PHYSICAL REVIEW D **84,** 086010 (2011)

Symmetry breaking phase transitions in the Aharony-Bergman-Jafferis-Maldacena theory with a finite $U(1)$ chemical potential

Dongsu Bak,[1,*] Kyung Kiu Kim,[2,†] and Sangheon Yun[2,‡]

[1]*Physics Department, University of Seoul, Seoul 130-743 Korea*

[2]*Institute for the Early Universe, Ewha Womans University, Seoul 120-750 Korea*

(Received 4 March 2011; revised manuscript received 20 May 2011; published 28 October 2011)

We consider the $U(1)$ charged sector of ABJM theory at finite temperature, which corresponds to the Reissner-Nordstrom AdS black hole in the dual type IIA supergravity description. Including back-reaction to the bulk geometry, we show that phase transitions occur to a broken phase where $SU(4)$ R-symmetry of the field theory is broken spontaneously by the condensation of dimension one or two operators. We construct the composite operators out of fields in ABJM theory and describe the phase transition with the dual gravity solutions. We show numerically and analytically that the relevant critical exponents for the dimension one operator agree precisely with those of mean field theory in the strongly coupled regime of the large N planar limit.

DOI: 10.1103/PhysRevD.84.086010 PACS numbers: 11.25.Mj, 04.70.Bw, 04.70.Dy, 11.10.Wx

I. INTRODUCTION

The Aharony-Bergman-Jafferis-Maldacena (ABJM) theory is the three dimensional $\mathcal{N} = 6$ $U(N) \times U(N)$ superconformal Chern-Simons theory with level $(k, -k)$ and dual to the type IIA string theory on $AdS_4 \times \mathbb{CP}_3$ background [1]. Some tests of this duality have been carried out, largely based on the integrability, with indication of some additional structures compared to the well-known AdS_5/CFT_4 counterpart [2–5]. The type IIA supergravity description is dual to the large N planar limit of the Chern-Simons theory where one takes N, $k \to \infty$ while holding 't Hooft coupling $\lambda = N/k$ fixed.

Some probes of the Chern-Simons plasma at finite temperature were carried out recently via the consistent, $\mathbb{CP}_3$ invariant dimensional reduction of the type IIA supergravity [6]. Alternatively a finite temperature plasma can be completely characterized by towers of static length (1/mass) scales. They arise as decaying spatial length scales of a perturbation theory when local operators are inserted at a certain point of the plasma. In Ref. [7], these scales are scanned for the low-lying bosonic modes from which the true mass gap m_g (the lowest in all) and the Debye screening mass m_D (the lowest in CT odd sector of the theory [8]) are found for the Chern-Simons plasma. Including Yang-Mills plasmas, these two scales are well representing the universal characteristics of a certain strongly coupled plasma. For instance, the ratio m_D/m_g for the $\mathcal{N} = 4$ SYM theory and the two-flavor model of QCD match each other in the strong coupling limit, supporting such a picture [9]. Leading thermodynamic corrections of ABJM theory in the small λ expansion are explored in Ref. [10].

In this paper, we note that the ABJM theory possesses $SU(4) \times U(1)$ R-symmetries and consider its particular sector in which one turns on a finite $U(1)$ number density. For the type IIA supergravity side, this sector is described by the $U(1)$ charged Reissner-Nordstrom (RN) AdS black brane solution whose physics will be our basic concern for the study of the field theory at a strong coupling [6]. We begin with the thermodynamic stability of the RN AdS black brane solution and prove that it is thermodynamically stable at all temperatures including the zero temperature limit. This is contrasted with the R-charged black holes in the type IIB supergravity which is dual to the R-charged sector of the $\mathcal{N} = 4$ super-Yang-Mills (SYM) theory. The R-charged solution becomes thermodynamically unstable below a certain temperature and the validity of the solution itself will be lost completely [11]. Thus the RN AdS solution of this note is, up to now, the only known example where one has the thermodynamic stability of the gravity solution at all temperatures and the dual field theory description is precisely known at the same time. In other words, we found that the RN AdS black brane solution is a relevant dual gravity description for the finite temperature ABJM theory with a chemical potential in the whole temperature region including the zero temperature, which is not the case for the $\mathcal{N} = 4$ SYM theory.

We go on to study phase structure and transitions occurring within this charged sector. These phase transitions turn out to be a second order type where the $SU(4)$ R-symmetry is broken spontaneously [12]. For this we scan behaviors of supergravity modes which are dual to the primary operators of some dimension Δ in the RN black hole background [13]. Among them, one finds the bosonic modes with mass squared ranged over $-9/4 \le m^2 < -3/2$ which are responsible for the phase transition. As will be further explained later on, the only possibility for the present case is $m^2 = -2$ corresponding to one $\Delta = 1$ operator of $SU(4)$ representation **15** and two $\Delta = 2$ operators of $SU(4)$

*dsbak@uos.ac.kr
†kimkyungkiu@gmail.com
‡sanhan@ewha.ac.kr

representations **15** and **84**, whose detailed operator contents will be specified below. The transition related to the condensate of the $\Delta = 1$ operator occurs at a critical temperature higher than that of the $\Delta = 2$ operators. Above the transition temperature, the scalar field has to be set to zero to satisfy the required boundary condition while the other part of the original RN black brane solution remains intact. Below the transition temperature the scalar field begins to develop a profile whose boundary behavior represents a condensation of operator expectation value without introducing any external source field. Of course the geometry will then be back-reacted accordingly. The solution represents the condensation of the operator expectation values below the transition temperature, which takes a particular direction in the space of the $SU(4)$ representation **15**. Thus the $SU(4)$ symmetry is spontaneously broken down to its little group. We then study the critical exponents of the phase transitions and show both analytically and numerically that the exponents precisely match with those of the mean field theory. We shall also show that there is further symmetry breaking phase transition of the same nature at a lower temperature which involves condensation of the $\Delta = 2$ operators.

In Section II, we shall discuss some relevant properties of the ABJM theory. Section III deals with the gravity description of the RN black brane and related physics in the field theory. In Section IV, our system described by RN black brane solution is shown to be thermodynamically stable. In Section V, we discuss the development of gravitational instabilities of scalar modes below the critical temperature. This will fix the critical temperature which depends on the dimension of the corresponding operators. In Section VI, we discuss the phase transition by studying the gravity solution representing the condensation of operator expectation values including the back-reaction to the bulk geometry. The critical exponents are shown to agree precisely with those of the mean field theory. The last section is devoted to the interpretations and concluding remarks.

II. THERMODYNAMICS OF ABJM THEORY AT SMALL λ

The on-shell degrees of the ABJM theory consist of bosonic and fermionic matter fields Y^I and $\Psi_I (I = 1, 2, 3, 4)$ together with two gauge fields A_m and $\bar{A}_m$. The complex scalar fields Y^I are in the representation of $(\mathbf{N}, \bar{\mathbf{N}}, \mathbf{4})$ of the $U(N) \times U(N)$ gauge as well as the $SU(4)$ R-symmetries. There are also the complexified Majorana fermions Ψ_I, which are in the representation of $(\mathbf{N}, \bar{\mathbf{N}}, \bar{\mathbf{4}})$. The gauge fields A_m and $\bar{A}_m$ in the adjoint representations of the first $U(N)$ and the second $U(N)$ respectively, are coupled to the matter fields $\Phi = (Y^I, \Psi_I)$ by

$$D_m \Phi = \partial_m \Phi + i A_m \Phi - i \Phi \bar{A}_m. \tag{2.1}$$

For further details of the Lagrangian and notation used in this note, see, for example, Ref. [3]. The theory possesses global $\mathcal{N} = 6$ 3d superconformal symmetries of $OSp(6|4)$ whose bosonic part is given by the 3d conformal symmetry of $SO(3, 2)$ multiplied by $SU(4)$ R-symmetry. This corresponds to the isometry of the $\mathrm{AdS}^4 \times \mathbb{CP}_3$ in the type IIA supergravity side. But one crucial difference from the $\mathcal{N} = 4$ case is the fact that there is an extra global $U(1)$ symmetry. The relevant charge is associated with the $U(1)$ phase transformation of complex fields by a overall phase factor. Denoting overall $U(1)$ parts of gauge fields by $A_m^{U(1)}$ and $\bar{A}_m^{U(1)}$, the above charge is only coupled to the relative $U(1)$ gauge field $A_m^{U(1)} - \bar{A}_m^{U(1)}$.

Now note that the fields Y^I and Ψ_I carry the $U(1)$ charges $(-1, +1)$ or -1 in terms of the relative one. Because of the Gauss law constraint of the Chern-Simons theory, these charges are always accompanied by $U(1)$ magnetic fluxes[1] $\frac{2\pi}{k}(+1, -1)$. Thus for general k including even the nonabelian contributions, the basic degrees in the field theory in the deconfined phase exhibit an anyonic nature due to the (generically nonabelian) statistical interactions between them. But due to the large N planar limit where we send N and k to infinity at the same time, the statistical interactions drop out since they are of higher order in $1/k$. Therefore, for instance, the total effective number of degrees in the weakly coupled small λ limit is simply proportional to N^2. Namely the entropy density at temperature T takes a value [1]

$$\mathcal{S}(\lambda \to 0) = \frac{21\zeta(3)}{\pi} N^2 T^2 \tag{2.2}$$

in the $\lambda \to 0$ limit. (The free energy density is always related to the entropy density by $\mathcal{F} = -\mathcal{S}T/3$ as dictated by the conformal symmetry.) On the other hand in the strongly coupled large λ region, the system is described by the black brane solution in the gravity side whose Bekenstein-Hawking entropy density reads [1]

$$\mathcal{S}(\lambda) = \frac{16\pi^2}{27\sqrt{2}} \frac{N^2 T^2}{\sqrt{\lambda}}. \tag{2.3}$$

The appearance of the $\frac{1}{\sqrt{\lambda}}$ suppression factor is not understood from the direct computation of the field theory. This drastic change of the number of degrees might be related to some remnant of anyonic interactions but we do not have any supporting evidence for this picture.

Our main concern in this paper is the sector of the ABJM theory where one turns on a finite $U(1)$ number density or equivalently the corresponding chemical potential μ. First consider the small λ region of planar limit and the high enough temperature with $T \gg \mu$. The fermions can be ignored in this limit. In the thermal circle compactified

[1] The unit flux for the current case with $e = 1$ is given by $h/e = 2\pi\hbar = 2\pi$ where we set $\hbar = 1$.

effective theory, the fermions can be integrated out at weak coupling and high temperature limit as their Matsubara frequency starts with πT. Of course the $U(1)$ current gets contributions from both bosonic and fermionic degrees of the theory. Their currents are conserved not separately but only in sum. Hence at weak coupling of small λ, instead of building up a Fermi surface, occupation of bosonic states is energetically preferred. The scalar fields in this effective 2d description acquire a mass [14],

$$m^2(T) = -\mu^2 + m_T^2, \tag{2.4}$$

where m_T^2 is the thermal mass correction. For $\lambda \ll 1$, the thermal mass has the expression [10]

$$m_T^2 = \frac{118}{3}\lambda^2(\log\lambda)^2T^2 + O(\lambda^2\log\lambda). \tag{2.5}$$

The theory lies in the unbroken phase if $m^2(T) \geq 0$. On the other hand, the system in the symmetric phase becomes unstable when $m^2(T) < 0$ or $\mu \ll T < \sqrt{\frac{3}{118}}\frac{\mu}{\lambda|\log\lambda|}$. As argued in Ref. [14], some of the operators of the field theory may acquire nonvanishing expectation values possibly leading to $SU(4)$ R-symmetry broken phase. But the precise fate of the system at weak coupling requires further study, which is beyond scope of the present paper.

III. RN BLACK BRANE AND TYPE IIA SUPERGRAVITY ON $ADS_4 \times \mathbb{CP}_3$

In the strongly coupled region of large λ, the description of the ABJM theory by the type IIA supergravity on $AdS_4 \times \mathbb{CP}_3$ is appropriate since geometry there is weakly curved. The type IIA spectra compactified on $\mathbb{CP}_3$ space was known quite some time ago [13]. Each mode of the resulting 4d supergravity is dual to a gauge invariant primary operator whose scaling dimension Δ is protected against quantum corrections. The presence of these bulk modes shows how the basic degrees of freedom are organized in the strongly coupled side of the ABJM theory. Thus study of these bulk modes for a given supergravity background will be our main tool probing the physics at the strong coupling.

The bulk modes consist of infinite towers of spectra from spin zero to spin two. We shall be here briefly describing some of the relevant low-lying modes for later discussions. Let us begin with the case of spin one: The lowest are two massless bulk gauge fields that are dual to the $\Delta = 2$ current operators in the field theory side. One is for the current of $SU(4)$ singlet representation [(000) in the $SU(4)$ Dynkin label notation], which is identified with that of the extra $U(1)$ global symmetry. In the gravity side the $U(1)$ is related to the $U(1)$ isometry of the M-theory circle from the 11d perspective and its charge is carried by D0 branes in a rough sense. As was shown explicitly in Ref. [6], this gauge field arises as a linear combination of

$$A_\mu = A_\mu^{D0} + 3A_\mu^{D4}, \tag{3.1}$$

where A_μ^{D0} and A_μ^{D4} respectively coupled to the D0 branes and $D4$ branes wrapping $\mathbb{CP}_2$ four cycle inside $\mathbb{CP}_3$. The other combination $\tilde{A} = A_\mu^{D0} - A_\mu^{D4}$ becomes massive by Higgs mechanism with $m^2 = 12$, which couples to the $\Delta = 5$ boundary current operator. It should be also noted that the monopole operator with overall field theory $U(1)$ charges $n(k, -k)(n \in \mathbf{Z})$ is the one example of heavy Bogomol'nyi-Prasad-Sommerfield (BPS) state coupled to this $U(1)$ bulk gauge field [1]. The second massless gauge field is in the adjoint **15** [(101)] of $SU(4)$ and couples to the boundary $SU(4)$ current operator of dimension $\Delta = 2$.

For the bulk scalar modes, the lowest ones with $m^2 = -2$ are relevant for our later discussions, which correspond to $\Delta = 1$, 2 operators. The $\Delta = 1$ scalar mode couples to the boundary operator of $SU(4)$ representation **15** [(101)] which takes the form

$$O_J^I = \mathrm{Tr} Y^I Y_J^\dagger - (\text{trace part}), \tag{3.2}$$

and has 15 independent components. The $\Delta = 2$ modes involve two field theory operators of $SU(4)$ representation **15** [(101)] and **84** [(202)] whose operator contents read

$$\begin{aligned} \tilde{O}_I^{\ J} &= \mathrm{Tr}\Psi_I \Psi^{\dagger J} - (\text{trace part}), \\ O_{KL}^{IJ} &= \mathrm{Tr} Y^{(I} Y_{(K}^\dagger Y^{J)} Y_{L)}^\dagger - (\text{trace part}) \end{aligned} \tag{3.3}$$

where the trace part denotes any contractions between the upper and the lower indices.

The fermionic bulk modes start with $|m| = 0$ corresponding to the operator dimension $\Delta = \frac{3}{2}$. These are boundary operators of $SU(4)$ representations **10** [(002)], **6** [(010)], and $\overline{\mathbf{10}}$ [(200)]. The low-lying modes up to $\Delta = 3$ are listed in Table I.

TABLE I. The low-lying spectra up to the operator dimension 3 are presented. The upper and lower indices denote, respectively, the parity and the mass squared value m^2 of the supergravity mode. The whole spectra of $\mathbb{CP}_3$ singlet sector are presented in addition.

	spin 0	spin $\frac{1}{2}$	spin 1	spin $\frac{3}{2}$	spin 2
$\Delta = 1$	$(101)^+_{-2}$				
$\Delta = \frac{3}{2}$		$(002)_0$ $(200)_0$ $(010)_0$			
$\Delta = 2$	$(202)^+_{-2}$ $(101)^-_{-2}$		$(000)^-_0$ $(101)^-_0$		
$\Delta = \frac{5}{2}$		$(103)_1$ $(301)_1$ $(111)_1$		$(010)_1$	
$\Delta = 3$	$(303)^+_0$ $(202)^-_0$ $(400)^-_0$ $(004)^-_0$ $(210)^-_0$		$(101)^-_2$ $(202)^-_2$		$(000)^+_0$
	$(012)^-_0$ $(020)^-_0$		$(210)^-_2$ $(012)^-_2$		
$\mathbb{CP}_3$ singlets	$(000)^+_4$ $(000)^-_{10}$ $(000)^+_{18}$		$(000)^-_0$ $(000)^-_{12}$		$(000)^+_0$

A few comments are in order. We note that there are no bulk supergravity modes that are charged under the massless $U(1)$ gauge field. The knowledge about the compactification spectra tells us about only linearized fluctuations of modes above the AdS_4 or the AdS_4 black brane solution. In Ref. [6], a consistent $\mathbb{CP}_3$ compactification keeping all $SU(4)$ invariant modes is carried out explicitly. Any solutions of this 4d system can be consistently embedded into the original 10d supergravity theory.

Our starting Lagrangian for the further discussion is

$$\mathcal{L} = \frac{1}{2\kappa^2}\left(\mathcal{R} + 6 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \sum_{a=1}^{n}((\nabla\phi_a)^2 + m_a^2\phi_a^2)\right) \tag{3.4}$$

where[2]

$$\kappa^{-2} = \frac{N^2}{6\pi\sqrt{2\lambda}}. \tag{3.5}$$

In this action, the Einstein Maxwell part with the negative cosmological constant is a fully consistent truncation [6] while the remaining scalar part is only valid up to quadratic order. Below we shall show that, above the critical temperature of the $\Delta = 1$ operator, the stable solution of the system is given by the RN black brane solution with vanishing scalar fields. This part of the solution is fully consistent as just stated. Below the critical temperature, the relevant scalar field begins to develop and the corresponding set of solutions is valid only if the magnitude of the scalar field is small enough. But the set of solutions in the near critical region carries all the information about universal natures of the phase transition including relevant critical exponents.

Similar models have been discussed many times in the bottom up approach [15–17]. However, it is hard to identify the dual field theory and the corresponding operator for condensation in this bottom up approach. In our work, on the other hand, the identification of the physics of boundary CFT is straightforward.

Below we take the following ansatz,

$$\begin{aligned} ds^2 &= e^{2A(r)}(-h(r)dt^2 + dx^2 + dy^2) + \frac{dr^2}{h(r)}, \\ A_t &= A_t(r), \qquad \phi_a = \phi_a(r), \end{aligned} \tag{3.6}$$

to describe a finite temperature system with plane plus time ($\mathbb{R}^2 \times \mathbb{R}$) translational symmetries. Plugging the above ansatz into the equations of motion, we are led to the following equations [18]:

$$\begin{aligned} &A'' + \frac{1}{2}\sum_{i=1}^{n}\phi_i'^2 = 0, \\ &h'' + 3A'h' - e^{-2A}F_{tr}^2 = 0, \\ &(e^A F_{tr})' = 0, \\ &h\phi_a'' + (3A'h + h')\phi_a' - m_a^2\phi_a = 0 \end{aligned} \tag{3.7}$$

with a constraint,

$$\begin{aligned} h\sum_{a=1}^{n}\phi_a'^2 - \frac{1}{2}e^{-2A}F_{tr}^2 - 2A'h' - 6h(A')^2 \\ = -6 + \sum_{a=1}^{n}m_a^2\phi_a^2. \end{aligned} \tag{3.8}$$

The third equation in (3.7) can be solved by

$$F_{rt} = 2qe^{-A(r)}, \tag{3.9}$$

leading to the set of equations

$$\begin{aligned} &A'' + \frac{1}{2}\sum_{i=1}^{n}\phi_i'^2 = 0, \\ &h'' + 3A'h' - 4q^2e^{-4A} = 0, \\ &h\phi_a'' + (3A'h + h')\phi_a' - m_a^2\phi_a = 0, \end{aligned} \tag{3.10}$$

which will be the starting point of our subsequent analysis.

The black brane solution,

$$A(r) = r, \qquad h(r) = 1 - e^{-3r+3r_H}, \qquad F_{rt} = \phi_a = 0, \tag{3.11}$$

describes the uncharged sector of the ABJM theory at finite temperature. Because of the conformal symmetry of the black brane background, the finite temperature phase here depends on only one dimensionful parameter which can be taken as the temperature T. Thus this uncharged sector possesses only one finite temperature phase corresponding to the high temperature limit. The temperature T is identified with the Hawking temperature of the black brane,

$$T = \frac{1}{4\pi}h'(r_H)e^{A(r_H)} = \frac{3}{4\pi}e^{r_H}, \tag{3.12}$$

where $r = r_H$ is the location of horizon. The Bekenstein-Hawking entropy density is given by the expression $\mathcal{S}$ in (2.3). The energy density, the free energy density and the pressure are related to the entropy density by $\mathcal{E} = \frac{2}{3}\mathcal{S}T = 2p = -2\mathcal{F}$ as simply dictated by the conformal symmetry of the background. Some probes of system are investigated in Ref. [6] by studying the response of the scalar and current operators to the external perturbation. The static length scales including the true mass gap as well as the Debye mass are further studied in Ref. [7].

Our main interest of this note is the RN black brane solution,

[2]In the bulk, any dimensionful quantity is measured with respect to the AdS radius scale ℓ which we set to be unity for the notational simplicity.

$$A(r) = r, \qquad h(r) = 1 - \varepsilon e^{-3r} + q^2 e^{-4r}, \qquad F_{rt} = 2qe^{-r}, \qquad \phi_a = 0, \tag{3.13}$$

which is an exact solution of the original 10d equations of motion. The parameters ε and q here are, respectively, proportional to the mass and the charge densities of the RN black brane. The horizon is located at $r = r_H$ with $h(r_H) = 0$ satisfying explicitly

$$\varepsilon e^{-3r_H} = 1 + q^2 e^{-4r_H}. \tag{3.14}$$

The minimum of $h(r)$ occurs at

$$e^{-r_m} = \frac{3\varepsilon}{4q^2}, \tag{3.15}$$

with $h'(r_m) = 0$. No nakedness condition for the mass and charge requires $h(r_m) \leq 0$ leading to the inequality

$$\left(\frac{\varepsilon}{4}\right)^2 \geq \left(\frac{q}{\sqrt{3}}\right)^3, \tag{3.16}$$

where the inequality is saturated at zero temperature. The Hawking temperature of the RN black brane becomes

$$T = \frac{3\varepsilon e^{-2r_H}}{4\pi}\left(1 - \frac{4q^2}{3\varepsilon} e^{-r_H}\right). \tag{3.17}$$

The entropy, the energy and the charge densities read

$$\mathcal{S} = \frac{2\pi}{\kappa^2} e^{2r_H}, \qquad \mathcal{E} = \frac{\varepsilon}{\kappa^2}, \qquad \rho = \frac{2q}{\kappa^2}, \tag{3.18}$$

with $\mathcal{F} = \mathcal{E} - T\mathcal{S}$ and $\mu_G = \frac{\partial \mathcal{F}(T,\rho)}{\partial \rho}$ where μ_G is proportional to the field theory chemical potential μ. Hence the no nakedness condition now takes a form

$$\left(\frac{\kappa^2 \mathcal{E}}{4}\right)^2 \geq \left(\frac{\kappa^2 \rho}{2\sqrt{3}}\right)^3. \tag{3.19}$$

In Ref. [19], this character is attributed to build-up of a Fermi surface for a finite number of the fermion number density. The weakly coupled massless fermions at finite temperature in general do satisfy an analogous inequality with precisely the same powers. Further support of the picture comes from the fact that the specific heat at low enough temperature is linear as

$$C_V = \gamma_V T, \tag{3.20}$$

which is another important characteristic of the Fermi surface.

However, there are some additional properties which cannot be understood from the fermion picture. The entropy density even at zero temperature remains finite; namely,

$$\mathcal{S}(T = 0) = \frac{\pi}{\sqrt{3}} \rho, \tag{3.21}$$

with a finite size of horizon at $e^{2r_H(T=0)} = \frac{q}{\sqrt{3}}$. Where this entropy comes from even at zero temperature is not clear. Recall further that the $U(1)$ current is conserved only in sum of bosonic and fermionic contributions together. Thus at weak coupling the build-up Fermi surface would not be possible if the bosons were in an unbroken phase and, hence, putting charges to the bosonic states were energetically favored. However the symmetric unbroken phase without condensation will be problematic as argued in the previous section for the weakly coupled case. Therefore the system should be at some unbroken phase with some bosonic condensate at least for the weakly coupled regime. In later sections, we would like to study the nature of phases occurring at the RN black holes, in the strongly coupled side, by the condensation of some operator expectation values.

Near boundary regions of large r, general asymptotically AdS solutions should behave as

$$\begin{aligned} A(r) &= a_1 r + a_0 + \cdots, \\ h(r) &= h(\infty) + h_3 e^{-3A} + \cdots, \\ F_{rt}(r) &= 2qe^{-A}, \end{aligned} \tag{3.22}$$

and the discussion of the boundary data for the scalar fields will be specified below. The entropy, energy, charge densities and the temperature have the expressions,

$$\begin{aligned} \mathcal{S} &= \frac{2\pi}{\kappa^2} e^{2A(r_H)}, \qquad \mathcal{E} = -\frac{h_3}{\kappa^2 h(\infty)}, \\ \rho &= \frac{2q}{\kappa^2}, \qquad T = \frac{1}{4\pi} e^{A(r_H)} \frac{h'(r_H)}{\sqrt{h(\infty)}}. \end{aligned} \tag{3.23}$$

In this computation all the thermodynamic quantities are measured with respect to the boundary time $\sqrt{h(\infty)}t$ such that one may bring the boundary metric of (3.6) in the standard form.

IV. THERMODYNAMIC STABILITY OF THE RN BLACK BRANE

In this section we shall discuss the thermodynamic stability of the RN black brane solution (3.13). The thermodynamic stability is ensured if the Hessian (second-derivative) matrix of the energy with respect to its thermodynamic variables has no negative eigenvalues. With any negative eigenvalue, the system becomes thermodynamically unstable under small fluctuations that drive the system toward some other stable point [20–22]. In the type IIB theory, $SO(6)$ R-charged black brane solution is available. But there it is observed that the R-charged black brane solution exhibits thermodynamic instabilities at a temperature lower than a certain critical value [11]. The fate of black brane in this unstable regime has not been known up to now.

Unlike the case of this type IIB counterpart, we find that our RN black brane solution is thermodynamically stable. To show this, note first that the energy density can be expressed as

$$\mathcal{E} = \frac{\kappa \mathcal{S}^{3/2}}{(2\pi)^{3/2}}\left(1 + \frac{\pi^2\rho^2}{\mathcal{S}^2}\right). \tag{4.1}$$

The components H_{ij} of the Hessian matrix are given by

$$\begin{aligned} H_{11} &= \frac{\partial^2 \mathcal{E}}{\partial \mathcal{S}^2} = \frac{3\kappa}{4(2\pi)^{3/2}\mathcal{S}^{1/2}}\left(1 + \frac{\pi^2\rho^2}{\mathcal{S}^2}\right), \\ H_{12} &= \frac{\partial^2 \mathcal{E}}{\partial \mathcal{S}\partial\rho} = -\frac{\kappa\pi^2}{(2\pi)^{3/2}}\frac{\rho}{\mathcal{S}^{3/2}}, \\ H_{22} &= \frac{\partial^2 \mathcal{E}}{\partial\rho^2} = \frac{2\kappa\pi^2}{(2\pi)^{3/2}\mathcal{S}^{1/2}}. \end{aligned} \tag{4.2}$$

The determinant of H_{ij} then becomes

$$\det H = \frac{\kappa^2}{16\pi\mathcal{S}}\left(3 + \frac{\pi^2\rho^2}{\mathcal{S}^2}\right) \tag{4.3}$$

together with $\mathrm{tr}H > 0$. This proves that the two eigenvalues are positive definite. One may also consider turning on small quantities of the three further charges $\rho_a (a = 1, 2, 3)$ in the Cartans of the $SU(4)$ R-symmetry. Including these contributions to the quadratic order, the energy density has the expression,

$$\mathcal{E} = \frac{\kappa \mathcal{S}^{3/2}}{(2\pi)^{3/2}}\left(1 + \frac{\pi^2}{\mathcal{S}^2}(\rho^2 + \rho_a\rho_a)\right). \tag{4.4}$$

Using now a 5×5 Hessian matrix, one may easily verify that the system at $\rho_a = 0$ is thermodynamically stable even in this enlarged space. Thus we conclude that the RN black brane solution is thermodynamically stable.

V. GEOMETRICAL INSTABILITY OF THE RN BLACK BRANE

In this section, we shall investigate possible geometrical instabilities of the RN black brane system with a particular probe mode turned on. Depending on the mass of the supergravity mode, the RN black brane may become geometrically unstable below a certain critical temperature T_c leading to a new black brane solution wearing nontrivial hairs.

For scalar modes, arising of the instability may be understood as follows [15–17]. Note that the usual geometrical stability condition for the AdS_{d+1} spacetime is given by the so called Breitenlohner-Freedman (BF) bound [23],

$$-\frac{d^2}{4} \le m^2, \tag{5.1}$$

which is indeed respected by any scalar mode of the present theory. This is the relevant condition for the stability of the near boundary region of the RN black brane solution that is asymptotic to AdS_4. On the other hand, the near horizon geometry of the extremal RN black brane in $d+1$ dimensions is given by $\mathrm{AdS}_2 \times \mathbb{R}^{d-1}$ with a scaled AdS radius of $1/\sqrt{d(d-1)}$. Hence for scalars, the BF bound of this region is violated if

$$m^2 < -\frac{d(d-1)}{4}. \tag{5.2}$$

Therefore for our case of $d = 3$, the instability below a certain temperature may occur if the mass squared of a bulk scalar is ranged in

$$-\frac{9}{4} \le m^2 < -\frac{3}{2}. \tag{5.3}$$

For the higher spin fields with spin $s \ge \frac{1}{2}$, one may show that there is no potential instability once they are neutral under the $U(1)$ gauge field of the RN black brane. Thus scanning the supergravity modes in Table I, one finds that the possible gravitational instabilities are limited to the case of scalar fields with $m^2 = -2$. As explained in Section III, these bulk scalars are dual to the field theory operators of dimension $\Delta = 1$ and 2.

To show the instability of the black hole, let us study the positivity condition of the energy functional for the scalar field. The energy of a scalar field reads

$$E = \int drdxdy\sqrt{-g}\Big[|g^{tt}|(\dot\phi)^2 + g^{rr}(\phi')^2 + g^{xx}(\partial_x\phi)^2 + g^{yy}(\partial_y\phi)^2 + m^2\phi^2\Big], \tag{5.4}$$

where dots and primes denote derivatives with respect to t and r respectively. If there exists any normalizable (probe) mode φ of the scalar field ϕ which makes this energy functional negative, the geometrical instability of the RN background can be triggered driving the system to some new stable configuration. In order to find a possible mode of instability, we take φ as a function of r only and look for the negative fluctuation mode satisfying

$$\left(e^{3A}h(r)\varphi'\right)' + 2e^{3A}\varphi - \kappa_0^2\varphi = 0, \tag{5.5}$$

where κ_0 is a real constant and we set $m^2 = -2$. Of course one may turn on the spatial fluctuation by considering the probe field depending on x and y by $e^{i(k_x x + k_y y)}$ but this will only increase the energy of the system. Hence the above consideration will be sufficient.

The boundary conditions are crucial for a determination of the solution of (5.5). For $r = \infty$, we note that the behavior of scalar fields in the near boundary region takes a form,

$$\varphi \sim s_\Delta(x)e^{-(3-\Delta)A(r)} + o_\Delta(x)e^{-\Delta A(r)} + \cdots, \tag{5.6}$$

where $\cdots$ denote higher order terms. From the standard dictionary of AdS/CFT, the presence of s_Δ corresponds to turning on an external source term for the dual operator $O_\Delta(x)$ while o_Δ represents the operator expectation value $\langle O_\Delta(x)\rangle_{s_\Delta}$ in the presence of the source $s_\Delta(x)$. In our present problem, we would like to consider the system without introducing the source term and, thus, our choice of the boundary condition here and below is $s_\Delta = 0$ at $r = \infty$. The other boundary condition is for the horizon of black brane. Basically we shall require nonsingularity of the configuration there. For this, we need to expand the Eq. (5.5) into a generalized power series in $r - r_H$, and then we can obtain boundary condition for the horizon as

$$\varphi'(r_H) = -\frac{2 - \kappa_0^2 e^{-3A(r_H)}}{h'(r_H)}\varphi(r_H),$$
$$\varphi''(r_H) = \frac{2 - \kappa_0^2 e^{-3A(r_H)}}{2(h'(r_H))^2}(3A'(r_H)h'(r_H) + h''(r_H) + 2 - \kappa_0^2 e^{-3A(r_H)})\varphi(r_H) - \frac{3\kappa_0^2 A'(r_H)e^{-3A(r_H)}}{2h'(r_H)}\varphi(r_H), \tag{5.7}$$

by setting the coefficients of the negative powers of $r - r_H$ to zero. Note that the denominators are given by $h'(r_H)$: If it is too small, we may have trouble in numerical computations. In order to avoid potential numerical instabilities, we use another form of RN solution given by

$$A(r) = (3 - q^2)r, \quad h(r) = \frac{1 - (1 + q^2)e^{-3A(r)} + q^2 e^{-4A(r)}}{(3 - q^2)^2},$$
$$A_t = -\frac{2q}{3 - q^2}(e^{-A(r)} - 1), \quad \phi_a = 0. \tag{5.8}$$

This solution can be related to (3.13) by the coordinate transformation,

$$r' = \frac{r - r_H}{3 - q^2 e^{-4r_H}}, \qquad t' = e^{r_H}(3 - q^2 e^{-4r_H})t$$
$$x' = e^{r_H}x, \qquad y' = e^{r_H}y, \tag{5.9}$$

with the redefinition of the parameters,

$$q' = qe^{-2r_H}, \qquad \varepsilon' = \varepsilon e^{-3r_H}. \tag{5.10}$$

After the transformation, we drop primes for the notational simplicity. The temperature for this system reads

$$T = \frac{1}{4\pi}(3 - q^2), \tag{5.11}$$

and the location of the horizon is at $r = 0$. Here and below, we shall use this rescaled background for the numerical analysis.

Now one can solve the probe Eq. (5.5) numerically to find the temperature below which the negative mode begins to develop. For the numerical analysis, we use the standard shooting method based on a MATHEMATICA coding. The results are as follows: The on-set of instability occurs at

$$T_c = 0.0395(7) \qquad [q_c = 1.582(0)] \tag{5.12}$$

for the $\Delta = 1$ operator. For the $\Delta = 2$ operators, we found

$$T_c = 0.0003(5) \qquad [q_c = 1.7307(7)]. \tag{5.13}$$

The differences in the numbers of significant digits of T_c and q_c arise due to the fact that T_c is proportional to $3 - q_c^2$.

VI. PHASE STRUCTURES AND CRITICAL EXPONENTS

In the previous section, we have established the instability of the RN black brane in the presence of bulk scalars with $m^2 = -2$. As temperature decreases below the critical temperature, the scalar fields begin to develop a nontrivial profile that affects the original RN black brane geometry. As we shall see later on, the boundary CFT undergoes a phase transition by the condensation of the expectation value of the dual field theory operator $O_\Delta(x)$.

In this section, we shall investigate this phase transition by looking at the supergravity solutions. Since the changes in the thermodynamic quantities like the entropy, energy and so on are encoded in the geometry, the probe analysis of the previous section alone is not sufficient and inclusion of the full back-reaction to the geometry will be essential. Our study of the relevant solutions will be mainly based on numerical analysis.

Our starting point of the analysis[3] is the set of equations in (3.9) and (3.10) where we turn on only one scalar field ϕ with $m^2 = -2$. To set up the problem completely, one has to also specify the boundary conditions. Let us begin with the horizon side. At the horizon, $h(r)$ has to be zero at some finite $r = r_H$ which is the coordinate location of the horizon. Using the translational freedom of the r coordinate, we shall choose

$$r_H = 0. \tag{6.1}$$

In addition, we note that one has the scaling freedom for $x_\mu = (t, x, y)$ and the coordinate r to generate a new solution. Using this we shall fix

$$A(0) = 0 \quad \text{and} \quad h'(0) = 1. \tag{6.2}$$

At the horizon we basically require that all the fields in (3.10) should be regular. Setting $\phi(0) = u$, the regularity leads to

$$\phi'(0) = -\frac{2u}{h'(0)}, \quad A'(0) = \frac{3 - q^2 e^{-4A(0)} + u^2}{h'(0)}, \tag{6.3}$$

and

$$h''(0) = -9 - 3u^2 + 7q^2 e^{-4A(0)},$$
$$A''(0) = -\frac{2u^2}{(h'(0))^2},$$
$$\phi''(0) = \frac{2u}{(h'(0))^2}(1 + 2q^2 e^{-4A(0)}). \tag{6.4}$$

(Solving (3.9) and (3.10) with the coordinate and boundary conditions in (6.1), (6.2), (6.3), and (6.4) for the case of

[3]We use here our original coordinates without performing the coordinate transformation in (5.9) and (5.10). Then, when the scalar field vanishes, we shall directly obtain the exact RN solution (5.8) by fixing some freedoms in our coordinate choice.

vanishing scalar field leads directly to the exact RN solution in (5.8) without need of the transformation in (5.9) and (5.10).) At $r = \infty$, we shall impose the behaviors of fields in (3.22) that are required for the asymptotically AdS spacetime. Again note that the scalar field in the near boundary region takes the form

$$\phi \sim s_\Delta(x) e^{-(3-\Delta)A(r)} + o_\Delta(x) e^{-\Delta A(r)} + \cdots, \tag{6.5}$$

where $\cdots$ denote higher order terms. (The interpretation of $s_\Delta(x)$ and $o_\Delta(x)$ in the boundary CFT is the same as that in the previous section.) We shall set the source term $s_\Delta(x) = 0$, which corresponds to the boundary system without an external source term. By this last condition, u will be determined as a function of q by $u = u(q)$.

Now adopting the shooting method based on a MATHEMATICA coding, we perform a numerical analysis for $\Delta = 1$ and 2 cases separately. For each case, we find one set of solutions that is parameterized by the value of q. The resulting functions $u(q)$ in (u, q) plane are drawn in Fig. 1. For each case, there is a critical value of q_c beyond which the corresponding scalar field begins to develop a nontrivial profile. Namely if $q \le q_c$, the exact RN solution in (5.8), which satisfies all the coordinate and boundary conditions in (6.1), (6.2), (6.3), and (6.4), remains intact. This part is depicted in Fig. 1 by the vertical blue solid line for $q \le q_c$. If $q > q_c$, the RN black brane is modified by wearing a nontrivial scalar hair with $u(q) \neq 0$; In Fig. 1, the blue dots represent the data set which we obtained by the numerical analysis. The blue solid curve represents the fitting function that is obtained by the standard curve fitting method in MATHEMATICA.

Thus the critical temperature may be evaluated by using the RN black brane solution (5.8) with $q = q_c$ leading to

$$T_c = \frac{1}{4\pi}(3 - q_c^2). \tag{6.6}$$

By the standard curve fitting method, we found

$$T_c = 0.0396(4) \qquad [q_c = 1.581(6)], \tag{6.7}$$

for the $\Delta = 1$ scalar field, and

$$T_c = 0.0003(4) \qquad [q_c = 1.7308(3)] \tag{6.8}$$

for the $\Delta = 2$ scalar fields. In principle, these critical values should agree precisely with those from the probe analysis of the previous section. Therefore the numerical precision of our analysis can be estimated by comparison of the numerical values from the two methods. We see that the $\Delta = 2$ critical temperature, whose value is closer to zero, has a poorer numerical accuracy.

In order to show further numerics, instead of using the charge density $\rho = 2q/\kappa^2$, we shall use simply q which differs from the actual charge density by a constant factor $2/\kappa^2$. (See (3.5) for the definition of κ.) In the above sets of solutions, both q (or the charge density) and the temperature $T = T(q)$ changes as one changes q along the transition. But what we want is to fix q (or the charge density) while changing the temperature of the system along the transition. In order to generate such sets of solutions, we use the coordinate transformation of the gravity system by

$$x^\mu \to x^\mu/\lambda_s. \tag{6.9}$$

Note that, by the scale transformation, the thermodynamic quantities transform as

$$\begin{aligned} &T \to \lambda_s T, \qquad q \to \lambda_s^2 q, \qquad \mathcal{E} \to \lambda_s^3 \mathcal{E}, \\ &\mathcal{S} \to \lambda_s^2 \mathcal{S}, \qquad o_\Delta \to \lambda_s^\Delta o_\Delta, \qquad s_\Delta \to \lambda_s^{3-\Delta} s_\Delta, \end{aligned} \tag{6.10}$$

and, hence,

$$\frac{\partial o_\Delta}{\partial s_\Delta} \to \lambda_s^{2\Delta-3} \frac{\partial o_\Delta}{\partial s_\Delta}. \tag{6.11}$$

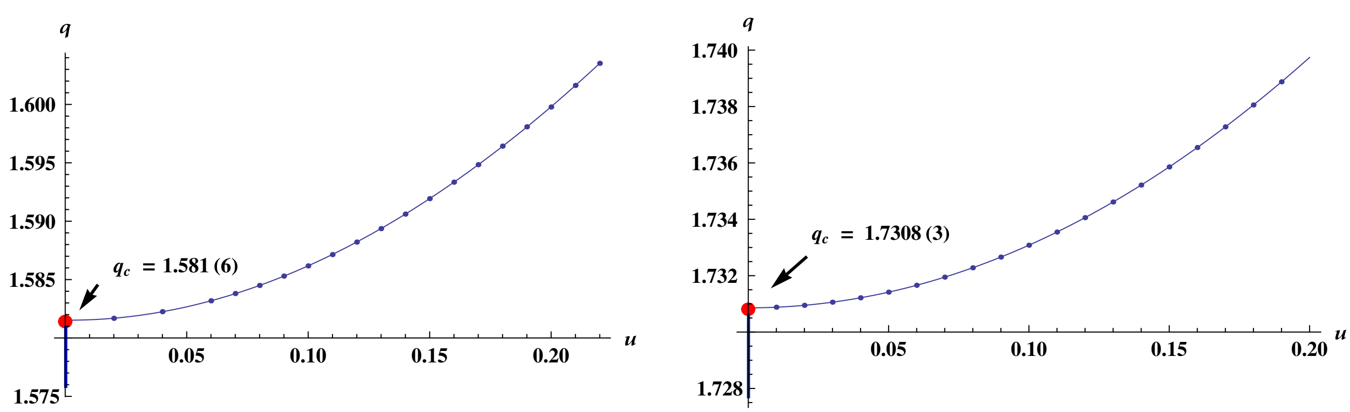

FIG. 1 (color online). The functions $u(q)$ in (u, q) plane are depicted in the left and the right sides, respectively, for the $\Delta = 1$ and the $\Delta = 2$ cases. For each case, the development of the scalar profile is represented by nonvanishing $u(q)$ for $q > q_c$; the dots represent our numerical data set while the solid curve is for the fitting function obtained by the standard curve fitting method in MATHEMATICA. Below q_c, $u(q) = 0$ is indicated by the blue solid line, which corresponds to the RN black holes in (5.8). In addition, we marked the numerical values of q_c by the red circles indicated by arrows, which are given by 1.581(6) and 1.7308(3), respectively, for the $\Delta = 1$ and the $\Delta = 2$ cases.

Using this scaling transformation, we shall fix $\tilde{q} = 1$ by choosing $\lambda_s = 1/\sqrt{q}$. (The quantities carrying a tilde are the ones after the scale transformation (6.10).) The new sets of scaled solutions are now parameterized by the rescaled temperature

$$\tilde{T} = \frac{T}{\sqrt{q}} \tag{6.12}$$

with fixed charge density $\tilde{\rho} = 2/\kappa^2$. Using the values in (6.7), we found the rescaled critical temperatures $\tilde{T}_c = T_c/\sqrt{q_c}$ as

$$\tilde{T}_c = 0.0315(6) \tag{6.13}$$

for the $\Delta = 1$ scalar field. For the $\Delta = 2$ scalar fields, we found

$$\tilde{T}_c = 0.0002(5) \tag{6.14}$$

using the values in (6.8).

Below the critical temperature, one finds a development of expectation value o_Δ which signals a phase transition. As we shall see details later on, this basically corresponds to the spontaneous symmetry breaking transition of the $SU(4)$ R-symmetry in which the condensate o_Δ plays the role of the order parameter. In terms of the scaled variable $\tilde{o}_\Delta = o_\Delta/q^{\Delta/2}$, the phase diagrams are depicted in Fig. 2 for the $\Delta = 1$ and the $\Delta = 2$ cases.

As expected, the transition for $\Delta = 1$ occurs at a higher critical temperature. Then the transition for $\Delta = 2$ cannot be treated separately. Namely one has to turn on both of the $\Delta = 1, 2$ scalar fields around the region of the $\Delta = 2$ transition in which the $\Delta = 1$ scalar field has developed some finite amount of profile representing the condensation. However remember that the scalar part of our starting Lagrangian is only valid up to quadratic orders. Hence our treatment loses its validity around the region of the $\Delta = 2$ transition. Though it is an interesting problem to clarify further, we shall leave it to future for investigation. For the remainder we shall focus on the transition involving the $\Delta = 1$ condensation.

The phase transition is of second order as in the usual cases of symmetry breaking transition. We note that natures of phase transitions are in general classified by their critical exponents. Here we compute numerically the exponents α, β and γ respectively defined by

$$\begin{aligned} &\frac{\partial \mathcal{E}}{\partial T} \sim |T - T_c|^{-\alpha} \\ &o_1 \sim |T - T_c|^{\beta} \\ &\chi_1 = \frac{\partial o_1}{\partial s_1}|_{s_1=0} \sim |T - T_c|^{-\gamma}. \end{aligned} \tag{6.15}$$

For the numerical estimation of the exponents α, β and γ, we use the numerical data sets, respectively, of the forms $(\log(T - T_c), \log(\partial E/\partial T))$, $(\log(T - T_c), \log o_1)$ and $(\log(T - T_c), \log \chi_1)$. Using the fitting with linear least squares, we estimated the relevant slopes as well as their standard deviations. From the corresponding data set for o_1 in Fig. 2, one finds

$$\beta_n = 0.04978 \pm 0.0032, \tag{6.16}$$

which is in a good agreement with the mean field value $\beta = 1/2$. Here and below the subscript n indicates that the relevant exponent is obtained by the numerical analysis. The left side of Fig. 3 shows the behavior of the energy as a function of temperature in the vicinity of the transition region. From the corresponding data set, the exponent for the specific heat is identified as

$$\alpha_n = 0.012 \pm 0.018, \tag{6.17}$$

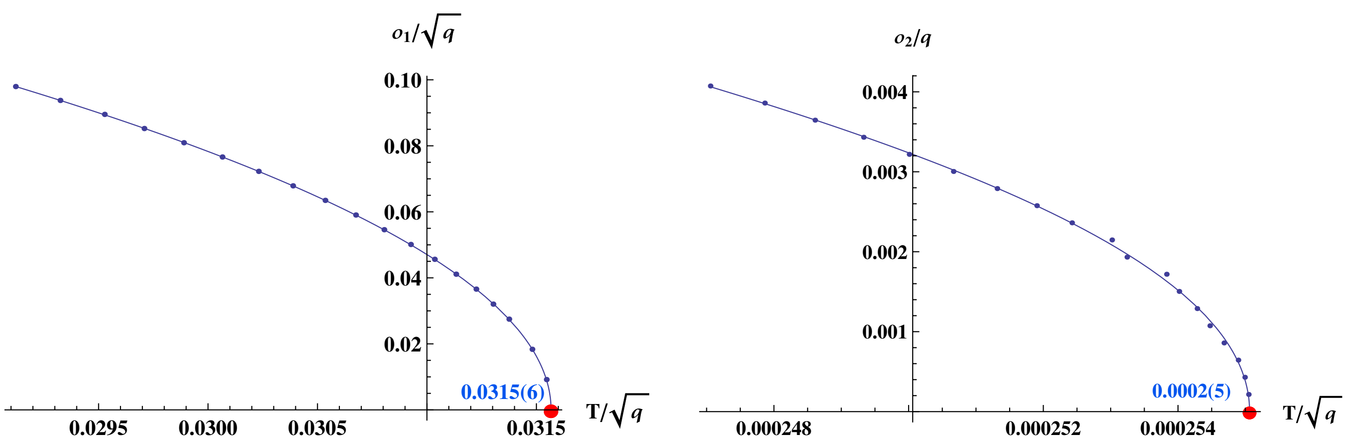

FIG. 2 (color online). The phase diagrams in the left and the right sides are, respectively, for the $\Delta = 1$ and the $\Delta = 2$ scalars. The system undergoes a symmetry breaking transition from a symmetric phase to a broken phase as the temperature is lowered below the critical temperature. For each case, the blue dots along the depicted curve represent our numerical data set and the blue solid curve itself depicts the fitting function that is obtained by the standard curve fitting method in MATHEMATICA. We marked the numerical values of $\tilde{T}_c$ by the red circles, placed on the rightmost side of the x axis, which are given by 0.315(6) and 0.0002(5) [see (6.13) and (6.14)], respectively, for the $\Delta = 1$ and the $\Delta = 2$ cases.

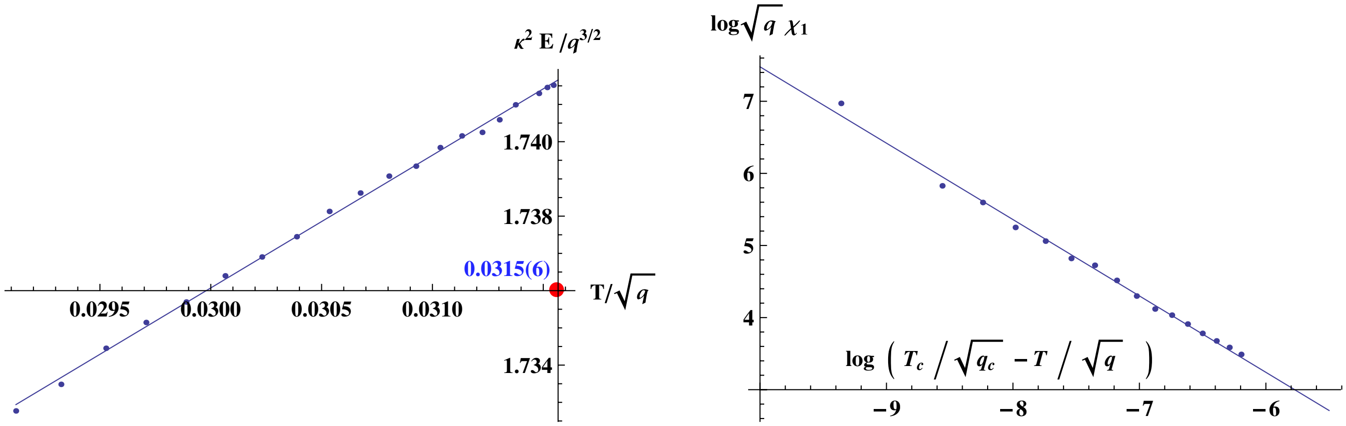

FIG. 3 (color online). The left figure shows the scaled energy density as a function of temperature. The right-hand side shows $\log(\sqrt{q}\chi_1)$ in (6.15) with respect to $\log(T/\sqrt{q} - T_c/\sqrt{q_c})$. For each case, the blue dots along the lines represent our numerical data set and the blue solid line depicts the fitting function that is obtained by the standard curve fitting method in MATHEMATICA.

which agrees well with the mean field value $\alpha = 0$. The right-hand side of Fig. 3 shows the temperature dependence of the susceptibility $\log\chi_1$ with respect to the logarithm of temperature. Again from the corresponding numerical data set, one finds

$$\gamma_n = 1.011 \pm 0.015, \tag{6.18}$$

which agrees well with the mean field value $\gamma = 1$. One can check that

$$(\alpha + 2\beta + \gamma)_n = 2.019 \pm 0.024 \tag{6.19}$$

which is consistent with the so-called Rushbrooke scaling law

$$\alpha + 2\beta + \gamma = 2. \tag{6.20}$$

Finally we shall reconfirm the above exponents by the analytic treatment. We note first that, due to the boundary condition $\phi(0) = u$, the scalar field behaves as $\phi \propto u$ when $u \ll 1$. Then inspecting the equations of motion (3.10) together with the boundary conditions (6.3) and (6.4), one finds that s_1 and o_1 should behave as

$$\begin{aligned} s_1 &= u(a_0(q) + a_2(q)u^2 + O(u^3)) \\ o_1 &= u(b_0(q) + b_2(q)u^2 + O(u^3)), \end{aligned} \tag{6.21}$$

before imposing the last boundary condition of $s_1 = 0$. For $|u| \ll 1$, the coefficients of the u^3 terms in s_1 and o_1 are basically controlled by the u^2 terms of $A'(0)$ and $h'(0)$ in (6.3) and (6.2), which one may argue by a careful examination of the equations of motion in (3.10): By this consideration, one may show that

$$a_2(q) = a_{20} + O(q - q_c), \qquad b_2(q) = b_{20} + O(q - q_c) \tag{6.22}$$

with $a_{20} \neq 0$ and $b_{20} \neq 0$. Then, since $s_1 = 0$ has a nontrivial solution $u(q) \neq 0$ only for $q > q_c$, $a_0(q)$ should behave as

$$a_0(q) = a_{01}(q - q_c) + O[(q - q_c)^2] \tag{6.23}$$

with $a_{01}a_{20} < 0$ for $|q - q_c| \ll 1$. (Here we assume the existence of q_c, which is justified by our numerical analysis.) Since we know that the $o_1 = 0$ boundary condition instead of $s_1 = 0$ leads to a different value for the critical charge, $b_0(q)$ has the expansion of the form

$$b_0(q) = b_{00} + O(q - q_c) \tag{6.24}$$

with $b_{00} \neq 0$. This argument shows that the $s_1 = 0$ condition leads to a solution

$$u = \begin{cases} \sqrt{\frac{-a_{01}}{a_{20}}}(q - q_c)^{1/2}(1 + O(q - q_c)) & q \geq q_c \\ 0 & q < q_c. \end{cases} \tag{6.25}$$

Thus,

$$o_1 \sim |T - T_c|^{1/2}, \tag{6.26}$$

which implies that $\beta = \frac{1}{2}$. Since the scalar field contribution to the energy density is of order u^2, the energy density has to be of the form

$$\mathcal{E} = e_0 + e_1(q - q_c) + e_2u^2 + \cdots \tag{6.27}$$

where the first two terms are from the original RN black brane with $u = 0$. We then conclude that $\alpha = 0$ since the leading power of specific heat is zero. By changing $u = u(q)$ to $u = u(q) + \delta u$, s_1 and o_1 in (6.21) vary by

$$\delta s_1 = 3a_{20}(u(q))^2\delta u, \qquad \delta o_1 = b_{00}\delta u \tag{6.28}$$

to the leading orders. Then the susceptibility behaves as

$$\frac{\delta o_1}{\delta s_1} = \frac{b_{00}}{3a_{20}} \frac{1}{(u(q))^2} \sim |T - T_c|^{-1}, \tag{6.29}$$

which implies that $\gamma = 1$. This proves that our exponents are those of the mean field theory.

VII. INTERPRETATIONS AND DISCUSSIONS

In the previous section, we have established the phase transition whose exponents belong to the universality class of the mean field theory. For the resulting expectation value of $\Delta = 1$ operator, let us introduce a notation

$$\mathcal{M}_{IJ}(x) = \langle O^I_J(x) \rangle \tag{7.1}$$

where O^I_J is given in (3.2). Since $\mathcal{M}_{IJ} = \mathcal{M}^*_{JI}$ and $\mathcal{M}_{II} = 0$, the 4×4 matrix $\mathcal{M}$ is Hermitian and traceless. We have seen that the boundary CFT undergoes a phase transition. Above the critical temperature T_c, the condensate is vanishing, i.e. $\mathcal{M} = 0$ while at lower temperature, $\mathcal{M} \neq 0$. This is the spontaneous symmetry breaking phase transition where the $SU(4)$ R-symmetry of the ABJM theory is broken by the presence of the condensate $\mathcal{M}$.

Using $\mathcal{M}$ as an order parameter, the phase transition may be effectively described by the Landau free energy F

$$F/T = \int d^2\vec{x}\Big(\mathrm{Tr}\nabla\mathcal{M}\cdot\nabla\mathcal{M} - \frac{1}{T}\mathrm{Tr}\mathcal{M}\mathcal{H} + 2r_0(T - T_c)\,\mathrm{Tr}\mathcal{M}^2 + g_0(\mathrm{Tr}\mathcal{M}^2)^2\Big), \tag{7.2}$$

where $\vec{x} = (x, y)$ and the 4×4 traceless Hermitian matrix $\mathcal{H}$ is for the external source term in the adjoint representation of $SU(4)$. The form of the quartic term is determined by the symmetry. Because of the symmetry of the gravity solution, it should be invariant under the transformation of $\mathcal{M}$ by the $SU(4)$ R-symmetry. The Landau free energy is minimized if

$$\mathcal{M} = \begin{cases} 0 & T > T_c \\ \sqrt{\frac{r_0}{g_0}}(T_c - T)^{1/2}\hat{n} & T \le T_c \end{cases} \tag{7.3}$$

where $\hat{n}$ is a 4×4 traceless Hermitian matrix with $\mathrm{Tr}\hat{n}^2 = 1$. This implies that $\beta = 1$. Noting $C_V = -T\frac{\partial^2 F}{\partial T^2}$ with

$$F \sim (T - T_c)^2, \tag{7.4}$$

one has $\alpha = 0$. Similarly one can check that $\gamma = 1$ from the definition of the susceptibility. One can further compute the correlation length scale ξ,

$$\xi = (2r_0(T_c - T))^{-(1/2)} \sim |T - T_c|^{-\nu} \quad (\text{for } T \le T_c), \tag{7.5}$$

from which one finds $\nu = 1/2$. It will be interesting to test the prediction of the mean field theory by the direct analysis of the bulk gravity system. But on top of the new background obtained from the numerical analysis, further numerical analysis of solving the scalar fluctuation equation has to be performed. However this requires some improvement of the current method to reach the required numerical precision.

As we described in Section III, our $U(1)$ charged black brane describes a system where the $U(1)$ number is carried by both bosonic and fermionic degrees. The build-up of the Fermi surface at the weak coupling side would not be possible if the bosonic degrees were in a unbroken phase. Indeed we have argued that there will be a condensation of the elementary degrees at weak coupling with low enough temperature. For the strong coupling side, we know that the basic degrees that are weakly coupled among themselves are now organized by the supergravity modes. We found that the symmetric phase of the RN black brane again becomes unstable when $T < T_c$. The effective mass squared of the Landau free energy description becomes negative in this region driving the symmetry breaking phase transition by the condensation of the operator expectation value. This partially explains the fate of the bosonic $U(1)$ charged state of the ABJM theory at strong coupling. But as observed before, there is a finite entropy of the extremal RN black brane at zero temperature. However due to the phase transition, ordering of degrees will lead to a reduction of entropy in general. Hence there can be a chance of having a zero-entropy system at zero temperature. One interesting point of study is then the modification of the RN black brane by the scalar condensation in the near zero temperature region. Since the scalar part of our starting Lagrangian is only valid up to quadratic orders, we cannot answer this question at the moment. Our understanding of higher-order scalar contribution to the Lagrangian must first be improved.

Another interesting aspect is the build-up of theFermi surface as mentioned before. The picture presented in Ref. [19] is seemingly plausible. It mostly concerns the region of near zero temperature, and works only for a pure RN black brane system. But now close to zero temperature, our black brane solution is significantly modified by the condensation of the nontrivial scalar profile. Further studies of the low temperature region appear interesting, in particular, with focus on the question whether the Fermi picture is still valid.

Finally we comment on the theorem by Coleman-Mermin-Wagner-Hohenberg [24]. The theorem states that a continuous global symmetry cannot be broken spontaneously in 1 + 1 dimensions at zero temperature and 2 + 1 dimensions at finite temperature. Our example appears to contradictwhat the above theorem states. However, we are working in the strict large N limit, which makes the dual gravity system completely classical. This violates some assumptions of the above theorem as pointed out in Ref. [25]. Our example here is reminiscent of the clash of the unitarity in the explicit time dependent black hole solution which describes a thermalization of a boundary field theory [26]. There again the trouble stems from the large N limit of the boundary field theory.

ACKNOWLEDGMENTS

K. K. Kim would like to thank Gungwon Kang for helpful discussions. D. B. was supported in part by NRF SRC-CQUeST-2005-0049409 and NRF Mid-Career Researcher Program 2011-0013228. K. K. and S. Y. were supported in part by WCU Grant No. R32-2008-000-10130-0.

---

[1] O. Aharony, O. Bergman, D. L. Jafferis, and J. Maldacena, J. High Energy Phys. 10 (2008) 091.
[2] J. A. Minahan and K. Zarembo, J. High Energy Phys. 09 (2008) 040.
[3] D. Bak and S. J. Rey, J. High Energy Phys. 10 (2008) 053.
[4] D. Bak, H. Min, and S. J. Rey, Nucl. Phys.**B827**, 381 (2010); Phys. Rev. D **81** 126004 (2010).
[5] J. A. Minahan, O. Ohlsson Sax, and C. Sieg, J. Phys. A **43**, 275402 (2010); J. A. Minahan, O. Ohlsson Sax, and C. Sieg, Nucl. Phys. **B846** 542-606 (2011);M. Leoni, A. Mauri, J. A. Minahan, O. O. Sax, A. Santambrogio, C. Sieg, and G. Tartaglino-Mazzucchelli, J. High Energy Phys. 1012 (2010) 074.
[6] D. Bak and S. Yun, Classical Quantum Gravity **27**, 215011 (2010).
[7] D. Bak, K. B. Fadafan, and H. Min, Phys. Lett. B **689**, 181 (2010).
[8] P. B. Arnold and L. G. Yaffe, Phys. Rev. D **52**, 7208 (1995).
[9] D. Bak, A. Karch, and L. G. Yaffe, J. High Energy Phys. 08 (2007) 049.
[10] M. Smedback, J. High Energy Phys. 04 (2010) 002.
[11] D. T. Son and A. O. Starinets, J. High Energy Phys. 03 (2006) 052.
[12] I. R. Klebanov and E. Witten, Nucl. Phys. B **556**, 89 (1999).
[13] B. E. W. Nilsson and C. N. Pope, Classical Quantum Gravity **1**, 499 (1984).
[14] D. Yamada and L. G. Yaffe, J. High Energy Phys. 09 (2006) 027.
[15] T. Faulkner, G. T.Horowitz, and M. M. Roberts, Classical Quantum Gravity **27**205007 (2010).
[16] T. Faulkner, G. T. Horowitz, and M. M. Roberts, J. High Energy Phys. 1104 (2011) 051.
[17] S. A. Hartnoll, C. P. Herzog, and G. T. Horowitz, J. High Energy Phys. 12 (2008) 015.
[18] S. S. Gubser and A. Nellore, J. High Energy Phys. 04 (2009) 008.
[19] S. J. Rey, Prog. Theor. Phys. Suppl. **177**, 128 (2009); D. Bak and S. J. Rey, J. High Energy Phys. 09 (2010) 032.
[20] S. S. Gubser and I. Mitra, J. High Energy Phys. 08 (2001) 018.
[21] V. E. Hubeny and M. Rangamani, J. High Energy Phys. 05 (2002) 027.
[22] T. Hirayama, G. Kang, and Y. Lee, Phys. Rev. D **67**, 024007 (2003).
[23] P. Breitenlohner and D. Z. Freedman, Phys. Lett. B **115**, 197 (1982).
[24] S. R. Coleman, Commun. Math. Phys. **31**, 259 (1973); N. D. Mermin and H. Wagner, Phys. Rev. Lett. **17**, 1133 (1966); P. C. Hohenberg, Phys. Rev. **158**, 383 (1967).
[25] D. Anninos, S. A. Hartnoll, and N. Iqbal, Phys. Rev. D **82**, 066008 (2010).
[26] D. Bak, M. Gutperle, and A. Karch, J. High Energy Phys. 12 (2007) 034; D. Bak, M. Gutperle, and S. Hirano, J. High Energy Phys. 02 (2007) 068