Some results on vector and tensor meson mixing in a generalized QCD-like theory

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We present results for the mixing of quark flavor eigenstates to form low-lying vector and tensor meson mass eigenstates in a generalized QCD-like theory with ℓ massless or light, degenerate quarks and one quark of substantial mass, m_Q. We show that as m_Q/Λ gets large, where Λ is the confinement scale, the angle governing this mixing approaches $\arctan(1/\sqrt{\ell})$. We give a generalization of the Gell-Mann Okubo mass relation for this theory. A comparison of the $\ell = 2$ results with data on ω-ϕ and $f_2(1275)$-$f_2'(1525)$ mixing is included.

I. INTRODUCTION

The mixing of the flavor-SU(3) singlet and octet states of vector and tensor mesons to form mass eigenstates is of fundamental importance in hadronic physics. In the case of the vector mesons, with $J^{PC} = 1^{--}$, the flavor-SU(3) singlet, ω_0 and octet, ω_8, mix through an angle $\theta_V \simeq 39°$ [1], close to the value that would lead to the complete decoupling of the light u and d quarks from the heavier s quark in the resultant mass eigenstates ω and ϕ. Throughout the years there have been many studies of this mixing [2–7]. In a modern context, some insight into this comes from the Appelquist-Carazzone decoupling theorem [8], according to which, in a vectorial theory, as the mass of a particle gets large compared with a relevant scale—here the scale of confinement and spontaneous chiral symmetry breaking (SχSB) in quantum chromodynamics (QCD), $\Lambda_{\mathrm{QCD}} \simeq 300$ MeV—one can integrate this particle out and define a low-energy effective field theory applicable below this scale. Evidently, even though m_s is not $\gg \Lambda_{\mathrm{QCD}}$, there is still a nearly complete decoupling, so that the physical ω is mostly comprised of the isospin-singlet combination $(u\bar{u} + d\bar{d})/\sqrt{2}$, while ϕ is mostly an $s\bar{s}$ state. The $s\bar{s}$ constituency of the ϕ is inferred from the fact that ϕ has a dominant (> 80%) branching ratio to K^+K^-, $K_L^0 K_S^0$, despite the fact that there is very little phase space for these decay modes, much less than the phase space available for $\phi \to \rho\pi$ and $\phi \to 3\pi$ decay modes. This motivated the Okubo-Zweig-Iizuka (OZI) rule [4]. A similar situation of near-ideal mixing occurs for the $J^{PC} = 2^{++}$ tensor mesons $f_2(1275)$-$f_2'(1525)$, and can also be understood in terms of approximate decoupling of the light $u\bar{u} + d\bar{d}$ state from the heavier $s\bar{s}$ state.

In this paper we discuss the mixing of self-conjugate mesons in a generalized QCD-like theory with emphasis on the role of decoupling. Specifically, we analyze the mixing of quark flavor eigenstates to form the lowest-lying mass eigenstates of vector and tensor mesons in a vectorial, asymptotically free, confining SU(N_c) gauge theory, where $N_c \geq 3$. We denote the scale of confinement and

spontaneous chiral symmetry breaking in this theory as Λ. The fermion content consists of N_f fermions transforming according to the fundamental representation of SU(N_c), i.e., the quarks, of which

$$\ell = N_f - 1 \tag{1.1}$$

are massless (or of small, equal mass $m_q \ll \Lambda$) and one has a larger mass, $m_Q > m_q$. We presume that electroweak interactions have been turned off, so that only color SU(N_c) interactions are present. We define the ratio

$$r_Q = \frac{m_Q}{\Lambda}. \tag{1.2}$$

We first derive the decoupling value of the mixing angle and then discuss how the mixing angle varies as a function of m_Q/Λ. For the special case $N_c = 3$ and $\ell = 2$, the framework for our analysis is a rough approximation to QCD with the heavy quarks, c, b, and t integrated out, since, the hard masses $m_{u,d} \ll \Lambda_{\mathrm{QCD}}$ [9–11], $m_s \sim 100$ MeV is smaller than, but comparable to Λ_{QCD}, and electroweak interactions are a small perturbation. Accordingly, we remark on the connection of our results with the mixing of the vector mesons ω and ϕ, and the tensor mesons $f_2(1275)$ and $f_2'(1525)$. We also derive a generalization of the Gell-Mann Okubo mass relation for this theory. Our work thus shows how these mixings and this relation can be understood in a more general theoretical framework.

In general, when analyzing flavor-singlet states, it is not just the quark flavor-eigenstates that one must include in the initial basis of states, but also pure gluonic states with the same J^{PC}. However, the masses of the lowest-lying glueballs with the relevant J^{PC} values, as determined by lattice measurements, are well above the masses of the relevant quark-antiquark states. Specifically, these lattice measurements yield (i) a value of ~4 GeV for the lowest-lying glueball with $J^{PC} = 1^{--}$, considerably greater than the masses of the ω and ϕ (and also above the mass of the J/ψ and ψ'); and (ii) a mass ~2.5 GeV for the lowest-lying glueball with $J^{PC} = 2^{++}$, which is substantially

larger than the masses of the $f_2(1275)$ and $f_2'(1525)$ tensor mesons [12]. Therefore, it should be a reasonable approximation to neglect mixing of quark-antiquark flavor eigenstates with these gluonic states to form the lowest-lying meson mass eigenstates. In contrast, as is well known, for the $J^{PC} = 0^{++}$ scalar mesons in the mass interval 1–2 GeV, the mixing of quark-antiquark and glue states is important.

II. MESON MIXING IN A GENERALIZED QCD-LIKE THEORY

In this section we discuss the mixing of vector and tensor meson flavor eigenstates to form the lowest-lying mass eigenstates in a generalized QCD-like $SU(N_c)$ gauge theory. We denote the ℓ quarks with zero mass or a common light mass $m_q \ll \Lambda$ as q_i, $i = 1, \ldots, \ell$, and the one with the larger mass, m_Q, as $q_{N_f} \equiv Q$. We will usually assume that $\ell \geq 2$, since for $\ell = 1$ the theory has no nontrivial isospin symmetry. It is assumed that N_f is sufficiently small that the theory is not only asymptotically free, but confines and spontaneously breaks chiral symmetry [13]. If the ℓ quarks are massless, this theory has a (nonanomalous) global flavor symmetry $G_{f,\ell} = SU(\ell)_L \otimes SU(\ell)_R \otimes U(1)_V$. (If the ℓ quarks have nonzero masses $m_q \ll \Lambda$, then the chiral part of this symmetry is approximate.) The condensates of the massless quarks, $\langle \bar{q}_i q_i \rangle$, where $\ell = 1, \ldots, \ell$, preserve the $U(1)_V$ and, since we neglect the small electromagnetic interactions [14], they are all equal. Hence, they spontaneously break the chiral part of $G_{f,\ell}$ to its diagonal, vectorial (isospin) subgroup, $SU(\ell)_V$. This residual $SU(\ell)_V$ isospin symmetry (as well as the $U(1)_V$ symmetry) is also present if the ℓ quarks have nonzero degenerate masses. The subscript V will henceforth be dropped when the meaning is clear. In the absence of an underlying dynamical mechanism to render the ℓ light quarks degenerate, one may regard the condition of degeneracy at a nonzero value of mass as artificial; the reason that we include it here is that in our application to QCD, we will make contact with quark model results, and these often take the u and d quarks to have a common effective mass m_q.

Now we treat m_Q as a variable, and start with it equal to its lower limit, $m_Q = m_q$. In this limit, the global chiral symmetry is $SU(N_f)_L \otimes SU(N_f)_R$, which is broken to $SU(N_f)_V$. The natural basis of states for $m_Q = m_q$ is the basis for $SU(N_f)_V$, with the (diagonal) generators of the $(N_f - 1)$-dimensional Cartan algebra of $SU(N_f)$ given by the $N_f \times N_f$ matrices

$$T_a, \qquad a = k^2 - 1 \quad \text{for } k = 1, \ldots, N_f. \tag{2.1}$$

For example, $T_3 = (1/2)\,\mathrm{diag}(1, -1, 0, \ldots, 0)$, $T_8 = (1/\sqrt{12})\,\mathrm{diag}(1, 1, -2, 0, \ldots, 0)$, etc. The corresponding neutral vector meson states are

$$|V_a\rangle = \sqrt{2}\,\bigg| \sum_{i,j=1}^{N_f} q_i(T^a)_{ij}\bar{q}_j \bigg\rangle. \tag{2.2}$$

Explicitly, with $\ell = N_f - 1$,

$$|V_a\rangle \equiv |V_{N_f^2-1}\rangle = \bigg| \frac{(\sum_{i=1}^{\ell} q_i\bar{q}_i) - \ell Q\bar{Q}}{\sqrt{\ell(\ell+1)}} \bigg\rangle. \tag{2.3}$$

We also define the flavor-$SU(n)$-singlet meson state

$$|(V_0)_n\rangle = \frac{1}{\sqrt{n}}\,\bigg| \sum_{i=1}^{n} q_i\bar{q}_i \bigg\rangle, \tag{2.4}$$

which will be relevant for $n = \ell$ and $n = N_f$.

We now increase m_Q from m_q to larger values. This breaks the $SU(N_f)$ isospin symmetry to $SU(\ell)$. For $m_q = 0$, the size of this breaking is determined by the ratio r_Q. For the case of $m_q \neq 0$, there are two relevant measures of this breaking, r_Q and m_Q/m_q, but if $m_q \ll \Lambda$, then one expects that the effect is primarily measured by r_Q. Of the N_f^2 vector mesons, ℓ^2 transform according to the irreducible representations of $SU(\ell)$ that occur in the Clebsch-Gordan decomposition $\square_\ell \times \bar{\square}_\ell = (\mathrm{adj})_\ell + 1_\ell$, where $\square_\ell$, adj_ℓ, and 1_ℓ denote the fundamental, adjoint and singlet of $SU(\ell)$. In the $SU(N_f)$ basis, the mixing of states must respect the $SU(\ell)$ symmetry. This only allows the mixing of the states corresponding to flavor-diagonal bilinears whose $SU(\ell)$ parts are proportional to $(V_0)_\ell$, i.e., $SU(\ell)$-singlets, and, in the quark-antiquark sector, there are only two such states, namely $|(V_0)_{N_f}\rangle$ and $|V_{N_f^2-1}\rangle$. Pure gluonic states are also $SU(\ell)$-singlets and can, in general, mix with these $SU(\ell)$-singlet quark-antiquark flavor eigenstates. However, as noted above, the masses of these pure gluonic states are sufficiently large that it should be reasonable to neglect them in our analysis. Accordingly, we focus on the mixing of the $|(V_0)_{N_f}\rangle$ and $|V_{N_f^2-1}\rangle$ states. We label the two resultant mass eigenstates as $|V_L\rangle$ and $|V_H\rangle$, with masses m_L and m_H, respectively, where the subscripts L and H stand for lighter and heavier.

We next discuss this diagonalization in greater detail. We analyze the mass-squared matrix M^2 in the basis of flavor eigenstates $(|(V_0)_{N_f}\rangle, |V_{N_f^2-1}\rangle)$ [15]. In general, one should take into account the fact that the particles involved are unstable, with strong decay widths that are substantial fractions of their masses. However, one may obtain an approximate description of the physics by neglecting the widths. In this case we can write M^2 as the real, symmetric matrix

$$M^2 = \begin{pmatrix} m_1^2 & \delta \\ \delta & m_a^2 \end{pmatrix}, \tag{2.5}$$

where the subscripts 1 and a refer to flavor $SU(N_f)$ singlet and adjoint. One can work backward from the observed masses and mixing angle to determine δ. Note that $|\delta|$ is bounded as to

$$|\delta| < m_1 m_a, \qquad (2.6)$$

since a value of $|\delta|$ as large as $m_1 m_a$ would have the consequence that the determinant $\det(M^2) = 0$ and hence that the lighter vector meson mass, m_L, would vanish. Indeed, since m_L is of order $2\Lambda_{\text{QCD}}$, it follows that $|\delta|$ is considerably smaller than this upper bound, and the explicit results satisfy this.

The matrix M^2 is diagonalized according to

$$R(\theta_V)M^2 R(\theta_V)^{-1} = M^2_{\text{diag.}} \qquad (2.7)$$

where

$$M^2_{\text{diag.}} = \begin{pmatrix} m_L^2 & 0 \\ 0 & m_H^2 \end{pmatrix} \qquad (2.8)$$

with the eigenvalues of M^2 given by

$$m^2_{H,L} = \frac{1}{2}[m_a^2 + m_1^2 \pm \sqrt{(m_a^2 - m_1^2)^2 + 4\delta^2}]. \qquad (2.9)$$

In Eq. (2.7), the orthogonal rotation matrix $R(\theta_V)$ is

$$R(\theta_V) = \begin{pmatrix} \cos\theta_V & \sin\theta_V \\ -\sin\theta_V & \cos\theta_V \end{pmatrix}, \qquad (2.10)$$

where

$$\tan(2\theta_V) = \frac{2\delta}{m_1^2 - m_a^2}. \qquad (2.11)$$

Note that this equation only determines θ_V up to a shift by $\pm\pi$. The masses m_L and m_H satisfy the trace and determinant relations

$$m_L^2 + m_H^2 = \text{Tr}(M^2) = m_1^2 + m_a^2 \qquad (2.12)$$

and

$$m_L^2 m_H^2 = \det(M^2) = m_1^2 m_a^2 - \delta^2. \qquad (2.13)$$

The flavor eigenstates are mapped to mass eigenstates according to

$$\begin{pmatrix} |V_L\rangle \\ |V_H\rangle \end{pmatrix} = R(\theta_V) \begin{pmatrix} |(V_0)_{N_f}\rangle \\ |V_{N_f^2-1}\rangle \end{pmatrix}. \qquad (2.14)$$

We first comment on the limit where $m_Q = m_q$, so that the SU(ℓ) flavor (generalized isospin) symmetry is increased to an SU$(\ell + 1) =$ SU(N_f) flavor symmetry. First, the SU(N_f) symmetry implies that $\delta = 0$, since the (1,2) and (2,1) elements of M^2 connect two different representations of SU(N_f) (the singlet and adjoint). The SU(N_f) flavor symmetry requires that all of the members of a given representation of SU(N_f) must be degenerate. Although it does not, by itself, imply that members of different SU(N_f) representations are degenerate, this is observed empirically to be approximately true in QCD. For example, the ρ and ω, which, to very good accuracy, are adjoint and singlet states with respect to isospin SU(2),

have nearly equal masses, with a fractional difference $(m_\omega - m_\rho)/[(1/2)(m_\omega + m_\rho)] \simeq 0.01$. In our generalized context, we may thus anticipate that the SU(N_f)-singlet and adjoint states $|(V_0)_{N_f}\rangle$ and $|V_{N_f^2-1}\rangle$ are approximately degenerate. Indeed, let us consider the case where these states are exactly degenerate, i.e., $m_1 = m_a$. In this case, any value of θ_V in Eq. (2.14) yields a corresponding set of degenerate mass eigenstates. This is reflected in the property that the right-hand side of Eq. (2.11) is proportional to the indeterminate ratio $0/0$.

One may also formally set $\delta = 0$ without setting $m_q = m_Q$ or $m_1 = m_a$. The $|(V_0)_{N_f}\rangle$ and $|V_{N_f^2-1}\rangle$ are then mass eigenstates, with masses $m((V_0)_{N_f}) = m_1$ and $m(V_{N_f^2-1}) = m_a$. The $|(V_0)_{N_f}\rangle$ state has a probability $1/N_f$ of being in one of the $|q_i\bar{q}_i\rangle$ states, for $i = 1, \ldots, \ell$, and hence a probability $\ell/N_f = \ell/(\ell + 1)$ of being in some $|q_i\bar{q}_i\rangle$ state, and a probability $1/N_f$ of being in the $|Q\bar{Q}\rangle$ state. In contrast, the $|V_{N_f^2-1}\rangle$ state has a probability $1/[\ell(\ell + 1)]$ of being in one of the $|q_i\bar{q}_i\rangle$ states, for a total probability of $1/N_f$ of being in some $|q_i\bar{q}_i\rangle$ state, and a probability $\ell/(\ell + 1) = \ell/N_f$ of being in the $|Q\bar{Q}\rangle$ state. The pair $Q\bar{Q}$ comprised of the heavier quark, thus enters the wave function for the $|V_{N_f^2-1}\rangle$ state with a probability ℓ/N_f which, for the $\ell \geq 2$ case under consideration here, is greater than the probability $1/N_f$ with which it enters into the wave function for the $|(V_0)_{N_f}\rangle$ state. If these considerations formed a complete analysis, they would imply that for $\ell \geq 2$, and $m_Q > m_q$, the $|V_{N_f^2-1}\rangle$ state would be heavier than the $|(V_0)_{N_f}\rangle$ state, i.e., $m_a > m_1$. This argument is borne out by the physical mass ordering in Eq. (4.14) below (and for tensor mesons, in Eq. (5.1)). We also mention an approach in which these mesons are modeled as dynamical fields in an effective Lagrangian, with both kinetic and mass terms. Although one cannot do perturbative calculations reliably because of the strongly coupled nature of the physics, one may note that there would be a particular kind of correction to the propagators of the $|(V_0)_{N_f}\rangle$ state, in which it makes a transition to a glueball state of the same J^{PC} and back again [6]. Insofar as one can use perturbative degrees of freedom to describe this transition, for a $J^{PC} = 1^{--}$ vector meson it would involve a graph in which the quark-antiquark pair annihilates into three or, more generally, an odd number of gluons (from C invariance) and then these create the quark-antiquark final state $|(V_0)_{N_f}\rangle$. For a $J^{PC} = 2^{++}$ tensor meson, the intermediate gluonic state would consist of an even number of gluons. Since our QCD-like theory is asymptotically free, as in QCD, the larger m_Q is, the smaller the gauge couplings multiplying the vertices involving gluon emission from the Q line [6]. This type of propagator correction would not occur for the $|V_{N_f^2-1}\rangle$, since it is not a flavor-singlet. As mentioned above, the

lowest-lying glueball state with $J^{PC} = 1^{--}$ has a mass considerably larger than the ω and ϕ masses, so it seems reasonable to neglect this propagator correction. With our ordering of the flavor eigenbasis as $(|(V_0)_{N_f}\rangle, |V_{N_f^2-1}\rangle)$ and the mass eigenbasis as $(|V_L\rangle, |V_H\rangle)$ and with $\delta = 0$, it follows that if $m_a > m_1$, then $\theta_V = 0$ and if $m_1 > m_a$, then $\theta = \pi/2$. (In each case, the other value of θ is excluded because it would map the lighter flavor state to $|V_H\rangle$ and the heavier flavor state to $|V_L\rangle$.)

We now proceed to the general case of nonzero δ. First, we observe that from Eq. (2.11) it follows that if $m_a > m_1$ ($m_1 < m_a$), then the sign of $\tan(2\theta_V)$ is opposite to (the same as) the sign of δ. The value of θ_V that produces exact decoupling (*dec.*) of the ℓ massless quarks q_i, with $i = 1, \ldots, \ell$ from the one heavier quark, Q, is determined by the condition that the mass eigenstates are $|V_L\rangle = |(V_0)_\ell\rangle$ and $|V_H\rangle = -|V_{Q\bar{Q}}\rangle$ (where the minus sign is included for convenience and has no physical significance). Imposing this decoupling condition on the transformation (2.14), we derive the decoupling value of the mixing angle,

$$\theta_{V,\text{dec.}} = \arctan\left(\frac{1}{\sqrt{\ell}}\right). \quad (2.15)$$

In this situation of exact decoupling, the $SU(N_f)_V$ eigenstates $|(V_0)_{N_f}\rangle$ and $|V_{N_f^2-1}\rangle$ are mapped to the mass eigenstates $|V_L\rangle = |(V_0)_\ell\rangle$ and $|V_H\rangle = -|V_{Q\bar{Q}}\rangle$ according to

$$\begin{pmatrix} |V_L\rangle \\ |V_H\rangle \end{pmatrix} = \begin{pmatrix} |(V_0)_\ell\rangle \\ -|V_{Q\bar{Q}}\rangle \end{pmatrix} = R(\theta_{V,\text{dec.}}) \begin{pmatrix} |(V_0)_{N_f}\rangle \\ |V_{N_f^2-1}\rangle \end{pmatrix}. \quad (2.16)$$

For the decoupling value $\theta_V = \theta_{V,\text{dec.}}$, we have

$$R(\theta_{V,\text{dec.}}) = \begin{pmatrix} \sqrt{\frac{\ell}{\ell+1}} & \frac{1}{\sqrt{\ell+1}} \\ -\frac{1}{\sqrt{\ell+1}} & \sqrt{\frac{\ell}{\ell+1}} \end{pmatrix}. \quad (2.17)$$

In the decoupling limit, the mass-squared matrix M^2 in the basis of flavor eigenstates $|(V_0)_{N_f}\rangle$ and $|V_{N_f^2-1}\rangle$ can be expressed as

$$\begin{aligned} (M^2)_{\text{dec.}} &= R(\theta_{\text{dec.}})^{-1} M^2_{\text{diag.}} R(\theta_{\text{dec.}}) \\ &= \begin{pmatrix} \frac{\ell m_L^2 + m_H^2}{\ell+1} & -\frac{\sqrt{\ell}(m_H^2 - m_L^2)}{\ell+1} \\ -\frac{\sqrt{\ell}(m_H^2 - m_L^2)}{\ell+1} & \frac{m_L^2 + \ell m_H^2}{\ell+1} \end{pmatrix}. \end{aligned} \quad (2.18)$$

Equating the (1, 1) and (2, 2) elements of $(M^2)_{\text{dec.}}$ in Eq. (2.18) to the corresponding elements of the general expression for M^2 in Eq. (2.5), we obtain relations for m_L^2 and m_H^2 in terms of m_1^2 and m_a^2 for this decoupling case (applicable for the nontrivial range $\ell \geq 2$), namely

$$m_L^2 = \frac{\ell m_1^2 - m_a^2}{\ell - 1} \quad (2.19)$$

and

$$m_H^2 = \frac{\ell m_a^2 - m_1^2}{\ell - 1}. \quad (2.20)$$

For given values of m_1 and m_a and for the nontrivial range $\ell \geq 2$ of interest here, one can determine the value of δ that leads to decoupling by setting $\theta_V = \theta_{V,\text{dec.}}$, substituting this into Eq. (2.11). This yields the result

$$\delta_{\text{dec.}} = -\left(\frac{\sqrt{\ell}}{\ell - 1}\right)(m_a^2 - m_1^2). \quad (2.21)$$

Combining the general inequality (2.6) and (2.21), we derive an upper bound on $|\delta_{\text{dec.}}|$, namely

$$\left(\frac{\sqrt{\ell}}{\ell - 1}\right)(m_a^2 - m_1^2) < m_1 m_a. \quad (2.22)$$

As with the general bound (2.6), one actually expects the left-hand side of this bound to be substantially smaller than the right-hand side, since otherwise, the diagonalization of M^2 in the decoupling limit would lead to the smaller squared-mass eigenvalue m_L^2 being much less than the typical QCD scale, Λ_{QCD}^2, which is unphysical, since the corresponding hadron is not a (pseudo)-Nambu-Goldstone boson.

The off-diagonal element of (2.18) gives an equivalent expression for $\delta_{\text{dec.}}$, viz., $\delta_{\text{dec.}} = -\sqrt{\ell}(m_H^2 - m_L^2)/(\ell + 1)$. Setting these two expressions equal, we derive a relation that applies if there is exact decoupling:

$$\frac{m_H^2 - m_L^2}{m_a^2 - m_1^2} = \frac{\ell + 1}{\ell - 1}. \quad (2.23)$$

As m_Q increases to values far above Λ_{QCD}, i.e., for $r_Q \gg 1$, one can define an effective low-energy field theory with the quark Q integrated out [8] and therefore the mixing of the quark-antiquark flavor eigenstates must lead to decoupling of this heavy quark. Imposing this decoupling condition on the transformation (2.14), we deduce that

$$\text{If } r_Q \to \infty, \text{ then } \theta_V \to \theta_{V,\text{dec.}} = \arctan\left(\frac{1}{\sqrt{\ell}}\right). \quad (2.24)$$

Our explicit calculations are in accord with this general result, since in this limit, $m_H/m_L \to \infty$, so that

$$M^2 \to \frac{m_H^2}{\ell + 1} \begin{pmatrix} 1 & -\sqrt{\ell} \\ -\sqrt{\ell} & \ell \end{pmatrix}. \quad (2.25)$$

The diagonalization of this limiting expression for M^2 yields $\tan(2\theta) = 2\sqrt{\ell}/(\ell - 1)$, i.e., $\tan\theta = \tan\theta_{\text{dec.}} = 1/\sqrt{\ell}$.

In general, as discussed above, the mixing involves not just quark-antiquark flavor-eigenstates, but also gluonic states of the same J^{PC}, to form the mass eigenstates. In this more general framework one can describe this mixing via the orthogonal 3×3 transformation that maps the quark-antiquark flavor eigenstates $|(V_0)_{N_f}\rangle$, $|V_{N_f^2-1}\rangle$, and

the glue state $|G\rangle$ with the same J^{PC} to the three resultant lowest-lying mass eigenstates, denoted $|V_L\rangle$, $|V_H\rangle$, and $|V_G\rangle$,

$$\begin{pmatrix} |V_L\rangle \\ |V_H\rangle \\ |V_G\rangle \end{pmatrix} = R_{\text{gen.}} \begin{pmatrix} |(V_0)_{N_f}\rangle \\ |V_{N_f^2-1}\rangle \\ |G\rangle \end{pmatrix}. \tag{2.26}$$

As an element of the orthogonal group $O(3)$, $R_{\text{gen.}}$ (with subscript *gen.* for "general") depends on three Euler angles. We write

$$R_{\text{gen.}} = \begin{pmatrix} \cos(\theta_{qG}) & 0 & \sin(\theta_{qG}) \\ 0 & 1 & 0 \\ -\sin(\theta_{qG}) & 0 & \cos(\theta_{qG}) \end{pmatrix}$$

$$\times \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta_{QG}) & \sin(\theta_{QG}) \\ 0 & -\sin(\theta_{QG}) & \cos(\theta_{QG}) \end{pmatrix}$$

$$\times \begin{pmatrix} \cos(\theta_V) & \sin(\theta_V) & 0 \\ -\sin(\theta_V) & \cos(\theta_V) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.27}$$

The decoupling theorem implies that as $r_Q \to \infty$ and the heavy quark, Q, is integrated out, the resultant low-energy effective field theory has the property that there is no mixing of a $|Q\bar{Q}\rangle$ state with either $|q\bar{q}\rangle$ or $|G\rangle$. Hence, reading from right to left in Eq. (2.27), (i) in the first transformation, the angle θ_V takes the value given by Eq. (2.24), so that the vector of states $(|(V_0)_{N_f}\rangle, |V_{N_f^2-1}\rangle, |G\rangle)^T$ is mapped to $(|(V_0)_\ell\rangle, -|V_{Q\bar{Q}}\rangle, |G\rangle)^T$; and (ii) in the second transformation,

$$\text{If} \quad r_Q \to \infty, \quad \text{then} \quad \theta_{QG} \to 0. \tag{2.28}$$

The third (i.e., left-most) transformation describes the mixing of the light-quark $|q\bar{q}\rangle$ and $|G\rangle$ states. This is not directly relevant to our present analysis on decoupling of the quark Q as $r_Q \to \infty$, but we note that, as discussed above, this mixing is small because of the considerable separation in mass between the gluonic states and the $|q\bar{q}\rangle$ states with the same J^{PC} values.

The mixings of $|(V_0)_{N_f}\rangle$, $|V_{N_f^2-1}\rangle$, and $|G\rangle$ depend in a complicated way on r_Q. As m_Q increases above m_q, the angle θ_V moves upward from 0 or downward from $\pi/2$, depending on its initial value for $m_Q = m_q$. On the basis of decoupling arguments, the variation of θ_V as a function of m_Q/Λ should be monotonic. The decoupling result (2.24) holds concerning the mixing of the $|Q\bar{Q}\rangle$ state with the lowest-lying $|q\bar{q}\rangle$ state of the same J^{PC}.

As m_Q increases sufficiently, the mass of a $|Q\bar{Q}\rangle$ state will become equal to the mass of the $|G\rangle$ state with the same J^{PC}, and there will be strong mixing between these two states, which will be reflected in θ_{QG}. Eventually, as

m_Q increases far above the confinement scale Λ, the mixing angles θ_V and θ_{QG} will approach their respective asymptotic values, (2.24) and (2.28). Decoupling arguments also imply that for m_Q above the mass of the lightest gluonic state, the approach of θ_{QV} to zero should also be monotonic. These statements apply for the mixings with the lowest-lying mass states, $(|V_L\rangle, |V_M\rangle, |V_H\rangle)^T$, in Eq. (2.26). It should be noted that as m_Q increases, the mass of a $|Q\bar{Q}\rangle$ state will ascend through values equal to the masses of various higher radial and orbital excitations of light-quark $|q\bar{q}\rangle$ states, and with excited gluonic states, with the same J^{PC}, so that one would have to take account of the mixing with these excited states also. Let us recall the standard spectroscopic notation for fermion-antifermion bound states, $n^{2S+1}L_J$, where n denotes the radial quantum number, and $\vec{S}$, $\vec{L}$, and $\vec{J}$ denote the total spin, orbital angular momentum, and total angular momentum of the state. As an example for vector mesons, as m_Q increases sufficiently, the mass of a $|Q\bar{Q}\rangle$ state would sequentially pass through the values of the masses of the n^3S_1 light-quark $|q\bar{q}\rangle$ states with $n \geq 2$, as well as the 1^3D_1 $|q\bar{q}\rangle$ state, resulting in significant mixing with these states. However, in general, for higher masses, the widths of these states become larger, which would have the effect of reducing the mixing.

III. GENERALIZED GELL-MANN OKUBO MASS RELATION

In this section we derive the generalization of the Gell-Mann Okubo (GMO) mass relation for vector and tensor mesons in our SU(N_c) QCD-like theory with $\ell = N_f - 1$ massless or light, degenerate quarks q_i, $i = 1, \ldots, \ell$ with mass m_q, and one quark Q of substantial mass, m_Q. The actual GMO relation assumes SU(2) isospin symmetry and incorporates SU(3) flavor symmetry-breaking. For the vector meson masses squared [15], the GMO relation is [3]

$$4m_{K^*}^2 = m_\rho^2 + 3m_a^2, \tag{3.1}$$

where m_ρ is the mass of the SU(2)-adjoint state, ρ, and m_a is the mass of the SU(3)-adjoint (octet) state, ω_8. Our generalization relates the mass squared of a $Q\bar{q}$ vector meson, where q is any of the ℓ q_is, to a linear combination of the squared masses of the meson that corresponds to the operator T_{ℓ^2-1} of the Lie algebra of SU(ℓ) and the meson that corresponds to the operator $T_{N_f^2-1}$ of the Lie algebra of SU(N_f). We denote these as V_{ℓ^2-1} and $V_{N_f^2-1} \equiv V_a$, respectively. In a simple quark-model approach, one can express the mass squared of each of these mesons as a sum of (i) a contribution E_g from the bound-state energy of the gluons, (ii) E_{qk} from the bound-state energies of the quarks, apart from their masses, and (iii) the contributions from the hard masses of the quark and antiquark. Here we are using the quark model as a simple method to derive the

generalization of the GMO relation that encompasses the relevant group-theoretic factors. The energies E_g and E_{qk} depend on the $n^{2S+1}L_J$ wave function for the state. We leave this wave-function dependence implicit henceforth. For the $Q\bar{q}$ meson, one thus writes

$$m^2_{Q\bar{q}} = \mu(E_0 + m_Q + m_q), \qquad (3.2)$$

where $\mu \sim 2\pi\Lambda$. For V_{ℓ^2-1}, we have

$$m^2_{V_{\ell^2-1}} = \mu(E_0 + 2m_q). \qquad (3.3)$$

The quark-model wave function of $V_{N_f^2-1}$ was given in Eq. (2.3). As was noted above, the $|V_{N_f^2-1}\rangle$ state has a probability $1/[\ell(\ell+1)]$ of being in one of the $|q_i\bar{q}_i\rangle$ states, for a total probability $1/(\ell+1)$ of being in some $|q_i\bar{q}_i\rangle$ state, and a probability $\ell/(\ell+1)$ of being in the $|Q\bar{Q}\rangle$ state. Hence, in the context of this quark-model approach,

$$m^2_{V_{N_f^2-1}} \equiv m^2_a = \mu\left[E_0 + \frac{2(m_q + \ell m_Q)}{\ell+1}\right]. \qquad (3.4)$$

We now write $x m^2_{Q\bar{q}} = y_1 m^2_{V_{\ell^2-1}} + y_2 m^2_{V_{N_f^2-1}}$, where x, y_1, and y_2 are constants to be determined. Without loss of generality, we can perform a rescaling to set $y_1 = 1$, and after this, we set $y_2 \equiv y$. Equating the coefficients of m_Q on either side of this equation, we find $x = 2\ell/(\ell+1)$. Equating the coefficients of m_q on either side of the equation and substituting the above value for x, we obtain $y = (\ell+1)/(\ell-1)$. With these coefficients, the E_0 terms are also equal on both sides of the equation. Multiplying through by the factor $\ell+1$, we thus derive the generalization of the Gell-Mann Okubo mass relation for mesons,

$$2\ell m^2_{Q\bar{q}} = (\ell-1)m^2_{V_{\ell^2-1}} + (\ell+1)m^2_{V_{N_f^2-1}}. \qquad (3.5)$$

One readily checks that for the case of actual QCD, with $\ell = 2$, $V_{Q\bar{q}} = K^*$, $V_{\ell^2-1} = V_3 = \rho$, and $V_{N_f^2-1} = \omega_8$, this relation reduces to the usual GMO relation, Eq. (3.1).

IV. COMPARISON WITH VECTOR MESON MIXING

In this section we revisit ω-ϕ mixing from the point of view of our general-ℓ analysis. It is appropriate first to recall the ways in which our abstract analysis with $\ell = 2$ (and $N_c = 3$) differs from real QCD. First, while m_u and m_d satisfy the criterion of being $\ll \Lambda_{\rm QCD}$, they are not degenerate, and, second, although m_s is substantially larger than m_u and m_d, it is not large compared to $\Lambda_{\rm QCD}$, indeed, $m_s/\Lambda_{\rm QCD} \simeq 1/3$. Therefore, *a priori*, one does not expect a large decoupling effect. Indeed, one of the most intriguing aspects of ω-ϕ mixing is how close this is to the decoupling limit even though $m_s/\Lambda_{\rm QCD}$ is not $\gg 1$. Moreover, in our abstract analysis, we have turned off electroweak interactions, and the inclusion of these, particularly electromagnetic

interactions, slightly modifies the results for general ℓ and, *a fortiori*, for $\ell = 2$.

We use the conventional notation for SU(3)-singlet and SU(3)-adjoint (octet) flavor eigenstates, which are both singlets under isospin SU(2), namely,

$$|\omega_0\rangle = |(V_0)_{N_f=3}\rangle = \left|\frac{u\bar{u} + d\bar{d} + s\bar{s}}{\sqrt{3}}\right\rangle, \qquad (4.1)$$

and

$$|\omega_8\rangle = |V_8\rangle = \left|\frac{u\bar{u} + d\bar{d} - 2s\bar{s}}{\sqrt{6}}\right\rangle. \qquad (4.2)$$

If the mixing were precisely as given by the decoupling result, then it would involve a rotation of these flavor SU(3)-singlet eigenstates through the angle mentioned in the Introduction, given by

$$\theta_{V,\rm dec.} = \arctan\left(\frac{1}{\sqrt{2}}\right) = 35.264° \quad \text{for } \ell = 2 \qquad (4.3)$$

to form the decoupled physical mass eigenstates

$$|\omega\rangle_{\rm dec.} = |(V_0)_{\ell=2}\rangle = \left|\frac{u\bar{u} + d\bar{d}}{\sqrt{2}}\right\rangle \qquad (4.4)$$

and $-|\phi\rangle_{\rm dec.}$, where

$$|\phi\rangle_{\rm dec.} = |V_{Q\bar{Q}}\rangle = |s\bar{s}\rangle. \qquad (4.5)$$

In this ideal (decoupling) mixing, Eq. (2.16) would take the form

$$\begin{pmatrix} |\omega\rangle_{\rm dec.} \\ -|\phi\rangle_{\rm dec.} \end{pmatrix} = \begin{pmatrix} \sqrt{2/3} & 1/\sqrt{3} \\ -1/\sqrt{3} & \sqrt{2/3} \end{pmatrix} \begin{pmatrix} |\omega_0\rangle \\ |\omega_8\rangle \end{pmatrix}, \qquad (4.6)$$

and the original mass-squared matrix in the $SU(3)_V$ flavor basis would be

$$(M^2)_{\rm dec.} = \begin{pmatrix} \frac{2m^2_\omega + m^2_\phi}{3} & -\frac{\sqrt{2}(m^2_\phi - m^2_\omega)}{3} \\ -\frac{\sqrt{2}(m^2_\phi - m^2_\omega)}{3} & \frac{m^2_\omega + 2m^2_\phi}{3} \end{pmatrix}. \qquad (4.7)$$

The actual physical angle, $\theta_{V,\rm ph}$, by which the SU(3)-singlet and SU(3)-octet states are rotated to form the actual physical mass eigenstates ω and ϕ, is close, but not exactly equal, to the decoupling value in Eq. (4.3). We recall the procedure for determining $\theta_{V,\rm ph}$, starting with the relation

$$\begin{pmatrix} |\omega\rangle \\ -|\phi\rangle \end{pmatrix} = R(\theta_{V,\rm ph}) \begin{pmatrix} |(V_0)_{N_f=3}\rangle \\ |V_8\rangle \end{pmatrix}. \qquad (4.8)$$

We next use the Gell-Mann Okubo mass relation, Eq. (3.1), which assumes SU(2) isospin symmetry but takes into account SU(3) breaking due to the strange quark mass [3]. Since SU(2) isospin symmetry is broken slightly (explicity) by both the nondegeneracy of the hard quark masses m_u and m_d and, separately, by electromagnetic

interactions, one expects the predictions of this relation to be accurate to a few. For our purposes it will suffice to use the central values of the relevant vector meson masses. From the measured values of the neutral vector mesons $m_{K^{*0}} = 895.9$ MeV and $m_\rho = 775.5$ MeV, one can solve Eq. (3.1) for m_a, obtaining the result

$$m_a = 932.6 \text{ MeV}. \tag{4.9}$$

Substituting this value for m_a together with the values

$$m_L = m_\omega = 782.65 \text{ MeV}$$
$$m_H = m_\phi = 1019.46 \text{ MeV} \tag{4.10}$$

in the trace relation, Eq. (2.12), one can solve for m_1, with the result

$$m_1 = 884.3 \text{ MeV}. \tag{4.11}$$

Next, substituting this into the determinant relation Eq. (2.13) gives δ up to sign, namely, $|\delta| = 0.2088 \text{ GeV}^2$, so that

$$\sqrt{|\delta|} = 457.0 \text{ MeV}. \tag{4.12}$$

These may be compared with the value of $|\delta|$ in the decoupling limit, as given by Eq. (2.21), which is (again quoting just the central value) $|\delta_{\text{dec.}}| = 1.241 \times 10^5 \text{ MeV}^2$, i.e.,

$$\sqrt{|\delta_{\text{dec.}}|} = 352.3 \text{ MeV}. \tag{4.13}$$

This decoupling value is somewhat smaller than the physical value in Eq. (4.12) and both are well below the upper bound from Eq. (2.6), $|\delta| < (908 \text{ MeV})^2$. In the flavor eigenbasis (again neglecting the nonzero widths), using a minus sign for δ to agree with Eq. (4.7), we obtain the following form for M^2, with masses quoted to the indicated accuracy:

$$M^2 = \begin{pmatrix} (0.884)^2 & -(0.457)^2 \\ -(0.457)^2 & (0.933)^2 \end{pmatrix} \text{ GeV}^2. \tag{4.14}$$

Note that $m_a > m_1$, in accord with the expectation from the quark constituency argument given above. It is also useful to express M^2 as the prefactor m_a^2 times a matrix of dimensionless entries to show their relative sizes:

$$M^2 = (0.870 \text{ GeV}^2)\begin{pmatrix} 0.899 & -0.240 \\ -0.240 & 1 \end{pmatrix}. \tag{4.15}$$

Substituting these values of m_a^2, m_1^2, and δ into Eq. (2.24) yields, for the physical mixing angle,

$$\theta_{V,\text{ph}} \simeq 39°, \tag{4.16}$$

in agreement, to the requisite accuracy, with the value $\theta_{V,\text{ph}} = 38.7°$ obtained in [1] from a similar quadratic fit and with the value $\theta_{V,\text{ph}} = (38.58 \pm 0.09)°$ obtained from a recent global fit by KLOE [16,17]. The fractional deviation of this value from the decoupling value is about 10%:

$$\frac{|\theta_{V,\text{ph}} - \theta_{\text{dec.}}|}{\theta_{V,\text{ph}}} \simeq \frac{4°}{39°} \simeq 0.1. \tag{4.17}$$

In the usual analysis reviewed above, the mixing angle θ_V is calculated in terms of physical meson masses, together with the Gell-Mann Okubo relation embodying the breaking of SU(3) symmetry. This analysis makes a connection with (the $\ell = 2$, $N_c = 3$ special case of) our general discussion in Sec. II, since these hadron masses depend on m_Q (with a weaker dependence on m_u and m_d, since $m_u, m_d \ll \Lambda_{\text{QCD}}$). Within the context of models such as the nonrelativistic quark model or the MIT bag model, one has analytic formulas for hadron masses in terms of $m_{u,d}$ and m_s. At present, the most reliable determination of hadron masses as functions of quark masses is via lattice QCD simulations. With either approximate analytic models or dedicated lattice simulations, one could carry out a calculation of the hadron masses in our generalized QCD-like theory with ℓ light or massless degenerate quarks and one quark of substantial mass and determine the dependence of θ_V on m_Q/Λ. The results would interpolate between the SU(N_f) limit and the asymptotic value of θ_V in Eq. (2.24).

V. COMPARISON WITH TENSOR MESON MIXING

We next briefly review tensor meson mixing and compare it with the $\ell = 2$ special case of our general-ℓ analysis. The observed $J^{PC} = 2^{++}$ tensor mesons $f_2(1275)$, $f_2'(1525)$, $a_2(1320)$ and $K_2^*(1430)$ form an SU(3) 1^3P_2 nonet. We consider the mixing between $f_2(1275)$ and $f_2'(1525)$. We use the same notation as for vector mesons with a subscript T added, so

$$M_T^2 = \begin{pmatrix} m_{1_T}^2 & \delta_T \\ \delta_T & m_{a_T}^2 \end{pmatrix}. \tag{5.1}$$

For this nonet the Gell-Mann Okubo relation reads

$$4m_{K_2^*}^2 = 3m_{a_T}^2 + m_{a_2}^2. \tag{5.2}$$

Again, for our purposes, it will suffice to use the central values of the measured masses. Substituting $m_{K_2^{*0}} = 1432.4$ MeV and $m_{a_2} = 1318.3$ MeV into Eq. (5.2), we obtain

$$m_{a_T} = 1468.5 \text{ MeV}. \tag{5.3}$$

Inserting this together with

$$m_{L_T} = m_{f_2} = 1275.1 \text{ MeV}$$
$$m_{H_T} = m_{f_2'} = 1525.0 \text{ MeV} \tag{5.4}$$

in the tensor meson trace relation $m_{L_T}^2 + m_{H_T}^2 = m_{1_T}^2 + m_{a_T}^2$ and solving for m_{1_T}, we obtain

$$m_{1_T} = 1339.8 \text{ MeV}. \tag{5.5}$$

Next, substituting this into the tensor meson determinant relation $m_{L_T}^2 m_{H_T}^2 = m_{1_T}^2 m_{a_T}^2 - \delta_T^2$ and solving for $|\delta|$, we get $|\delta_T| = 0.29964$ GeV^2, so that

$$\sqrt{|\delta|} = 457.0 \text{ MeV}. \tag{5.6}$$

This is slightly less than the value for complete decoupling, which, as calculated from the tensor-meson analogue of Eq. (2.21), is $\sqrt{|\delta_{T,\text{dec.}}|} = 510.9$ MeV. In the flavor basis for the tensor mesons, neglecting the nonzero widths, M_T^2 thus has the numerical form, to the indicated accuracy,

$$
\begin{aligned}
M_T^2 &= \begin{pmatrix} (1.34)^2 & -(0.547)^2 \\ -(0.547)^2 & (1.47)^2 \end{pmatrix} \text{GeV}^2 \\
&= (2.16 \text{ GeV}^2) \begin{pmatrix} 0.832 & -0.139 \\ -0.139 & 1 \end{pmatrix}. \tag{5.7}
\end{aligned}
$$

Substituting these values of $m_{a_T}^2$, $m_{1_T}^2$, and δ_T into the tensor-meson analogue of Eq. (2.24) yields, for the physical mixing angle,

$$\theta_{T,\text{ph}} \simeq 29.5°, \tag{5.8}$$

in agreement, to the requisite accuracy, with the value $\theta_{T,\text{ph}} = 29.6°$ obtained in [1] from a similar quadratic fit. The fractional deviation of this value from the decoupling value is about 15%. In contrast to the vector-meson case, the physical $f_2 - f_2'$ mixing angle is thus smaller than the decoupling value; $\theta_{T,\text{ph}} - \theta_{\text{dec.}} = 29.5° - 35.26° \simeq -5.8°$. As in the vector-meson case, the fact that $\theta_{T,\text{ph}}$ is close to $\theta_{\text{dec.}}$ means that the tensor meson $f_2(1275)$ is predominantly an isospin-singlet combination of the light quarks, $(u\bar{u} + d\bar{d})/\sqrt{2}$, while $f_2'(1525)$ is predominantly an $s\bar{s}$ state. This is in accord with the fact that $f_2(1275) \to \pi\pi$ and $f_2'(1525) \to K\bar{K}$ are experimentally the dominant decay modes for f_2 and f_2', respectively.

VI. DEPENDENCE OF MIXING ANGLE IN A SIMPLE MODEL

In this section we present a simple model to relate the mixing angles θ_V and θ_T in our generalized QCD-like theory to the heavier quark mass, m_Q. Since the general results apply to both vector and tensor meson mixing, we suppress the indices V and T in the notation. We emphasize at the outset that the actual calculation of hadron masses in QCD as functions of the hard quark masses, in particular, m_s, is arguably best done with lattice gauge theoretic methods, and the quark model that we use here is clearly a simplification of the actual physics. As before, we neglect electroweak interactions. It will be shown that the initial model that we use is, indeed, too simplistic to describe the physics accurately, and we will then modify it to give reasonable results. We recall Eq. (3.4) for the mass squared

of the $|V_{V_{N_f^2-1}}\rangle$ state. By similar arguments concerning the quark constituency of the wave function, we have,

$$m_{(V_0)_{N_f}}^2 \equiv m_1^2 = \mu\left[E_0 + \frac{2(\ell m_q + m_Q)}{\ell + 1} \right]. \tag{6.1}$$

Thus,

$$\text{Tr}(M^2) = m_H^2 + m_L^2 = 2E_0 + 2(m_q + m_Q). \tag{6.2}$$

From the determinant condition (2.13), we can then determine the square of the mixing entry, δ^2. Taking δ to be negative, we thus obtain

$$\delta = -\frac{2\sqrt{\ell}\mu(m_Q - m_q)}{\ell + 1} \tag{6.3}$$

and put $(M^2)_{12} = \delta$. With these values of m_1^2, m_a^2, and δ, the diagonalization of the M^2 matrix would yield $\tan(2\theta) = 2\sqrt{\ell}(\ell - 1)^{-1}(m_Q - m_q)/(m_Q - m_q)$. If $m_Q = m_q$, this is proportional to $0/0$, and hence indeterminate. As discussed above, this result is expected, since in this case there is complete SU(N_f) symmetry, and the $|(V_0)_{N_f}\rangle$ and $|V_{N_f^2-1}\rangle$ are degenerate in mass and may thus be rotated by an $R(\theta)$ with any mixing angle θ to form mass eigenstates. However, if $m_Q > m_q$, then we may cancel the $(m_Q - m_q)$ factor through and obtain $\tan(2\theta) = 2\sqrt{\ell}/(\ell - 1)$, i.e., $\theta = \theta_{\text{dec.}}$. This result is unphysical, since it predicts that there is complete decoupling of the heavier quark regardless of how small the nonzero mass different $m_Q - m_q$ is. This unphysical result shows that the initial model is too simplistic. To remedy this defect, one takes account of the fact that there is a propagator correction (for both the kinetic and mass squared terms) in which the SU(N_f) flavor-singlet state $|(V_0)_{N_f}\rangle$ annihilates to an intermediate virtual purely gluonic state and then goes back to itself again. This annihilation process cannot occur for the flavor-SU(N_f) adjoint state, $|(V_0)_{N_f^2-1}\rangle$. Modifying $m_1^2 \to m_1^2 + x_{\text{an}}$, where x_{an} denotes the annihilation contribution to the squared mass, we then have, in this quark-model approach,

$$\mu^{-1}M^2 = \begin{pmatrix} E_0 + \frac{2(\ell m_q + m_Q)}{\ell+1} + x_{\text{an}} & -\frac{2\sqrt{\ell}(m_Q - m_q)}{\ell+1} \\ -\frac{2\sqrt{\ell}(m_Q - m_q)}{\ell+1} & E_0 + \frac{2(m_q + \ell m_Q)}{\ell+1} \end{pmatrix}. \tag{6.4}$$

The diagonalization of this matrix yields

$$\tan(2\theta) = \frac{2\sqrt{\ell}}{\ell - 1 - \xi} \tag{6.5}$$

where

$$\xi \equiv \frac{(\ell + 1)x_{\text{an}}}{2(m_Q - m_q)}. \tag{6.6}$$

Provided that $x_{an} \neq 0$, it follows that (i) as $(m_Q - m_q)/x_{an} \to 0$, $\xi \to \infty$ and hence $\tan(2\theta) \to 0$, so that $\theta \to 0$ or $\pi/2$, and (ii) as $(m_Q - m_q)/x_{an} \to \infty$, $\xi \to 0$, and $\theta \to \theta_{dec.}$ This prediction makes physical sense and agrees with our general result in Eq. (2.24). This analysis, by itself, does not determine the sign of x_{an}. If $x_{an} < 0$, then θ increases monotonically from 0 to $\theta_{dec.}$ as $(m_Q - m_q)/x_{an}$ increases from 0 to ∞. If $x_{an} > 0$ and $\theta = \pi/2$ for $m_Q - m_q \to 0^+$, then as $(m_Q - m_q)/x_{an}$ increases from 0^+ to ∞, θ decreases monotonically from $\pi/2$ to $\theta_{dec.}$, passing through $\pi/4$ as ξ increases through the value $\ell - 1$.

VII. OTHER MESON MIXINGS

Similar mixings of SU(3)-singlet and SU(3)-octet states occur for other mesons. As is well known, the mixing of the $J^{PC} = 1^{-+}$ pseudoscalar mesons η and η' from the $1\,^1S_0$ nonet is complicated by several effects, including the fact that the η is an approximate Nambu-Goldstone boson (NGB), and the fact that there is a splitting between the octet of approximate Nambu-Goldstone bosons and the η' due to the breaking of the flavor-singlet axial vector global symmetry by instantons, which has the consequence that the η' is not an approximate NGB [18]. An additional effect involves mixing of the η and η' with the 0^{-+} glueball. Since we have focused here on mixing in a generalized QCD-like theory and its connection with decoupling, the illustration with vector and tensor mesons is sufficient for our purposes.

A comment on axial vector meson mixing is also in order here. In the quark model, two nonets of $J^P = 1^+$ axial-vector mesons are expected as orbitally excited quark-antiquark bound states. In the usual spectroscopic notation, these are $1\,^3P_1$ and $1\,^1P_1$. These two nonets have different C quantum numbers for their respective neutral mesons, namely $C = +$ and $C = -$. Experimentally, the $1\,^3P_1$ nonet consists of $a_1(1260)$, $f_1(1285)$, $f_1(1420)$ and K_{1A}, while the $1\,^1P_1$ nonet contains $b_1(1235)$, $h_1(1170)$, $h_1(1380)$ and K_{1B}. The nonstrange axial vector mesons, for example, the neutral $a_1(1260)$ and $b_1(1235)$ cannot mix because of their opposite C-parities. In contrast, K_{1A} and K_{1B} do mix to form corresponding physical mass eigenstates. This complicates the analysis of the mixings of the SU(3)-singlet and SU(3)-octet mesons in the $1\,^3P_1$ and $1\,^1P_1$ nonets. For recent discussions of these mixings, see [19] and references therein.

VIII. CONCLUSIONS

In this paper we have discussed the mixing of quark flavor-eigenstates to form low-lying vector and tensor meson mass eigenstates in a QCD-like theory with ℓ massless or light, degenerate quarks and one quark of substantial mass, m_Q. We have derived the asymptotic value of the mixing angle in the limit as m_Q/Λ gets large. We have also presented a generalization of the Gell-Mann Okubo mass relation for this theory. Finally, we have remarked on how the $\ell = 2$ special case of our results relate to the observed ω-ϕ and $f_2(1275)$-$f_2'(1525)$ mixings.

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[11] In our discussion of meson mixing, we rely on the empirical fact that QCD conserves P, C, and CP. This is automatic if $m_q = 0$, since then the CP-violating angle $\bar{\theta}_{CPV} = \theta_{CPV} - \arg(\det(M_u)) - \arg(\det(M_d))$ can be rotated away, where θ_{CPV} is the coefficient of the instanton-induced topological term $g_s^2 (32\pi^2)^{-1} G_{\mu\nu}^a \tilde{G}^{a\mu\nu}$ in the QCD Lagrangian. For $m_q \neq 0$, we implicitly assume a solution of the strong CP problem.

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[13] The determination of the value of $N_{f,cr}$ separating the interval of smaller N_f with SχSB from the interval of larger $N_f < (11/2)N_c$ where the theory is still asymptotically free but does not have SχSB has been the subject of intensive study recently using lattice methods. See, e.g., T. Appelquist, G. T. Fleming, and E. T. Neil, Phys. Rev. Lett. **100**, 171607 (2008); X.-Y. Jin and R. D. Mawhinney, Proc. Sci., LAT2009 (2009) 049; Z. Fodor *et al.*, arXiv:1104.3124; T. Appelquist *et al.*, Phys. Rev. D **84**, 054501 (2011); and talks at Lattice 2011, https://latt11.llnl.gov.

[14] Although electromagnetic interactions are a small perturbation, their presence, together with vacuum alignment arguments, implies the absence of charge-violating quark condensates such as $\langle \bar{u}d \rangle$, $\langle \bar{u}s \rangle$, etc.

[15] Here and below, e.g., when using the Gell-Mann Okubo mass relation and giving various quark model formulas, we follow the common practice of using the squared masses of the mesons rather than the masses themselves, because for bosons it is the squared masses that appear in effective Lagrangians.

[16] As an auxiliary part of a global fit to the η-η' mixing and η'-gluonium mixing, a recent value of $\theta_{V,ph}$ was reported in F. Ambrosino *et al.* (KLOE Collab.), J. High Energy Phys. 07 (2009) 105, in terms of a quantity $\psi_V \equiv \theta_{V,ph} - \theta_{V,dec.}$. Their value, $\psi_V = (3.32 \pm 0.09)°$ gives $\theta_{V,ph} = (38.58 \pm 0.09)°$, in agreement with [1].

[17] The fitted values of $\theta_{V,ph}$ (and $\theta_{T,ph}$) in [1] are fixed constants. It may be noted that the 2000 paper of Benyaoun *et al.* [7] discussed an effective energy-dependent $\theta_{V,ph}$.

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