\epsilon^{1-2c}}{(1-2c)(1+2c)}\bigg]+\mathcal{O}(z^4)\bigg\},$$
$$\frac{\partial R^{c,\epsilon}_{(\pm\mp)}}{\partial z}(z) = -\frac{2\epsilon^{c-(1/2)}}{z^2\Gamma(\frac{1}{2}-c)\Gamma(\frac{1}{2}+c)}\bigg\{1+\frac{z^2}{2}\bigg[\frac{1}{1-2c}+\frac{\epsilon^2}{1+2c}-\frac{2\epsilon^{1-2c}}{(1-2c)(1+2c)}\bigg]+\mathcal{O}(z^4)\bigg\}. \tag{116}$$

Analogously, one derives

$$\Sigma^{L,c}_{(\pm\mp)}(t,t') = \sum_{n=1}^{\infty} \frac{[f^{L,c}_{(+-)}(y^{c(n)}_{(\pm\mp)},t)][f^{L,c}_{(+-)}(y^{c(n)}_{(\pm\mp)},t')]}{[y^{c(n)}_{(\pm\mp)}]^2}$$
$$= -\frac{1}{2}\,\mathrm{Res}\bigg\{\bigg[\frac{2}{z^3} + \frac{1}{z^2 R^{c,\epsilon}_{(\pm\mp)}(z)}\frac{\partial R^{c,\epsilon}_{(\pm\mp)}}{\partial z}(z)\bigg][f^L_{(+-)}(z,t)][f^L_{(+-)}(z,t')],\ z=0\bigg\}$$
$$= -\frac{\epsilon\ln\epsilon}{8\pi}\bigg(\frac{(1-2c)^2(3+2c)}{(1-\epsilon^{1-2c})[\zeta^{(-)}_{(ir)}(c,\epsilon)]}\bigg)^{1/2}\bigg\{\theta(t-t')t^{1+c}t'^{-c}|1-t^{-1-2c}|\bigg[\frac{2}{1-2c}+\frac{2\epsilon^2}{1+2c}-\frac{4\epsilon^{1-2c}}{(1-2c)(1+2c)}$$
$$+\frac{\epsilon^{3+2c}+\epsilon^{1-2c}-2\epsilon^2-\epsilon^4}{[\zeta^{(-)}_{(ir)}(c,\epsilon)]} - \frac{3(1+2c)^2}{(3-2c)(5+2c)[\zeta^{(-)}_{(ir)}(c,\epsilon)]} + \frac{(1-2c)\epsilon^{5+2c}}{(5+2c)[\zeta^{(-)}_{(ir)}(c,\epsilon)]} + \frac{(3+2c)\epsilon^{3-2c}}{(3-2c)[\zeta^{(-)}_{(ir)}(c,\epsilon)]}$$
$$+\frac{(3-2c)\epsilon^{1-2c}-2(1-2c)\epsilon^{3-2c}}{(3-2c)(1+2c)(1-\epsilon^{1-2c})} + \frac{1-t^{1-2c}}{(1-2c)(1-t^{-1-2c})} + \frac{t^2-t^{-1-2c}}{(3+2c)(1-t^{-1-2c})} + \frac{t'^2}{1-2c}$$
$$-\frac{2\epsilon^{1-2c}t'^{1+2c}}{(1-2c)(1+2c)}\bigg] + (t\leftrightarrow t')\bigg\}. \tag{117}$$

Similarly, one can verify that $\Sigma^{L,c}_{(\pm\mp)}(t,t')$ satisfies the corresponding BCs on UV and IR branes, respectively,

$$\Sigma^{L,c}_{(\pm\mp)}(t,t')|_{t_{ir}=1}=0, \qquad \bigg[t_{\rm uv}\frac{\partial\Sigma^{L,c}_{(\pm\mp)}}{\partial t_{\rm uv}}+c\Sigma^{L,c}_{(\pm\mp)}\bigg](t,t')|_{t_{\rm uv}=\epsilon}=0, \tag{118}$$

and the function $\Sigma^{L,c}_{(\pm\mp)}(t,t)$ is always non-negative for any real bulk mass c and $t\in[\epsilon,1]$. When the left-handed fermions satisfy $(-+)$ BCs, the Laurent series of $R^{c,\epsilon}_{\mp\pm}(z)$ at $z=0$ is

$$\begin{aligned} R^{c,\epsilon}_{(\mp\pm)}(z)&=\frac{2\epsilon^{-c-(1/2)}}{z\Gamma(\frac{1}{2}-c)\Gamma(\frac{1}{2}+c)}\bigg\{1-\frac{z^2}{2}\bigg[\frac{1}{1+2c}+\frac{\epsilon^2}{1-2c}-\frac{2\epsilon^{1+2c}}{(1-2c)(1+2c)}\bigg]+\mathcal{O}(z^4)\bigg\},\\ \frac{\partial R^{c,\epsilon}_{(\mp\pm)}}{\partial z}(z)&=-\frac{2\epsilon^{-c-(1/2)}}{z^2\Gamma(\frac{1}{2}-c)\Gamma(\frac{1}{2}+c)}\bigg\{1+\frac{z^2}{2}\bigg[\frac{1}{1+2c}+\frac{\epsilon^2}{1-2c}-\frac{2\epsilon^{1+2c}}{(1-2c)(1+2c)}\bigg]+\mathcal{O}(z^4)\bigg\}. \end{aligned} \tag{119}$$

Correspondingly, the summing over infinite KK excitations is formulated as

$$\begin{aligned} \Sigma^{L,c}_{(\mp\pm)}(t,t')&=\sum_{n=1}^{\infty}\frac{[f^{L,c}_{(-+)}(y^{c(n)}_{(\mp\pm)},t)][f^{L,c}_{(-+)}(y^{c(n)}_{(\mp\pm)},t')]}{[y^{c(n)}_{(\mp\pm)}]^2}\\ &=-\frac{1}{2}\,\mathrm{Res}\bigg\{\bigg[\frac{2}{z^3}+\frac{1}{z^2R^{c,\epsilon}_{(\mp\pm)}(z)}\frac{\partial R^{c,\epsilon}_{(\mp\pm)}}{\partial z}(z)\bigg][f^{L}_{(-+)}(z,t)][f^{L}_{(-+)}(z,t')],z=0\bigg\}\\ &=-\frac{\epsilon^{3/2+c}\ln\epsilon}{8\pi}\bigg(\frac{(1-2c)^2(3+2c)}{(1-\epsilon^{1-2c})[\zeta^{(-)}_{(\rm uv)}(c,\epsilon)]}\bigg)^{1/2}\bigg\{\theta(t-t')t^{-c}t'^{1+c}|t'^{-1-2c}\\ &\quad-\epsilon^{-1-2c}|\bigg[\frac{2}{1+2c}+\frac{2\epsilon^2}{1-2c}-\frac{4\epsilon^{1+2c}}{(1-2c)(1+2c)}+\frac{2(1-2c)+(1+2c)\epsilon^{3-2c}-(3-2c)\epsilon^2}{(3-2c)(1+2c)(1-\epsilon^{1-2c})}\\ &\quad-\frac{1+2\epsilon^2-\epsilon^{1-2c}-\epsilon^{3+2c}}{[\zeta^{(-)}_{(\rm uv)}(c,\epsilon)]}-\frac{3(1+2c)^2\epsilon^4}{(3-2c)(5+2c)[\zeta^{(-)}_{(\rm uv)}(c,\epsilon)]}+\frac{(1-2c)\epsilon^{-1-2c}}{(5+2c)[\zeta^{(-)}_{(\rm uv)}(c,\epsilon)]}\\ &\quad+\frac{(3+2c)\epsilon^{1+2c}}{(3-2c)[\zeta^{(-)}_{(\rm uv)}(c,\epsilon)]}+\frac{t^2}{1-2c}-\frac{2t^{1+2c}}{(1-2c)(1+2c)}+\frac{t'^{1-2c}-\epsilon^{1-2c}}{(1-2c)(t'^{-1-2c}-\epsilon^{-1-2c})}\\ &\quad+\frac{\epsilon^2t'^{-1-2c}-\epsilon^{-1-2c}t'^2}{(3+2c)(t'^{-1-2c}-\epsilon^{-1-2c})}\bigg]+(t\leftrightarrow t')\bigg\}. \end{aligned} \tag{120}$$

As $t\neq t'$, $\Sigma^{L,c}_{(\mp\pm)}(t,t')$ satisfies the corresponding BCs on UV and IR branes, respectively,

$$\bigg[t_{ir}\frac{\partial\Sigma^{L,c}_{(\mp\pm)}}{\partial t_{ir}}+c\Sigma^{L,c}_{(\mp\pm)}\bigg](t,t')|_{t_{ir}=1}=0, \qquad \Sigma^{L,c}_{(\mp\pm)}(t,t')|_{t_{\rm uv}=\epsilon}=0. \tag{121}$$

Meanwhile the function $\Sigma^{L,c}_{(\mp\pm)}(t,t)$ is always non-negative for any real bulk mass c and $t\in[\epsilon,1]$.

In a similar way, one analogously has

$$\begin{aligned} \Sigma^{R,c}_{(\mp\mp)}(t,t')&=\sum_{n=1}^{\infty}\frac{[f^{R,c}_{(++)}(y^{c(n)}_{(\mp\mp)},t)][f^{R,c}_{(++)}(y^{c(n)}_{(\mp\mp)},t')]}{[y^{c(n)}_{(\mp\mp)}]^2}=\Sigma^{L,-c}_{(\pm\pm)}(t,t'),\\ \Sigma^{R,c}_{(\pm\pm)}(t,t')&=\sum_{n=1}^{\infty}\frac{[f^{R,c}_{(--)}(y^{c(n)}_{(\mp\mp)},t)][f^{R,c}_{(--)}(y^{c(n)}_{(\pm\pm)},t')]}{[y^{c(n)}_{(\pm\pm)}]^2}=\Sigma^{L,-c}_{(\mp\mp)}(t,t'),\\ \Sigma^{R,c}_{(\mp\pm)}(t,t')&=\sum_{n=1}^{\infty}\frac{[f^{R,c}_{(+-)}(y^{c(n)}_{(\mp\mp)},t)][f^{R,c}_{(+-)}(y^{c(n)}_{(\mp\pm)},t')]}{[y^{c(n)}_{(\mp\pm)}]^2}=\Sigma^{L,-c}_{(\pm\mp)}(t,t'),\\ \Sigma^{R,c}_{(\pm\mp)}(t,t')&=\sum_{n=1}^{\infty}\frac{[f^{R,c}_{(-+)}(y^{c(n)}_{(\pm\mp)},t)][f^{R,c}_{(-+)}(y^{c(n)}_{(\pm\mp)},t')]}{[y^{c(n)}_{(\pm\mp)}]^2}=\Sigma^{L,-c}_{(\mp\pm)}(t,t'). \end{aligned} \tag{122}$$

V. SUMMING OVER INFINITE SERIES OF KK MODES IN A UNIVERSAL EXTRA DIMENSION

In this section, we depart from the main line above and discuss how to sum over the infinite series of KK modes in a universal extra dimension. In these models all fields can propagate in all available dimensions and the SM particles correspond to zero modes in the KK decomposition of five-dimensional fields with $(++)$ BCs. Towers of KK partners for the SM particles and additional towers of KK modes for five-dimensional fields with $(--)$ BCs do not correspond to any fields in the SM [67,68]. The simplest model of this type is proposed by Appelquist, Cheng, and Dobrescu [69] in which the only additional free parameter relating to the SM is the compactification scale $1/r$.

Assuming that topology of the fifth dimension is the orbifold S^1/Z_2 and the coordinate $y \equiv x^5$ runs from 0 to $2\pi r$, one can write the KK expansions of five-dimensional fields, respectively, as [70]

$$\begin{aligned}
\psi^+(x,y) &= \frac{1}{\sqrt{2\pi r}}\psi_{R(0)}(x) \\
&\quad + \frac{1}{\sqrt{\pi r}}\sum_{n=1}^{\infty}\Big\{\psi_{R(n)}(x)\cos\frac{ny}{r} + \psi_{L(n)}(x)\sin\frac{ny}{r}\Big\}, \\
\psi^-(x,y) &= \frac{1}{\sqrt{2\pi r}}\psi_{L(0)}(x) \\
&\quad + \frac{1}{\sqrt{\pi r}}\sum_{n=1}^{\infty}\Big\{\psi_{L(n)}(x)\cos\frac{ny}{r} + \psi_{R(n)}(x)\sin\frac{ny}{r}\Big\}, \\
A^\mu(x,y) &= \frac{1}{\sqrt{2\pi r}}A^\mu_{(0)}(x) + \frac{1}{\sqrt{\pi r}}\sum_{n=1}^{\infty}A^\mu_{(n)}(x)\cos\frac{ny}{r}, \\
A^5(x,y) &= \frac{1}{\sqrt{\pi r}}\sum_{n=1}^{\infty}A^5_{(n)}(x)\sin\frac{ny}{r}, \\
\phi^+(x,y) &= \frac{1}{\sqrt{2\pi r}}\phi_{(0)}(x) + \frac{1}{\sqrt{\pi r}}\sum_{n=1}^{\infty}\phi^+_{(n)}(x)\cos\frac{ny}{r}, \\
\phi^-(x,y) &= \frac{1}{\sqrt{\pi r}}\sum_{n=1}^{\infty}\phi^-_{(n)}(x)\sin\frac{ny}{r}.
\end{aligned} \tag{123}$$

Here, the Dirac spinors $\psi^\pm = (P_R + P_L)\psi^\pm = \psi^\pm_L + \psi^\pm_R$ satisfy the following BCs:

$$\begin{aligned}
&\partial_y\psi^+_R|_{y=\{0,\pi r\}} = 0, \qquad \psi^+_L|_{y=\{0,\pi r\}} = 0, \quad \text{or} \\
&\partial_y\psi^-_L|_{y=\{0,\pi r\}} = 0, \qquad \psi^-_R|_{y=\{0,\pi r\}} = 0,
\end{aligned} \tag{124}$$

with the chirality projectors $P_{R/L} = (1 \pm \gamma_5)/2$. Furthermore, the vector fields satisfy the BCs

$$\partial_y A^\mu|_{y=\{0,\pi r\}} = 0, \qquad A^5|_{y=\{0,\pi r\}} = 0, \tag{125}$$

and the scalar fields $\phi^\pm$ satisfy the BCs

$$\partial_y\phi^+|_{y=\{0,\pi r\}} = 0, \qquad \phi^-|_{y=\{0,\pi r\}} = 0. \tag{126}$$

In a universal extra dimension, the couplings involving KK excitations do not depend on bulk profiles since integral over the fifth coordinate y can be integrated out explicitly. Correspondingly, the summing over KK excitations is simplified drastically because it is unnecessary to extend the square of the bulk profile to the complex plane when we apply the residue theorem. Actually, the authors of Ref. [70] have applied the relation

$$\sum_{n=1}^{\infty}\frac{b}{n^2+c} = \frac{b(\sqrt{c}\pi\coth(\sqrt{c}\pi) - 1)}{2c} \tag{127}$$

to perform the summing over KK excitations in some lower energy processes. In a universal extra dimension, the KK mass n/r is the zero of order one of the function $\sin(\pi rz)$, and the residue of the function $G(z) = \pi r\cos(\pi rz)/\sin(\pi rz)$ at $z = n/r$ is uniform one. Obviously, the function

$$f(z) = \frac{B}{z^2 + C} \tag{128}$$

is uniformly bounded because

$$\lim_{|z|\to\infty}|zf(z)| = 0. \tag{129}$$

Here,

$$B = \frac{b}{r^2}, \qquad C = \frac{c}{r^2}. \tag{130}$$

Using the residue theorem, one obtains obviously

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{b}{n^2+c} &= \sum_{n=1}^{\infty}\frac{B}{(\frac{n}{r})^2 + C} \\
&= -\frac{1}{2}\,\mathrm{Res}\Big\{\frac{\pi r\cos(\pi rz)}{\sin\pi rz}\frac{B}{z^2+C}, z = 0\Big\} \\
&\quad - \frac{1}{2}\,\mathrm{Res}\Big\{\frac{\pi r\cos(\pi rz)}{\sin\pi rz}\frac{B}{z^2+C}, z = i\sqrt{C}\Big\} \\
&\quad - \frac{1}{2}\,\mathrm{Res}\Big\{\frac{\pi r\cos(\pi rz)}{\sin\pi rz}\frac{B}{z^2+C}, z = -i\sqrt{C}\Big\} \\
&= -\frac{B}{2C} - \frac{B\pi r\cos(i\sqrt{C}\pi r)}{2\sqrt{C}\sin(i\sqrt{C}\pi r)} \\
&= \frac{b(\sqrt{c}\pi\coth(\sqrt{c}\pi) - 1)}{2c},
\end{aligned} \tag{131}$$

which is the equation presented in Eq. (127). Furthermore, the residue theorem can be applied to sum over the infinite series of KK excitations in more complicate forms.

VI. THE CORRECTIONS TO $\bar{B} \to X_s\gamma$ IN A WARPED EXTRA DIMENSION

In the framework with a warped extra dimension and the custodial symmetry, the neutral Higgs field located on the IR brane induces the mixing between states with the same electric charge. A consequence of the mixing is that the FCNC transitions are also mediated by the neutral Higgs field, the KK excitations of gluon, photon, and neutral electroweak gauge bosons, besides the charged electroweak gauge bosons $W^{\pm}$ together with their KK partners. In this section, we present one-loop radiative corrections to the rare decay $b \to s + \gamma$ in the extension of the SM with a warped extra dimension and the custodial symmetry, and we analyze the possible constraint on the parameter space of new physics from experimental data for the branching ratio of $\bar{B} \to X_s \gamma$.

The effective Hamiltonian for $\bar{B} \to X_s \gamma$ at scales $\mu_b = \mathcal{O}(m_b)$ is given by [71]

$$\begin{aligned}\mathcal{H}_{b\to s\gamma}(\mu_b) = -\frac{4G_F}{\sqrt{2}}\bigg\{&\sum_{i=1}^{6} C_i(\mu_b)Q_i + \sum_{i=3}^{6}\tilde{C}_i(\mu_b)\tilde{Q}_i \\ &+ \sum_{i=1}^{2}\sum_{f}[C_{f,i}^{LL}(\mu_b)Q_{f,i}^{LL} + C_{f,i}^{LR}(\mu_b)Q_{f,i}^{LR} \\ &+ C_{f,i}^{RL}(\mu_b)Q_{f,i}^{RL} + C_{f,i}^{RR}(\mu_b)Q_{f,i}^{RR}] \\ &+ \sum_{i=1}^{2}[\hat{C}_{d,i}^{LL}(\mu_b)\hat{Q}_{d,i}^{LL} + \hat{C}_{d,i}^{LR}(\mu_b)\hat{Q}_{d,i}^{LR} \\ &+ \hat{C}_{d,i}^{RL}(\mu_b)\hat{Q}_{d,i}^{RL} + \hat{C}_{d,i}^{RR}(\mu_b)\hat{Q}_{d,i}^{RR}] \\ &+ C_{7\gamma}(\mu_b)Q_{7\gamma} + C_{8G}(\mu_b)Q_{8G} \\ &+ \tilde{C}_{7\gamma}(\mu_b)\tilde{Q}_{7\gamma} + \tilde{C}_{8G}(\mu_b)\tilde{Q}_{8G}\bigg\}, \end{aligned} \tag{132}$$

with G_F denoting the Fermi constant and $f = u, c, d, s, b$. The magnetic dipole moment operators are

$$\begin{aligned} Q_{7\gamma} &= \frac{e}{16\pi^2} m_b \bar{s}_\alpha \sigma^{\mu\nu} P_R b_\alpha F_{\mu\nu}, \\ Q_{8G} &= \frac{g_s}{16\pi^2} m_b \bar{s}_\alpha T^a_{\alpha\beta} \sigma^{\mu\nu} P_R b_\beta G^a_{\mu\nu}, \end{aligned} \tag{133}$$

the concrete expressions of dimension-six operators $Q_i (i = 1, \cdots, 6)$ can be found in Ref. [71], and tilde operators are obtained from Q_i, $Q_{7\gamma}$, Q_{8G} after interchanging the right-handed projector $P_R = (1 + \gamma_5)/2$ with the left-handed one $P_L = (1 - \gamma_5)/2$. Here α, $\beta = 1, 2, 3$ denote the color indices of quarks, $F_{\mu\nu}$ and $G^a_{\mu\nu} (a = 1, \cdots, 8)$ are the electromagnetic and strong field strength tensors, respectively. The neutral current-current operators [72,73] are defined through

$$\begin{aligned} Q_{u,1}^{AB} &= (\bar{s}_\alpha \gamma_\mu P_A b_\beta)(\bar{u}_\beta \gamma_\mu P_B u_\alpha), \\ Q_{u,2}^{AB} &= (\bar{s}_\alpha \gamma_\mu P_A b_\alpha)(\bar{u}_\beta \gamma_\mu P_B u_\beta), \\ \hat{Q}_{d,1}^{AB} &= (\bar{s}_\alpha \gamma_\mu P_A d_\beta)(\bar{d}_\beta \gamma_\mu P_B b_\alpha), \\ \hat{Q}_{d,2}^{AB} &= (\bar{s}_\alpha \gamma_\mu P_A d_\alpha)(\bar{d}_\beta \gamma_\mu P_B b_\beta), \end{aligned} \tag{134}$$

with A, $B = L$, R.

In Eq. (132), the Fermi constant G_F is extracted from the muon decay $\mu^- \to e^- \nu_\mu \bar{\nu}_e$. At tree level in the framework with a warped extra dimension and the custodial symmetry, this process is mediated by the exchange of charged and neutral gauge bosons together with corresponding KK excitations. For light leptons involved in the decay, we can neglect those nonuniversal effects suppressed by relevant fermion profiles near the IR brane [18] using the ansatz for anarchic Yukawa couplings. This leaves a universal correction not depending on the lepton flavors in $\mu^- \to e^- \nu_\mu \bar{\nu}_e$,

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8m_W^2}\{1 + \Delta G_F\} \tag{135}$$

with

$$\Delta G_F = \frac{m_W^2}{2\Lambda_{\rm KK}^2}\bigg[1 + \frac{1}{2\ln\epsilon}\bigg]. \tag{136}$$

The magnetic penguin operators $Q_{7\gamma}$, Q_{8G}, $\tilde{Q}_{7\gamma}$, $\tilde{Q}_{8G}$ in the effective Hamiltonian are induced by virtual heavy freedoms through one-loop diagrams at electroweak scale, and the relevant Feynman diagrams are drawn in Fig. 1 when we adopt the background gauge [64].

The couplings between scalar/gauge bosons and fermions depend on bulk profiles of relevant fields and corresponding mixing matrices. To obtain approximately the mixing between zero modes of charged $2/3$, $-1/3$ quarks and corresponding KK excitations, we write the infinite-dimensional column vectors for quarks in the chirality basis as [63]

$$\begin{aligned} \Psi_L(2/3) &= (q_{u_L}^{i(0)}(++), \cdots, q_{u_L}^{i(n)}(++), U_L^{i(n)}(+-), \tilde{U}_L^{i(n)}(+-), \chi_{d_L}^{i(n)}(-+), u_L^{i(n)}(--), \cdots)^T, \\ \Psi_R(2/3) &= (u_R^{i(0)}(++), \cdots, q_{u_R}^{i(n)}(--), U_R^{i(n)}(-+), \tilde{U}_R^{i(n)}(-+), \chi_{d_R}^{i(n)}(+-), u_R^{i(n)}(++), \cdots)^T, \\ \Psi_L(-1/3) &= (q_{d_L}^{i(0)}(++), \cdots, q_{d_L}^{i(n)}(++), D_L^{i(n)}(+-), d_L^{i(n)}(--), \cdots)^T, \\ \Psi_R(-1/3) &= (d_R^{i(0)}(++), \cdots, q_{d_R}^{i(n)}(--), D_R^{i(n)}(-+), d_R^{i(n)}(++), \cdots)^T, \end{aligned} \tag{137}$$

where $i = 1, 2, 3$ is the index of generation, $n = 1, 2, \cdots, \infty$ is the index of KK exciting modes, and the signs in parentheses denote the BCs satisfied by corresponding fields on UV and IR branes, respectively. In chirality basis Eq. (137), the mass matrix of charged $2/3$ quarks M_U is given in Table I, where $M_{\rm (BCs)}^{c(n)} = \Lambda_{\rm KK} y_{\rm (BCs)}^{c(n)}$ and the concrete expressions for other nonzero elements are presented in Appendix A. Similarly, the mass matrix of charged $-1/3$ quarks M_D in chirality basis is given in Table II.

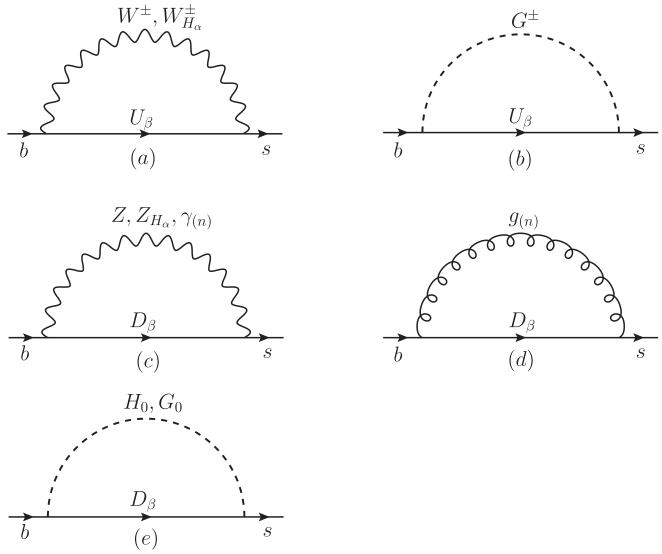

FIG. 1. The Feynman diagrams for $b \to s\gamma$ and $b \to sg$ in a warped extra dimension with custodial symmetry; the photon and gluon can be attached in all possible ways. Where Z, $W^{\pm}$, H_0, G_0, $G^{\pm}$, $U_i = u_i$, $D_i = d_i$ ($i = 1, 2, 3$) denote the normally neutral and charged gauge bosons, neutral Higgs boson, neutral and charged Goldstone bosons, and up- and down-type quarks for three generations, Z_{H_α}, $W^{\pm}_{H_\alpha}$, $\gamma_{(n)}$, $g_{(n)}$, $U_{(3+\beta)}$, $D_{(3+\beta)}$, (n, α, $\beta = 1, 2, \cdots, \infty$) denote those heavy gauge bosons together with up- and down-type quarks, respectively.

We then formally diagonalize the mass matrices of charged $2/3$ and $-1/3$ quarks and write the mass eigenstates separately as

$$U_{\alpha,L} = [\mathcal{U}_L^\dagger \Psi_L(2/3)]_\alpha, \qquad U_{\alpha,R} = [\mathcal{U}_R^\dagger \Psi_R(2/3)]_\alpha,$$
$$D_{\alpha,L} = [\mathcal{D}_L^\dagger \Psi_L(-1/3)]_\alpha, \qquad D_{\alpha,R} = [\mathcal{D}_R^\dagger \Psi_R(-1/3)]_\alpha. \tag{138}$$

Here, the charged $2/3$ quarks U_1, U_2, U_3 are identified as up-type quarks u, c, t, and the charged $-1/3$ quarks D_1, D_2, D_3 are identified as the down-type quarks d, s, b in the SM, respectively.

In the gauge sector, the mass matrices for charged and neutral gauge bosons can be determined from the Higgs kinetic term

$$M^2_{W^\pm} = \begin{pmatrix} \frac{e^2v^2}{4s_W^2}, & \cdots, & [\mathcal{M}^2_{W^\pm_{L,L}}]_{0,n'}, & [\mathcal{M}^2_{W^\pm_{L,R}}]_{0,n'}, & \cdots \\ \cdots, & \cdots, & \cdots, & \cdots, & \cdots \\ [\mathcal{M}^2_{W^\pm_{L,L}}]_{0,n}, & \cdots, & [\mathcal{M}^2_{W^\pm_{L,L}}]_{n,n'}, & [\mathcal{M}^2_{W^\pm_{L,R}}]_{n,n'}, & \cdots \\ [\mathcal{M}^2_{W^\pm_{L,R}}]_{0,n}, & \cdots, & [\mathcal{M}^2_{W^\pm_{L,R}}]_{n,n'}, & [\mathcal{M}^2_{W^\pm_{R,R}}]_{n,n'}, & \cdots \end{pmatrix},$$
$$M^2_Z = \begin{pmatrix} \frac{e^2v^2}{4s_W^2c_W^2}, & \cdots, & [\mathcal{M}^2_{Z_{L,L}}]_{0,n'}, & [\mathcal{M}^2_{Z_{L,X}}]_{0,n'}, & \cdots \\ \cdots, & \cdots, & \cdots, & \cdots, & \cdots \\ [\mathcal{M}^2_{Z_{L,L}}]_{0,n}, & \cdots, & [\mathcal{M}^2_{Z_{L,L}}]_{n,n'}, & [\mathcal{M}^2_{Z_{L,X}}]_{n,n'}, & \cdots \\ [\mathcal{M}^2_{Z_{L,X}}]_{0,n}, & \cdots, & [\mathcal{M}^2_{Z_{L,X}}]_{n,n'}, & [\mathcal{M}^2_{Z_{X,X}}]_{n,n'}, & \cdots \end{pmatrix}, \tag{139}$$

TABLE I. Mass matrix of charged $2/3$ quarks

$[\mathcal{M}^{qu}_{0,0}]_{ij}$,	$\cdots$,	0,	$[\mathcal{M}^{qU}_{0,n'}]_{ij}$,	$[\mathcal{M}^{q\tilde{U}}_{0,n'}]_{ij}$,	0,	$[\mathcal{M}^{qu}_{0,n'}]_{ij}$,	$\cdots$
$\cdots$,	$\cdots$,	$\cdots$,	$\cdots$,	$\cdots$,	$\cdots$,	$\cdots$,	$\cdots$
$[\mathcal{M}^{qu}_{n,0}]_{ij}$,	$\cdots$,	$[M^{c^i_B(n)}_{(\pm\pm)}]\delta_{nn'}\delta_{ij}$,	$[\mathcal{M}^{qU}_{n,n'}]_{ij}$,	$[\mathcal{M}^{q\tilde{U}}_{n,n'}]_{ij}$,	0,	$[\mathcal{M}^{qu}_{n,n'}]_{ij}$,	$\cdots$
0,	$\cdots$,	0,	$[M^{c^i_T(n)}_{(\pm\mp)}]\delta_{nn'}\delta_{ij}$,	0,	0,	0,	$\cdots$
0,	$\cdots$,	0,	0,	$[M^{c^i_T(n)}_{(\pm\mp)}]\delta_{nn'}\delta_{ij}$,	0,	0,	$\cdots$
$[\mathcal{M}^{\chi u}_{n,0}]_{ij}$,	$\cdots$,	0,	$[\mathcal{M}^{\chi U}_{n,n'}]_{ij}$,	$[\mathcal{M}^{\chi\tilde{U}}_{n,n'}]_{ij}$,	$[M^{c^i_B(n)}_{(\mp\pm)}]\delta_{nn'}\delta_{ij}$,	$[\mathcal{M}^{\chi u}_{n,n'}]_{ij}$,	$\cdots$
0,	$\cdots$,	0,	0,	0,	0,	$[M^{c^i_S(n)}_{(\mp\mp)}]\delta_{nn'}\delta_{ij}$,	$\cdots$
$\cdots$,	$\cdots$,	$\cdots$,	$\cdots$,	$\cdots$,	$\cdots$,	$\cdots$,	$\cdots$

TABLE II. Mass matrix of charged $-1/3$ quarks.

$[\mathcal{M}^{qd}_{0,0}]_{ij}$,	$\cdots$,	0,	$[\mathcal{M}^{qD}_{0,n'}]_{ij}$,	$[\mathcal{M}^{qd}_{0,n'}]_{ij}$,	$\cdots$
$\cdots$,	$\cdots$,	$\cdots$,	$\cdots$,	$\cdots$,	$\cdots$
$[\mathcal{M}^{qd}_{n,0}]_{ij}$,	$\cdots$,	$[M^{c^i_B(n)}_{(\pm\pm)}]\delta_{nn'}\delta_{ij}$,	$[\mathcal{M}^{qD}_{n,n'}]_{ij}$,	$[\mathcal{M}^{qd}_{n,n'}]_{ij}$,	$\cdots$
0,	$\cdots$,	0,	$[M^{c^i_T(n)}_{(\pm\mp)}]\delta_{nn'}\delta_{ij}$,	0,	$\cdots$
0,	$\cdots$,	0,	0,	$[M^{c^i_T(n)}_{(\mp\mp)}]\delta_{nn'}\delta_{ij}$,	$\cdots$
$\cdots$,	$\cdots$,	$\cdots$,	$\cdots$,	$\cdots$,	$\cdots$

where υ is the nonzero vacuum expectation value (VEV) of the Higgs located on IR-brane, the abbreviations $s_w = \sin\theta_W$, $c_w = \cos\theta_W$ with θ_W denoting the Weinberg angle. Meanwhile, elements of the matrices above are presented in Appendix A.

Formally, we can express interaction eigenstates of charged and neutral electroweak gauge bosons in linear combination of the mass eigenstates as

$$\begin{aligned}
W_L^{(0)\pm} &= (Z_W)_{0,0} W^\pm + \sum_{\alpha=1}^{\infty} (Z_W)_{0,\alpha} W_{H_\alpha}^\pm, \\
W_L^{(n)\pm} &= (Z_W)_{2n-1,0} W^\pm + \sum_{\alpha=1}^{\infty} (Z_W)_{2n-1,\alpha} W_{H_\alpha}^\pm, \\
W_R^{(n)\pm} &= (Z_W)_{2n,0} W^\pm + \sum_{\alpha=1}^{\infty} (Z_W)_{2n,\alpha} W_{H_\alpha}^\pm, \\
Z^{(0)} &= (Z_Z)_{0,0} Z + \sum_{\alpha=1}^{\infty} (Z_Z)_{0,\alpha} Z_{H_\alpha}, \\
Z^{(n)} &= (Z_Z)_{2n-1,0} Z + \sum_{\alpha=1}^{\infty} (Z_Z)_{2n-1,\alpha} Z_{H_\alpha}, \\
Z_X^{(n)} &= (Z_Z)_{2n,0} Z + \sum_{\alpha=1}^{\infty} (Z_Z)_{2n,\alpha} Z_{H_\alpha},
\end{aligned} \tag{140}$$

in which Z_W, Z_Z respectively denote the mixing matrices for charged as well as neutral electroweak gauge bosons, and Z, $W^\pm$ are identified as the corresponding gauge bosons in the SM.

When one-loop diagrams are composed by virtual intermediate charged gauge bosons and charged 2/3 quarks, the corrections to Wilson coefficients at the electroweak scale $\mu_{\rm EW}$ are formulated as

$$\begin{aligned}
\frac{G_F}{\sqrt{2}} C_{7\gamma}^{(a)}(\mu_{\rm EW}) &= \frac{e^2}{8\mu_{\rm EW}^2 s_W^2} \sum_{\beta=1}^{\infty} \Big\{ (\xi_{W^\pm}^L)_{s,\beta}^\dagger (\xi_{W^\pm}^L)_{\beta,b} F_{1,\gamma}^{(a)}(x_{U_\beta}, x_{W^\pm}) + \frac{m_{U_\beta}}{m_b} (\xi_{W^\pm}^L)_{s,\beta}^\dagger (\xi_{W^\pm}^R)_{\beta,b} F_{2,\gamma}^{(a)}(x_{U_\beta}, x_{W^\pm}) \\
&\quad + \sum_{\alpha=1}^{\infty} (\xi_{W_{H_\alpha}^\pm}^L)_{s,\beta}^\dagger (\xi_{W_{H_\alpha}^\pm}^L)_{\beta,b} F_{1,\gamma}^{(a)}(x_{U_\beta}, x_{W_{H_\alpha}^\pm}) + \frac{m_{U_\beta}}{m_b} \sum_{\alpha=1}^{\infty} (\xi_{W_{H_\alpha}^\pm}^L)_{s,\beta}^\dagger (\xi_{W_{H_\alpha}^\pm}^R)_{\beta,b} F_{2,\gamma}^{(a)}(x_{U_\beta}, x_{W_{H_\alpha}^\pm}) \Big\}, \\
\frac{G_F}{\sqrt{2}} C_{8G}^{(a)}(\mu_{\rm EW}) &= \frac{e^2}{8\mu_{\rm EW}^2 s_W^2} \sum_{\beta=1}^{\infty} \Big\{ (\xi_{W^\pm}^L)_{s,\beta}^\dagger (\xi_{W^\pm}^L)_{\beta,b} F_{1,g}^{(a)}(x_{U_\beta}, x_{W^\pm}) + \frac{m_{U_\beta}}{m_b} (\xi_{W^\pm}^L)_{s,\beta}^\dagger (\xi_{W^\pm}^R)_{\beta,b} F_{2,g}^{(a)}(x_{U_\beta}, x_{W^\pm}) \\
&\quad + \sum_{\alpha=1}^{\infty} (\xi_{W_{H_\alpha}^\pm}^L)_{s,\beta}^\dagger (\xi_{W_{H_\alpha}^\pm}^L)_{\beta,b} F_{1,g}^{(a)}(x_{U_\beta}, x_{W_{H_\alpha}^\pm}) + \frac{m_{U_\beta}}{m_b} \sum_{\alpha=1}^{\infty} (\xi_{W_{H_\alpha}^\pm}^L)_{s,\beta}^\dagger (\xi_{W_{H_\alpha}^\pm}^R)_{\beta,b} F_{2,g}^{(a)}(x_{U_\beta}, x_{W_{H_\alpha}^\pm}) \Big\}, \\
\frac{G_F}{\sqrt{2}} \tilde{C}_{7\gamma}^{(a)}(\mu_{\rm EW}) &= \frac{G_F}{\sqrt{2}} C_{7\gamma}^{(a)}(\mu_{\rm EW}) (\xi_{W^\pm}^L \leftrightarrow \xi_{W^\pm}^R, \xi_{W_{H_\alpha}^\pm}^L \leftrightarrow \xi_{W_{H_\alpha}^\pm}^R), \\
\frac{G_F}{\sqrt{2}} \tilde{C}_{8G}^{(a)}(\mu_{\rm EW}) &= \frac{G_F}{\sqrt{2}} C_{8G}^{(a)}(\mu_{\rm EW}) (\xi_{W^\pm}^L \leftrightarrow \xi_{W^\pm}^R, \xi_{W_{H_\alpha}^\pm}^L \leftrightarrow \xi_{W_{H_\alpha}^\pm}^R),
\end{aligned} \tag{141}$$

with $x_i = m_i^2/\mu_{\rm EW}^2$. The concrete expressions of relevant couplings are presented in Appendix B, and the form factors are explicitly given by

$$\begin{aligned}
F_{1,\gamma}^{(a)}(x,y) &= \Big[-\frac{1}{36} \frac{\partial^3 \varrho_{3,1}}{\partial y^3} - \frac{1}{4} \frac{\partial^2 \varrho_{2,1}}{\partial y^2} - \frac{1}{3} \frac{\partial \varrho_{1,1}}{\partial y} \Big](x,y), \qquad & F_{2,\gamma}^{(a)}(x,y) &= \Big[\frac{1}{3} \frac{\partial^2 \varrho_{2,1}}{\partial y^2} + \frac{4}{3} \frac{\partial \varrho_{1,1}}{\partial y} \Big](x,y), \\
F_{1,g}^{(a)}(x,y) &= \Big[\frac{1}{12} \frac{\partial^3 \varrho_{3,1}}{\partial y^3} - \frac{1}{2} \frac{\partial \varrho_{1,1}}{\partial y} \Big](x,y), & F_{2,g}^{(a)}(x,y) &= \Big[-\frac{\partial^2 \varrho_{2,1}}{\partial y^2} + 2 \frac{\partial \varrho_{1,1}}{\partial y} \Big](x,y).
\end{aligned} \tag{142}$$

Here, the function $\varrho_{m,n}(x,y)$ is defined through

$$\varrho_{m,n}(x,y) = \frac{x^m \ln^n x - y^m \ln^n y}{x - y}. \tag{143}$$

Similarly, we can write down the corrections to Wilson coefficients at the electroweak scale $\mu_{\rm EW}$ from one-loop diagrams which are composed by the virtual charged Goldstone and charged $2/3$ quarks

$$\begin{aligned}
\frac{G_F}{\sqrt{2}}C_{7\gamma}^{(b)}(\mu_{\rm EW}) &= \frac{1}{4\mu_{\rm EW}^2}\sum_{\beta=1}^{\infty}\Big\{(\eta_{G^\pm}^L)_{s,\beta}^\dagger(\eta_{G^\pm}^L)_{\beta,b}F_{1,\gamma}^{(b)}(x_{U_\beta}, x_{W^\pm}) + \frac{m_{U_\beta}}{m_b}(\eta_{G^\pm}^L)_{s,\beta}^\dagger(\eta_{G^\pm}^R)_{\beta,b}F_{2,\gamma}^{(b)}(x_{U_\beta}, x_{W^\pm})\Big\},\\
\frac{G_F}{\sqrt{2}}C_{8G}^{(b)}(\mu_{\rm EW}) &= \frac{1}{4\mu_{\rm EW}^2}\sum_{\beta=1}^{\infty}\Big\{(\eta_{G^\pm}^L)_{s,\beta}^\dagger(\eta_{G^\pm}^L)_{\beta,b}F_{1,g}^{(b)}(x_{U_\beta}, x_{W^\pm}) + \frac{m_{U_\beta}}{m_b}(\eta_{G^\pm}^L)_{s,\beta}^\dagger(\eta_{G^\pm}^R)_{\beta,b}F_{2,g}^{(b)}(x_{U_\beta}, x_{W^\pm})\Big\},\\
\frac{G_F}{\sqrt{2}}\tilde{C}_{7\gamma}^{(b)}(\mu_{\rm EW}) &= \frac{G_F}{\sqrt{2}}C_{7\gamma}^{(b)}(\mu_{\rm EW})(\eta_{G^\pm}^L \leftrightarrow \eta_{G^\pm}^R),\\
\frac{G_F}{\sqrt{2}}\tilde{C}_{8G}^{(b)}(\mu_{\rm EW}) &= \frac{G_F}{\sqrt{2}}C_{8G}^{(b)}(\mu_{\rm EW})(\eta_{G^\pm}^L \leftrightarrow \eta_{G^\pm}^R),
\end{aligned} \tag{144}$$

where the expressions of relevant couplings can be found in Appendix B, and those form factors are given by

$$\begin{aligned}
F_{1,\gamma}^{(b)}(x,y) &= \Big[-\frac{1}{72}\frac{\partial^3\varrho_{3,1}}{\partial y^3} - \frac{1}{24}\frac{\partial^2\varrho_{2,1}}{\partial y^2} + \frac{1}{6}\frac{\partial\varrho_{1,1}}{\partial y}\Big](x,y),\\
F_{2,\gamma}^{(b)}(x,y) &= \Big[\frac{1}{12}\frac{\partial^2\varrho_{2,1}}{\partial y^2} - \frac{1}{6}\frac{\partial\varrho_{1,1}}{\partial y} - \frac{1}{3}\frac{\partial\varrho_{1,1}}{\partial x}\Big](x,y),\\
F_{1,g}^{(b)}(x,y) &= \Big[\frac{1}{24}\frac{\partial^3\varrho_{3,1}}{\partial y^3} - \frac{1}{4}\frac{\partial^2\varrho_{2,1}}{\partial y^2} + \frac{1}{4}\frac{\partial\varrho_{1,1}}{\partial y}\Big](x,y),\\
F_{2,g}^{(b)}(x,y) &= \Big[-\frac{1}{4}\frac{\partial^2\varrho_{2,1}}{\partial y^2} + \frac{1}{2}\frac{\partial\varrho_{1,1}}{\partial y} - \frac{1}{2}\frac{\partial\varrho_{1,1}}{\partial x}\Big](x,y).
\end{aligned} \tag{145}$$

For the Feynman diagrams drawn in 1(c), intermediate virtual particles involve the neutral gauge bosons Z, Z_{H_α}, $\gamma_{(n)}$, and charged $-1/3$ quarks and the corresponding corrections to Wilson coefficients at electroweak scale are expressed as

$$\begin{aligned}
\frac{G_F}{\sqrt{2}}C_{7\gamma}^{(c)}(\mu_{\rm EW}) = &-\frac{e^2}{48\mu_{\rm EW}^2 s_W^2 c_W^2}\sum_{\beta=1}^{\infty}\Big\{(\xi_Z^L)_{s,\beta}^\dagger(\xi_Z^L)_{\beta,b}F_{1,g}^{(a)}(x_{D_\beta}, x_Z) + \frac{m_{D_\beta}}{m_b}(\xi_Z^L)_{s,\beta}^\dagger(\xi_Z^R)_{\beta,b}F_{2,g}^{(a)}(x_{D_\beta}, x_Z)\\
&+ \sum_{\alpha=1}^{\infty}(\xi_{Z_{H_\alpha}}^L)_{s,\beta}^\dagger(\xi_{Z_{H_\alpha}}^L)_{\beta,b}F_{1,g}^{(a)}(x_{D_\beta}, x_{Z_{H_\alpha}}) + \frac{m_{D_\beta}}{m_b}\sum_{\alpha=1}^{\infty}(\xi_{Z_{H_\alpha}}^L)_{s,\beta}^\dagger(\xi_{Z_{H_\alpha}}^R)_{\beta,b}F_{2,g}^{(a)}(x_{D_\beta}, x_{Z_{H_\alpha}})\Big\}\\
&- \frac{e^2}{108\mu_{\rm EW}^2}\sum_{n=1}^{\infty}\sum_{\beta=1}^{\infty}\Big\{(\xi_{\gamma_{(n)}}^L)_{s,\beta}^\dagger(\xi_{\gamma_{(n)}}^L)_{\beta,b}F_{1,g}^{(a)}(x_{D_\beta}, x_{\gamma_{(n)}}) + \frac{m_{D_\beta}}{m_b}(\xi_{\gamma_{(n)}}^L)_{s,\beta}^\dagger(\xi_{\gamma_{(n)}}^R)_{\beta,b}F_{2,g}^{(a)}(x_{D_\beta}, x_{\gamma_{(n)}})\Big\},\\
\frac{G_F}{\sqrt{2}}C_{8G}^{(c)}(\mu_{\rm EW}) = &-\frac{3G_F}{\sqrt{2}}C_{7\gamma}^{(c)}(\mu_{\rm EW}),\\
\frac{G_F}{\sqrt{2}}\tilde{C}_{7\gamma}^{(c)}(\mu_{\rm EW}) = &\frac{G_F}{\sqrt{2}}C_{7\gamma}^{(c)}(\mu_{\rm EW})(\xi_Z^L \leftrightarrow \xi_Z^R, \xi_{Z_{H_\alpha}}^L \leftrightarrow \xi_{Z_{H_\alpha}}^R, \xi_{\gamma_{(n)}}^L \leftrightarrow \xi_{\gamma_{(n)}}^R),\\
\frac{G_F}{\sqrt{2}}\tilde{C}_{8G}^{(c)}(\mu_{\rm EW}) = &\frac{G_F}{\sqrt{2}}C_{8G}^{(c)}(\mu_{\rm EW})(\xi_Z^L \leftrightarrow \xi_Z^R, \xi_{Z_{H_\alpha}}^L \leftrightarrow \xi_{Z_{H_\alpha}}^R, \xi_{\gamma_{(n)}}^L \leftrightarrow \xi_{\gamma_{(n)}}^R).
\end{aligned} \tag{146}$$

Similarly, the couplings between neutral gauge bosons and charged $-1/3$ quarks are collected in Appendix B.

Correspondingly, the contributions to Wilson coefficients at electroweak scale from Fig. 1(d) are

$$\begin{aligned}
\frac{G_F}{\sqrt{2}}C_{7\gamma}^{(d)}(\mu_{\rm EW}) &= -\frac{g_s^2}{9\mu_{\rm EW}^2}\sum_{n=1}^{\infty}\sum_{\beta=1}^{\infty}\Big\{(\xi_{g_{(n)}}^L)_{s,\beta}^{\dagger}(\xi_{g_{(n)}}^L)_{\beta,b}F_{1,g}^{(a)}(x_{D_\beta},x_{g_{(n)}}) + \frac{m_{D_\beta}}{m_b}(\xi_{g_{(n)}}^L)_{s,\beta}^{\dagger}(\xi_{g_{(n)}}^R)_{\beta,b}F_{2,g}^{(a)}(x_{D_\beta},x_{g_{(n)}})\Big\},\\
\frac{G_F}{\sqrt{2}}C_{8G}^{(d)}(\mu_{\rm EW}) &= \frac{g_s^2}{4\mu_{\rm EW}^2}\sum_{n=1}^{\infty}\sum_{\beta=1}^{\infty}\Big\{(\xi_{g_{(n)}}^L)_{s,\beta}^{\dagger}(\xi_{g_{(n)}}^L)_{\beta,b}F_{1,g}^{(d)}(x_{D_\beta},x_{g_{(n)}}) + \frac{m_{D_\beta}}{m_b}(\xi_{g_{(n)}}^L)_{s,\beta}^{\dagger}(\xi_{g_{(n)}}^R)_{\beta,b}F_{2,g}^{(d)}(x_{D_\beta},x_{g_{(n)}})\Big\},\\
\frac{G_F}{\sqrt{2}}\tilde{C}_{7\gamma}^{(d)}(\mu_{\rm EW}) &= \frac{G_F}{\sqrt{2}}C_{7\gamma}^{(d)}(\mu_{\rm EW})(\xi_{g_{(n)}}^L \leftrightarrow \xi_{g_{(n)}}^R),\\
\frac{G_F}{\sqrt{2}}\tilde{C}_{8G}^{(d)}(\mu_{\rm EW}) &= \frac{G_F}{\sqrt{2}}C_{8G}^{(d)}(\mu_{\rm EW})(\xi_{g_{(n)}}^L \leftrightarrow \xi_{g_{(n)}}^R),
\end{aligned} \tag{147}$$

where the form factors are defined as

$$F_{1,g}^{(d)}(x,y) = \Big[-\frac{5}{36}\frac{\partial^3\varrho_{3,1}}{\partial y^3} - \frac{3}{8}\frac{\partial^2\varrho_{2,1}}{\partial y^2} + \frac{1}{12}\frac{\partial\varrho_{1,1}}{\partial y}\Big](x,y), \qquad F_{2,g}^{(d)}(x,y) = \Big[\frac{5}{3}\frac{\partial^2\varrho_{2,1}}{\partial y^2} - \frac{1}{3}\frac{\partial\varrho_{1,1}}{\partial y}\Big](x,y). \tag{148}$$

As intermediate virtual particles are neutral Higgs/Goldstone and charged $-1/3$ quarks, the corresponding corrections to relevant Wilson coefficients can be written as

$$\begin{aligned}
\frac{G_F}{\sqrt{2}}C_{7\gamma}^{(e)}(\mu_{\rm EW}) = &-\frac{1}{12\mu_{\rm EW}^2}\sum_{\beta=1}^{\infty}\Big\{(\eta_{H_0})_{s,\beta}^{\dagger}(\eta_{H_0})_{\beta,b}F_{1,g}^{(b)}(x_{D_\beta},x_{H_0}) + \frac{m_{D_\beta}}{m_b}(\eta_{H_0})_{s,\beta}^{\dagger}(\eta_{H_0})_{\beta,b}^{\dagger}F_{2,g}^{(b)}(x_{D_\beta},x_{H_0})\\
&+ (\eta_{G_0})_{s,\beta}^{\dagger}(\eta_{G_0})_{\beta,b}F_{1,g}^{(b)}(x_{D_\beta},x_Z) - \frac{m_{D_\beta}}{m_b}(\eta_{G_0})_{s,\beta}^{\dagger}(\eta_{G_0})_{\beta,b}^{\dagger}F_{2,g}^{(b)}(x_{D_\beta},x_Z)\Big\},\\
\frac{G_F}{\sqrt{2}}C_{8G}^{(e)}(\mu_{\rm EW}) = &-\frac{3G_F}{\sqrt{2}}C_{7\gamma}^{(e)}(\mu_{\rm EW}),\\
\frac{G_F}{\sqrt{2}}\tilde{C}_{7\gamma}^{(e)}(\mu_{\rm EW}) = &\frac{G_F}{\sqrt{2}}C_{7\gamma}^{(e)}(\mu_{\rm EW})(\eta_{H_0}\leftrightarrow(\eta_{H_0})^{\dagger},\ \eta_{G_0}\leftrightarrow-(\eta_{G_0})^{\dagger}),\\
\frac{G_F}{\sqrt{2}}\tilde{C}_{8G}^{(e)}(\mu_{\rm EW}) = &\frac{G_F}{\sqrt{2}}C_{8G}^{(e)}(\mu_{\rm EW})(\eta_{H_0}\leftrightarrow(\eta_{H_0})^{\dagger},\ \eta_{G_0}\leftrightarrow-(\eta_{G_0})^{\dagger}).
\end{aligned} \tag{149}$$

The tedious expressions of relevant couplings are collected in Appendix B also.

The contributions from standard electroweak interaction and new physics to the effective Hamiltonian of $b \to s\gamma$ at electroweak energy scale are presented exactly in Eq. (141), (144), (146), (147), and (149). However, the corrections of new physics cannot be divorced from those of the SM explicitly in those equations because we have no means of obtaining the mixing matrices $\mathcal{U}_{L,R}$, $\mathcal{D}_{L,R}$ exactly for infinite-dimensional column vectors. Meanwhile we cannot point out obviously which among those new physics contributions probably gives potentially substantial corrections to theoretical prediction on the branching ratio of $\bar{B} \to X_s\gamma$, and which among those new physics contributions can be neglected safely after considering experimental observations.

Fortunately the present experimental data all indicate the energy scale of low-lying KK excitations $\Lambda_{\rm KK} \gg \mu_{\rm EW} \sim v$. In order to write our formulas above in transparent forms, we expand the Wilson coefficients presented in Eq. (141), etc., to the order $\mathcal{O}(v^2/\Lambda_{\rm KK}^2)$. When the intermediate lines of Feynman diagrams represent virtual SM particles, we approximate the corresponding effective couplings up to the order $\mathcal{O}(v^2/\Lambda_{\rm KK}^2)$. Because the intermediate virtual particles of Feynman diagrams involve KK excitations, we approximate the corresponding effective couplings to $\mathcal{O}(1)$ since the corresponding form factors already contain the global suppression factor $v^2/\Lambda_{\rm KK}^2$, compared with that from the one-loop diagrams which only involve virtual SM particles. Keeping this point in mind, we approach the nontrivial elements of left- and right-handed mixing matrices of charged $2/3$ quarks as

$$
\begin{aligned}
(\mathcal{U}_L)_{ij} &= (\mathcal{U}_L^{(0)} \mathcal{Z}_L^u)_{ij} - \frac{1}{2\Lambda_{\rm KK}^2} \sum_{k=1}^{3} \sum_{n=1}^{\infty} [\mathcal{M}_{0,n}^{qU}]_{ik} [y_{(\pm\mp)}^{c_T^k(n)}]^{-2} ([\mathcal{M}_{0,n}^{qU}]^\dagger \mathcal{U}_L^{(0)})_{kj} \\
&\quad - \frac{1}{2\Lambda_{\rm KK}^2} \sum_{k=1}^{3} \sum_{n=1}^{\infty} [\mathcal{M}_{0,n}^{q\tilde{U}}]_{ik} [y_{(\pm\mp)}^{c_T^k(n)}]^{-2} ([\mathcal{M}_{0,n}^{q\tilde{U}}]^\dagger \mathcal{U}_L^{(0)})_{kj} \\
&\quad - \frac{1}{2\Lambda_{\rm KK}^2} \sum_{k=1}^{3} \sum_{n=1}^{\infty} [\mathcal{M}_{0,n}^{qu}]_{ik} [y_{(\mp\mp)}^{c_S^k(n)}]^{-2} ([\mathcal{M}_{0,n}^{qu}]^\dagger \mathcal{U}_L^{(0)})_{kj} + \mathcal{O}\left(\frac{v^3}{\Lambda_{\rm KK}^3}\right), \\
(\mathcal{U}_L)_{(15n-9+i)j} &= -\frac{1}{\Lambda_{\rm KK}} [y_{(\pm\mp)}^{c_T^i(n)}]^{-1} ([\mathcal{M}_{0,n}^{qU}] \mathcal{U}_L^{(0)})_{ij} + \mathcal{O}\left(\frac{v^3}{\Lambda_{\rm KK}^3}\right), \\
(\mathcal{U}_L)_{(15n-6+i)j} &= -\frac{1}{\Lambda_{\rm KK}} [y_{(\pm\mp)}^{c_T^i(n)}]^{-1} ([\mathcal{M}_{0,n}^{q\tilde{U}}] \mathcal{U}_L^{(0)})_{ij} + \mathcal{O}\left(\frac{v^3}{\Lambda_{\rm KK}^3}\right), \\
(\mathcal{U}_L)_{(15n+i)j} &= -\frac{1}{\Lambda_{\rm KK}} [y_{(\mp\mp)}^{c_S^i(n)}]^{-1} ([\mathcal{M}_{0,n}^{qu}] \mathcal{U}_L^{(0)})_{ij} + \mathcal{O}\left(\frac{v^3}{\Lambda_{\rm KK}^3}\right), \\
(\mathcal{U}_L)_{i(15n-9+j)} &= \frac{1}{\Lambda_{\rm KK}} [\mathcal{M}_{0,n}^{qU}]_{ij} [y_{(\pm\mp)}^{c_T^j(n)}]^{-1} + \mathcal{O}\left(\frac{v^3}{\Lambda_{\rm KK}^3}\right), \\
(\mathcal{U}_L)_{i(15n-6+j)} &= \frac{1}{\Lambda_{\rm KK}} [\mathcal{M}_{0,n}^{q\tilde{U}}]_{ij} [y_{(\pm\mp)}^{c_T^j(n)}]^{-1} + \mathcal{O}\left(\frac{v^3}{\Lambda_{\rm KK}^3}\right), \\
(\mathcal{U}_L)_{i(15n+j)} &= \frac{1}{\Lambda_{\rm KK}} [\mathcal{M}_{0,n}^{qu}]_{ij} [y_{(\mp\mp)}^{c_S^j(n)}]^{-1} + \mathcal{O}\left(\frac{v^3}{\Lambda_{\rm KK}^3}\right), \\
(\mathcal{U}_L)_{IJ} &= \delta_{IJ} + \mathcal{O}\left(\frac{v}{\Lambda_{\rm KK}}\right), \qquad (I, J \geq 4), \\
(\mathcal{U}_R)_{ij} &= (\mathcal{U}_R^{(0)} \mathcal{Z}_R^u)_{ij} - \frac{1}{2\Lambda_{\rm KK}^2} \sum_{k=1}^{3} \sum_{n=1}^{\infty} [\mathcal{M}_{n,0}^{qu}]_{ik}^\dagger [y_{(\pm\pm)}^{c_B^k(n)}]^{-2} ([\mathcal{M}_{n,0}^{qu}] \mathcal{U}_R^{(0)})_{kj} \\
&\quad - \frac{1}{2\Lambda_{\rm KK}^2} \sum_{k=1}^{3} \sum_{n=1}^{\infty} [\mathcal{M}_{n,0}^{\chi u}]_{ik}^\dagger [y_{(\mp\pm)}^{c_B^k(n)}]^{-2} ([\mathcal{M}_{n,0}^{\chi u}] \mathcal{U}_R^{(0)})_{kj} + \mathcal{O}\left(\frac{v^3}{\Lambda_{\rm KK}^3}\right), \\
(\mathcal{U}_R)_{(15n-12+i)j} &= -\frac{1}{\Lambda_{\rm KK}} [y_{(\pm\pm)}^{c_B^i(n)}]^{-1} ([\mathcal{M}_{n,0}^{qu}] \mathcal{U}_R^{(0)})_{ij} + \mathcal{O}\left(\frac{v^3}{\Lambda_{\rm KK}^3}\right), \\
(\mathcal{U}_R)_{(15n-3+i)j} &= -\frac{1}{\Lambda_{\rm KK}} [y_{(\mp\pm)}^{c_B^i(n)}]^{-1} ([\mathcal{M}_{n,0}^{\chi u}] \mathcal{U}_R^{(0)})_{ij} + \mathcal{O}\left(\frac{v^3}{\Lambda_{\rm KK}^3}\right), \\
(\mathcal{U}_R)_{i(15n-12+j)} &= \frac{1}{\Lambda_{\rm KK}} [\mathcal{M}_{n,0}^{qu}]_{ij}^\dagger [y_{(\pm\pm)}^{c_B^j(n)}]^{-1} + \mathcal{O}\left(\frac{v^3}{\Lambda_{\rm KK}^3}\right), \\
(\mathcal{U}_R)_{i(15n-3+j)} &= \frac{1}{\Lambda_{\rm KK}} [\mathcal{M}_{n,0}^{\chi u}]_{ij}^\dagger [y_{(\mp\pm)}^{c_B^j(n)}]^{-1} + \mathcal{O}\left(\frac{v^3}{\Lambda_{\rm KK}^3}\right), \\
(\mathcal{U}_R)_{IJ} &= \delta_{IJ} + \mathcal{O}\left(\frac{v}{\Lambda_{\rm KK}}\right), \qquad (I, J \geq 4).
\end{aligned} \tag{150}
$$

Here, the 3×3 matrices $\mathcal{U}_{L,R}^{(0)}$ denote the rotation from chirality eigenstates to quark mass eigenstates in the absence of mixing between zero modes and corresponding KK excitations,

$$
\mathcal{U}_L^{(0)\dagger} [\mathcal{M}_{0,0}^{qu}] \mathcal{U}_R^{(0)} = \mathrm{diag}(m_u^{(0)}, m_c^{(0)}, m_t^{(0)}). \tag{151}
$$

Meanwhile, the 3×3 matrices $\mathcal{Z}_{L,R}^u$ diagonalize the following matrix:

$$\begin{aligned} \mathcal{Z}_L^{u\dagger}[\mathrm{diag}(m_u^{(0)}, m_c^{(0)}, m_t^{(0)}) + \delta\mathcal{M}^u]\mathcal{Z}_R^u &= \mathrm{diag}\{|m_u^{(0)} + (\delta\mathcal{M}^u)_{11}|, |m_c^{(0)} + (\delta\mathcal{M}^u)_{22}|, |m_t^{(0)} + (\delta\mathcal{M}^u)_{33}|\} + \mathcal{O}\Big(\frac{v^4}{\Lambda_{\mathrm{KK}}^4}\Big) \\ &= \mathrm{diag}\{m_u, m_c, m_t\}, \end{aligned} \tag{152}$$

with

$$\begin{aligned} (\mathcal{U}_L^{(0)}\delta\mathcal{M}^u\mathcal{U}_R^{(0)\dagger})_{ij} = &-\sum_{n=1}^{\infty}\Big\{\frac{1}{2\Lambda_{\mathrm{KK}}^2}\sum_{k=1}^{3}[\mathcal{M}_{0,n}^{qU}]_{ik}[y_{(\pm\mp)}^{c_T^k(n)}]^{-2}([\mathcal{M}_{0,n}^{qU}]^\dagger[\mathcal{M}_{0,0}^{qu}])_{kj} \\ &+ \frac{1}{2\Lambda_{\mathrm{KK}}^2}\sum_{k=1}^{3}[\mathcal{M}_{0,n}^{q\tilde{U}}]_{ik}[y_{(\pm\mp)}^{c_T^k(n)}]^{-2}([\mathcal{M}_{0,n}^{q\tilde{U}}]^\dagger[\mathcal{M}_{0,0}^{qu}])_{kj} \\ &+ \frac{1}{2\Lambda_{\mathrm{KK}}^2}\sum_{k=1}^{3}[\mathcal{M}_{0,n}^{qu}]_{ik}[y_{(\mp\mp)}^{c_S^k(n)}]^{-2}([\mathcal{M}_{0,n}^{qu}]^\dagger[\mathcal{M}_{0,0}^{qu}])_{kj} \\ &+ \frac{1}{2\Lambda_{\mathrm{KK}}^2}\sum_{k=1}^{3}([\mathcal{M}_{0,0}^{qu}][\mathcal{M}_{n,0}^{qu}]^\dagger)_{ik}[y_{(\pm\pm)}^{c_B^k(n)}]^{-2}[\mathcal{M}_{n,0}^{qu}]_{kj} \\ &+ \frac{1}{2\Lambda_{\mathrm{KK}}^2}\sum_{k=1}^{3}([\mathcal{M}_{0,0}^{qu}][\mathcal{M}_{n,0}^{\chi u}]^\dagger)_{ik}[y_{(\mp\pm)}^{c_B^k(n)}]^{-2}[\mathcal{M}_{n,0}^{\chi u}]_{kj}\Big\} + \mathcal{O}\Big(\frac{v^4}{\Lambda_{\mathrm{KK}}^4}\Big) \\ = &-\frac{v^2}{4\Lambda_{\mathrm{KK}}^2}\sum_{k,l=1}^{3}\{f_{(++)}^{L,c_B^i}(0,1)[Y_{ik}^d\{\Sigma_{(\pm\mp)}^{R,c_T^k}(1,1)\}Y_{kl}^{d\dagger} + Y_{ik}^u\{\Sigma_{(\mp\mp)}^{R,c_S^k}(1,1)\}Y_{kl}^{u\dagger}]f_{(++)}^{L,c_B^l}(0,1)[\mathcal{M}_{0,0}^{qu}]_{lj} \\ &+ [\mathcal{M}_{0,0}^{qu}]_{ik}^\dagger f_{(++)}^{R,c_S^k}(0,1)[Y_{kl}^{u\dagger}\{\Sigma_{(\pm\pm)}^{L,c_B^l}(1,1)\}Y_{lj}^u + Y_{kl}^{u\dagger}\{\Sigma_{(\mp\pm)}^{L,c_B^l}(1,1)\}Y_{lj}^u]f_{(++)}^{R,c_S^j}(0,1)\} + \mathcal{O}\Big(\frac{v^4}{\Lambda_{\mathrm{KK}}^4}\Big). \end{aligned} \tag{153}$$

Here, we assume that the parameters $m_u^{(0)}, m_c^{(0)}, m_t^{(0)}$ are real because this requirement can always be achieved through the redefinitions of relevant quark fields. To formulate our formulas concisely, we adopt the abbreviations $\Sigma_{(\mathrm{BCs})}^{G}(t, t')$, $\Sigma_{(\mathrm{BCs})}^{L,c}(t, t')$, and $\Sigma_{(\mathrm{BCs})}^{R,c}(t, t')$ denoting the summing over corresponding KK excitations; their concrete expressions are given in Eqs. (99), (102), (110), (114), (117), (120), and (122), respectively.

Assuming the mixing matrices $\mathcal{Z}_{L,R}^u = 1 + \delta\mathcal{Z}_{L,R}^u$, then $\delta\mathcal{Z}_{L,R}^u$ are approximated as

$$\begin{aligned} \delta\mathcal{Z}_L^u &= \begin{pmatrix} 0 & \frac{m_u\delta\mathcal{M}_{21}^{u*}+m_c\delta\mathcal{M}_{12}^u}{m_c^2-m_u^2} & \frac{m_u\delta\mathcal{M}_{31}^{u*}+m_t\delta\mathcal{M}_{13}^u}{m_t^2-m_u^2} \\ -\frac{m_u\delta\mathcal{M}_{21}^u+m_c\delta\mathcal{M}_{12}^{u*}}{m_c^2-m_u^2} & 0 & \frac{m_c\delta\mathcal{M}_{32}^{u*}+m_t\delta\mathcal{M}_{23}^u}{m_t^2-m_c^2} \\ -\frac{m_u\delta\mathcal{M}_{31}^u+m_t\delta\mathcal{M}_{13}^{u*}}{m_t^2-m_u^2} & -\frac{m_c\delta\mathcal{M}_{32}^u+m_t\delta\mathcal{M}_{23}^{u*}}{m_t^2-m_c^2} & 0 \end{pmatrix} + \mathcal{O}\Big(\frac{v^4}{\Lambda_{\mathrm{KK}}^4}\Big), \\ \delta\mathcal{Z}_R^u &= \begin{pmatrix} 0 & \frac{m_c\delta\mathcal{M}_{21}^{u*}+m_u\delta\mathcal{M}_{12}^u}{m_c^2-m_u^2} & \frac{m_t\delta\mathcal{M}_{31}^{u*}+m_u\delta\mathcal{M}_{13}^u}{m_t^2-m_u^2} \\ -\frac{m_c\delta\mathcal{M}_{21}^u+m_u\delta\mathcal{M}_{12}^{u*}}{m_c^2-m_u^2} & 0 & \frac{m_t\delta\mathcal{M}_{32}^{u*}+m_c\delta\mathcal{M}_{23}^u}{m_t^2-m_c^2} \\ -\frac{m_t\delta\mathcal{M}_{31}^u+m_u\delta\mathcal{M}_{13}^{u*}}{m_t^2-m_u^2} & -\frac{m_t\delta\mathcal{M}_{32}^u+m_c\delta\mathcal{M}_{23}^{u*}}{m_t^2-m_c^2} & 0 \end{pmatrix} + \mathcal{O}\Big(\frac{v^4}{\Lambda_{\mathrm{KK}}^4}\Big). \end{aligned} \tag{154}$$

Applying Eqs. (150)–(152), we have

$$\begin{aligned} (\mathcal{U}_L^\dagger M_U \mathcal{U}_R)_{ij} &= \mathrm{diag}\{m_u, m_c, m_t\} + \mathcal{O}\Big(\frac{v^4}{\Lambda_{\mathrm{KK}}^4}\Big), \qquad (i, j = 1, 2, 3), \\ (\mathcal{U}_L^\dagger M_U \mathcal{U}_R)_{iJ} &= \mathcal{O}\Big(\frac{v^3}{\Lambda_{\mathrm{KK}}^3}\Big), \\ (\mathcal{U}_L^\dagger M_U \mathcal{U}_R)_{Ij} &= \mathcal{O}\Big(\frac{v^3}{\Lambda_{\mathrm{KK}}^3}\Big), \qquad (I, J \geq 4). \end{aligned} \tag{155}$$

The emphasized point here is that the formulas presented in Eqs. (150) and (153) are coincided with the results derived by the effective Lagrangian approach in Refs. [49,74]. In a similar way, the nontrivial elements of left- and right-handed mixing matrices in the charged $-1/3$ quark sector are approximated as

$$
\begin{aligned}
(\mathcal{D}_L)_{ij} &= (\mathcal{D}_L^{(0)} \mathcal{Z}_L^d)_{ij} - \frac{1}{2\Lambda_{\rm KK}^2} \sum_{k=1}^{3} \sum_{n=1}^{\infty} [\mathcal{M}_{0,n}^{qD}]_{ik} [y_{(\pm\mp)}^{c_T^k(n)}]^{-2} ([\mathcal{M}_{0,n}^{qD}]^\dagger \mathcal{D}_L^{(0)})_{kj} \\
&\quad - \frac{1}{2\Lambda_{\rm KK}^2} \sum_{k=1}^{3} \sum_{n=1}^{\infty} [\mathcal{M}_{0,n}^{qd}]_{ik} [y_{(\mp\mp)}^{c_T^k(n)}]^{-2} ([\mathcal{M}_{0,n}^{qd}]^\dagger \mathcal{D}_L^{(0)})_{kj} + \mathcal{O}\left(\frac{v^3}{\Lambda_{\rm KK}^3}\right), \\
(\mathcal{D}_L)_{(9n-3+i)j} &= -\frac{1}{\Lambda_{\rm KK}} [y_{(\pm\mp)}^{c_T^i(n)}]^{-1} ([\mathcal{M}_{0,n}^{qD}]^\dagger \mathcal{D}_L^{(0)})_{ij} + \mathcal{O}\left(\frac{v^3}{\Lambda_{\rm KK}^3}\right), \\
(\mathcal{D}_L)_{(9n+i)j} &= -\frac{1}{\Lambda_{\rm KK}} [y_{(\mp\mp)}^{c_T^i(n)}]^{-1} ([\mathcal{M}_{0,n}^{qd}]^\dagger \mathcal{D}_L^{(0)})_{ij} + \mathcal{O}\left(\frac{v^3}{\Lambda_{\rm KK}^3}\right), \\
(\mathcal{D}_L)_{i(9n-3+j)} &= \frac{1}{\Lambda_{\rm KK}} [\mathcal{M}_{0,n}^{qD}]_{ij} [y_{(\pm\mp)}^{c_T^j(n)}]^{-1} + \mathcal{O}\left(\frac{v^3}{\Lambda_{\rm KK}^3}\right), \\
(\mathcal{D}_L)_{i(9n+j)} &= \frac{1}{\Lambda_{\rm KK}} [\mathcal{M}_{0,n}^{qd}]_{ij} [y_{(\mp\mp)}^{c_T^j(n)}]^{-1} + \mathcal{O}\left(\frac{v^3}{\Lambda_{\rm KK}^3}\right), \\
(\mathcal{D}_L)_{IJ} &= \delta_{IJ} + \mathcal{O}\left(\frac{v}{\Lambda_{\rm KK}}\right), \qquad (I, J \geq 4), \\
(\mathcal{D}_R)_{ij} &= (\mathcal{D}_R^{(0)} \mathcal{Z}_R^d)_{ij} - \frac{1}{2\Lambda_{\rm KK}^2} \sum_{k=1}^{3} \sum_{n=1}^{\infty} [\mathcal{M}_{n,0}^{qd}]_{ik}^\dagger [y_{(\pm\pm)}^{c_B^k(n)}]^{-2} ([\mathcal{M}_{n,0}^{qd}] \mathcal{D}_R^{(0)})_{kj} + \mathcal{O}\left(\frac{v^3}{\Lambda_{\rm KK}^3}\right), \\
(\mathcal{D}_R)_{(9n-6+i)j} &= -\frac{1}{\Lambda_{\rm KK}} [y_{(\pm\pm)}^{c_B^i(n)}]^{-1} ([\mathcal{M}_{n,0}^{qd}] \mathcal{D}_R^{(0)})_{ij} + \mathcal{O}\left(\frac{v^3}{\Lambda_{\rm KK}^3}\right), \\
(\mathcal{D}_R)_{i(9n-6+j)} &= \frac{1}{\Lambda_{\rm KK}} [\mathcal{M}_{n,0}^{qd}]_{ij}^\dagger [y_{(\pm\pm)}^{c_B^j(n)}]^{-1} + \mathcal{O}\left(\frac{v^3}{\Lambda_{\rm KK}^3}\right), \\
(\mathcal{D}_R)_{IJ} &= \delta_{IJ} + \mathcal{O}\left(\frac{v}{\Lambda_{\rm KK}}\right), \qquad (I, J \geq 4),
\end{aligned} \tag{156}
$$

where the 3×3 matrices $\mathcal{D}_{L,R}^{(0)}$ denote the rotation from chirality eigenstates to quark mass eigenstates in the absence of mixing between zero modes and corresponding KK excitations,

$$
\mathcal{D}_L^{(0)\dagger} [\mathcal{M}_{0,0}^{qd}] \mathcal{D}_R^{(0)} = \mathrm{diag}(m_d^{(0)}, m_s^{(0)}, m_b^{(0)}). \tag{157}
$$

Analogously, the 3×3 matrices $\mathcal{Z}_{L,R}^d$ diagonalize the following matrix:

$$
\begin{aligned}
\mathcal{Z}_L^{d\dagger} [\mathrm{diag}(m_d^{(0)}, m_s^{(0)}, m_b^{(0)}) + \delta\mathcal{M}^d] \mathcal{Z}_R^d &= \mathrm{diag}\{|m_d^{(0)} + (\delta\mathcal{M}^d)_{11}|, |m_s^{(0)} + (\delta\mathcal{M}^d)_{22}|, |m_b^{(0)} + (\delta\mathcal{M}^d)_{33}|\} + \mathcal{O}\left(\frac{v^4}{\Lambda_{\rm KK}^4}\right) \\
&= \mathrm{diag}\{m_d, m_s, m_b\}
\end{aligned} \tag{158}
$$

with

$$
\begin{aligned}
(\mathcal{D}_L^{(0)}\delta\mathcal{M}^d\mathcal{D}_R^{(0)\dagger})_{ij} &= -\sum_{n=1}^{\infty}\Big\{\frac{1}{2\Lambda_{\rm KK}^2}\sum_{k=1}^{3}[\mathcal{M}_{0,n}^{qD}]_{ik}[y_{(\pm\mp)}^{c_T^k(n)}]^{-2}([\mathcal{M}_{0,n}^{qD}]^\dagger[\mathcal{M}_{0,0}^{qd}])_{kj} \\
&\quad + \frac{1}{2\Lambda_{\rm KK}^2}\sum_{k=1}^{3}[\mathcal{M}_{0,n}^{qd}]_{ik}[y_{(\mp\mp)}^{c_T^k(n)}]^{-2}([\mathcal{M}_{0,n}^{qd}]^\dagger[\mathcal{M}_{0,0}^{qd}])_{kj} \\
&\quad + \frac{1}{2\Lambda_{\rm KK}^2}\sum_{k=1}^{3}([\mathcal{M}_{0,0}^{qd}][\mathcal{M}_{n,0}^{qd}]^\dagger)_{ik}[y_{(\pm\pm)}^{c_B^k(n)}]^{-2}[\mathcal{M}_{n,0}^{qd}]_{kj}\Big\} + \mathcal{O}\Big(\frac{v^4}{\Lambda_{\rm KK}^4}\Big) \\
&= -\frac{v^2}{4\Lambda_{\rm KK}^2}\sum_{k,l=1}^{3}\{f_{(++)}^{L,c_B^i}(0,1)[Y_{ik}^d\{\Sigma_{(\pm\mp)}^{R,c_T^k}(1,1)\}Y_{kl}^{d\dagger} + Y_{ik}^d\{\Sigma_{(\mp\mp)}^{R,c_T^k}(1,1)\}Y_{kl}^{d\dagger}]f_{(++)}^{L,c_B^l}(0,1)[\mathcal{M}_{0,0}^{qd}]_{lj} \\
&\quad + [\mathcal{M}_{0,0}^{qd}]_{ik}^\dagger f_{(++)}^{R,c_T^k}(0,1)Y_{kl}^{d\dagger}\{\Sigma_{(\pm\pm)}^{L,c_B^l}(1,1)\}Y_{lj}^u f_{(++)}^{R,c_T^j}(0,1)\Big\} + \mathcal{O}\Big(\frac{v^4}{\Lambda_{\rm KK}^4}\Big). \qquad (159)
\end{aligned}
$$

Here, we also assume that the parameters $m_d^{(0)}$, $m_s^{(0)}$, $m_b^{(0)}$ are real. In a similar way, we approach the mixing matrices as $\mathcal{Z}_{L,R}^d = 1 + \delta\mathcal{Z}_{L,R}^d$ with

$$
\begin{aligned}
\delta\mathcal{Z}_L^d &= \begin{pmatrix} 0 & \frac{m_d\delta\mathcal{M}_{21}^{d*}+m_s\delta\mathcal{M}_{12}^d}{m_s^2-m_d^2} & \frac{m_d\delta\mathcal{M}_{31}^{d*}+m_b\delta\mathcal{M}_{13}^d}{m_b^2-m_d^2} \\ -\frac{m_d\delta\mathcal{M}_{21}^d+m_s\delta\mathcal{M}_{12}^{d*}}{m_s^2-m_d^2} & 0 & \frac{m_s\delta\mathcal{M}_{32}^{d*}+m_b\delta\mathcal{M}_{23}^d}{m_b^2-m_s^2} \\ -\frac{m_d\delta\mathcal{M}_{31}^d+m_b\delta\mathcal{M}_{13}^{d*}}{m_b^2-m_d^2} & -\frac{m_s\delta\mathcal{M}_{32}^d+m_b\delta\mathcal{M}_{23}^{d*}}{m_b^2-m_s^2} & 0 \end{pmatrix} + \mathcal{O}\Big(\frac{v^4}{\Lambda_{\rm KK}^4}\Big), \\
\delta\mathcal{Z}_R^d &= \begin{pmatrix} 0 & \frac{m_s\delta\mathcal{M}_{21}^{d*}+m_d\delta\mathcal{M}_{12}^d}{m_s^2-m_d^2} & \frac{m_b\delta\mathcal{M}_{31}^{d*}+m_d\delta\mathcal{M}_{13}^d}{m_b^2-m_d^2} \\ -\frac{m_s\delta\mathcal{M}_{21}^d+m_d\delta\mathcal{M}_{12}^{d*}}{m_s^2-m_d^2} & 0 & \frac{m_b\delta\mathcal{M}_{32}^{d*}+m_s\delta\mathcal{M}_{23}^d}{m_b^2-m_s^2} \\ -\frac{m_b\delta\mathcal{M}_{31}^d+m_d\delta\mathcal{M}_{13}^{d*}}{m_b^2-m_d^2} & -\frac{m_b\delta\mathcal{M}_{32}^d+m_s\delta\mathcal{M}_{23}^{d*}}{m_b^2-m_s^2} & 0 \end{pmatrix} + \mathcal{O}\Big(\frac{v^4}{\Lambda_{\rm KK}^4}\Big).
\end{aligned} \qquad (160)
$$

Generally, we define the Cabibbo-Kobayashi-Maskawa (CKM) matrix as

$$
(V_{\rm CKM})_{tb} = (\mathcal{U}_L^\dagger\mathcal{D}_L)_{tb} = (\mathcal{U}_L^{(0)\dagger}\mathcal{D}_L^{(0)})_{tb} + \frac{v^2}{\Lambda_{\rm KK}^2}(\Delta_{\rm KK}^{(2)})_{tb} + \mathcal{O}\Big(\frac{v^3}{\Lambda_{\rm KK}^3}\Big) = (V_{\rm CKM}^{(0)})_{tb} + \frac{v^2}{\Lambda_{\rm KK}^2}(\Delta_{\rm KK}^{(2)})_{tb} + \mathcal{O}\Big(\frac{v^3}{\Lambda_{\rm KK}^3}\Big), \qquad (161)
$$

with $V_{\rm CKM}^{(0)} = \mathcal{U}_L^{(0)\dagger}\mathcal{D}_L^{(0)}$ being a unitary 3×3 matrix, and the leading-order correction $\Delta_{\rm KK}^{(2)}$ together with other higher order corrections from heavy KK excitations break down the unitary property of $V_{\rm CKM}$ [49].

To the order $\mathcal{O}(v^2/\Lambda_{\rm KK}^2)$, we approach the mixing matrices of gauge bosons as

$$
\begin{aligned}
&(\mathcal{Z}_W)_{0,0} = 1 + \mathcal{O}\Big(\frac{v^4}{\Lambda_{\rm KK}^4}\Big), && (\mathcal{Z}_W)_{0,(2n-1)} = \frac{1}{\Lambda_{\rm KK}^2}[\mathcal{M}_{W_{L,L}^\pm}^2]_{0,n}[y_{(++)}^{W_L(n)}]^{-2} + \mathcal{O}\Big(\frac{v^4}{\Lambda_{\rm KK}^4}\Big), \\
&(\mathcal{Z}_W)_{0,(2n)} = \frac{1}{\Lambda_{\rm KK}^2}[\mathcal{M}_{W_{L,R}^\pm}^2]_{0,n}[y_{(-+)}^{W_R(n)}]^{-2} + \mathcal{O}\Big(\frac{v^4}{\Lambda_{\rm KK}^4}\Big), && (\mathcal{Z}_W)_{(2n-1),0} = -\frac{1}{\Lambda_{\rm KK}^2}[\mathcal{M}_{W_{L,L}^\pm}^2]_{0,n}[y_{(++)}^{W_L(n)}]^{-2} + \mathcal{O}\Big(\frac{v^4}{\Lambda_{\rm KK}^4}\Big), \\
&(\mathcal{Z}_W)_{(2n),0} = -\frac{1}{\Lambda_{\rm KK}^2}[\mathcal{M}_{W_{L,R}^\pm}^2]_{0,n}[y_{(-+)}^{W_R(n)}]^{-2} + \mathcal{O}\Big(\frac{v^4}{\Lambda_{\rm KK}^4}\Big), && (\mathcal{Z}_Z)_{0,0} = 1 + \mathcal{O}\Big(\frac{v^4}{\Lambda_{\rm KK}^4}\Big), \\
&(\mathcal{Z}_Z)_{0,(2n-1)} = \frac{1}{\Lambda_{\rm KK}^2}[\mathcal{M}_{Z_{L,L}}^2]_{0,n}[y_{(++)}^{Z(n)}]^{-2} + \mathcal{O}\Big(\frac{v^4}{\Lambda_{\rm KK}^4}\Big), && (\mathcal{Z}_Z)_{0,(2n)} = \frac{1}{\Lambda_{\rm KK}^2}[\mathcal{M}_{Z_{L,X}}^2]_{0,n}[y_{(-+)}^{Z_X(n)}]^{-2} + \mathcal{O}\Big(\frac{v^4}{\Lambda_{\rm KK}^4}\Big), \\
&(\mathcal{Z}_Z)_{(2n-1),0} = -\frac{1}{\Lambda_{\rm KK}^2}[\mathcal{M}_{Z_{L,L}}^2]_{0,n}[y_{(++)}^{Z(n)}]^{-2} + \mathcal{O}\Big(\frac{v^4}{\Lambda_{\rm KK}^4}\Big), && (\mathcal{Z}_Z)_{(2n),0} = -\frac{1}{\Lambda_{\rm KK}^2}[\mathcal{M}_{Z_{L,X}}^2]_{0,n}[y_{(-+)}^{Z_X(n)}]^{-2} + \mathcal{O}\Big(\frac{v^4}{\Lambda_{\rm KK}^4}\Big). \quad (162)
\end{aligned}
$$

Meanwhile, the masses of lightest charged and neutral gauge bosons are, respectively, given as

$$m_W^2 = \frac{e^2 v^2}{4s_w^2}\left\{1 + \frac{\pi e^2 v^2}{2s_w^2\Lambda_{\rm KK}^2}[\{\Sigma_{(++)}^G(1,1)\} + \{\Sigma_{(-+)}^G(1,1)\}] + \mathcal{O}\left(\frac{v^4}{\Lambda_{\rm KK}^4}\right)\right\},$$
$$m_Z^2 = \frac{e^2 v^2}{4s_w^2 c_w^2}\left\{1 + \frac{\pi e^2 v^2}{2s_w^2 c_w^2\Lambda_{\rm KK}^2}[\{\Sigma_{(++)}^G(1,1)\} + (1-2s_w^2)\{\Sigma_{(-+)}^G(1,1)\}] + \mathcal{O}\left(\frac{v^4}{\Lambda_{\rm KK}^4}\right)\right\}. \quad (163)$$

With the preparation above, we approach relevant nontrivial couplings in Fig. 1(a) to the order $\mathcal{O}(v^2/\Lambda_{\rm KK}^2)$ as

$$(\xi_{W^\pm}^L)_{t,b} = (V_{\rm CKM}^{(0)})_{tb} + (\delta Z_L^{u\dagger} V_{\rm CKM}^{(0)})_{tb} + (V_{\rm CKM}^{(0)}\delta Z_L^d)_{tb} - \frac{v^2}{4\Lambda_{\rm KK}^2}(\Delta_{W^\pm}^L)_{tb}, + \mathcal{O}\left(\frac{v^3}{\Lambda_{\rm KK}^3}\right),$$
$$(\xi_{W^\pm}^R)_{t,b} = \frac{v^2}{2\Lambda_{\rm KK}^2}(\Delta_{W^\pm}^R)_{tb} + \mathcal{O}\left(\frac{v^3}{\Lambda_{\rm KK}^3}\right),$$
$$(\xi_{W_{H(2n-1)}^\pm}^L)_{t,b} = \frac{4\sqrt{2\pi}}{kr\epsilon}\sum_{i=1}^3(\mathcal{U}_L^{(0)})_{ti}^\dagger\int_\epsilon^1 dt\chi_{(++)}^{W_L}(y_{(++)}^{W_L(n)},t)[f_{(++)}^{L,c_B^i}(0,t)]^2(\mathcal{D}_L^{(0)})_{ib} + \mathcal{O}\left(\frac{v^3}{\Lambda_{\rm KK}^3}\right),$$
$$(\xi_{W_{H(2n-1)}^\pm}^L)_{\alpha,b} = \frac{4\sqrt{2\pi}}{kr\epsilon}\sum_{i=1}^3\sum_{n'=1}^\infty \delta_{\alpha(15n'-12+i)} \times\int_\epsilon^1 dt\chi_{(++)}^{W_L}(y_{(++)}^{W_L(n)},t)[f_{(++)}^{L,c_B^i}(y_{(\pm\pm)}^{c_B^i(n')},t)][f_{(++)}^{L,c_B^i}(0,t)](\mathcal{D}_L^{(0)})_{ib} + \mathcal{O}\left(\frac{v^2}{\Lambda_{\rm KK}^2}\right), \qquad (\alpha\ge 4),$$
$$(\xi_{W_{H(2n)}^\pm}^L)_{\alpha,b} = \frac{4\sqrt{2\pi}}{kr\epsilon}\sum_{i=1}^3\sum_{n'=1}^\infty \delta_{\alpha(15n'-3+i)} \times\int_\epsilon^1 dt\chi_{(-+)}^{W_R}(y_{(-+)}^{W_R(n)},t)[f_{(-+)}^{L,c_B^i}(y_{(\mp\pm)}^{c_B^i(n')},t)][f_{(++)}^{L,c_B^i}(0,t)](\mathcal{D}_L^{(0)})_{ib} + \mathcal{O}\left(\frac{v^2}{\Lambda_{\rm KK}^2}\right), \qquad (\alpha\ge 4), \quad (164)$$

with

$$(\Delta_{W^\pm}^L)_{tb} = -\sum_{i,j,k=1}^3(\mathcal{U}_L^{(0)})_{ti}^\dagger f_{(++)}^{L,c_B^i}(0,1)\{2Y_{ik}^d[\Sigma_{(\pm\mp)}^{R,c_T^k}(1,1)]Y_{kj}^{d\dagger} + Y_{ik}^d[\Sigma_{(\mp\mp)}^{R,c_T^k}(1,1)]Y_{kj}^{d\dagger} + Y_{ik}^u[\Sigma_{(\mp\mp)}^{R,c_S^k}(1,1)]Y_{kj}^{u\dagger}\}f_{(++)}^{L,c_B^j}(0,1)(\mathcal{D}_L^{(0)})_{jb}$$
$$-\frac{8\pi e^2}{s_W^2}\sum_{i=1}^3(\mathcal{U}_L^{(0)})_{ti}^\dagger\left\{\frac{1}{kr\epsilon}\int_\epsilon^1 dt[f_{(++)}^{L,c_B^i}(0,t)]^2[\Sigma_{(++)}^G(t,1)](\mathcal{D}_L^{(0)})_{ib}\right\},$$
$$(\Delta_{W^\pm}^R)_{tb} = \sum_{i,j,k=1}^3(\mathcal{U}_R^{(0)})_{ti}^\dagger f_{(++)}^{R,c_S^i}(0,1)Y_{ik}^{d\dagger}[\Sigma_{(\pm\pm)}^{L,c_B^k}(1,1)]Y_{kj}^d f_{(++)}^{R,c_T^j}(0,1)(\mathcal{D}_R^{(0)})_{jb}. \quad (165)$$

Then we can approximate the corrections to relevant Wilson coefficients from Fig. 1(a) to the order $\mathcal{O}(v^2/\Lambda_{\rm KK}^2)$ as

$$C_{7\gamma}^{(a)}(\mu_{\rm EW}) = (1-\Delta G_F)C_{7\gamma}^{{\rm SM}(a)}(\mu_{\rm EW}) + \sum_{i=1}^3(\Upsilon_{i,1}^{(a)})_{sb}x_{W^\pm}F_{1,\gamma}^{(a)}(x_{u_i},x_{W^\pm}) + \frac{2m_W^4}{\mu_{\rm EW}^2\Lambda_{\rm KK}^2 e^2}\sum_{i=1}^3\frac{m_{u_i}}{m_b}(V_{\rm CKM}^{(0)})_{si}^\dagger(\Delta_{W^\pm}^R)_{ib}F_{2,\gamma}^{(a)}(x_{u_i},x_{W^\pm}) + \mathcal{O}\left(\frac{v^4}{\Lambda_{\rm KK}^4}\right),$$
$$C_{8G}^{(a)}(\mu_{\rm EW}) = (1-\Delta G_F)C_{8G}^{SM(a)}(\mu_{\rm EW}) + \sum_{i=1}^3(\Upsilon_{i,1}^{(a)})_{sb}x_{W^\pm}F_{1,g}^{(a)}(x_{u_i},x_{W^\pm}) + \frac{2m_W^4}{\mu_{\rm EW}^2\Lambda_{\rm KK}^2 e^2}\sum_{i=1}^3\frac{m_{u_i}}{m_b}(V_{\rm CKM}^{(0)})_{si}^\dagger(\Delta_{W^\pm}^R)_{ib}F_{2,g}^{(a)}(x_{u_i},x_{W^\pm}) + \mathcal{O}\left(\frac{v^4}{\Lambda_{\rm KK}^4}\right),$$
$$\tilde{C}_{7\gamma}^{(a)}(\mu_{\rm EW}) = \frac{2m_W^4}{\mu_{\rm EW}^2\Lambda_{\rm KK}^2 e^2}\sum_{i=1}^3\frac{m_{u_i}}{m_b}(\Delta_{W^\pm}^R)_{si}^\dagger(V_{CKM}^{(0)})_{ib}F_{2,\gamma}^{(a)}(x_{u_i},x_{W^\pm}) + \mathcal{O}\left(\frac{v^4}{\Lambda_{\rm KK}^4}\right),$$
$$\tilde{C}_{8G}^{(a)}(\mu_{\rm EW}) = \frac{2m_W^4}{\mu_{\rm EW}^2\Lambda_{\rm KK}^2 e^2}\sum_{i=1}^3\frac{m_{u_i}}{m_b}(\Delta_{W^\pm}^R)_{si}^\dagger(V_{\rm CKM}^{(0)})_{ib}F_{2,g}^{(a)}(x_{u_i},x_{W^\pm}) + \mathcal{O}\left(\frac{v^4}{\Lambda_{\rm KK}^4}\right), \quad (166)$$

with

$$C_{7\gamma}^{\mathrm{SM}(a)}(\mu_{\mathrm{EW}}) = x_{W^\pm}\sum_{i=1}^{3}(V_{\mathrm{CKM}}^{(0)})_{si}^{\dagger}(V_{\mathrm{CKM}}^{(0)})_{ib}\left(F_{1,\gamma}^{(a)}(x_{u_i}, x_{W^\pm}) + \frac{23}{36x_{W^\pm}}\right),$$
$$C_{8G}^{\mathrm{SM}(a)}(\mu_{\mathrm{EW}}) = x_{W^\pm}\sum_{i=1}^{3}(V_{\mathrm{CKM}}^{(0)})_{si}^{\dagger}(V_{\mathrm{CKM}}^{(0)})_{ib}\left(F_{1,g}^{(a)}(x_{u_i}, x_{W^\pm}) + \frac{1}{3x_{W^\pm}}\right) \tag{167}$$

representing the SM corrections from Fig. 1(a) at electroweak energy scale. Here, we define the abbreviation

$$(\Upsilon_{i,1}^{(a)})_{sb} = (V_{\mathrm{CKM}}^{(0)})_{si}^{\dagger}[(\delta Z_L^{u\dagger} + \delta Z_L^u)(V_{\mathrm{CKM}}^{(0)})]_{ib} + (V_{\mathrm{CKM}}^{(0)})_{si}^{\dagger}(V_{\mathrm{CKM}}^{(0)}\delta Z_L^d)_{ib} + (V_{\mathrm{CKM}}^{(0)}\delta Z_L^d)_{si}^{\dagger}(V_{\mathrm{CKM}}^{(0)})_{ib}$$
$$+ \frac{v^2}{4\Lambda_{\mathrm{KK}}^2}[(V_{\mathrm{CKM}}^{(0)})_{si}^{\dagger}(\Delta_{W^\pm}^L)_{ib} + (\Delta_{W^\pm}^L)_{si}^{\dagger}(V_{\mathrm{CKM}}^{(0)})_{ib}]. \tag{168}$$

It is worth emphasizing that

$$\lim_{x_{u_i}\to 0} F_{1,\gamma}^{(a)}(x_{u_i}, x_{W^\pm}) = -\frac{23}{36x_{W^\pm}}, \qquad \lim_{x_{u_i}\to 0} F_{1,g}^{(a)}(x_{u_i}, x_{W^\pm}) = -\frac{1}{3x_{W^\pm}}. \tag{169}$$

Using the unitary property of $V_{\mathrm{CKM}}^{(0)}$, one easily find that those terms are canceled after summing over the index of generation. To the order $\mathcal{O}(v^2/\Lambda_{\mathrm{KK}}^2)$, we find that the corrections from the diagrams composed by virtual charged gauge boson KK partners and charged 2/3 quark KK partners simultaneously cancel exactly the corresponding corrections from the diagrams composed by virtual standard charged gauge boson and charged 2/3 quark KK partners after summing over infinite KK excitation series. Similarly, the nontrivial couplings involving in Fig. 1(b) are approached as

$$(\eta_{G^\pm}^L)_{t,b} = \frac{e}{\sqrt{2}s_w}\left\{\frac{m_t}{m_W}(V_{\mathrm{CKM}}^{(0)})_{tb} - \frac{(\delta\mathcal{M}^u)_{33}}{m_W}(V_{\mathrm{CKM}}^{(0)})_{tb} - \frac{\pi m_t m_W}{\Lambda_{\mathrm{KK}}^2}[\{\Sigma_{(++)}^G(1,1)\right\} + \{\Sigma_{(-+)}^G(1,1)\}](V_{\mathrm{CKM}}^{(0)})_{tb}$$
$$+ \sum_{i=1}^{3}\left[\frac{m_{u_i}}{m_W}(\delta Z_R^u)_{ti}(V_{\mathrm{CKM}}^{(0)})_{ib} + \frac{m_t}{m_W}(V_{\mathrm{CKM}}^{(0)})_{ti}(\delta Z_L^d)_{ib}\right] + \frac{v^2}{4\Lambda_{\mathrm{KK}}^2}(\Delta_{G^\pm}^L)_{tb}\} + \mathcal{O}\left(\frac{v^3}{\Lambda_{\mathrm{KK}}^3}\right),$$
$$(\eta_{G^\pm}^L)_{\alpha,b} = \sum_{i,j=1}^{3}\sum_{n=1}^{\infty}\{\delta_{\alpha,(15n+i)}f_{(++)}^{R,c_S^i}(y_{(\mp\mp)}^{c_S^i(n)},1)Y_{ij}^{u\dagger} + \delta_{\alpha,(15n-9+i)}f_{(-+)}^{R,c_T^i}(y_{(\pm\mp)}^{c_T^i(n)},1)Y_{ij}^{d\dagger}$$
$$+ \delta_{\alpha,(15n-6+i)}f_{(-+)}^{R,c_T^i}(y_{(\pm\mp)}^{c_T^i(n)},1)Y_{ij}^{d\dagger}\}f_{(++)}^{L,c_B^j}(0,1)(\mathcal{D}_L^{(0)})_{j,b} + \mathcal{O}\left(\frac{v^2}{\Lambda_{\mathrm{KK}}^2}\right),$$
$$(\eta_{G^\pm}^R)_{t,b} = \frac{e}{\sqrt{2}s_w}\left\{(V_{\mathrm{CKM}}^{(0)})_{tb}\frac{m_b}{m_W} - \frac{(\delta\mathcal{M}^d)_{33}}{m_W}(V_{\mathrm{CKM}}^{(0)})_{tb} - \frac{\pi m_t m_W}{\Lambda_{\mathrm{KK}}^2}[\{\Sigma_{(++)}^G(1,1)\} + \{\Sigma_{(-+)}^G(1,1)\}](V_{\mathrm{CKM}}^{(0)})_{tb}\right.$$
$$\left.+ \sum_{i=1}^{3}\left[\frac{m_b}{m_W}(\delta Z_R^u)_{ti}(V_{\mathrm{CKM}}^{(0)})_{ib} + \frac{m_{d_i}}{m_W}(V_{\mathrm{CKM}}^{(0)})_{ti}(\delta Z_L^d)_{ib}\right] + \frac{v^2}{4\Lambda_{\mathrm{KK}}^2}(\Delta_{G^\pm}^R)_{tb}\right\} + \mathcal{O}\left(\frac{v^3}{\Lambda_{\mathrm{KK}}^3}\right),$$
$$(\eta_{G^\pm}^R)_{\alpha,b} = \sum_{i,j=1}^{3}\sum_{n=1}^{\infty}\delta_{\alpha,(15n-12+i)}f_{(++)}^{L,c_B^i}(y_{(\pm\pm)}^{c_B^i(n)},1)Y_{ij}^{d}f_{(++)}^{R,c_T^j}(0,1)(\mathcal{D}_R^{(0)})_{j,b} + \mathcal{O}\left(\frac{v^2}{\Lambda_{\mathrm{KK}}^2}\right), \tag{170}$$

with

$$(\Delta^L_{G^\pm})_{tb} = -\sum_{i,j,k,l=1}^{3}\frac{m_{u_l}}{m_W}(\mathcal{U}_R^{(0)})^\dagger_{ti}f^{R,c_S^i}_{(++)}(0,1)Y^{u\dagger}_{ik}[\{\Sigma^{L,c_B^k}_{(\pm\pm)}(1,1)\}+\{\Sigma^{L,c_B^k}_{(\mp\pm)}(1,1)\}]Y^u_{kj}f^{R,c_S^j}_{(++)}(0,1)(\mathcal{U}_R^{(0)})_{jl}(V^{(0)}_{\rm CKM})_{lb}$$
$$-\sum_{i,j,k,l=1}^{3}\frac{m_t}{m_W}(\mathcal{U}_L^{(0)})^\dagger_{ti}f^{L,c_B^i}_{(++)}(0,1)Y^d_{ik}[\{\Sigma^{R,c_T^k}_{(\pm\mp)}(1,1)\}+\{\Sigma^{R,c_T^k}_{(\mp\mp)}(1,1)\}]Y^{d\dagger}_{kj}f^{L,c_B^j}_{(++)}(0,1)(\mathcal{U}_L^{(0)})_{jl}(V^{(0)}_{\rm CKM})_{lb},$$
$$(\Delta^R_{G^\pm})_{tb} = -\sum_{i,j,k,l=1}^{3}\frac{m_{d_i}}{m_W}(V^{(0)}_{\rm CKM})_{ti}(\mathcal{D}_R^{(0)})^\dagger_{ik}f^{R,c_T^k}_{(++)}(0,1)Y^{d\dagger}_{kl}[\Sigma^{L,c_B^l}_{(\pm\pm)}(1,1)]\times Y^d_{lj}f^{R,c_S^j}_{(++)}(0,1)(\mathcal{D}_R^{(0)})_{jb}$$
$$-\sum_{i,j,k,l=1}^{3}\frac{m_b}{m_W}(V^{(0)}_{\rm CKM})_{ti}(\mathcal{D}_L^{(0)})^\dagger_{ik}f^{L,c_B^k}_{(++)}(0,1)[Y^d_{kl}\{\Sigma^{R,c_T^l}_{(\pm\mp)}(1,1)\}Y^{d\dagger}_{lj}+Y^u_{kl}\{\Sigma^{R,c_T^l}_{(\mp\mp)}(1,1)\}Y^{u\dagger}_{lj}]f^{L,c_B^j}_{(++)}(0,1)(\mathcal{D}_L^{(0)})_{jb}. \quad (171)$$

The corrections to Wilson coefficients from this sector are correspondingly formulated to the order $\mathcal{O}(v^2/\Lambda^2_{\rm KK})$ as

$$C^{(b)}_{7\gamma}(\mu_{\rm EW}) = (1-\Delta G_F)C^{{\rm SM}(b)}_{7\gamma}(\mu_{\rm EW}) + \sum_{i=1}^{3}\{(\Upsilon^{(b)}_{i,1})_{sb}x_{W^\pm}F^{(b)}_{1,\gamma}(x_{u_i},x_{W^\pm}) + (\Upsilon^{(b)}_{i,2})_{sb}x_{W^\pm}F^{(b)}_{2,\gamma}(x_{u_i},x_{W^\pm})\}$$
$$+\frac{2m_W^2s_w^2}{9\Lambda^2_{\rm KK}e^2}\sum_{i,j,k=1}^{3}(\mathcal{D}_L^{(0)})^\dagger_{si}[f^{L,c_B^i}_{(++)}(0,1)]\{Y^u_{ik}[\Sigma^{R,c_S^k}_{(\mp\mp)}(1,1)]Y^{u\dagger}_{kj}+2Y^d_{ik}[\Sigma^{R,c_T^k}_{(\pm\mp)}(1,1)]Y^{d\dagger}_{kj}\}[f^{L,c_B^j}_{(++)}(0,1)](\mathcal{D}_L^{(0)})_{jb}$$
$$+\mathcal{O}\left(\frac{v^4}{\Lambda^4_{\rm KK}}\right),$$
$$C^{(b)}_{8G}(\mu_{\rm EW}) = (1-\Delta G_F)C^{{\rm SM}(b)}_{8G}(\mu_{\rm EW}) + \sum_{i=1}^{3}\{(\Upsilon^{(b)}_{i,1})_{sb}x_{W^\pm}F^{(b)}_{1,g}(x_{u_i},x_{W^\pm}) + (\Upsilon^{(b)}_{i,2})_{sb}x_{W^\pm}F^{(b)}_{2,g}(x_{u_i},x_{W^\pm})\}$$
$$+\frac{m_W^2s_w^2}{12\Lambda^2_{\rm KK}e^2}\sum_{i,j,k=1}^{3}(\mathcal{D}_L^{(0)})^\dagger_{si}[f^{L,c_B^i}_{(++)}(0,1)]\{Y^u_{ik}[\Sigma^{R,c_S^k}_{(\mp\mp)}(1,1)]Y^{u\dagger}_{kj}+2Y^d_{ik}[\Sigma^{R,c_T^k}_{(\pm\mp)}(1,1)]Y^{d\dagger}_{kj}\}[f^{L,c_B^j}_{(++)}(0,1)](\mathcal{D}_L^{(0)})_{jb}$$
$$+\mathcal{O}\left(\frac{v^4}{\Lambda^4_{\rm KK}}\right), \tilde{C}^{(b)}_{7\gamma}(\mu_{\rm EW})$$
$$=\sum_{i=1}^{3}(\Upsilon^{(b)}_{i,3})_{sb}x_{u_i}F^{(b)}_{2,\gamma}(x_{u_i},x_{W^\pm}) + \frac{2m_W^2s_w^2}{9\Lambda^2_{\rm KK}e^2}\sum_{i,j,k=1}^{3}(\mathcal{D}_R^{(0)})^\dagger_{si}[f^{R,c_T^i}_{(++)}(0,1)]Y^d_{ik}[\Sigma^{L,c_B^k}_{(\pm\pm)}(1,1)]Y^{d\dagger}_{kj}[f^{R,c_T^j}_{(++)}(0,1)](\mathcal{D}_R^{(0)})_{jb}$$
$$+\mathcal{O}\left(\frac{v^4}{\Lambda^4_{\rm KK}}\right),$$
$$\tilde{C}^{(b)}_{8G}(\mu_{\rm EW}) = \sum_{i=1}^{3}(\Upsilon^{(b)}_{i,3})_{sb}x_{u_i}F^{(b)}_{2,g}(x_{u_i},x_{W^\pm}) + \frac{m_W^2s_w^2}{12\Lambda^2_{\rm KK}e^2}\sum_{i,j,k=1}^{3}(\mathcal{D}_R^{(0)})^\dagger_{si}[f^{R,c_T^i}_{(++)}(0,1)]Y^d_{ik}[\Sigma^{L,c_B^k}_{(\pm\pm)}(1,1)]Y^{d\dagger}_{kj}[f^{R,c_T^j}_{(++)}(0,1)](\mathcal{D}_R^{(0)})_{jb}$$
$$+\mathcal{O}\left(\frac{v^4}{\Lambda^4_{\rm KK}}\right), \quad (172)$$

where the SM corrections from Fig. 1(b) to Wilson coefficients are written as

$$C^{{\rm SM}(b)}_{7\gamma}(\mu_{\rm EW}) = \sum_{i=1}^{3}(V^{(0)}_{\rm CKM})^\dagger_{si}(V^{(0)}_{\rm CKM})_{ib}x_{u_i}(F^{(b)}_{1,\gamma}+F^{(b)}_{2,\gamma})(x_{u_i},x_{W^\pm}),$$
$$C^{{\rm SM}(b)}_{8G}(\mu_{\rm EW}) = \sum_{i=1}^{3}(V^{(0)}_{\rm CKM})^\dagger_{si}(V^{(0)}_{\rm CKM})_{ib}x_{u_i}(F^{(b)}_{1,g}+F^{(b)}_{2,g})(x_{u_i},x_{W^\pm}). \quad (173)$$

Here the short cutting notations are defined through

$$
\begin{aligned}
(\Upsilon^{(b)}_{i,1})_{sb} = & -\frac{m_{u_i}}{m_W^2}(V^{(0)}_{\rm CKM})^\dagger_{si}(V^{(0)}_{\rm CKM})_{ib}[(\delta\mathcal{M}^u)_{ii}+(\delta\mathcal{M}^u)^*_{ii}] \\
& -\frac{2\pi m^2_{u_i}}{\Lambda^2_{\rm KK}}(V^{(0)}_{\rm CKM})^\dagger_{si}(V^{(0)}_{\rm CKM})_{ib}\{[\Sigma^G_{(++)}(1,1)]+[\Sigma^G_{(-+)}(1,1)]\} \\
& +\sum_{j=1}^3\frac{m_{u_i}m_{u_j}}{m_W^2}(V^{(0)}_{\rm CKM})^\dagger_{si}[(\delta Z^u_R)_{ij}+(\delta Z^u_R)^\dagger_{ij}](V^{(0)}_{\rm CKM})_{jb} \\
& +\sum_{j=1}^3\frac{m^2_{u_i}}{m_W^2}[(V^{(0)}_{\rm CKM})^\dagger_{si}(V^{(0)}_{\rm CKM})_{ij}(\delta Z^d_L)_{jb}+(\delta Z^d_L)^\dagger_{sj}(V^{(0)}_{\rm CKM})^\dagger_{ji}(V^{(0)}_{\rm CKM})_{ib}] \\
& +\frac{v^2}{4\Lambda^2_{\rm KK}}[(V^{(0)}_{\rm CKM})^\dagger_{si}(\Delta^L_{G^\pm})_{ib}+(\Delta^L_{G^\pm})^\dagger_{si}(V^{(0)}_{\rm CKM})_{ib}], \\
(\Upsilon^{(b)}_{i,2})_{sb} = & -(V^{(0)}_{\rm CKM})^\dagger_{si}(V^{(0)}_{\rm CKM})_{ib}\Big[\frac{m^2_{u_i}}{m^2_{W^\pm}m_b}(\delta\mathcal{M}^d)_{ii}+\frac{m_{u_i}}{m^2_{W^\pm}}(\delta\mathcal{M}^d)^*_{ii}\Big] \\
& -\frac{\pi m^2_{u_i}}{\Lambda^2_{\rm KK}}(V^{(0)}_{\rm CKM})^\dagger_{si}(V^{(0)}_{\rm CKM})_{ib}\Big(1+\frac{m_{u_i}}{m_b}\Big)\{[\Sigma^G_{(++)}(1,1)]+[\Sigma^G_{(-+)}(1,1)]\} \\
& +\sum_{j=1}^3\frac{m^2_{u_i}}{m_W^2}(V^{(0)}_{\rm CKM})^\dagger_{si}[(\delta Z^u_R)_{jb}(V^{(0)}_{\rm CKM})_{jb}+\frac{m_{d_j}}{m_b}(V^{(0)}_{\rm CKM})_{ij}(\delta Z^d_L)_{jb}] \\
& +\sum_{j=1}^3\Big[\frac{m_{u_i}m_{u_j}}{m_W^2}(V^{(0)}_{\rm CKM})^\dagger_{sj}(\delta Z^u_R)^\dagger_{ji}+\frac{m^2_{u_i}}{m_W^2}(\delta Z^d_L)^\dagger_{sj}(V^{(0)}_{\rm CKM})^\dagger_{ji}\Big](V^{(0)}_{\rm CKM})_{ib} \\
& +\frac{v^2 m_{u_i}}{4\Lambda^2_{\rm KK}m_b}[(V^{(0)}_{\rm CKM})^\dagger_{si}(\Delta^R_{G^\pm})_{ib}+(\Delta^L_{G^\pm})^\dagger_{si}(V^{(0)}_{\rm CKM})_{ib}], \\
(\Upsilon^{(b)}_{i,3})_{sb} = & -\frac{(\delta\mathcal{M}^d)^*_{22}}{m_b}(V^{(0)}_{\rm CKM})^\dagger_{si}(V^{(0)}_{\rm CKM})_{ib} \\
& -\frac{\pi m_{u_i}m_W^2}{\Lambda^2_{\rm KK}m_b}(V^{(0)}_{\rm CKM})^\dagger_{si}(V^{(0)}_{\rm CKM})_{ib}\{[\Sigma^G_{(++)}(1,1)]+[\Sigma^G_{(-+)}(1,1)]\} \\
& +\sum_{j=1}^3\Big[\frac{m_s}{m_b}(V^{(0)}_{\rm CKM})^\dagger_{sj}(\delta Z^u_R)^\dagger_{ji}+\frac{m_{d_j}}{m_b}(\delta Z^d_L)^\dagger_{sj}(V^{(0)}_{\rm CKM})^\dagger_{ji}\Big](V^{(0)}_{\rm CKM})_{ib}+\frac{v^2}{4\Lambda^2_{\rm KK}}(\Delta^R_{G^\pm})^\dagger_{si}(V^{(0)}_{\rm CKM})_{ib}.
\end{aligned} \tag{174}
$$

Choosing $\mu_{\rm EW}=m_W$, one directly finds that the sum of Eqs. (167) and (173) recovers the theoretical predictions on Wilson coefficients of dipole operators in the SM at electroweak energy scale.

As the FCNC transitions are mediated by the massive neutral gauge bosons Z, Z_{H_α}, $\gamma_{(n)}$, relevant couplings are expanded according $v^2/\Lambda^2_{\rm KK}$ as

$$
\begin{aligned}
(\xi^L_Z)_{s,b} &= \Big(1-\frac{2}{3}s_w^2\Big)\Big[\delta_{sb}+(\delta Z^d_L)^\dagger_{sb}+(\delta Z^d_L)_{sb}+\frac{v^2}{2\Lambda^2_{\rm KK}}(\Delta^L_Z)_{sb}\Big]+\mathcal{O}\Big(\frac{v^3}{\Lambda^3_{\rm KK}}\Big), \\
(\xi^R_Z)_{s,b} &= -\frac{2}{3}s_w^2\Big[\delta_{sb}+(\delta Z^d_R)^\dagger_{sb}+(\delta Z^d_R)_{sb}+\frac{v^2}{2\Lambda^2_{\rm KK}}(\Delta^R_Z)_{sb}\Big]+\mathcal{O}\Big(\frac{v^3}{\Lambda^3_{\rm KK}}\Big), \\
(\xi^L_{Z_{H_{(2n-1)}}})_{s,b} &= \Big(1-\frac{2}{3}s_w^2\Big)\frac{4\sqrt{2\pi}}{kr\epsilon}\sum_{i=1}^3(\mathcal{D}^{(0)}_L)^\dagger_{si}\int_\epsilon^1 dt\chi^Z_{(++)}(y^{Z(n)}_{(++)},t)[f^{L,c^i_B}_{(++)}(0,t)]^2(\mathcal{D}^{(0)}_L)_{ib}+\mathcal{O}\Big(\frac{v}{\Lambda_{\rm KK}}\Big), \\
(\xi^L_{Z_{H_{(2n)}}})_{s,b} &= \frac{3-2s_w^2}{3\sqrt{1-2s_w^2}}\frac{4\sqrt{2\pi}}{kr\epsilon}\sum_{i=1}^3(\mathcal{D}^{(0)}_L)^\dagger_{si}\int_\epsilon^1 dt\chi^{Z_X}_{(-+)}(y^{Z_X(n)}_{(-+)},t)[f^{L,c^i_B}_{(++)}(0,t)]^2(\mathcal{D}^{(0)}_L)_{ib}+\mathcal{O}\Big(\frac{v}{\Lambda_{\rm KK}}\Big), \\
(\xi^R_{Z_{H_{(2n-1)}}})_{s,b} &= -\frac{2}{3}s_w^2\frac{4\sqrt{2\pi}}{kr\epsilon}\sum_{i=1}^3(\mathcal{D}^{(0)}_R)^\dagger_{si}\int_\epsilon^1 dt\chi^Z_{(++)}(y^{Z(n)}_{(++)},t)[f^{R,c^i_T}_{(++)}(0,t)]^2(\mathcal{D}^{(0)}_R)_{ib}+\mathcal{O}\Big(\frac{v}{\Lambda_{\rm KK}}\Big), \\
(\xi^R_{Z_{H_{(2n)}}})_{s,b} &= -\frac{3-4s_w^2}{3\sqrt{1-2s_w^2}}\frac{4\sqrt{2\pi}}{kr\epsilon}\sum_{i=1}^3(\mathcal{D}^{(0)}_R)^\dagger_{si}\int_\epsilon^1 dt\chi^{Z_X}_{(-+)}(y^{Z_X(n)}_{(-+)},t)[f^{R,c^i_T}_{(++)}(0,t)]^2(\mathcal{D}^{(0)}_R)_{ib}+\mathcal{O}\Big(\frac{v}{\Lambda_{\rm KK}}\Big),
\end{aligned}
$$

$$
\begin{aligned}
(\xi^L_{Z_{H_{(2n-1)}}})_{\alpha,b} &= \Big(1-\frac{2}{3}s_w^2\Big)\frac{4\sqrt{2\pi}}{kr\epsilon}\sum_{i=1}^{3}\sum_{n'=1}^{\infty}\delta_{\alpha(9n'-6+i)} \\
&\quad\times\int_\epsilon^1 dt\chi^Z_{(++)}(y^{Z(n)}_{(++)},t)\times[f^{L,c_B^i}_{(++)}(0,t)][f^{L,c_B^i}_{(++)}(y^{c_B^i(n')}_{(\pm\pm)},t)](\mathcal{D}_L^{(0)})_{ib}+\mathcal{O}\Big(\frac{v}{\Lambda_{\rm KK}}\Big), \\
(\xi^L_{Z_{H_{(2n)}}})_{\alpha,b} &= \frac{3-2s_w^2}{3\sqrt{1-2s_w^2}}\frac{4\sqrt{2\pi}}{kr\epsilon}\sum_{i=1}^{3}\sum_{n'=1}^{\infty}\delta_{\alpha(9n'-6+i)} \\
&\quad\times\int_\epsilon^1 dt\chi^{Z_X}_{(-+)}(y^{Z_X(n)}_{(-+)},t)\times[f^{L,c_B^i}_{(++)}(0,t)][f^{L,c_B^i}_{(++)}(y^{c_B^i(n')}_{(\pm\pm)},t)](\mathcal{D}_L^{(0)})_{ib}+\mathcal{O}\Big(\frac{v}{\Lambda_{\rm KK}}\Big), \\
(\xi^R_{Z_{H_{(2n-1)}}})_{\alpha,b} &= -\frac{2}{3}s_w^2\frac{4\sqrt{2\pi}}{kr\epsilon}\sum_{i=1}^{3}\sum_{n'=1}^{\infty}\delta_{\alpha(9n'+i)} \\
&\quad\times\int_\epsilon^1 dt\chi^Z_{(++)}(y^{Z(n)}_{(++)},t)[f^{R,c_T^i}_{(++)}(0,t)][f^{R,c_T^i}_{(++)}(y^{c_T^i(n')}_{(\mp\mp)},t)](\mathcal{D}_R^{(0)})_{ib}+\mathcal{O}\Big(\frac{v}{\Lambda_{\rm KK}}\Big), \\
(\xi^R_{Z_{H_{(2n)}}})_{\alpha,b} &= -\frac{3-4s_w^2}{3\sqrt{1-2s_w^2}}\frac{4\sqrt{2\pi}}{kr\epsilon}\sum_{i=1}^{3}\sum_{n'=1}^{\infty}\delta_{\alpha(9n'+i)} \\
&\quad\times\int_\epsilon^1 dt\chi^{Z_X}_{(-+)}(y^{Z_X(n)}_{(-+)},t)[f^{R,c_T^i}_{(++)}(0,t)][f^{R,c_T^i}_{(++)}(y^{c_T^i(n')}_{(\mp\mp)},t)](\mathcal{D}_R^{(0)})_{ib}+\mathcal{O}\Big(\frac{v}{\Lambda_{\rm KK}}\Big), \\
(\xi^L_{\gamma_{(n)}})_{s,b} &= -\frac{4\sqrt{2\pi}}{3kr\epsilon}\sum_{i=1}^{3}(\mathcal{D}_L^{(0)})^\dagger_{s,i}\int_\epsilon^1 dt\chi^A_{(++)}(y^{A(n)}_{(++)},t)[f^{L,c_B^i}_{(++)}(0,t)]^2(\mathcal{D}_L^{(0)})_{i,b}+\mathcal{O}\Big(\frac{v}{\Lambda_{\rm KK}}\Big), \\
(\xi^R_{\gamma_{(n)}})_{s,b} &= -\frac{4\sqrt{2\pi}}{3kr\epsilon}\sum_{i=1}^{3}(\mathcal{D}_R^{(0)})^\dagger_{s,i}\int_\epsilon^1 dt\chi^A_{(++)}(y^{A(n)}_{(++)},t)[f^{R,c_T^i}_{(++)}(0,t)]^2(\mathcal{D}_R^{(0)})_{i,b}+\mathcal{O}\Big(\frac{v}{\Lambda_{\rm KK}}\Big), \\
(\xi^L_{\gamma_{(n)}})_{\alpha,b} &= -\frac{4\sqrt{2\pi}}{3kr\epsilon}\sum_{i=1}^{3}\sum_{n'=1}^{\infty}\delta_{\alpha,(9n'-6+i)} \\
&\quad\times\int_\epsilon^1 dt\chi^A_{(++)}(y^{A(n)}_{(++)},t)\{f^{L,c_B^i}_{(++)}(0,t)f^{L,c_B^i}_{(++)}(y^{c_B^i(n')}_{(\pm\pm)},t)\}(\mathcal{D}_L^{(0)})_{i,b}+\mathcal{O}\Big(\frac{v}{\Lambda_{\rm KK}}\Big), \\
(\xi^R_{\gamma_{(n)}})_{\alpha,b} &= -\frac{4\sqrt{2\pi}}{3kr\epsilon}\sum_{i=1}^{3}\sum_{n'=1}^{\infty}\delta_{\alpha,(9n'+i)} \\
&\quad\times\int_\epsilon^1 dt\chi^A_{(++)}(y^{A(n)}_{(++)},t)\{f^{R,c_T^i}_{(++)}(0,t)f^{R,c_T^i}_{(++)}(y^{c_T^i(n')}_{(\pm\pm)},t)\}(\mathcal{D}_R^{(0)})_{i,b}+\mathcal{O}\Big(\frac{v}{\Lambda_{\rm KK}}\Big).
\end{aligned}
\tag{175}
$$

Here the short cutting notations Δ^L_Z, Δ^R_Z are defined by

$$
\begin{aligned}
(\Delta^L_Z)_{sb} &= \sum_{i,j,k=1}^{3}(\mathcal{D}_L^{(0)})^\dagger_{si}f^{L,c_B^i}_{(++)}(0,1)Y^d_{ik}([\Sigma^{R,c_T^k}_{(\pm\mp)}(1,1)]-\frac{3}{3-2s_w^2}[\Sigma^{R,c_T^k}_{(\mp\mp)}(1,1)])Y^{d\dagger}_{kj}f^{L,c_B^j}_{(++)}(0,1)(\mathcal{D}_L^{(0)})_{jb} \\
&\quad-\frac{4\pi e^2}{s_w^2c_w^2kr\epsilon}\sum_{i=1}^{3}(\mathcal{D}_L^{(0)})^\dagger_{si}\Big(\int_\epsilon^1 dt\{[\Sigma^G_{(++)}(t,1)]+[\Sigma^G_{(-+)}(t,1)]\}[f^{L,c_B^i}_{(++)}(0,t)]^2\Big)(\mathcal{D}_L^{(0)})_{ib}, \\
(\Delta^R_Z)_{sb} &= -\frac{3}{2s_w^2}\sum_{i,j,k=1}^{3}(\mathcal{D}_R^{(0)})^\dagger_{si}f^{R,c_T^i}_{(++)}(0,1)Y^{d\dagger}_{ik}[\Sigma^{L,c_B^k}_{(\pm\pm)}(1,1)]Y^d_{kj}f^{R,c_T^j}_{(++)}(0,1)(\mathcal{D}_R^{(0)})_{jb} \\
&\quad-\frac{4\pi e^2}{s_w^2c_w^2kr\epsilon}\sum_{i,j=1}^{3}(\mathcal{D}_R^{(0)})^\dagger_{si}\Big(\int_\epsilon^1 dt\{[\Sigma^G_{(++)}(t,1)]+\Big(\frac{3}{s_w^2}-4\Big)[\Sigma^G_{(-+)}(t,1)]\}[f^{R,c_T^i}_{(++)}(0,t)]^2\Big)(\mathcal{D}_R^{(0)})_{jb}.
\end{aligned}
\tag{176}
$$

Then the corrections to relevant Wilson coefficients from Fig. 1(c) are analogously formulated to the order $\mathcal{O}(v^2/\Lambda^2_{\rm KK})$ as

$$
\begin{aligned}
C_{7\gamma}^{(c)}(\mu_{\rm EW}) =& \bigg\{\frac{(3-2s_W^2)^2}{162 s_W^2}\bigg[2(\delta Z_L^d)_{sb}^\dagger + 2(\delta Z_L^d)_{sb} + \frac{v^2}{2\Lambda_{\rm KK}^2}(\Delta_Z^L)_{sb} + \frac{v^2}{2\Lambda_{\rm KK}^2}(\Delta_Z^L)_{sb}^\dagger\bigg] \\
&+ \frac{1}{9}\bigg(1-\frac{2}{3}s_W^2\bigg)\bigg[(\delta Z_L^d)_{sb}^\dagger + (\delta Z_L^d)_{sb} + \frac{v^2}{2\Lambda_{\rm KK}^2}(\Delta_Z^L)_{sb}^\dagger\bigg] \\
&+ \frac{1}{9}\bigg(1-\frac{2}{3}s_W^2\bigg)\frac{m_s}{m_b}\bigg[(\delta Z_R^d)_{sb}^\dagger + (\delta Z_R^d)_{sb} + \frac{v^2}{2\Lambda_{\rm KK}^2}(\Delta_Z^R)_{sb}\bigg]\bigg\} \\
&+ \frac{32\pi s_W^2 m_W^2}{27\Lambda_{\rm KK}^2(kr\epsilon)^2}\sum_{i,j,k=1}^{3}\bigg\{\frac{m_{d_k}}{m_b}(\mathcal{D}_L^{(0)})_{si}^\dagger(\mathcal{D}_L^{(0)})_{ik}(\mathcal{D}_R^{(0)})_{kj}^\dagger(\mathcal{D}_R^{(0)})_{jb}\int_\epsilon^1 dt\int_\epsilon^1 dt'\bigg(\frac{25-16s_W^2}{9c_W^2}[\Sigma_{(++)}^G(t,t')] \\
&+ \frac{(3-2s_W^2)(3-4s_W^2)}{s_W^2 c_W^2(1-2s_W^2)}[\Sigma_{(-+)}^G(t,t')]\bigg)[f_{(++)}^{L,c_B^i}(0,t)]^2[f_{(++)}^{R,c_T^j}(0,t')]^2\bigg\} + \mathcal{O}\bigg(\frac{v^4}{\Lambda_{\rm KK}^4}\bigg), \\
C_{8G}^{(c)}(\mu_{\rm EW}) =& -3C_{7\gamma}^{(c)}(\mu_{\rm EW}), \tilde{C}_{7\gamma}^{(c)}(\mu_{\rm EW}) \\
=& \bigg\{\frac{2}{81}s_W^2\bigg[2(\delta Z_R^d)_{sb}^\dagger + 2(\delta Z_R^d)_{sb} + \frac{v^2}{2\Lambda_{\rm KK}^2}(\Delta_Z^R)_{sb} + \frac{v^2}{2\Lambda_{\rm KK}^2}(\Delta_Z^R)_{sb}^\dagger\bigg] \\
&+ \frac{1}{9}\bigg(1-\frac{2}{3}s_W^2\bigg)\frac{m_s}{m_b}\bigg[(\delta Z_L^d)_{sb}^\dagger + (\delta Z_L^d)_{sb} + \frac{v^2}{2\Lambda_{\rm KK}^2}(\Delta_Z^L)_{sb}\bigg] \\
&+ \frac{1}{9}\bigg(1-\frac{2}{3}s_W^2\bigg)\bigg[(\delta Z_R^d)_{sb}^\dagger + (\delta Z_R^d)_{sb} + \frac{v^2}{2\Lambda_{\rm KK}^2}(\Delta_Z^R)_{sb}^\dagger\bigg]\bigg\} \\
&+ \frac{32\pi s_W^2 m_W^2}{9\Lambda_{\rm KK}^2(kr\epsilon)^2}\sum_{i,j,k=1}^{3}\bigg\{\frac{m_{d_k}}{m_b}(\mathcal{D}_R^{(0)})_{si}^\dagger(\mathcal{D}_R^{(0)})_{ik}(\mathcal{D}_L^{(0)})_{kj}^\dagger(\mathcal{D}_L^{(0)})_{jb} \\
&\times\int_\epsilon^1 dt\int_\epsilon^1 dt'\bigg(\frac{79-52s_W^2}{27c_W^2}[\Sigma_{(++)}^G(t,t')] + \frac{(3-2s_W^2)(3-4s_W^2)^2}{s_W^2c_W^2(1-2s_W^2)}[\Sigma_{(-+)}^G(t,t')]\bigg)[f_{(++)}^{R,c_T^i}(0,t)]^2[f_{(++)}^{L,c_B^j}(0,t')]^2\bigg\} \\
&+ \mathcal{O}\bigg(\frac{v^4}{\Lambda_{\rm KK}^4}\bigg), \\
\tilde{C}_{8G}^{(c)}(\mu_{\rm EW}) =& -3\tilde{C}_{7\gamma}^{(c)}(\mu_{\rm EW}).
\end{aligned} \tag{177}
$$

As mentioned above, the KK exciting modes of gluon also arise the FCNC transitions in the SM extension with a warped extra dimension and the custodial symmetry. The relevant couplings are approached as

$$
\begin{aligned}
(\xi_{g_{(n)}}^L)_{s,b} &= \frac{4\sqrt{2\pi}}{kr\epsilon}\sum_{i=1}^{3}(\mathcal{D}_L^{(0)})_{s,i}^\dagger\int_\epsilon^1 dt\chi_{(++)}^A(y_{(++)}^{A(n)},t)[f_{(++)}^{L,c_B^i}(0,t)]^2(\mathcal{D}_L^{(0)})_{i,b} + \mathcal{O}\bigg(\frac{v}{\Lambda_{\rm KK}}\bigg), \\
(\xi_{g_{(n)}}^R)_{s,b} &= \frac{4\sqrt{2\pi}}{kr\epsilon}\sum_{i=1}^{3}(\mathcal{D}_R^{(0)})_{s,i}^\dagger\int_\epsilon^1 dt\chi_{(++)}^A(y_{(++)}^{A(n)},t)[f_{(++)}^{R,c_T^i}(0,t)]^2(\mathcal{D}_R^{(0)})_{i,b} + \mathcal{O}\bigg(\frac{v}{\Lambda_{\rm KK}}\bigg), \\
(\xi_{g_{(n)}}^L)_{\alpha,b} &= \frac{4\sqrt{2\pi}}{kr\epsilon}\sum_{i=1}^{3}\sum_{n'=1}^{\infty}\delta_{\alpha,(9n'-6+i)}\int_\epsilon^1 dt\chi_{(++)}^A(y_{(++)}^{A(n)},t)\{f_{(++)}^{L,c_B^i}(0,t)f_{(++)}^{L,c_B^i}(y_{(\pm\pm)}^{c_B^i(n')},t)\}(\mathcal{D}_L^{(0)})_{i,b} + \mathcal{O}\bigg(\frac{v}{\Lambda_{\rm KK}}\bigg), \\
(\xi_{g_{(n)}}^R)_{\alpha,b} &= \frac{4\sqrt{2\pi}}{kr\epsilon}\sum_{i=1}^{3}\sum_{n'=1}^{\infty}\delta_{\alpha,(9n'+i)}\int_\epsilon^1 dt\chi_{(++)}^A(y_{(++)}^{A(n)},t)\{f_{(++)}^{R,c_T^i}(0,t)f_{(++)}^{R,c_T^i}(y_{(\pm\pm)}^{c_T^i(n')},t)\}(\mathcal{D}_R^{(0)})_{i,b} + \mathcal{O}\bigg(\frac{v}{\Lambda_{\rm KK}}\bigg).
\end{aligned} \tag{178}
$$

The corresponding corrections from this sector to Wilson coefficients of dipole moment operators at electroweak energy scale are given by

$$
\begin{aligned}
C_{7\gamma}^{(d)}(\mu_{\rm EW}) &= -\frac{256\pi s_W^2 m_W^2 g_s^2}{9\Lambda_{\rm KK}^2 (kr\epsilon)^2 e^2} \sum_{i,j,k=1}^{3} \Big\{ \frac{m_{d_k}}{m_b} (\mathcal{D}_L^{(0)})^\dagger_{si} (\mathcal{D}_L^{(0)})_{ik} (\mathcal{D}_R^{(0)})^\dagger_{kj} (\mathcal{D}_R^{(0)})_{jb} \int_\epsilon^1 dt \int_\epsilon^1 dt' [\Sigma^G_{(++)}(t,t')][f^{L,c_B^i}_{(++)}(0,t)]^2 \\
&\quad \times [f^{R,c_T^j}_{(++)}(0,t')]^2 \Big\} + \mathcal{O}\Big(\frac{v^4}{\Lambda_{\rm KK}^4}\Big), \\
C_{8G}^{(d)}(\mu_{\rm EW}) &= \frac{256\pi s_W^2 m_W^2 g_s^2}{3\Lambda_{\rm KK}^2 (kr\epsilon)^2 e^2} \sum_{i,j,k=1}^{3} \Big\{ \frac{m_{d_k}}{m_b} (\mathcal{D}_L^{(0)})^\dagger_{si} (\mathcal{D}_L^{(0)})_{ik} (\mathcal{D}_R^{(0)})^\dagger_{kj} (\mathcal{D}_R^{(0)})_{jb} \int_\epsilon^1 dt \int_\epsilon^1 dt' [\Sigma^G_{(++)}(t,t')][f^{L,c_B^i}_{(++)}(0,t)]^2 \\
&\quad \times [f^{R,c_T^j}_{(++)}(0,t')]^2 \Big\} + \mathcal{O}\Big(\frac{v^4}{\Lambda_{\rm KK}^4}\Big), \tilde{C}_{7\gamma}^{(d)}(\mu_{\rm EW}) \\
&= -\frac{256\pi s_W^2 m_W^2 g_s^2}{9\Lambda_{\rm KK}^2 (kr\epsilon)^2 e^2} \sum_{i,j,k=1}^{3} \Big\{ \frac{m_{d_k}}{m_b} (\mathcal{D}_R^{(0)})^\dagger_{si} (\mathcal{D}_R^{(0)})_{ik} (\mathcal{D}_L^{(0)})^\dagger_{kj} (\mathcal{D}_L^{(0)})_{jb} \int_\epsilon^1 dt \int_\epsilon^1 dt' [\Sigma^G_{(++)}(t,t')][f^{R,c_T^i}_{(++)}(0,t)]^2 \\
&\quad \times [f^{L,c_B^j}_{(++)}(0,t')]^2 \Big\} + \mathcal{O}\Big(\frac{v^4}{\Lambda_{\rm KK}^4}\Big), \\
\tilde{C}_{8G}^{(d)}(\mu_{\rm EW}) &= \frac{256\pi s_W^2 m_W^2 g_s^2}{3\Lambda_{\rm KK}^2 (kr\epsilon)^2 e^2} \sum_{i,j,k=1}^{3} \Big\{ \frac{m_{d_k}}{m_b} (\mathcal{D}_R^{(0)})^\dagger_{si} (\mathcal{D}_R^{(0)})_{ik} (\mathcal{D}_L^{(0)})^\dagger_{kj} (\mathcal{D}_L^{(0)})_{jb} \int_\epsilon^1 dt \int_\epsilon^1 dt' [\Sigma^G_{(++)}(t,t')][f^{R,c_T^i}_{(++)}(0,t)]^2 \\
&\quad \times [f^{L,c_B^j}_{(++)}(0,t')]^2 \Big\} + \mathcal{O}\Big(\frac{v^4}{\Lambda_{\rm KK}^4}\Big).
\end{aligned}
\tag{179}
$$

In Eqs. (166), (172), (177), and (179), the corrections from KK exciting states contain a global suppression factor $v^2/\Lambda_{\rm KK}^2$ comparing with the SM contributions presented in Eq. (167) and (173).

Finally, the relevant FCNC couplings mediated by neutral Higgs and Goldstone are

$$
\begin{aligned}
(\eta^L_{H_0})_{s,b} &= -\frac{e}{\sqrt{2}s_w} \Big\{ \frac{m_b}{m_W}\delta_{sb} - \frac{(\delta\mathcal{M}^d)_{33}}{m_W}\delta_{sb} + \frac{m_b}{m_W}(\delta Z_R^d)^\dagger_{sb} + \frac{m_s}{m_W}(\delta Z_L^d)_{sb} \\
&\quad - \delta_{sb}\frac{\pi m_b m_W}{\Lambda_{\rm KK}^2}[\{\Sigma^G_{(++)}(1,1)\} + \{\Sigma^G_{(-+)}(1,1)\}] \\
&\quad + \frac{v^2}{4\Lambda_{\rm KK}^2}\Big[\frac{m_b}{m_W}(\Delta^{(1)}_{H_0})_{sb} + \frac{m_s}{m_W}(\Delta^{(2)}_{H_0})_{sb}\Big]\Big\} + \mathcal{O}\Big(\frac{v^3}{\Lambda_{\rm KK}^3}\Big), \\
(\eta^R_{H_0})_{s,b} &= (\eta^L_{H_0})^\dagger_{s,b}, (\eta^L_{H_0})_{\alpha,b} = \sum_{i,j=1}^{3}\Big\{ -\sum_{n=1}^{\infty}(\delta_{\alpha,(9n+i)} f^{R,c_T^i}_{(++)}(y^{c_T^i(n)}_{(\mp\mp)},1) Y^{d\dagger}_{ij} f^{L,c_B^j}_{(++)}(0,1)(\mathcal{D}_L^{(0)})_{j,b} \\
&\quad + \sum_{n=1}^{\infty}(\delta_{\alpha,(9n-3+i)} f^{R,c_T^i}_{(-+)}(y^{c_T^i(n)}_{(\pm\mp)},1) Y^{d\dagger}_{ij} f^{L,c_B^j}_{(++)}(0,1)(\mathcal{D}_L^{(0)})_{j,b}\Big\} + \mathcal{O}\Big(\frac{v}{\Lambda_{\rm KK}}\Big), \\
(\eta^R_{H_0})_{\alpha,b} &= -\sum_{i,j=1}^{3}\sum_{n=1}^{\infty}\delta_{\alpha,(9n-6+i)} f^{L,c_B^i}_{(++)}(y^{c_B^i(n)}_{(\pm\pm)},1) Y^d_{ij} f^{R,c_T^j}_{(++)}(0,1)(\mathcal{D}_R^{(0)})_{j,b} + \mathcal{O}\Big(\frac{v}{\Lambda_{\rm KK}}\Big), \\
(\eta^L_{G_0})_{s,b} &= -(\eta^L_{H_0})_{s,b}, (\eta^R_{G_0})_{s,b} = -(\eta^L_{G_0})^\dagger_{s,b}, (\eta^L_{G_0})_{\alpha,b} \\
&= \sum_{i,j=1}^{3}\Big\{\sum_{n=1}^{\infty}(\delta_{\alpha,(9n+i)} f^{R,c_T^i}_{(++)}(y^{c_T^i(n)}_{(\mp\mp)},1) Y^{d\dagger}_{ij} f^{L,c_B^j}_{(++)}(0,1)(\mathcal{D}_L^{(0)})_{j,b} \\
&\quad + \sum_{n=1}^{\infty}(\delta_{\alpha,(9n-3+i)} f^{R,c_T^i}_{(-+)}(y^{c_T^i(n)}_{(\pm\mp)},1) Y^{d\dagger}_{ij} f^{L,c_B^j}_{(++)}(0,1)(\mathcal{D}_L^{(0)})_{j,b}\Big\} + \mathcal{O}\Big(\frac{v}{\Lambda_{\rm KK}}\Big), \\
(\eta^R_{G_0})_{\alpha,b} &= \sum_{i,j=1}^{3}\sum_{n=1}^{\infty}\delta_{\alpha,(9n-6+i)} f^{L,c_B^i}_{(++)}(y^{c_B^i(n)}_{(\pm\pm)},1) Y^d_{ij} f^{R,c_T^j}_{(++)}(0,1)(\mathcal{D}_R^{(0)})_{j,b} + \mathcal{O}\Big(\frac{v}{\Lambda_{\rm KK}}\Big),
\end{aligned}
\tag{180}
$$

where the abbreviations are given by

$$(\Delta_{H_0}^{(1)})_{sb} = -\sum_{i,j,k,l=1}^{3} (\mathcal{D}_R^{(0)})_{si}^{\dagger} f_{(++)}^{R,c_T^i}(0,1) Y_{ik}^{d\dagger} [\Sigma_{(\pm\pm)}^{L,c_B^k}(1,1)] Y_{kj}^{d} f_{(++)}^{R,c_T^j}(0,1) (\mathcal{D}_L^{(0)})_{jb},$$
$$(\Delta_{H_0}^{(2)})_{sb} = -\sum_{i,j,k,l=1}^{3} (\mathcal{D}_R^{(0)})_{si}^{\dagger} f_{(++)}^{L,c_B^i}(0,1) Y_{ik}^{d} [\{\Sigma_{(\pm\mp)}^{R,c_T^k}(1,1)\} + \{\Sigma_{(\mp\mp)}^{R,c_T^k}(1,1)\}] Y_{kj}^{d\dagger} f_{(++)}^{L,c_B^j}(0,1) (\mathcal{D}_L^{(0)})_{jb}. \quad (181)$$

As virtual intermediate fields in Fig. 1(e) are neutral Higgs/Goldstone and standard charged $-1/3$ quarks and the corresponding corrections to Wilson coefficients of dipole moment operators from this sector contain additional suppression factor $m_b^3 m_s/m_W^4$ comparing with the corrections presented in Eq. (167) and (173). The possible substantial corrections from this sector originate from the one-loop diagrams in which virtual intermediate fields are neutral Higgs/Goldstone and KK excitations of charged $-1/3$ quarks,

$$\begin{aligned}
C_{7\gamma}^{(e)}(\mu_{\rm EW}) &= -\frac{s_W^2 m_W^2}{18\Lambda_{\rm KK}^2 e^2} \sum_{i,j,k=1}^{3} \{(\mathcal{D}_L^{(0)})_{si}^{\dagger} f_{(++)}^{L,c_B^i}(0,1) Y_{ik}^{d} ([\Sigma_{(\mp\mp)}^{R,c_T^k}(1,1)] \\
&\quad + [\Sigma_{(\pm\mp)}^{R,c_T^k}(1,1)]) Y_{kj}^{d\dagger} f_{(++)}^{L,c_B^j}(0,1) (\mathcal{D}_L^{(0)})_{jb}\} + \mathcal{O}\Big(\frac{v^4}{\Lambda_{\rm KK}^4}\Big) \\
&= -\frac{m_b m_s}{36\Lambda_{\rm KK}^2} \sum_{k=1}^{3} (\mathcal{D}_R^{(0)})_{sk}^{\dagger} \frac{[\Sigma_{(\mp\mp)}^{R,c_T^k}(1,1)] + [\Sigma_{(\pm\mp)}^{R,c_T^k}(1,1)]}{[f_{(++)}^{R,c_T^k}(0,1)]^2} (\mathcal{D}_R^{(0)})_{kb} + \mathcal{O}\Big(\frac{v^4}{\Lambda_{\rm KK}^4}\Big), \\
C_{8G}^{(e)}(\mu_{\rm EW}) &= -3C_{7\gamma}^{(e)}(\mu_{\rm EW}), \\
\tilde{C}_{7\gamma}^{(e)}(\mu_{\rm EW}) &= -\frac{s_W^2 m_W^2}{18\Lambda_{\rm KK}^2 e^2} \sum_{i,j,k=1}^{3} \{(\mathcal{D}_R^{(0)})_{si}^{\dagger} f_{(++)}^{R,c_T^i}(0,1) Y_{ik}^{d\dagger} [\Sigma_{(\pm\pm)}^{L,c_B^k}(1,1)] Y_{kj}^{d} f_{(++)}^{R,c_T^j}(0,1) (\mathcal{D}_R^{(0)})_{jb} + \mathcal{O}\Big(\frac{v^4}{\Lambda_{\rm KK}^4}\Big) \\
&= -\frac{m_b m_s}{36\Lambda_{\rm KK}^2} \sum_{k=1}^{3} (\mathcal{D}_L^{(0)})_{sk}^{\dagger} \frac{[\Sigma_{(\pm\pm)}^{L,c_B^k}(1,1)]}{[f_{(++)}^{L,c_B^k}(0,1)]^2} (\mathcal{D}_L^{(0)})_{kb} + \mathcal{O}\Big(\frac{v^4}{\Lambda_{\rm KK}^4}\Big), \\
\tilde{C}_{8G}^{(e)}(\mu_{\rm EW}) &= -3\tilde{C}_{7\gamma}^{(e)}(\mu_{\rm EW}).
\end{aligned} \quad (182)$$

Comparing with the corrections from other sectors, the contributions from neutral Higgs and KK excitations of charged $-1/3$ quarks in Eq. (182) contain an additional suppression factor $m_b m_s/m_W^2$ besides the global suppression factor $v^2/\Lambda_{\rm KK}^2$.

In order to complete analysis of the decay $b \to s\gamma$ we have to include QCD corrections which originate dominantly from the mixing of charged current-current operators into dipole operators and to a smaller extent from the mixing with QCD-penguin operators [75]. Formally we can define two column vectors composed by the Wilson coefficients in Eq. (132) as

$$(\mathbf{C}^{\rm SM})^T = \{C_i, C_{7\gamma}, C_{8G}, C_{f,j}^{LL}, C_{f,j}^{LR}, \hat{C}_{d,j}^{LL}, \hat{C}_{d,j}^{LR}\}^T,$$
$$(\mathbf{C}^{NP})^T = \{\tilde{C}_k, \tilde{C}_{7\gamma}, \tilde{C}_{8G}, C_{f,j}^{RL}, C_{f,j}^{RR}, \hat{C}_{d,j}^{RL}, \hat{C}_{d,j}^{RR}\}^T, \quad (183)$$

with $i = 1, 2, \cdots, 6$, $j = 1, 2$, $k = 3, \cdots, 6$, and $f = u, c, d, s, b$. Then, the Wilson coefficients at hadron scale μ_b are given as

$$C_\alpha^{\rm SM}(\mu_b) = \sum_\beta U_{\alpha\beta}(\mu_b, \mu_{\rm EW}) C_\beta^{\rm SM}(\mu_{\rm EW}),$$
$$C_\alpha^{NP}(\mu_b) = \sum_\beta W_{\alpha\beta}(\mu_b, \mu_{\rm EW}) C_\beta^{NP}(\mu_{\rm EW}). \quad (184)$$

Here $U_{\alpha\beta}(\mu_b, \mu_{\rm EW})$, $W_{\alpha\beta}(\mu_b, \mu_{\rm EW})$ are elements of the evolution matrices which can be expressed in terms of the anomalous dimension matrices and the QCD-β-functions [72].

For charged current-current operators, the Wilson coefficients are approximated as

$$C_1(\mu_{\rm EW}) = C_1^{\rm SM}(\mu_{\rm EW}) + \delta C_1(\mu_{\rm EW}),$$
$$C_2(\mu_{\rm EW}) = C_2^{\rm SM}(\mu_{\rm EW}) + \delta C_2(\mu_{\rm EW}), \quad (185)$$

where the concrete expressions of $C_{1,2}^{\rm SM}(\mu_W)$ can be found in Ref. [75], and the corrections from KK excitations to the order $\mathcal{O}(v^2/\Lambda_{\rm KK}^2)$ are formulated as

$$\delta C_1(\mu_{\rm EW}) = 0,$$

$$\delta C_2(\mu_{\rm EW}) = \frac{1}{(V^{(0)\dagger}_{\rm CKM})_{st}(V^{(0)}_{\rm CKM})_{tb}}\Big\{(V^{(0)\dagger}_{\rm CKM})_{sc}[(\delta Z^u_L V^{(0)}_{\rm CKM})_{cb} + (V^{(0)}_{\rm CKM}\delta Z^d_L)_{cb}] + [(\delta Z^{d\dagger}_L V^{(0)\dagger}_{\rm CKM})_{sc} + (V^{(0)\dagger}_{\rm CKM}\delta Z^{u\dagger}_L)_{sc}](V^{(0)}_{\rm CKM})_{cb} - \frac{v^2}{4\Lambda^2_{\rm KK}}[(V^{(0)\dagger}_{\rm CKM})_{sc}(\Delta^L_{W^\pm})_{cb} + (\Delta^L_{W^\pm})^\dagger_{sc}(V^{(0)}_{\rm CKM})_{cb}] + \frac{32\pi m_W^2}{\Lambda^2_{\rm KK}}\sum_{i,j=1}^{3}(\mathcal{D}^{(0)}_L)^\dagger_{si}(\mathcal{U}^{(0)}_L)_{ic}(\mathcal{U}^{(0)}_L)^\dagger_{cj}(\mathcal{D}^{(0)}_L)_{jb}\frac{1}{(kr\epsilon)^2} \times \int_\epsilon^1 dt\int_\epsilon^1 dt'[f^{L,c^i_B}_{(++)}(0,t)]^2[\Sigma^G_{(++)}(t,t')][f^{L,c^j_B}_{(++)}(0,t')]^2\Big\}. \tag{186}$$

Similarly, the corrections from exciting KK modes to the Wilson coefficients of neutral current-current operators are presented in Appendix C.

In the presence of new physics the expression for the branching ratio of $\bar{B}\to X_s\gamma$ is given as follows:

$$\mathrm{Br}(\bar{B}\to X_s\gamma) = R(|C_{7\gamma}(\mu_b)|^2 + |\tilde{C}_{7\gamma}(\mu_b)|^2 + N(E_\gamma)), \tag{187}$$

where the overall factor $R = 2.47\times 10^{-3}$, and the non-perturbative contribution $N(E_\gamma) = (3.6\pm 0.6)\times 10^{-3}$ [72]. In our numerical analysis, we choose the hadron scale $\mu_b = 2.5$ GeV, and include the SM contribution at NNLO level $C_{7\gamma}(\mu_b) = -0.3523$ [59–61]. Meanwhile we approach the corrections from KK excitations in the leading-order approximation.

Assuming anarchic Yukawa couplings, i.e. complex-valued matrices Y^u, Y^d with random elements, we can reproduce the up- and down-type quark mass hierarchies with the ansatz for hierarchical structures of the profiles of zero modes on IR brane [18],

$$\begin{aligned}
&[f^{L,c^1_B}_{(++)}(0,1)] < [f^{L,c^2_B}_{(++)}(0,1)] < [f^{L,c^3_B}_{(++)}(0,1)],\\
&[f^{R,c^1_T}_{(++)}(0,1)] < [f^{R,c^2_T}_{(++)}(0,1)] < [f^{R,c^3_T}_{(++)}(0,1)],\\
&[f^{R,c^1_S}_{(++)}(0,1)] < [f^{R,c^2_S}_{(++)}(0,1)] < [f^{R,c^3_S}_{(++)}(0,1)].
\end{aligned} \tag{188}$$

Applying the Froggatt-Nielsen mechanism [76] in the SM extension with a warped extra dimension and the custodial symmetry, one obtains

$$\begin{aligned}
m_u &= \frac{v}{\sqrt{2}}\frac{|\det(Y^u)|}{|\mathcal{Y}^u_{11}|}|[f^{L,c^1_B}_{(++)}(0,1)][f^{R,c^1_S}_{(++)}(0,1)]| + (\delta\mathcal{M}^u)_{11},\\
m_d &= \frac{v}{\sqrt{2}}\frac{|\det(Y^d)|}{|\mathcal{Y}^d_{11}|}|[f^{L,c^1_B}_{(++)}(0,1)][f^{R,c^1_T}_{(++)}(0,1)]| + (\delta\mathcal{M}^d)_{11},\\
m_c &= \frac{v}{\sqrt{2}}\frac{|\mathcal{Y}^u_{11}|}{|Y^u_{33}|}|[f^{L,c^2_B}_{(++)}(0,1)][f^{R,c^2_S}_{(++)}(0,1)]| + (\delta\mathcal{M}^u)_{22},\\
m_s &= \frac{v}{\sqrt{2}}\frac{|\mathcal{Y}^d_{11}|}{|Y^d_{33}|}|[f^{L,c^2_B}_{(++)}(0,1)][f^{R,c^2_T}_{(++)}(0,1)]| + (\delta\mathcal{M}^d)_{22},\\
m_t &= \frac{v}{\sqrt{2}}|Y^u_{33}||[f^{L,c^3_B}_{(++)}(0,1)][f^{R,c^3_S}_{(++)}(0,1)]| + (\delta\mathcal{M}^u)_{33},\\
m_b &= \frac{v}{\sqrt{2}}|Y^d_{33}||[f^{L,c^3_B}_{(++)}(0,1)][f^{R,c^3_T}_{(++)}(0,1)]| + (\delta\mathcal{M}^d)_{33}.
\end{aligned} \tag{189}$$

Here $\mathcal{Y}^q_{ij}$ denotes the minor of Y^d, i.e. the determinant of the square matrix formed by removing the ith row and the jth column from Y^q. In a similar way, we can give expressions for the left- and right-handed mixing matrices $\mathcal{U}^{(0)}_{L,R}$, $\mathcal{D}^{(0)}_{L,R}$ in terms of Y^q, $\mathcal{Y}^q$ together with relevant bulk profiles on IR brane [18]. Then the Wolfenstein parameters of the CKM matrix can be written as

$$\lambda = \frac{|[f^{L,c^1_B}_{(++)}(0,1)]|}{|[f^{L,c^2_B}_{(++)}(0,1)]|}\left|\frac{\mathcal{Y}^d_{21}}{\mathcal{Y}^d_{11}} - \frac{\mathcal{Y}^u_{21}}{\mathcal{Y}^u_{11}}\right|, \qquad A = \frac{|[f^{L,c^2_B}_{(++)}(0,1)]|^3}{|[f^{L,c^1_B}_{(++)}(0,1)]|^2|[f^{L,c^3_B}_{(++)}(0,1)]|}\frac{|\frac{Y^d_{23}}{Y^d_{33}} - \frac{Y^u_{23}}{Y^u_{33}}|}{|[\frac{\mathcal{Y}^d_{21}}{\mathcal{Y}^d_{11}} - \frac{\mathcal{Y}^u_{21}}{\mathcal{Y}^u_{11}}]^2|},$$
$$\bar{\rho} - i\bar{\eta} = \frac{Y^d_{33}\mathcal{Y}^u_{31} - Y^d_{23}\mathcal{Y}^u_{21} + Y^d_{13}\mathcal{Y}^u_{11}}{Y^d_{33}\mathcal{Y}^u_{11}[\frac{Y^d_{23}}{Y^d_{33}} - \frac{Y^u_{23}}{Y^u_{33}}][\frac{\mathcal{Y}^d_{21}}{\mathcal{Y}^d_{11}} - \frac{\mathcal{Y}^u_{21}}{\mathcal{Y}^u_{11}}]}. \tag{190}$$

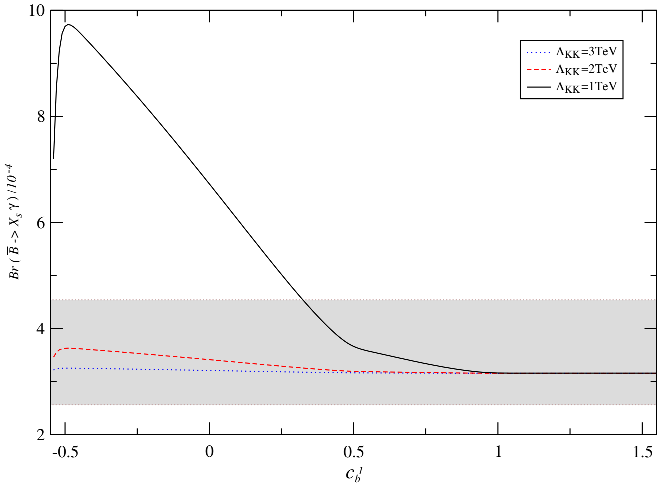

FIG. 2 (color online). Assuming $c_T^1 = c_S^1 = -0.75$, $c_T^2 = c_S^2 = -0.55$, $c_T^3 = c_S^3 = -0.35$, and $c_B^2 = -0.5 + c_B^1$, $c_B^3 = -1 + c_B^1$, we present the branching ratio of $\bar{B} \to X_s\gamma$ varying with the bulk mass c_B^1. The solid line represents the energy scale of low-lying KK mode $\Lambda_{\rm KK} = 1$ TeV, the dashed line represents $\Lambda_{\rm KK} = 2$ TeV, and the dotted line represents $\Lambda_{\rm KK} = 3$ TeV, respectively. In addition, the gray band denotes the experimental data with 3σ deviation.

Through Eqs. (163) and (188)–(190), the present experimental observations impose direct constraints on the parameter space of extension of the SM with a warped extra dimension and the custodial symmetry.

The inputs of the SM sector are [77] $\alpha_{\rm EW} = 1./128.8$, $m_W = 80.23$ GeV, $m_Z = 91.18$ GeV, $\alpha_s(m_Z) = 0.117$, $m_u = 0.003$ GeV, $m_c = 0.62$ GeV, $m_t^{\rm pole} = 173.8$ GeV, $m_d = 0.006$ GeV, $m_s = 0.115$ GeV, $m_b^{\rm pole} = 4.8$ GeV. For the Wolfenstein parameters of the CKM matrix, we take $\lambda = 0.22$, $A = 0.81$, $\bar{\rho} = 0.13$, $\bar{\eta} = 0.34$. Without losing generality, we choose the Yukawa couplings $Y_{ij}^u = 0.01$ $(i \neq j,\ i,\ j = 1, 2, 3)$, $Y_{21}^d = Y_{31}^d = Y_{32}^d = 0.01$. Fixing the bulk masses c_B^i, c_S^i, c_T^i, we derive the other elements of Yukawa couplings numerically through Eqs. (189) and (190).

Taking $c_T^1 = c_S^1 = -0.75$, $c_T^2 = c_S^2 = -0.55$, and $c_T^3 = c_S^3 = -0.35$, we present the branching ratio of $\bar{B} \to X_s\gamma$ varying with the bulk mass c_B^1 in Fig. 2, where the solid line represents the energy scale of low-lying KK mode $\Lambda_{\rm KK} = 1$ TeV, the dashed line represents $\Lambda_{\rm KK} = 2$ TeV, and the dotted line represents $\Lambda_{\rm KK} = 3$ TeV, respectively. In addition, we also assume $c_B^2 = -0.5 + c_B^1$, $c_B^3 = -1 + c_B^1$ to guarantee that the profiles of zero modes on IR brane satisfy the hierarchical structures Eq. (188). Besides the global suppression factor $\upsilon^2/\Lambda_{\rm KK}^2$, the dominating corrections from KK excitations to the branching ratio of $\bar{B} \to X_s\gamma$ depend on the bulk masses c_B^i $(i = 1, 2, 3)$ in terms of $[f_{(++)}^{L,c_B^i}(0, t)][f_{(++)}^{L,c_B^j}(0, t)]$. Because of this, the contributions from new physics to the branching ratio of $\bar{B} \to X_s\gamma$ decrease quickly as $c_B^1 \geq 1$, and can be neglected safely comparing with the contributions from the SM to the branching ratio of $\bar{B} \to X_s\gamma$. Actually, the function $[f_{(++)}^{L,c}(0, 1)]$ tends to zero steeply as $c > 0.5$.

Taking $c_S^1 = -0.75$, $c_S^2 = -0.55$, $c_S^3 = -0.35$, $c_B^1 = 0.55$, $c_B^2 = 0.25$, and $c_B^3 = -0.05$, we present the branching ratio of $\bar{B} \to X_s\gamma$ varying with the bulk mass c_T^1 in Fig. 3, where the solid line represents the energy scale of

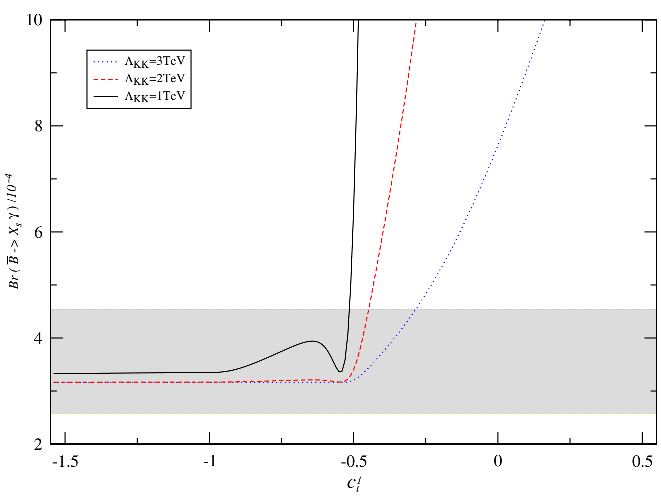

FIG. 3 (color online). Assuming $c_S^1 = -0.75$, $c_S^2 = -0.55$, $c_S^3 = -0.35$, $c_B^1 = 0.55$, $c_B^2 = 0.25$ $c_B^3 = -0.05$, and $c_T^2 = 0.5 + c_T^1$, $c_T^3 = 1 + c_T^1$, we present the branching ratio of $\bar{B} \to X_s\gamma$ varying with the bulk mass c_T^1. The solid line represents the energy scale of low-lying KK mode $\Lambda_{\rm KK} = 1$ TeV, the dashed line represents $\Lambda_{\rm KK} = 2$ TeV, and the dotted line represents $\Lambda_{\rm KK} = 3$ TeV, respectively. In addition, the gray band denotes the experimental data with 3σ deviation.

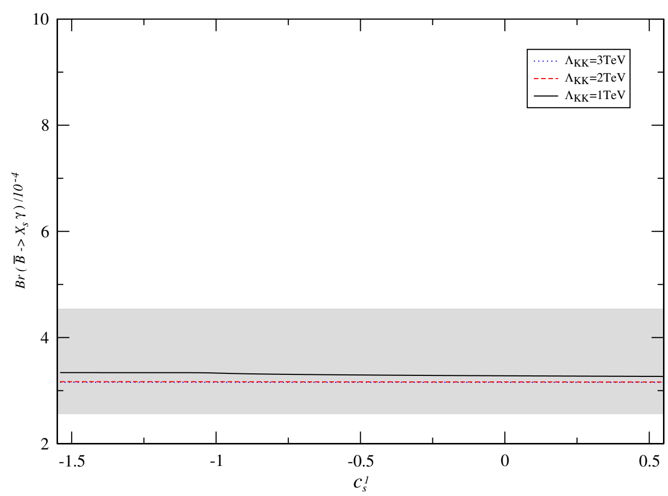

FIG. 4 (color online). Assuming $c_T^1 = -0.75$, $c_T^2 = -0.55$, $c_T^3 = -0.35$, $c_B^1 = 0.55$, $c_B^2 = 0.25$ $c_B^3 = -0.05$, and $c_S^2 = 0.5 + c_S^1$, $c_S^3 = 1 + c_S^1$, we present the branching ratio of $\bar{B} \to X_s\gamma$ varying with the bulk mass c_T^1. The solid line represents the energy scale of low-lying KK mode $\Lambda_{\rm KK} = 1$ TeV, the dashed line represents $\Lambda_{\rm KK} = 2$ TeV, and the dotted line represents $\Lambda_{\rm KK} = 3$ TeV, respectively. In addition, the gray band denotes the experimental data with 3σ deviation.

low-lying KK mode $\Lambda_{\rm KK} = 1$ TeV, the dashed line represents $\Lambda_{\rm KK} = 2$ TeV, and the dotted line represents $\Lambda_{\rm KK} = 3$ TeV, respectively. In addition, we also assume $c_T^2 = 0.5 + c_T^1$, $c_T^3 = 1 + c_T^1$ to guarantee the profiles of zero modes on IR brane satisfy the hierarchical structures Eq. (188). Besides the global suppression factor $\upsilon^2/\Lambda_{\rm KK}^2$, the dominating corrections from KK excitations to the branching ratio of $\bar{B} \to X_s \gamma$ depend on the bulk masses c_T^i $(i = 1, 2, 3)$ in terms of $[f_{(++)}^{R,c_T^i}(0,t)][f_{(++)}^{R,c_T^j}(0,t)]$. Because of this reason, the contributions from new physics to the branching ratio of $\bar{B} \to X_s \gamma$ increase quickly as $c_T^1 \geq -0.5$, and dominate the theoretical prediction on the branching ratio of $\bar{B} \to X_s \gamma$. Actually, the function $[f_{(++)}^{R,c}(0,1)]$ amplifies steeply as $c > 0.5$.

Taking $c_T^1 = -0.75$, $c_T^2 = -0.55$, $c_T^3 = -0.35$, $c_B^1 = 0.55$, $c_B^2 = 0.25$, and $c_B^3 = -0.05$, we present the branching ratio of $\bar{B} \to X_s \gamma$ varying with the bulk mass c_S^1 in Fig. 4, where the solid line represents the energy scale of low-lying KK mode $\Lambda_{\rm KK} = 1$ TeV, the dashed line represents $\Lambda_{\rm KK} = 2$ TeV, and the dotted line represents $\Lambda_{\rm KK} = 3$ TeV, respectively. In addition, we also assume $c_S^2 = 0.5 + c_S^1$, $c_S^3 = 1 + c_S^1$ to guarantee that the profiles of zero modes on IR brane satisfy the hierarchical structures Eq. (188). Differing from the dependence of the branching ratio of $\bar{B} \to X_s \gamma$ on the bulk masses c_B^1, c_T^1, the dependence of the branching ratio of $\bar{B} \to X_s \gamma$ on the bulk mass c_S^1 is very mild.

VII. SUMMARY

In this work, we verify that the eigenvalues of KK excitations in the SM extension with a warped extra dimension and the custodial symmetry are real, and are symmetrically distributed contrasting to the origin in the complex plane. We also present the sufficient condition to judge if the infinite series of KK excitations is convergent. Applying the residue theorem, we sum over the infinitely series of KK modes, and analyze the possible relation between summation of the product of KK mode propagator with the corresponding bulk profiles in four-dimensional effective theory and the propagator of field in five-dimensional full theory. We sum over the infinitely series of KK modes for the gauge boson with $(++)$ BCs, and recover the results in the literature which are obtained through the equation of motion and the completeness relation of bulk profiles for KK modes. Additionally, we also present the summing over infinite KK exciting series $\Sigma_{(-+)}^G(t,t')$, $\Sigma_{(\rm BCs)}^{L,c}(t,t')$, and $\Sigma_{(\rm BCs)}^{R,c}(t,t')$, which satisfy the corresponding BCs on IR and UV branes, respectively. We extend this method to sum over the KK modes in a universal extra dimension, and obtain the equation applied extensively in the literature. As an example, we present the radiative correction to the rare decay $b \to s + \gamma$ in the SM extension with a warped extra dimension and the custodial symmetry, and analyze the possible constraint on the parameter space of new physics from experimental observation of the branching ratio of $\bar{B} \to X_s \gamma$.

ACKNOWLEDGMENTS

The work has been supported by the National Natural Science Foundation of China (NNSFC) with Grant No. 10975027.

APPENDIX A: NONZERO ELEMENTS IN THE MASS MATRICES

$$
\begin{aligned}
[\mathcal{M}_{0,0}^{qu}]_{ij} &= f_{(++)}^{L,c_B^i}(0,1)\frac{\upsilon Y_{ij}^u}{\sqrt{2}} f_{(++)}^{R,c_S^j}(0,1),\\
[\mathcal{M}_{0,n}^{qU}]_{ij} &= -f_{(++)}^{L,c_B^i}(0,1)\frac{\upsilon Y_{ij}^d}{2} f_{(-+)}^{R,c_T^j}(y_{(\pm\mp)}^{c_T^j(n)},1),\\
[\mathcal{M}_{0,n}^{q\tilde{U}}]_{ij} &= -[\mathcal{M}_{0,n}^{qU}]_{ij}\\
[\mathcal{M}_{0,n}^{qu}]_{ij} &= f_{(++)}^{L,c_B^i}(0,1)\frac{\upsilon Y_{ij}^u}{\sqrt{2}} f_{(++)}^{R,c_S^j}(y_{(\mp\mp)}^{c_S^j(n)},1),\\
[\mathcal{M}_{n,0}^{qu}]_{ij} &= f_{(++)}^{L,c_B^i}(y_{(\pm\pm)}^{c_B^i(n)},1)\frac{\upsilon Y_{ij}^u}{\sqrt{2}} f_{(++)}^{R,c_S^j}(0,1),\\
[\mathcal{M}_{n,n'}^{qU}]_{ij} &= -f_{(++)}^{L,c_B^i}(y_{(\pm\pm)}^{c_B^i(n)},1)\frac{\upsilon Y_{ij}^d}{2} f_{(-+)}^{R,c_T^j}(y_{(\pm\mp)}^{c_T^j(n')},1),\\
[\mathcal{M}_{n,n'}^{q\tilde{U}}]_{ij} &= -[\mathcal{M}_{n,n'}^{qU}]_{ij},\\
[\mathcal{M}_{n,n'}^{qu}]_{ij} &= f_{(++)}^{L,c_B^i}(y_{(\pm\pm)}^{c_B^i(n)},1)\frac{\upsilon Y_{ij}^u}{\sqrt{2}} f_{(++)}^{R,c_S^j}(y_{(\mp\mp)}^{c_S^j(n')},1),\\
[\mathcal{M}_{n,0}^{\chi u}]_{ij} &= -f_{(-+)}^{L,c_B^i}(y_{(\mp\pm)}^{c_B^i(n)},1)\frac{\upsilon Y_{ij}^u}{\sqrt{2}} f_{(++)}^{R,c_S^j}(0,1),\\
[\mathcal{M}_{n,n'}^{\chi U}]_{ij} &= -f_{(-+)}^{L,c_B^i}(y_{(\mp\pm)}^{c_B^i(n)},1)\frac{\upsilon Y_{ij}^d}{2} f_{(-+)}^{R,c_T^j}(y_{(\pm\mp)}^{c_T^j(n')},1),\\
[\mathcal{M}_{n,n'}^{\chi\tilde{U}}]_{ij} &= -[\mathcal{M}_{n,n'}^{\chi U}]_{ij},\\
[\mathcal{M}_{n,n'}^{\chi u}]_{ij} &= -f_{(-+)}^{L,c_B^i}(y_{(\mp\pm)}^{c_B^i(n)},1)\frac{\upsilon Y_{ij}^u}{\sqrt{2}} f_{(++)}^{R,c_S^j}(y_{(\mp\mp)}^{c_S^j(n')},1),\\
[\mathcal{M}_{0,0}^{qd}]_{ij} &= f_{(++)}^{L,c_B^i}(0,1)\frac{\upsilon Y_{ij}^d}{\sqrt{2}} f_{(++)}^{R,c_T^j}(0,1),\\
[\mathcal{M}_{0,n}^{qD}]_{ij} &= -f_{(++)}^{L,c_B^i}(0,1)\frac{\upsilon Y_{ij}^d}{\sqrt{2}} f_{(-+)}^{R,c_T^j}(y_{(\pm\mp)}^{c_T^j(n)},1),\\
[\mathcal{M}_{0,n}^{qd}]_{ij} &= f_{(++)}^{L,c_B^i}(0,1)\frac{\upsilon Y_{ij}^d}{\sqrt{2}} f_{(++)}^{R,c_T^j}(y_{(\mp\mp)}^{c_T^j(n)},1),\\
[\mathcal{M}_{n,0}^{qd}]_{ij} &= f_{(++)}^{L,c_B^i}(y_{(\pm\pm)}^{c_B^i(n)},1)\frac{\upsilon Y_{ij}^d}{\sqrt{2}} f_{(++)}^{R,c_T^j}(0,1),\\
[\mathcal{M}_{n,n'}^{qD}]_{ij} &= -f_{(++)}^{L,c_B^i}(y_{(\pm\pm)}^{c_B^i(n)},1)\frac{\upsilon Y_{ij}^d}{\sqrt{2}} f_{(-+)}^{R,c_T^j}(y_{(\pm\mp)}^{c_T^j(n')},1),\\
[\mathcal{M}_{n,n'}^{qd}]_{ij} &= f_{(++)}^{L,c_B^i}(y_{(\pm\pm)}^{c_B^i(n)},1)\frac{\upsilon Y_{ij}^d}{\sqrt{2}} f_{(++)}^{R,c_T^j}(y_{(\mp\mp)}^{c_T^j(n')},1).
\end{aligned}
\tag{A1}
$$

$$
\begin{aligned}
[\mathcal{M}^2_{W^\pm_{L,L}}]_{0,n} &= \frac{\sqrt{2\pi}e^2 v^2}{4s_W^2}\chi^{W_L}_{(++)}(y^{W_L(n)}_{(++)},1),\\
[\mathcal{M}^2_{W^\pm_{L,R}}]_{0,n} &= \frac{\sqrt{2\pi}e^2 v^2}{4s_W^2}\chi^{W_R}_{(-+)}(y^{W_R(n)}_{(-+)},1),\\
[\mathcal{M}^2_{W^\pm_{L,R}}]_{n,n'} &= \frac{2\pi e^2 v^2}{4s_W^2}\chi^{W_L}_{(++)}(y^{W_L(n)}_{(++)},1)\chi^{W_R}_{(-+)}(y^{W_R(n')}_{(-+)},1),\\
[\mathcal{M}^2_{W^\pm_{L,L}}]_{n,n'} &= \frac{2\pi e^2 v^2}{4s_W^2}\chi^{W_L}_{(++)}(y^{W_L(n)}_{(++)},1)\chi^{W_L}_{(++)}(y^{W_L(n')}_{(++)},1)+[\Lambda_{\rm KK}y^{W_L(n)}_{(++)}]^2\delta_{nn'},\\
[\mathcal{M}^2_{W^\pm_{R,R}}]_{n,n'} &= \frac{2\pi e^2 v^2}{4s_W^2}\chi^{W_R}_{(-+)}(y^{W_R(n)}_{(-+)},1)\chi^{W_R}_{(-+)}(y^{W_R(n')}_{(-+)},1)+[\Lambda_{\rm KK}y^{W_R(n)}_{(-+)}]^2\delta_{nn'},\\
[\mathcal{M}^2_{Z_{L,L}}]_{0,n} &= \frac{\sqrt{2\pi}e^2 v^2}{4s_W^2c_W^2}\chi^{Z}_{(++)}(y^{Z(n)}_{(++)},1),\\
[\mathcal{M}^2_{Z_{L,X}}]_{0,n} &= \frac{\sqrt{2\pi(1-2s_W^2)}e^2 v^2}{4s_W^2c_W^2}\chi^{Z_X}_{(-+)}(y^{Z_X(n)}_{(-+)},1),\\
[\mathcal{M}^2_{Z_{L,X}}]_{n,n'} &= \frac{2\pi(1-2s_W^2)e^2 v^2}{4s_W^2c_W^2}\chi^{Z}_{(++)}(y^{Z(n)}_{(++)},1)\chi^{Z_X}_{(-+)}(y^{Z_X(n')}_{(-+)},1),\\
[\mathcal{M}^2_{Z_{L,L}}]_{n,n'} &= \frac{2\pi(1-2s_W^2)e^2 v^2}{4s_W^2c_W^2}\chi^{Z}_{(++)}(y^{Z(n)}_{(++)},1)\chi^{Z}_{(++)}(y^{Z(n')}_{(++)},1)+[\Lambda_{\rm KK}y^{Z(n)}_{(++)}]^2\delta_{nn'},\\
[\mathcal{M}^2_{Z_{X,X}}]_{n,n'} &= \frac{2\pi(1-2s_W^2)e^2 v^2}{4s_W^2c_W^2}\chi^{Z_X}_{(-+)}(y^{Z_X(n)}_{(-+)},1)\chi^{Z_X}_{(-+)}(y^{Z_X(n')}_{(-+)},1)+[\Lambda_{\rm KK}y^{Z_X(n)}_{(-+)}]^2\delta_{nn'}.
\end{aligned}
\tag{A2}
$$

APPENDIX B: THE COUPLINGS BETWEEN BOSONS AND QUARKS

$$
\begin{aligned}
(\xi^L_{W^\pm})_{\alpha_1,\alpha_2} &= (Z_W)_{0,0}\sum_{i=1}^{3}\Big\{(\mathcal{U}_L^\dagger)_{\alpha_1,i}(\mathcal{D}_L)_{i,\alpha_2}+\sum_{n=1}^{\infty}(\mathcal{U}_L^\dagger)_{\alpha_1,(15n-12+i)}(\mathcal{D}_L)_{(9n-6+i),\alpha_2}\Big\}\\
&+\frac{4\sqrt{2\pi}}{kr\epsilon}\sum_{i=1}^{3}\sum_{n_0=1}^{\infty}(Z_W)_{(2n_0-1),0}\int_\epsilon^1 dt\,\chi^{W_L}_{(++)}(y^{W_L(n_0)}_{(++)},t)\Big\{f^{L,c_B^i}_{(++)}(0,t)f^{L,c_B^i}_{(++)}(0,t)(\mathcal{U}_L^\dagger)_{\alpha_1,i}(\mathcal{D}_L)_{i,\alpha_2}\\
&+\sum_{n'=1}^{\infty}f^{L,c_B^i}_{(++)}(0,t)f^{L,c_B^i}_{(++)}(y^{c_B^i(n')}_{(\pm\pm)},t)(\mathcal{U}_L^\dagger)_{\alpha_1,i}(\mathcal{D}_L)_{(9n'-6+i),\alpha_2}\\
&+\sum_{n=1}^{\infty}f^{L,c_B^i}_{(++)}(y^{c_B^i(n)}_{(\pm\pm)},t)f^{L,c_B^i}_{(++)}(0,t)(\mathcal{U}_L^\dagger)_{\alpha_1,(15n-12+i)}(\mathcal{D}_L)_{i,\alpha_2}\\
&+\sum_{n=1}^{\infty}\sum_{n'=1}^{\infty}f^{L,c_B^i}_{(++)}(y^{c_B^i(n)}_{(\pm\pm)},t)f^{L,c_B^i}_{(++)}(y^{c_B^i(n')}_{(\pm\pm)},t)(\mathcal{U}_L^\dagger)_{\alpha_1,(15n-12+i)}(\mathcal{D}_L)_{(9n'-6+i),\alpha_2}\\
&+\sum_{n=1}^{\infty}\sum_{n'=1}^{\infty}f^{L,c_T^i}_{(+-)}(y^{c_T^i(n)}_{(\pm\mp)},t)f^{L,c_T^i}_{(+-)}(y^{c_T^i(n')}_{(\pm\mp)},t)(\mathcal{U}_L^\dagger)_{\alpha_1,(15n-9+i)}(\mathcal{D}_L)_{(9n'-3+i),\alpha_2}\Big\}\\
&+\frac{4\sqrt{2\pi}e}{kr\epsilon}\sum_{i=1}^{3}\sum_{n_0=1}^{\infty}(Z_W)_{2n_0,0}\int_\epsilon^1 dt\,\chi^{W_R}_{(-+)}(y^{W_R(n_0)}_{(-+)},t)\Big\{\sum_{n=1}^{\infty}f^{L,c_B^i}_{(-+)}(y^{c_B^i(n)}_{(\mp\pm)},t)f^{L,c_B^i}_{(++)}(0,t)(\mathcal{U}_L^\dagger)_{\alpha_1,(15n-3+i)}(\mathcal{D}_L)_{i,\alpha_2}\\
&+\sum_{n=1}^{\infty}\sum_{n'=1}^{\infty}f^{L,c_B^i}_{(-+)}(y^{c_B^i(n)}_{(\mp\pm)},t)f^{L,c_B^i}_{(++)}(y^{c_B^i(n')}_{(\pm\pm)},t)(\mathcal{U}_L^\dagger)_{\alpha_1,(15n-3+i)}(\mathcal{D}_L)_{(9n'-6+i),\alpha_2}\\
&+\sum_{n=1}^{\infty}\sum_{n'=1}^{\infty}f^{L,c_T^i}_{(+-)}(y^{c_T^i(n)}_{(\pm\mp)},t)f^{L,c_T^i}_{(--)}(y^{c_T^i(n')}_{(\mp\mp)},t)(\mathcal{U}_L^\dagger)_{\alpha_1,(15n-6+i)}(\mathcal{D}_L)_{(9n'+i),\alpha_2}\Big\},
\end{aligned}
$$

$$(\xi^R_{W^\pm})_{\alpha_1,\alpha_2} = (Z_W)_{0,0}\sum_{i=1}^{3}\sum_{n=1}^{\infty}(\mathcal{U}^\dagger_R)_{\alpha_1,(15n-12+i)}(\mathcal{D}_R)_{(9n-6+i),\alpha_2} + \frac{4\sqrt{2\pi}}{kr\epsilon}\sum_{i=1}^{3}\sum_{n_0=1}^{\infty}\sum_{n=1}^{\infty}\sum_{n'=1}^{\infty}(Z_W)_{(2n_0-1),0}$$
$$\times\int_\epsilon^1 dt\chi^{W_L}_{(++)}(y^{W_L(n_0)}_{(++)},t)\Big\{f^{R,c^i_B}_{(--)}(y^{c^i_B(n)}_{(\pm\pm)},t)f^{R,c^i_B}_{(--)}(y^{c^i_B(n')}_{(\pm\pm)},t)(\mathcal{U}^\dagger_R)_{\alpha_1,(15n-12+i)}(\mathcal{D}_R)_{(9n'-6+i),\alpha_2}$$
$$+ f^{R,c^i_T}_{(-+)}(y^{c^i_T(n)}_{(\pm\mp)},t)f^{R,c^i_T}_{(-+)}(y^{c^i_T(n')}_{(\pm\mp)},t)(\mathcal{U}^\dagger_R)_{\alpha_1,(15n-9+i)}(\mathcal{D}_R)_{(9n'-3+i),\alpha_2}\Big\} + \frac{4\sqrt{2\pi}}{kr\epsilon}\sum_{i=1}^{3}\sum_{n_0=1}^{\infty}\sum_{n=1}^{\infty}\sum_{n'=1}^{\infty}(Z_W)_{2n_0,0}$$
$$\times\int_\epsilon^1 dt\chi^{W_R}_{(-+)}(y^{W_R(n_0)}_{(-+)},t)\Big\{f^{R,c^i_B}_{(+-)}(y^{c^i_B(n)}_{(\mp\pm)},t)f^{R,c^i_B}_{(--)}(y^{c^i_B(n')}_{(\pm\pm)},t)(\mathcal{U}^\dagger_R)_{\alpha_1,(15n-3+i)}(\mathcal{D}_R)_{(9n'-6+i),\alpha_2}$$
$$+ f^{R,c^i_T}_{(-+)}(y^{c^i_T(n)}_{(\pm\mp)},t)f^{R,c^i_T}_{(++)}(y^{c^i_T(n')}_{(\mp\mp)},t)(\mathcal{U}^\dagger_R)_{\alpha_1,(15n-6+i)}(\mathcal{D}_R)_{(9n'+i),\alpha_2}\Big\},$$

$$(\xi^L_{W^\pm_{H_\alpha}})_{\alpha_1,\ \alpha_2} = (\xi^L_{W^\pm})_{\alpha_1,\alpha_2}\{(Z_W)_{0,0}\to(Z_W)_{0,\alpha},(Z_W)_{(2n_0-1),0}\to(Z_W)_{(2n_0-1),\alpha},(Z_W)_{2n_0),0}\to(Z_W)_{2n_0,\alpha}\},$$
$$(\xi^R_{W^\pm_{H_\alpha}})_{\alpha_1,\alpha_2} = (\xi^R_{W^\pm})_{\alpha_1,\alpha_2}\{(Z_W)_{0,0}\to(Z_W)_{0,\alpha},(Z_W)_{(2n_0-1),0}\to(Z_W)_{(2n_0-1),\alpha},(Z_W)_{2n_0),0}\to(Z_W)_{2n_0,\alpha}\},$$

$$(\xi^L_Z)_{\alpha_1,\alpha_2} = (Z_Z)_{0,0}\sum_{i=1}^{3}\Big\{\Big(1-\frac{2}{3}s^2_W\Big)\Big[(\mathcal{D}^\dagger_L)_{\alpha_1,i}(\mathcal{D}_L)_{i,\alpha_2} + \sum_{n=1}^{\infty}(\mathcal{D}^\dagger_L)_{\alpha_1,(9n-6+i)}(\mathcal{D}_L)_{(9n-6+i),\alpha_2}$$
$$+ 2\sum_{n=1}^{\infty}(\mathcal{D}^\dagger_L)_{\alpha_1,(9n-3+i)}(\mathcal{D}_L)_{(9n-3+i),\alpha_2}\Big] - \frac{2}{3}s^2_W\sum_{n=1}^{\infty}(\mathcal{D}^\dagger_L)_{\alpha_1,(9n+i)}(\mathcal{D}_L)_{(9n+i),\alpha_2}\Big\}$$
$$+ \frac{4\sqrt{2\pi}}{kr\epsilon}\sum_{n_0=1}^{\infty}\sum_{i=1}^{3}(Z_Z)_{(2n_0-1),0}\int_\epsilon^1 dt[\chi^Z_{(++)}(y^{Z(n_0)}_{(++)},t)]\Big\{\Big(1-\frac{2}{3}s^2_W\Big)\Big[f^{L,c^i_B}_{(++)}(0,t)f^{L,c^i_B}_{(++)}(0,t)(\mathcal{D}^\dagger_L)_{\alpha_1,i}(\mathcal{D}_L)_{i,\alpha_2}$$
$$+ \sum_{n=1}^{\infty}f^{L,c^i_B}_{(++)}(0,t)f^{L,c^i_B}_{(++)}(y^{c^i_B(n)}_{(\pm\pm)},t)((\mathcal{D}^\dagger_L)_{\alpha_1,i}(\mathcal{D}_L)_{(9n-6+i),\alpha_2}$$
$$+ (\mathcal{D}^\dagger_L)_{\alpha_1,(9n-6+i)}(\mathcal{D}_L)_{i,\alpha_2})\sum_{n,n'=1}^{\infty}f^{L,c^i_T}_{(--)}(y^{c^i_T(n)}_{(\mp\mp)},t)f^{L,c^i_T}_{(--)}(y^{c^i_T(n')}_{(\mp\mp)},t)(\mathcal{D}^\dagger_L)_{\alpha_1,(9n+i)}(\mathcal{D}_L)_{(9n'+i),\alpha_2}\Big\}$$
$$+ \frac{4\sqrt{2\pi}}{3\sqrt{2c^2_W-1}kr\epsilon}\sum_{n_0=1}^{\infty}\sum_{i=1}^{3}(Z_Z)_{2n_0,0}\int_\epsilon^1 dt\chi^{Z_X}_{(-+)}(y^{Z_X(n_0)}_{(-+)},t)\{(1+2c^2_W)[f^{L,c^i_B}_{(++)}(0,t)f^{L,c^i_B}_{(++)}(0,t)(\mathcal{D}^\dagger_L)_{\alpha_1,i}(\mathcal{D}_L)_{i,\alpha_2}$$
$$+ \sum_{n=1}^{\infty}f^{L,c^i_B}_{(++)}(0,t)f^{L,c^i_B}_{(++)}(y^{c^i_B(n)}_{(\pm\pm)},t)((\mathcal{D}^\dagger_L)_{\alpha_1,i}(\mathcal{D}_L)_{(9n-6+i),\alpha_2} + (\mathcal{D}^\dagger_L)_{\alpha_1,(9n-6+i)}(\mathcal{D}_L)_{i,\alpha_2})$$
$$+ \sum_{n,n'=1}^{\infty}f^{L,c^i_B}_{(++)}(y^{c^i_B(n)}_{(\pm\pm)},t)f^{L,c^i_B}_{(++)}(y^{c^i_B(n')}_{(\pm\pm)},t)(\mathcal{D}^\dagger_L)_{\alpha_1,(9n-6+i)}(\mathcal{D}_L)_{(9n'-6+i),\alpha_2}]$$
$$+ 4s^2_W\sum_{n,n'=1}^{\infty}f^{L,c^i_T}_{(+-)}(y^{c^i_T(n)}_{(\pm\mp)},t)f^{L,c^i_T}_{(+-)}(y^{c^i_T(n')}_{(\pm\mp)},t)(\mathcal{D}^\dagger_L)_{\alpha_1,(9n-3+i)}(\mathcal{D}_L)_{(9n'-3+i),\alpha_2}$$
$$- 2(3-4s^2_W)\sum_{n,n'=1}^{\infty}f^{L,c^i_T}_{(--)}(y^{c^i_T(n)}_{(\mp\mp)},t)f^{L,c^i_T}_{(--)}(y^{c^i_T(n')}_{(\mp\mp)},t)(\mathcal{D}^\dagger_L)_{\alpha_1,(9n+i)}(\mathcal{D}_L)_{(9n'+i),\alpha_2}\Big\},$$

$$(\xi_Z^R)_{\alpha_1,\alpha_2} = (Z_Z)_{0,0}\sum_{i=1}^{3}\bigg\{\bigg(1-\frac{2}{3}s_W^2\bigg)\sum_{n=1}^{\infty}[(\mathcal{D}_R^\dagger)_{\alpha_1,(9n-6+i)}(\mathcal{D}_R)_{(9n-6+i),\alpha_2}+2(\mathcal{D}_R^\dagger)_{\alpha_1,(9n-3+i)}(\mathcal{D}_R)_{(9n-3+i),\alpha_2}]$$
$$-\frac{2}{3}s_W^2[(\mathcal{D}_R^\dagger)_{\alpha_1,i}(\mathcal{D}_R)_{i,\alpha_2}+\sum_{n=1}^{\infty}(\mathcal{D}_R^\dagger)_{\alpha_1,(9n+i)}(\mathcal{D}_R)_{(9n+i),\alpha_2}]\bigg\}$$
$$+\frac{4\sqrt{2\pi}}{kr\epsilon}\sum_{n_0=1}^{\infty}\sum_{i=1}^{3}(Z_Z)_{(2n_0-1),0}\int_\epsilon^1 dt[\chi_{(++)}^{Z}(y_{(++)}^{Z(n_0)},t)]\bigg\{-\frac{2}{3}s_W^2\bigg[f_{(++)}^{R,c_T^i}(0,t)f_{(++)}^{R,c_T^i}(0,t)(\mathcal{D}_R^\dagger)_{\alpha_1,i}(\mathcal{D}_R)_{i,\alpha_2}$$
$$+\sum_{n=1}^{\infty}f_{(++)}^{R,c_T^i}(0,t)f_{(++)}^{R,c_T^i}(y_{(\mp\mp)}^{c_T^i(n)},t)((\mathcal{D}_R^\dagger)_{\alpha_1,i}(\mathcal{D}_R)_{(9n+i),\alpha_2}+(\mathcal{D}_R^\dagger)_{\alpha_1,(9n+i)}(\mathcal{D}_R)_{i,\alpha_2})$$
$$+\sum_{n,n'=1}^{\infty}f_{(++)}^{R,c_T^i}(y_{(\mp\mp)}^{c_T^i(n)},t)f_{(++)}^{R,c_T^i}(y_{(\mp\mp)}^{c_T^i(n')},t)(\mathcal{D}_R^\dagger)_{\alpha_1,(9n+i)}(\mathcal{D}_R)_{(9n'+i),\alpha_2}\bigg]$$
$$+\bigg(1-\frac{2}{3}s_W^2\bigg)\sum_{n,n'=1}^{\infty}f_{(--)}^{R,c_B^i}(y_{(\pm\pm)}^{c_B^i(n)},t)f_{(--)}^{R,c_B^i}(y_{(\pm\pm)}^{c_B^i(n')},t)(\mathcal{D}_R^\dagger)_{\alpha_1,(9n-6+i)}(\mathcal{D}_R)_{(9n'-6+i),\alpha_2}$$
$$+2\sum_{n,n'=1}^{\infty}f_{(-+)}^{R,c_T^i}(y_{(\pm\mp)}^{c_T^i(n)},t)f_{(-+)}^{R,c_T^i}(y_{(\pm\mp)}^{c_T^i(n')},t)(\mathcal{D}_R^\dagger)_{\alpha_1,(9n-3+i)}(\mathcal{D}_R)_{(9n'-3+i),\alpha_2}\bigg]\bigg\}$$
$$+\frac{4\sqrt{2\pi}e}{3\sqrt{2c_W^2-1}kr\epsilon}\sum_{n_0=1}^{\infty}\sum_{i=1}^{3}(Z_Z)_{2n_0,0}\int_\epsilon^1 dt\chi_{(-+)}^{Z_X}(y_{(-+)}^{Z_X(n_0)},t)\bigg\{-2(3-4s_W^2)\bigg[f_{(++)}^{R,c_T^i}(0,t)f_{(++)}^{R,c_T^i}(0,t)(\mathcal{D}_R^\dagger)_{\alpha_1,i}(\mathcal{D}_R)_{i,\alpha_2}$$
$$+\sum_{n=1}^{\infty}f_{(++)}^{R,c_T^i}(0,t)f_{(++)}^{R,c_T^i}(y_{(\mp\mp)}^{c_T^i(n)},t)((\mathcal{D}_R^\dagger)_{\alpha_1,i}(\mathcal{D}_R)_{(9n+i),\alpha_2}+(\mathcal{D}_R^\dagger)_{\alpha_1,(9n+i)}(\mathcal{D}_R)_{i,\alpha_2})$$
$$+\sum_{n,n'=1}^{\infty}f_{(++)}^{R,c_T^i}(y_{(\mp\mp)}^{c_T^i(n)},t)f_{(++)}^{R,c_T^i}(y_{(\mp\mp)}^{c_T^i(n')},t)(\mathcal{D}_R^\dagger)_{\alpha_1,(9n+i)}(\mathcal{D}_R)_{(9n'+i),\alpha_2}\bigg]$$
$$+(3-2s_W^2)\sum_{n,n'=1}^{\infty}f_{(--)}^{R,c_B^i}(y_{(\pm\pm)}^{c_B^i(n)},t)f_{(--)}^{R,c_B^i}(y_{(\pm\pm)}^{c_B^i(n')},t)(\mathcal{D}_R^\dagger)_{\alpha_1,(9n-6+i)}(\mathcal{D}_R)_{(9n'-6+i),\alpha_2}$$
$$+4s_W^2\sum_{n,n'=1}^{\infty}f_{(-+)}^{R,c_T^i}(y_{(\pm\mp)}^{c_T^i(n)},t)f_{(-+)}^{R,c_T^i}(y_{(\pm\mp)}^{c_T^i(n')},t)(\mathcal{D}_R^\dagger)_{\alpha_1,(9n-3+i)}(\mathcal{D}_R)_{(9n'-3+i),\alpha_2}\bigg\},$$

$$(\xi_{Z_{H_\alpha}}^L)_{\alpha_1,\alpha_2} = (\xi_Z^L)_{\alpha_1,\alpha_2}\{(Z_Z)_{0,0}\to(Z_Z)_{0,\alpha},(Z_Z)_{(2n_0-1),0}\to(Z_Z)_{(2n_0-1),\alpha},(Z_Z)_{2n_0),0}\to(Z_Z)_{2n_0,\alpha}\},$$
$$(\xi_{Z_{H_\alpha}}^R)_{\alpha_1,\alpha_2} = (\xi_Z^R)_{\alpha_1,\alpha_2}\{(Z_Z)_{0,0}\to(Z_Z)_{0,\alpha},(Z_Z)_{(2n_0-1),0}\to(Z_Z)_{(2n_0-1),\alpha},(Z_Z)_{2n_0),0}\to(Z_Z)_{2n_0,\alpha}\},$$

$$(\xi_{\gamma_{(n)}}^L)_{\alpha_1,\alpha_2} = \frac{4\sqrt{2\pi}}{3kr\epsilon}\sum_{i=1}^{3}\int_\epsilon^1 dt\chi_{(++)}^{A}(y_{(++)}^{A(n)},t)\{f_{(++)}^{L,c_B^i}(0,t)f_{(++)}^{L,c_B^i}(0,t)(\mathcal{D}_L^\dagger)_{\alpha_1,i}(\mathcal{D}_L)_{i,\alpha_2}$$
$$+\sum_{n=1}^{\infty}f_{(++)}^{L,c_B^i}(0,t)f_{(++)}^{L,c_B^i}(y_{(\pm\pm)}^{c_B^i(n)},t)((\mathcal{D}_L^\dagger)_{\alpha_1,i}(\mathcal{D}_L)_{(9n-6+i),\alpha_2}+(\mathcal{D}_L^\dagger)_{\alpha_1,(9n-6+i)}(\mathcal{D}_L)_{i,\alpha_2})$$
$$+\sum_{n,n'=1}^{\infty}f_{(++)}^{L,c_B^i}(y_{(\pm\pm)}^{c_B^i(n)},t)f_{(++)}^{L,c_B^i}(y_{(\pm\pm)}^{c_B^i(n')},t)(\mathcal{D}_L^\dagger)_{\alpha_1,(9n-6+i)}(\mathcal{D}_L)_{(9n'-6+i),\alpha_2}$$
$$+\sum_{n,n'=1}^{\infty}f_{(+-)}^{L,c_T^i}(y_{(\pm\mp)}^{c_T^i(n)},t)f_{(+-)}^{L,c_T^i}(y_{(\pm\mp)}^{c_T^i(n')},t)(\mathcal{D}_L^\dagger)_{\alpha_1,(9n-3+i)}(\mathcal{D}_L)_{(9n'-3+i),\alpha_2}$$
$$+\sum_{n,n'=1}^{\infty}f_{(--)}^{L,c_T^i}(y_{(\mp\mp)}^{c_T^i(n)},t)f_{(--)}^{L,c_T^i}(y_{(\mp\mp)}^{c_T^i(n')},t)(\mathcal{D}_L^\dagger)_{\alpha_1,(9n+i)}(\mathcal{D}_L)_{(9n'+i),\alpha_2}\bigg\},$$

$$
\begin{aligned}
(\xi^R_{\gamma_{(n)}})_{\alpha_1,\alpha_2} = \frac{4\sqrt{2\pi}}{3kr\epsilon}\sum_{i=1}^{3}\int_\epsilon^1 dt\chi^A_{(++)}(y^{A(n)}_{(++)},t)\Big\{ & f^{R,c^i_T}_{(++)}(0,t)f^{R,c^i_T}_{(++)}(0,t)(\mathcal{D}^\dagger_R)_{\alpha_1,i}(\mathcal{D}_R)_{i,\alpha_2} \\
&+\sum_{n=1}^{\infty} f^{R,c^i_T}_{(++)}(0,t)f^{R,c^i_T}_{(++)}(y^{c^i_T(n)}_{(\mp\mp)},t)((\mathcal{D}^\dagger_R)_{\alpha_1,i}(\mathcal{D}_R)_{(9n+i),\alpha_2}+(\mathcal{D}^\dagger_R)_{\alpha_1,(9n+i)}(\mathcal{D}_R)_{i,\alpha_2}) \\
&+\sum_{n,n'=1}^{\infty} f^{R,c^i_T}_{(++)}(y^{c^i_T(n)}_{(\mp\mp)},t)f^{R,c^i_T}_{(++)}(y^{c^i_T(n')}_{(\mp\mp)},t)(\mathcal{D}^\dagger_R)_{\alpha_1,(9n+i)}(\mathcal{D}_R)_{(9n'+i),\alpha_2} \\
&+\sum_{n,n'=1}^{\infty} f^{R,c^i_B}_{(--)}(y^{c^i_B(n)}_{(\pm\pm)},t)f^{R,c^i_B}_{(--)}(y^{c^i_B(n')}_{(\pm\pm)},t)(\mathcal{D}^\dagger_R)_{\alpha_1,(9n-6+i)}(\mathcal{D}_R)_{(9n'-6+i),\alpha_2} \\
&+\sum_{n,n'=1}^{\infty} f^{R,c^i_T}_{(-+)}(y^{c^i_T(n)}_{(\pm\mp)},t)f^{R,c^i_T}_{(-+)}(y^{c^i_T(n')}_{(\pm\mp)},t)(\mathcal{D}^\dagger_R)_{\alpha_1,(9n-3+i)}(\mathcal{D}_R)_{(9n'-3+i),\alpha_2}\Big\},
\end{aligned}
$$

$$
\begin{aligned}
(\xi^L_{g_{(n)}})_{\alpha_1,\alpha_2} = \frac{4\sqrt{2\pi}}{kr\epsilon}\sum_{i=1}^{3}\int_\epsilon^1 dt\chi^g_{(++)}(y^{g(n)}_{(++)},t)\Big\{ & f^{L,c^i_B}_{(++)}(0,t)f^{L,c^i_B}_{(++)}(0,t)(\mathcal{D}^\dagger_L)_{\alpha_1,i}(\mathcal{D}_L)_{i,\alpha_2} \\
&+\sum_{n=1}^{\infty} f^{L,c^i_B}_{(++)}(0,t)f^{L,c^i_B}_{(++)}(y^{c^i_B(n)}_{(\pm\pm)},t)((\mathcal{D}^\dagger_L)_{\alpha_1,i}(\mathcal{D}_L)_{(9n-6+i),\alpha_2}+(\mathcal{D}^\dagger_L)_{\alpha_1,(9n-6+i)}(\mathcal{D}_L)_{i,\alpha_2}) \\
&+\sum_{n,n'=1}^{\infty} f^{L,c^i_B}_{(++)}(y^{c^i_B(n)}_{(\pm\pm)},t)f^{L,c^i_B}_{(++)}(y^{c^i_B(n')}_{(\pm\pm)},t)(\mathcal{D}^\dagger_L)_{\alpha_1,(9n-6+i)}(\mathcal{D}_L)_{(9n'-6+i),\alpha_2} \\
&+\sum_{n,n'=1}^{\infty} f^{L,c^i_T}_{(+-)}(y^{c^i_T(n)}_{(\pm\mp)},t)f^{L,c^i_T}_{(+-)}(y^{c^i_T(n')}_{(\pm\mp)},t)(\mathcal{D}^\dagger_L)_{\alpha_1,(9n-3+i)}(\mathcal{D}_L)_{(9n'-3+i),\alpha_2} \\
&+\sum_{n,n'=1}^{\infty} f^{L,c^i_T}_{(--)}(y^{c^i_T(n)}_{(\mp\mp)},t)f^{L,c^i_T}_{(--)}(y^{c^i_T(n')}_{(\mp\mp)},t)(\mathcal{D}^\dagger_L)_{\alpha_1,(9n+i)}(\mathcal{D}_L)_{(9n'+i),\alpha_2}\Big\},
\end{aligned}
$$

$$
\begin{aligned}
(\xi^R_{g_{(n)}})_{\alpha_1,\alpha_2} = \frac{4\sqrt{2\pi}}{kr\epsilon}\sum_{i=1}^{3}\int_\epsilon^1 dt\chi^g_{(++)}(y^{g(n)}_{(++)},t)\Big\{ & f^{R,c^i_T}_{(++)}(0,t)f^{R,c^i_T}_{(++)}(0,t)(\mathcal{D}^\dagger_R)_{\alpha_1,i}(\mathcal{D}_R)_{i,\alpha_2} \\
&+\sum_{n=1}^{\infty} f^{R,c^i_T}_{(++)}(0,t)f^{R,c^i_T}_{(++)}(y^{c^i_T(n)}_{(\mp\mp)},t)((\mathcal{D}^\dagger_R)_{\alpha_1,i}(\mathcal{D}_R)_{(9n+i),\alpha_2}+(\mathcal{D}^\dagger_R)_{\alpha_1,(9n+i)}(\mathcal{D}_R)_{i,\alpha_2}) \\
&+\sum_{n,n'=1}^{\infty} f^{R,c^i_T}_{(++)}(y^{c^i_T(n)}_{(\mp\mp)},t)f^{R,c^i_T}_{(++)}(y^{c^i_T(n')}_{(\mp\mp)},t)(\mathcal{D}^\dagger_R)_{\alpha_1,(9n+i)}(\mathcal{D}_R)_{(9n'+i),\alpha_2} \\
&+\sum_{n,n'=1}^{\infty} f^{R,c^i_B}_{(--)}(y^{c^i_B(n)}_{(\pm\pm)},t)f^{R,c^i_B}_{(--)}(y^{c^i_B(n')}_{(\pm\pm)},t)(\mathcal{D}^\dagger_R)_{\alpha_1,(9n-6+i)}(\mathcal{D}_R)_{(9n'-6+i),\alpha_2} \\
&+\sum_{n,n'=1}^{\infty} f^{R,c^i_T}_{(-+)}(y^{c^i_T(n)}_{(\pm\mp)},t)f^{R,c^i_T}_{(-+)}(y^{c^i_T(n')}_{(\pm\mp)},t)(\mathcal{D}^\dagger_R)_{\alpha_1,(9n-3+i)}(\mathcal{D}_R)_{(9n'-3+i),\alpha_2}\Big\},
\end{aligned}
$$

$$
\begin{aligned}
(\eta^L_{G^\pm})_{\alpha_1,\alpha_2} = \sum_{i,j=1}^{3}\Big\{ & (\mathcal{U}^\dagger_R)_{\alpha_1,i}f^{R,c^i_S}_{(++)}(0,1)Y^{u\dagger}_{ij}f^{L,c^j_B}_{(++)}(0,1)(\mathcal{D}_L)_{j,\alpha_2}+\sum_{n=1}^{\infty}(\mathcal{U}^\dagger_R)_{\alpha_1,(15n+i)}f^{R,c^i_S}_{(++)}(y^{c^i_S(n)}_{(\mp\mp)},1)Y^{u\dagger}_{ij}f^{L,c^j_B}_{(++)}(0,1)(\mathcal{D}_L)_{j,\alpha_2} \\
&+\sum_{n=1}^{\infty}(\mathcal{U}^\dagger_R)_{\alpha_1,i}f^{R,c^i_S}_{(++)}(0,1)Y^{u\dagger}_{ij}f^{L,c^j_B}_{(++)}(y^{c^j_B(n)}_{(\pm\pm)},1)(\mathcal{D}_L)_{(9n-6+j),\alpha_2} \\
&+\sum_{n,n'=1}^{\infty} f^{R,c^i_B}_{(--)}(y^{c^i_B(n)}_{(\pm\pm)},t)f^{R,c^i_B}_{(--)}(y^{c^i_B(n')}_{(\pm\pm)},t)(\mathcal{D}^\dagger_R)_{\alpha_1,(9n-6+i)}(\mathcal{D}_R)_{(9n'-6+i),\alpha_2} \\
&+\sum_{n,n'=1}^{\infty} f^{R,c^i_T}_{(-+)}(y^{c^i_T(n)}_{(\pm\mp)},t)f^{R,c^i_T}_{(-+)}(y^{c^i_T(n')}_{(\pm\mp)},t)(\mathcal{D}^\dagger_R)_{\alpha_1,(9n-3+i)}(\mathcal{D}_R)_{(9n'-3+i),\alpha_2}\Big\},
\end{aligned}
$$

$$
\begin{aligned}
(\eta^L_{G^\pm})_{\alpha_1,\alpha_2} = \sum_{i,j=1}^{3}\Big\{&(\mathcal{U}_R^\dagger)_{\alpha_1,i} f^{R,c_S^i}_{(++)}(0,1) Y^{u\dagger}_{ij} f^{L,c_B^j}_{(++)}(0,1)(\mathcal{D}_L)_{j,\alpha_2} \\
&+ \sum_{n=1}^{\infty}(\mathcal{U}_R^\dagger)_{\alpha_1,(15n+i)} f^{R,c_S^i}_{(++)}(y^{c_S^i(n)}_{(\mp\mp)},1) Y^{u\dagger}_{ij} f^{L,c_B^j}_{(++)}(0,1)(\mathcal{D}_L)_{j,\alpha_2} \\
&+ \sum_{n=1}^{\infty}(\mathcal{U}_R^\dagger)_{\alpha_1,i} f^{R,c_S^i}_{(++)}(0,1) Y^{u\dagger}_{ij} f^{L,c_B^j}_{(++)}(y^{c_B^j(n)}_{(\pm\pm)},1)(\mathcal{D}_L)_{(9n-6+j),\alpha_2} \\
&+ \sum_{n,n'=1}^{\infty}(\mathcal{U}_R^\dagger)_{\alpha_1,(15n+i)} f^{R,c_S^i}_{(++)}(y^{c_S^i(n)}_{(\mp\mp)},1) Y^{u\dagger}_{ij} f^{L,c_B^j}_{(++)}(y^{c_B^j(n')}_{(\pm\pm)},1)(\mathcal{D}_L)_{(9n'-6+j),\alpha_2} \\
&+ \sum_{n=1}^{\infty}(\mathcal{U}_R^\dagger)_{\alpha_1,(15n-9+i)} f^{R,c_T^i}_{(-+)}(y^{c_T^i(n)}_{(\mp\pm)},1) Y^{d\dagger}_{ij} f^{L,c_B^j}_{(++)}(0,1)(\mathcal{D}_L)_{j,\alpha_2} \\
&+ \sum_{n,n'=1}^{\infty}(\mathcal{U}_R^\dagger)_{\alpha_1,(15n-9+i)} f^{R,c_T^i}_{(-+)}(y^{c_T^i(n)}_{(\pm\mp)},1) Y^{d\dagger}_{ij} f^{L,c_B^j}_{(++)}(y^{c_B^j(n')}_{(\pm\pm)},1)(\mathcal{D}_L)_{(9n'-6+j),\alpha_2} \\
&+ \sum_{n=1}^{\infty}(\mathcal{U}_R^\dagger)_{\alpha_1,(15n-6+i)} f^{R,c_T^i}_{(-+)}(y^{c_T^i(n)}_{(\pm\mp)},1) Y^{d\dagger}_{ij} f^{L,c_B^j}_{(++)}(0,1)(\mathcal{D}_L)_{j,\alpha_2} \\
&+ \sum_{n,n'=1}^{\infty}(\mathcal{U}_R^\dagger)_{\alpha_1,(15n-6+i)} f^{R,c_T^i}_{(-+)}(y^{c_B^i(n)}_{(\pm\mp)},1) Y^{d\dagger}_{ij} f^{L,c_B^j}_{(++)}(y^{c_B^j(n')}_{(\pm\pm)},1)(\mathcal{D}_L)_{(9n'-6+j),\alpha_2}\Big\},
\end{aligned}
$$

$$
\begin{aligned}
(\eta^R_{G^\pm})_{\alpha_1,\alpha_2} = \sum_{i,j=1}^{3}\Big\{&(\mathcal{U}_L^\dagger)_{\alpha_1,i} f^{L,c_B^i}_{(++)}(0,1) Y^{d}_{ij} f^{R,c_T^j}_{(++)}(0,1)(\mathcal{D}_R)_{j,\alpha_2} \\
&+ \sum_{n=1}^{\infty}(\mathcal{U}_L^\dagger)_{\alpha_1,(15n-12+i)} f^{L,c_B^i}_{(++)}(y^{c_B^i(n)}_{(\pm\pm)},1) Y^{d}_{ij} f^{R,c_T^j}_{(++)}(0,1)(\mathcal{D}_R)_{j,\alpha_2} \\
&+ \sum_{n=1}^{\infty}(\mathcal{U}_L^\dagger)_{\alpha_1,i} f^{L,c_B^i}_{(++)}(0,1) Y^{d}_{ij} f^{R,c_T^j}_{(++)}(y^{c_T^j(n)}_{(\mp\mp)},1)(\mathcal{D}_R)_{(9n+j),\alpha_2} \\
&+ \sum_{n,n'=1}^{\infty}(\mathcal{U}_L^\dagger)_{\alpha_1,(15n-12+i)} f^{L,c_B^i}_{(++)}(y^{c_B^i(n)}_{(\pm\pm)},1) Y^{d}_{ij} f^{R,c_T^j}_{(++)}(y^{c_T^j(n')}_{(\mp\mp)},1)(\mathcal{D}_R)_{(9n'+j),\alpha_2} \\
&+ \sum_{n,n'=1}^{\infty}(\mathcal{U}_L^\dagger)_{\alpha_1,(15n-3+i)} f^{L,c_B^i}_{(-+)}(y^{c_B^i(n)}_{(\mp\pm)},1) Y^{d}_{ij} f^{R,c_T^j}_{(-+)}(y^{c_T^j(n')}_{(\pm\mp)},1)(\mathcal{D}_R)_{(9n'-3+j),\alpha_2}\Big\},
\end{aligned}
$$

$$
\begin{aligned}
(\eta^L_{H_0})_{\alpha_1,\alpha_2} = \sum_{i,j=1}^{3}\Big\{&-(\mathcal{D}_R^\dagger)_{\alpha_1,i} f^{R,c_T^i}_{(++)}(0,1) Y^{d\dagger}_{ij} f^{L,c_B^j}_{(++)}(0,1)(\mathcal{D}_L)_{j,\alpha_2} \\
&- \sum_{n=1}^{\infty}(\mathcal{D}_R^\dagger)_{\alpha_1,(9n+i)} f^{R,c_T^i}_{(++)}(y^{c_T^i(n)}_{(\mp\mp)},1) Y^{d\dagger}_{ij} f^{L,c_B^j}_{(++)}(0,1)(\mathcal{D}_L)_{j,\alpha_2} \\
&- \sum_{n=1}^{\infty}(\mathcal{D}_R^\dagger)_{\alpha_1,i} f^{R,c_T^i}_{(++)}(0,1) Y^{d\dagger}_{ij} f^{L,c_B^j}_{(++)}(y^{c_B^j(n)}_{(\pm\pm)},1)(\mathcal{D}_L)_{(9n-6+j),\alpha_2} \\
&- \sum_{n,n'=1}^{\infty}(\mathcal{D}_R^\dagger)_{\alpha_1,(9n+i)} f^{R,c_T^i}_{(++)}(y^{c_T^i(n)}_{(\mp\mp)},1) Y^{d\dagger}_{ij} f^{L,c_B^j}_{(++)}(y^{c_B^j(n')}_{(\pm\pm)},1)(\mathcal{D}_L)_{(9n'-6+j),\alpha_2} \\
&+ \sum_{n=1}^{\infty}(\mathcal{D}_R^\dagger)_{\alpha_1,(9n-3+i)} f^{R,c_T^i}_{(-+)}(y^{c_T^i(n)}_{(\pm\mp)},1) Y^{d\dagger}_{ij} f^{L,c_B^j}_{(++)}(0,1)(\mathcal{D}_L)_{j,\alpha_2} \\
&+ \sum_{n,n'=1}^{\infty}(\mathcal{D}_R^\dagger)_{\alpha_1,(9n-3+i)} f^{R,c_T^i}_{(-+)}(y^{c_T^i(n)}_{(\pm\mp)},1) Y^{d\dagger}_{ij} f^{L,c_B^j}_{(++)}(y^{c_B^j(n')}_{(\pm\pm)},1)(\mathcal{D}_L)_{(9n'-6+j),\alpha_2}\Big\},
\end{aligned}
$$

$$
\begin{aligned}
(\eta^R_{H_0})_{\alpha_1,\alpha_2} &= (\eta^L_{H_0})^\dagger_{\alpha_1,\alpha_2},\\
(\eta^L_{G_0})_{\alpha_1,\alpha_2} &= \sum_{i,j=1}^{3}\bigg\{(\mathcal{D}^\dagger_R)_{\alpha_1,i} f^{R,c^i_T}_{(++)}(0,1) Y^{d\dagger}_{ij} f^{L,c^j_B}_{(++)}(0,1)(\mathcal{D}_L)_{j,\alpha_2} + \sum_{n=1}^{\infty}(\mathcal{D}^\dagger_R)_{\alpha_1,(9n+i)} f^{R,c^i_T}_{(++)}(y^{c^i_T(n)}_{(\mp\mp)},1) Y^{d\dagger}_{ij} f^{L,c^j_B}_{(++)}(0,1)(\mathcal{D}_L)_{j,\alpha_2}\\
&\quad + \sum_{n=1}^{\infty}(\mathcal{D}^\dagger_R)_{\alpha_1,i} f^{R,c^i_T}_{(++)}(0,1) Y^{d\dagger}_{ij} f^{L,c^j_B}_{(++)}(y^{c^j_B(n)}_{(\pm\pm)},1)(\mathcal{D}_L)_{(9n-6+j),\alpha_2}\\
&\quad + \sum_{n,n'=1}^{\infty}(\mathcal{D}^\dagger_R)_{\alpha_1,(9n+i)} f^{R,c^i_T}_{(++)}(y^{c^i_T(n)}_{(\mp\mp)},1) Y^{d\dagger}_{ij} f^{L,c^j_B}_{(++)}(y^{c^j_B(n')}_{(\pm\pm)},1)(\mathcal{D}_L)_{(9n'-6+j),\alpha_2}\\
&\quad + \sum_{n=1}^{\infty}(\mathcal{D}^\dagger_R)_{\alpha_1,(9n-3+i)} f^{R,c^i_T}_{(-+)}(y^{c^i_T(n)}_{(\pm\mp)},1) Y^{d\dagger}_{ij} f^{L,c^j_B}_{(++)}(0,1)(\mathcal{D}_L)_{j,\alpha_2}\\
&\quad + \sum_{n,n'=1}^{\infty}(\mathcal{D}^\dagger_R)_{\alpha_1,(9n-3+i)} f^{R,c^i_T}_{(-+)}(y^{c^i_T(n)}_{(\pm\mp)},1) Y^{d\dagger}_{ij} f^{L,c^j_B}_{(++)}(y^{c^j_B(n')}_{(\pm\pm)},1)(\mathcal{D}_L)_{(9n'-6+j),\alpha_2}\bigg\},\\
(\eta^R_{G_0})_{\alpha_1,\alpha_2} &= -(\eta^L_{G_0})^\dagger_{\alpha_1,\alpha_2}.
\end{aligned}
\tag{B1}
$$

APPENDIX C: THE WILSON COEFFICIENTS OF NEUTRAL CURRENT-CURRENT OPERATORS TO THE ORDER $\mathcal{O}(v^2/\Lambda^2_{\rm KK})$

$$
\begin{aligned}
C^{LL}_{u,1}(\mu_{\rm EW}) &= \frac{32\pi m_Z^2 s_w^2 c_w^2 g_s^2}{\Lambda^2_{\rm KK} e^2 (kr\epsilon)^2}\sum_{i,j=1}^{3}(\mathcal{D}^{(0)}_L)^\dagger_{si}(\mathcal{D}^{(0)}_L)_{ib}(\mathcal{U}^{(0)}_L)^\dagger_{uj}(\mathcal{U}^{(0)}_L)_{ju}\int_\epsilon^1 dt\int_\epsilon^1 dt'[f^{L,c^i_B}_{(++)}(0,t)]^2[\Sigma^G_{(++)}(t,t')][f^{L,c^j_B}_{(++)}(0,t')]^2,\\
C^{LL}_{u,2}(\mu_{\rm EW}) &= -\frac{1}{2}\left(1-\frac{2}{3}s_w^2\right)\left(1-\frac{4}{3}s_w^2\right)\left[(\delta Z^d_L)_{sb}+(\delta Z^d_L)^\dagger_{sb}+\frac{v^2}{2\Lambda^2_{\rm KK}}(\Delta^L_Z)_{sb}\right]\\
&\quad + \frac{32\pi m_Z^2}{\Lambda^2_{\rm KK}(kr\epsilon)^2}\sum_{i,j=1}^{3}(\mathcal{D}^{(0)}_L)^\dagger_{si}(\mathcal{D}^{(0)}_L)_{ib}(\mathcal{U}^{(0)}_L)^\dagger_{uj}(\mathcal{U}^{(0)}_L)_{ju}\int_\epsilon^1 dt\int_\epsilon^1 dt'[f^{L,c^i_B}_{(++)}(0,t)]^2\\
&\quad \times\left\{-\left[\frac{(3-2s_w^2)(3-4s_w^2)}{18}+\frac{s_w^2c_w^2}{9}\left(4+\frac{3g_s^2}{e^2}\right)\right][\Sigma^G_{(++)}(t,t')]+\frac{(3-2s_w^2)^2}{18(1-2s_w^2)}[\Sigma^G_{(-+)}(t,t')]\right\}[f^{L,c^j_B}_{(++)}(0,t')]^2,\\
C^{LR}_{u,1}(\mu_{\rm EW}) &= \frac{32\pi m_Z^2 s_w^2 c_w^2 g_s^2}{\Lambda^2_{\rm KK} e^2 (kr\epsilon)^2}\sum_{i,j=1}^{3}(\mathcal{D}^{(0)}_L)^\dagger_{si}(\mathcal{D}^{(0)}_L)_{ib}(\mathcal{U}^{(0)}_R)^\dagger_{uj}(\mathcal{U}^{(0)}_R)_{ju}\int_\epsilon^1 dt\int_\epsilon^1 dt'[f^{L,c^i_B}_{(++)}(0,t)]^2[\Sigma^G_{(++)}(t,t')][f^{R,c^j_S}_{(++)}(0,t')]^2,\\
C^{LR}_{u,2}(\mu_{\rm EW}) &= \frac{2s_w^2}{3}\left(1-\frac{2}{3}s_w^2\right)\left[(\delta Z^d_L)_{sb}+(\delta Z^d_L)^\dagger_{sb}+\frac{v^2}{2\Lambda^2_{\rm KK}}(\Delta^L_Z)_{sb}\right]\\
&\quad + \frac{32\pi m_Z^2}{\Lambda^2_{\rm KK}(kr\epsilon)^2}\sum_{i,j=1}^{3}(\mathcal{D}^{(0)}_L)^\dagger_{si}(\mathcal{D}^{(0)}_L)_{ib}(\mathcal{U}^{(0)}_R)^\dagger_{uj}(\mathcal{U}^{(0)}_R)_{ju}\int_\epsilon^1 dt\int_\epsilon^1 dt'[f^{L,c^i_B}_{(++)}(0,t)]^2\\
&\quad \times\left\{\left[\frac{2s_w^2(3-2s_w^2)}{9}-\frac{s_w^2c_w^2}{9}\left(4+\frac{3g_s^2}{e^2}\right)\right][\Sigma^G_{(++)}(t,t')]+\frac{2s_w^2(3-2s_w^2)}{9(1-2s_w^2)}[\Sigma^G_{(-+)}(t,t')]\right\}[f^{R,c^j_S}_{(++)}(0,t')]^2,\\
C^{RL}_{u,1}(\mu_{\rm EW}) &= \frac{32\pi m_Z^2 s_w^2 c_w^2 g_s^2}{\Lambda^2_{\rm KK} e^2 (kr\epsilon)^2}\sum_{i,j=1}^{3}(\mathcal{D}^{(0)}_R)^\dagger_{si}(\mathcal{D}^{(0)}_R)_{ib}(\mathcal{U}^{(0)}_L)^\dagger_{uj}(\mathcal{U}^{(0)}_L)_{ju}\int_\epsilon^1 dt\int_\epsilon^1 dt'[f^{R,c^i_T}_{(++)}(0,t)]^2[\Sigma^G_{(++)}(t,t')][f^{L,c^j_B}_{(++)}(0,t')]^2,
\end{aligned}
$$

$$C^{RL}_{u,2}(\mu_{\rm EW}) = \frac{s_w^2}{3}\left(1-\frac{4}{3}s_w^2\right)\left[(\delta Z_R^d)_{sb} + (\delta Z_R^d)^\dagger_{sb} + \frac{v^2}{2\Lambda_{\rm KK}^2}(\Delta_Z^R)_{sb}\right] + \frac{32\pi m_Z^2}{\Lambda_{\rm KK}^2(kr\epsilon)^2}\sum_{i,j=1}^3(\mathcal{D}_R^{(0)})^\dagger_{si}(\mathcal{D}_R^{(0)})_{ib}(\mathcal{U}_L^{(0)})^\dagger_{uj}(\mathcal{U}_L^{(0)})_{ju} \times \int_\epsilon^1 dt\int_\epsilon^1 dt'[f^{R,c_T^i}_{(++)}(0,t)]^2\left\{\left[\frac{s_w^2(3-4s_w^2)}{9} - \frac{s_w^2c_w^2}{9}\left(4+\frac{3g_s^2}{e^2}\right)\right][\Sigma^G_{(++)}(t,t')] - \frac{(3-2s_w^2)(3-4s_w^2)}{18(1-2s_w^2)}[\Sigma^G_{(-+)}(t,t')]\right\}[f^{L,c_B^j}_{(++)}(0,t')]^2,$$

$$C^{RR}_{u,1}(\mu_{\rm EW}) = \frac{32\pi m_Z^2 s_w^2c_w^2g_s^2}{\Lambda_{\rm KK}^2e^2(kr\epsilon)^2}\sum_{i,j=1}^3(\mathcal{D}_R^{(0)})^\dagger_{si}(\mathcal{D}_R^{(0)})_{ib}(\mathcal{U}_R^{(0)})^\dagger_{uj}(\mathcal{U}_R^{(0)})_{ju}\int_\epsilon^1 dt\int_\epsilon^1 dt'[f^{R,c_T^i}_{(++)}(0,t)]^2[\Sigma^G_{(++)}(t,t')][f^{R,c_S^j}_{(++)}(0,t')]^2,$$

$$C^{RR}_{u,2}(\mu_{\rm EW}) = -\frac{4s_w^4}{9}\left[(\delta Z_R^d)_{sb} + (\delta Z_R^d)^\dagger_{sb} + \frac{v^2}{2\Lambda_{\rm KK}^2}(\Delta_Z^R)_{sb}\right] - \frac{32\pi m_Z^2}{\Lambda_{\rm KK}^2(kr\epsilon)^2}\sum_{i,j=1}^3(\mathcal{D}_R^{(0)})^\dagger_{si}(\mathcal{D}_R^{(0)})_{ib}(\mathcal{U}_R^{(0)})^\dagger_{uj}(\mathcal{U}_R^{(0)})_{ju} \times \int_\epsilon^1 dt\int_\epsilon^1 dt'[f^{R,c_T^i}_{(++)}(0,t)]^2\left\{\left[\frac{4s_w^4}{9} + \frac{s_w^2c_w^2}{9}\left(4+\frac{3g_s^2}{e^2}\right)\right][\Sigma^G_{(++)}(t,t')] + \frac{2s_w^2(3-4s_w^2)}{9(1-2s_w^2)}[\Sigma^G_{(-+)}(t,t')]\right\}[f^{R,c_S^j}_{(++)}(0,t')]^2,$$

$$C^{LL}_{d,1}(\mu_{\rm EW}) = \frac{32\pi m_Z^2 s_w^2c_w^2g_s^2}{\Lambda_{\rm KK}^2e^2(kr\epsilon)^2}\sum_{i,j=1}^3(\mathcal{D}_L^{(0)})^\dagger_{si}(\mathcal{D}_L^{(0)})_{ib}(\mathcal{D}_L^{(0)})^\dagger_{dj}(\mathcal{D}_L^{(0)})_{jd}\int_\epsilon^1 dt\int_\epsilon^1 dt'[f^{L,c_B^i}_{(++)}(0,t)]^2[\Sigma^G_{(++)}(t,t')][f^{L,c_B^j}_{(++)}(0,t')]^2,$$

$$C^{LL}_{d,2}(\mu_{\rm EW}) = \frac{1}{2}\left(1-\frac{2}{3}s_w^2\right)^2\left[(\delta Z_L^d)_{sb} + (\delta Z_L^d)^\dagger_{sb} + \frac{v^2}{2\Lambda_{\rm KK}^2}(\Delta_Z^L)_{sb}\right] + \frac{32\pi m_Z^2}{\Lambda_{\rm KK}^2(kr\epsilon)^2}\sum_{i,j=1}^3(\mathcal{D}_L^{(0)})^\dagger_{si}(\mathcal{D}_L^{(0)})_{ib}(\mathcal{D}_L^{(0)})^\dagger_{dj}(\mathcal{D}_L^{(0)})_{jd} \times \int_\epsilon^1 dt\int_\epsilon^1 dt'[f^{L,c_B^i}_{(++)}(0,t)]^2\left\{\left[\frac{(3-2s_w^2)^2}{18} + \frac{s_w^2c_w^2}{9}\left(2-\frac{3g_s^2}{e^2}\right)\right][\Sigma^G_{(++)}(t,t')] + \frac{(3-2s_w^2)^2}{18(1-2s_w^2)}[\Sigma^G_{(-+)}(t,t')]\right\}[f^{L,c_B^j}_{(++)}(0,t')]^2,$$

$$C^{LR}_{d,1}(\mu_{\rm EW}) = \frac{32\pi m_Z^2 s_w^2c_w^2g_s^2}{\Lambda_{\rm KK}^2e^2(kr\epsilon)^2}\sum_{i,j=1}^3(\mathcal{D}_L^{(0)})^\dagger_{si}(\mathcal{D}_L^{(0)})_{ib}(\mathcal{D}_R^{(0)})^\dagger_{dj}(\mathcal{D}_R^{(0)})_{jd}\int_\epsilon^1 dt\int_\epsilon^1 dt'[f^{L,c_B^i}_{(++)}(0,t)]^2[\Sigma^G_{(++)}(t,t')][f^{R,c_T^j}_{(++)}(0,t')]^2,$$

$$C^{LR}_{d,2}(\mu_{\rm EW}) = -\frac{s_w^2}{3}\left(1-\frac{2}{3}s_w^2\right)\left[(\delta Z_L^d)_{sb} + (\delta Z_L^d)^\dagger_{sb} + \frac{v^2}{2\Lambda_{\rm KK}^2}(\Delta_Z^L)_{sb}\right] + \frac{32\pi m_Z^2}{\Lambda_{\rm KK}^2(kr\epsilon)^2}\sum_{i,j=1}^3(\mathcal{D}_L^{(0)})^\dagger_{si}(\mathcal{D}_L^{(0)})_{ib}(\mathcal{D}_R^{(0)})^\dagger_{dj}(\mathcal{D}_R^{(0)})_{jd} \times \int_\epsilon^1 dt\int_\epsilon^1 dt'[f^{L,c_B^i}_{(++)}(0,t)]^2\left\{\left[-\frac{s_w^2(3-2s_w^2)}{9} + \frac{s_w^2c_w^2}{9}\left(2-\frac{3g_s^2}{e^2}\right)\right][\Sigma^G_{(++)}(t,t')] - \frac{(3-2s_w^2)(3-4s_w^2)}{18(1-2s_w^2)}[\Sigma^G_{(-+)}(t,t')]\right\}\left[f^{R,c_T^j}_{(++)}(0,t')\right]^2,$$

$$C^{RL}_{d,1}(\mu_{\rm EW}) = \frac{32\pi m_Z^2 s_w^2c_w^2g_s^2}{\Lambda_{\rm KK}^2e^2(kr\epsilon)^2}\sum_{i,j=1}^3(\mathcal{D}_R^{(0)})^\dagger_{si}(\mathcal{D}_R^{(0)})_{ib}(\mathcal{D}_L^{(0)})^\dagger_{dj}(\mathcal{D}_L^{(0)})_{jd}\int_\epsilon^1 dt\int_\epsilon^1 dt'[f^{R,c_T^i}_{(++)}(0,t)]^2[\Sigma^G_{(++)}(t,t')]\left[f^{L,c_B^j}_{(++)}(0,t')\right]^2,$$

$$C^{RL}_{d,2}(\mu_{\rm EW}) = -\frac{s_w^2}{3}\left(1-\frac{2}{3}s_w^2\right)\left[(\delta Z_R^d)_{sb} + (\delta Z_R^d)^\dagger_{sb} + \frac{v^2}{2\Lambda_{\rm KK}^2}(\Delta_Z^R)_{sb}\right] + \frac{32\pi m_Z^2}{\Lambda_{\rm KK}^2(kr\epsilon)^2}\sum_{i,j=1}^{3}(\mathcal{D}_R^{(0)})^\dagger_{si}(\mathcal{D}_R^{(0)})_{ib}(\mathcal{D}_L^{(0)})^\dagger_{dj}(\mathcal{D}_L^{(0)})_{jd}$$
$$\times\int_\epsilon^1 dt\int_\epsilon^1 dt'[f^{R,c_T^i}_{(++)}(0,t)]^2\left\{\left[-\frac{s_w^2(3-2s_w^2)}{9}+\frac{s_w^2c_w^2}{9}\left(2-\frac{3g_s^2}{e^2}\right)\right][\Sigma^G_{(++)}(t,t')]\right.$$
$$\left.-\frac{(3-2s_w^2)(3-4s_w^2)}{18(1-2s_w^2)}[\Sigma^G_{(-+)}(t,t')]\right\}[f^{L,c_B^j}_{(++)}(0,t')]^2,$$

$$C^{RR}_{d,1}(\mu_{\rm EW}) = \frac{32\pi m_Z^2 s_w^2 c_w^2 g_s^2}{\Lambda_{\rm KK}^2 e^2 (kr\epsilon)^2}\sum_{i,j=1}^{3}(\mathcal{D}_R^{(0)})^\dagger_{si}(\mathcal{D}_R^{(0)})_{ib}(\mathcal{D}_R^{(0)})^\dagger_{dj}(\mathcal{D}_R^{(0)})_{jd}\int_\epsilon^1 dt\int_\epsilon^1 dt'[f^{R,c_T^i}_{(++)}(0,t)]^2[\Sigma^G_{(++)}(t,t')][f^{R,c_T^j}_{(++)}(0,t')]^2,$$

$$C^{RR}_{d,2}(\mu_W) = \frac{2s_w^4}{9}\left[(\delta Z_R^d)_{sb} + (\delta Z_R^d)^\dagger_{sb} + \frac{v^2}{2\Lambda_{\rm KK}^2}(\Delta_Z^R)_{sb}\right] + \frac{32\pi m_Z^2}{\Lambda_{\rm KK}^2(kr\epsilon)^2}\sum_{i,j=1}^{3}(\mathcal{D}_R^{(0)})^\dagger_{si}(\mathcal{D}_R^{(0)})_{ib}(\mathcal{D}_R^{(0)})^\dagger_{dj}(\mathcal{D}_R^{(0)})_{jd}$$
$$\times\int_\epsilon^1 dt\int_\epsilon^1 dt'[f^{R,c_T^i}_{(++)}(0,t)]^2\left\{\left[\frac{2s_w^4)}{9}+\frac{s_w^2c_w^2}{9}\left(2-\frac{3g_s^2}{e^2}\right)\right][\Sigma^G_{(++)}(t,t')]\right.$$
$$\left.-\frac{(3-4s_w^2)^2}{18(1-2s_w^2)}[\Sigma^G_{(-+)}(t,t')]\right\}\left[f^{R,c_T^j}_{(++)}(0,t')\right]^2. \tag{C1}$$

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