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Noncommutative magnetic moment of charged particles

T. C. Adorno,[1] D. M. Gitman,[1] A. E. Shabad,[2] and D. V. Vassilevich[3,4]
[1]*Instituto de Física, Universidade de São Paulo, Brazil*
[2]*P.N. Lebedev Physics Institute, Moscow, Russia*
[3]*CMCC - Universidade Federal do ABC, Santo André, S.P., Brazil*
[4]*Department of Physics, St. Petersburg State University, Russia*
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It has been argued that in noncommutative field theories, the sizes of physical objects cannot be taken smaller than an ''elementary length'' related to noncommutativity parameters. By gauge covariantly extending field equations of noncommutative $U(1)_\star$ theory to cover the presence of external sources, we find electric and magnetic fields produced by an extended static charge. We find that such a charge, apart from being an ordinary electric monopole, is also a magnetic dipole. By writing off the existing experimental clearance in the value of the lepton magnetic moments for the present effect, we get the bound on noncommutativity at the level of 10^4 TeV.

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The noncommutative (NC) field theory suggests a very profound revision of the idea of space and time by referring to 4-coordinates X^μ as operatorial, noncommuting quantities, $[X^\mu, X^\nu] = i\theta^{\mu\nu}$. Usually the antisymmetric NC tensor $\theta^{\mu\nu}$ is taken as constant and small in its magnitude; this will be our choice, too. Because of the uncertainty relation intrinsic to the noncommutativity [1], various components of the coordinate 4-vector cannot be simultaneously given definite values. This implies that sizes of physical objects in this theory cannot be taken smaller than an ''elementary length.'' Throughout this paper we consider the space-space noncommutativity. This means that a reference frame [2] is admitted to exist, wherein $\theta^{0\nu} = 0$, so that the remaining NC parameters can be combined in the 3-vector $\theta^i \equiv (1/2)\varepsilon^{ijk}\theta^{jk}$ in that frame.

The characteristic length is defined through the absolute value of this vector as $l_{\rm NC}^2 = |\theta| \equiv (\hbar c/\Lambda_{\rm NC})^2$, where $\Lambda_{\rm NC}$ is the corresponding energy scale. Hence, all the sources should not be pointlike, but rather have a characteristic size of the order of $l_{\rm NC}$, see also Ref. [3].

The aim of this work is to study the field produced by a finite-size static charge in NC electrodynamics. To treat such charges we need to avoid the difficulty caused by the gauge-invariance violation by a classical external source, analogous to the trouble encountered in non-Abelian field theories [4]. This will be achieved by extending the Seiberg-Witten (SW) map [5] to the case when external currents are present in the lowest nontrivial order with respect to the NC parameters. With this extension in hands, we find corrections to the electromagnetic potentials produced by a finite-size static electric charge. Solutions, regular everywhere, neither can nor should have a point-charge limit. By selecting such solutions we essentially part from the standard commutative case. We find that a static electric charge eZ distributed in a spherically symmetric way over a sphere of a finite radius a, apart from being an ordinary electric monopole, is also a magnetic dipole. Its magnetic moment is directed along the NC vector $\boldsymbol{\theta}$ and its value is quadratic in the charge eZ and depends on the size a of the latter. In this way we define the NC contribution to the magnetic moment of an elementary particle viewed upon as a classical particle with its electric charge distributed according to the electromagnetic form factor. This NC contribution appears to be proportional to $l_{\rm NC}^2/a$, with a being the charge radius. Then, a comparison with experimental results allows us to establish restriction on $l_{\rm NC}$ (or on $\Lambda_{\rm NC}$). The strongest bounds are coming from the measurements of the anomalous magnetic moments of leptons under the assumption that the charge radius is given by the noncommutativity length, $a \sim l_{\rm NC}$.

Previously, an NC magnetic solution for the field of a static electric charge was found by Stern [6], who, in contrast to our work, assumed that the charge is truly pointlike (of zero radius). The results then differ drastically from ours, and we shall present a comparison between the two approaches.

As an NC space we take the Moyal plane equipped with the Moyal star product $\check{f}(x) \star \check{g}(x) = \check{f}(x) \times \exp[(i/2)\overleftarrow{\partial}_\mu \theta^{\mu\nu} \overrightarrow{\partial}_\nu]\check{g}(x)$. We refer to the non-Abelian action of an NC $U(1)_\star$ gauge theory $\check{S} = \check{S}_{\rm A} + \check{S}_{jA}$,

$$\check{S}_{\rm A} = -\frac{1}{16\pi c}\int dx \check{F}_{\mu\nu} \star \check{F}^{\mu\nu}, \qquad \check{S}_{jA} = -\frac{1}{c^2}\int dx \check{j}^\mu \star \check{A}_\mu \tag{1}$$

that consists of the standard gauge-invariant part $\check{S}_{\rm A}$, where $\check{F}_{\mu\nu} = \partial_\mu \check{A}_\nu - \partial_\nu \check{A}_\mu + ig[\check{A}_\mu \overset{\star}{,} \check{A}_\nu]$, and the part $\check{S}_{jA}$ responsible for the interaction of the electromagnetic field potential $\check{A}_\mu$ with an external current $\check{j}^\mu$. Here and in what follows the designation $[\overset{\star}{,}]$ means the Moyal commutator, while the gauge coupling constant g is, as usual, identified with the elementary electric charge $g = e/(\hbar c)$ (see e.g. Guralnik *et al.* [7]) in order that the interaction strength between the electromagnetic and a complex, say, spinor field

$\check{\psi}$ might be fixed in a gauge-invariant way as $\int dx \bar{\check{\psi}} \star \gamma^\mu(\partial_\mu - ie/(\hbar c)\check{A}_\mu\star)\check{\psi}$. We shall still be distinguishing the constants e and g until their explicit mutual identification is needed. The compatibility condition $\check{D}_\mu \delta\check{S}/\delta\check{A}_\mu = 0$ of the equations of motion $\delta\check{S}/\delta\check{A}_\mu = 0$ requires that the current and the field be related by the equation of covariant current-conservation $\check{D}_\mu \check{j}^\mu = \partial_\mu \check{j}^\mu + ig[\check{A}_\mu \stackrel{\star}{,} \check{j}^\mu] = 0$. This cannot provide the vanishing of the variation $\delta\check{S}_{jA} = -(1/gc^2)\int dx\{(\partial_\mu \check{j}^\mu) \star \check{\lambda}\}$ under a gauge transformation with the parameter $\check{\lambda}$, because it would require the conservation law $\partial_\mu \check{j}^\mu = 0$, incompatible with the equations of motion, see [8]. Hence, the total action $\check{S}$ is not gauge invariant. To handle this difficulty we shall in what follows be basing on the field equation $\delta\check{S}/\delta\check{A}_\mu = 0$, which is gauge covariant.

The $U(1)_\star$ gauge theory is consistent as it satisfies the criteria of Ref. [9]. Therefore, it is not necessary to consider SW map or to make the gauge transformations twisted [10]. However, for studying phenomenological aspects of an NC theory it is advisable [7] to perform the SW map, since it allows one to work with commuting electromagnetic fields A^μ that have standard $U(1)$ gauge transformation properties. It is known that in the lowest nontrivial approximation in the NC parameter θ, to which approximation we shall henceforth restrict ourselves, the field $\check{A}_\mu$ is SW-mapped as

$$\check{A}_\mu = A_\mu + \frac{g}{2}\theta^{\alpha\beta}A_\alpha[\partial_\beta A_\mu + f_{\beta\mu}], \tag{2}$$

where $f_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. In our case Eq. (2) should be supplemented by the SW map for currents [8,11]

$$\check{j}^\mu = j^\mu + g\theta^{\alpha\beta}A_\alpha\partial_\beta j^\mu, \tag{3}$$

that is deduced from the requirement that the external current should gauge transform covariantly $\delta\check{j}^\mu = i[\check{\lambda} \stackrel{\star}{,} \check{j}^\mu]$, the same as the current of charged particles, e.g. $\bar{\check{\psi}}\gamma_\mu\check{\psi}$, does. The SW map is not unique, but one can show [8], that the corresponding ambiguity does not affect corrections to the potential of a static charge to the first order in θ. After the SW map (2) and (3) is applied to the equations of motion $\delta\check{S}/\delta\check{A}_\mu = 0$ and $\check{D}_\mu\delta\check{S}/\delta\check{A}_\mu = 0$ one gets the nonlinear field equations with external current, valid to the first order in $\theta^{\mu\nu}$,

$$\begin{aligned} &\partial_\nu f^{\nu\mu} - g\theta^{\alpha\beta}\Big[\partial_\nu(f_\alpha{}^\nu f_\beta{}^\mu) - f_{\nu\alpha}\partial_\beta f^{\nu\mu} \\ &\quad - A_\alpha\partial_\beta\Big(\partial_\nu f^{\nu\mu} - \frac{4\pi}{c}j^\mu\Big)\Big] = \frac{4\pi}{c}j^\mu, \\ &\partial_\mu j^\mu + g\theta^{\alpha\beta}(f_{\mu\alpha}\partial_\beta j^\mu + A_\alpha\partial_\beta\partial_\mu j^\mu) = 0. \end{aligned} \tag{4}$$

The explicit presence of potentials in (4) may look disturbing, but this difficulty is easily solved. To restore covariance we consider a perturbative solution of Eqs. (4) by expanding it in the noncommutative parameter. Explicitly, starting with the zeroth approximation $A^{(0)}$, $j^{(0)}$ that satisfies the standard Maxwell $\partial_\nu f^{(0)\nu\mu} = (4\pi/c)j^{(0)\mu}$ and current-conservation $\partial_\mu j^{(0)\mu} = 0$ equations, we obtain for the first-order corrections $A^{(1)}$, $j^{(1)}$

$$\begin{aligned} &\partial_\nu f^{(1)\nu\mu} - g\theta^{\alpha\beta}(\partial_\nu(f_\alpha^{(0)\nu}f_\beta^{(0)\mu}) - f_{\nu\alpha}^{(0)}\partial_\beta f^{(0)\nu\mu}) = \frac{4\pi}{c}j^{(1)\mu}, \\ &\partial_\mu j^{(1)\mu} + g\theta^{\alpha\beta}f_{\mu\alpha}^{(0)}\partial_\beta j^{(0)\mu} = 0. \end{aligned} \tag{5}$$

In what follows we shall study solutions to (5) produced by a static spherically symmetric charge distribution. It is defined in two regions, I: $r < a$ and II: $r > a$, $r = |\mathbf{x}|$,

$$j^{(0)\mu} = (c\rho, 0),\ \rho_{\rm I}(\mathbf{x}) = \frac{3}{4\pi}\frac{Ze}{a^3}, \qquad \rho_{\rm II}(\mathbf{x}) = 0. \tag{6}$$

The uniform charge distribution inside the sphere, whose radius is a, is taken for simplicity. Extensions to arbitrary spherical symmetric distributions, continuous ones included, may be also considered, when necessary. The charge density (6) tends to the delta function in the point-charge limit: $\rho(\mathbf{x}) = Ze\delta^3(\mathbf{x})$, as $a \to 0$. We shall argue, however, that the corresponding point-source solution (the Green function) does not exist even as a standard generalized function. Once no spherical *physical object* should be taken with its radius smaller than the elementary length, we will restrict our consideration to the values $a > l_{\rm NC}$.

We use the Coulomb gauge $\partial_i A^i = 0$ for the stationary solutions, to which we confine our consideration. Then the standard Maxwell equations provide the following spherically symmetric, $A^{(0)\mu}(\mathbf{x}) = A^{(0)\mu}(r)$, electromagnetic potential $A^{(0)\mu} = (A^{(0)0}, 0)$,

$$A_{\rm I}^{(0)0}(r) = -\frac{Ze}{2a^3}r^2 + \frac{3}{2}\frac{Ze}{a}, \qquad A_{\rm II}^{(0)0}(r) = \frac{Ze}{r}, \tag{7}$$

which satisfies the smoothness conditions $A_{\rm I}^0(r)|_{r=a} = A_{\rm II}^0(r)|_{r=a}$, $\partial_r A_{\rm I}^0(r)|_{r=a} = \partial_r A_{\rm II}^0(r)|_{r=a}$ at the boundary of the sphere, is regular in the origin $A^0(0) \neq \infty$, and falls off at infinity $A^0(r)|_{r\to+\infty} = 0$.

The analysis presented above is valid for arbitrary constant $\theta^{\mu\nu}$. Henceforward we restrict ourselves to the space-space noncommutativity ($\theta^{0\mu} = 0$). Because of the spherical symmetry and to the stationarity, the second equation in (5) is satisfied by $j^{(1)\mu} = 0$, no correction to the current is required. This implies that the current remains dynamically intact, $j^\mu = j^{(0)\mu}$, so we may refer to it as a fixed external current, as this is customary in an $U(1)$ theory. The NC Maxwell equation (5) for the zeroth component ($\mu = 0$) now reduces to $\nabla^2 A^{(1)0} = 0$, so that there are no first-order corrections $A^{(1)0}(\mathbf{x})$ to the potential, that would satisfy the same boundary conditions. (Such corrections appear, if a background magnetic field is added to the zero-order solution (7), see [8].) However, for the spatial components ($\mu = k = 1, 2, 3$) we obtain the inhomogeneous Laplace equations

$$\nabla^2 A_{\rm I}^{(1)k}(\mathbf{x}) = -g\left(\frac{Ze}{a^3}\right)^2 \theta^{ik} x^i, \quad \nabla^2 A_{\rm II}^{(1)k}(\mathbf{x}) = -g\left(\frac{Ze}{r^3}\right)^2 \theta^{ik} x^i. \tag{8}$$

Their only smooth solution, regular in $r = 0$ and decreasing for $r \to \infty$ is

$$\begin{aligned} A_{\rm I}^{(1)k}(\mathbf{x}) &= -\frac{g}{4}\left(\frac{Ze}{a^2}\right)^2\left(\frac{2}{5}\frac{r^2}{a^2} - 1\right)\theta^{ik} x^i, \\ A_{\rm II}^{(1)k}(\mathbf{x}) &= \frac{g}{4}\left(\frac{Ze}{r^2}\right)^2\left(\frac{8}{5}\frac{r}{a} - 1\right)\theta^{ik} x^i. \end{aligned} \tag{9}$$

This solution neither has nor should have the point-source limit at $a \to 0$. The leading long-distance part of the vector-potential $\mathbf{A}_{\rm II}^{(1)}$ behaves like that of a magnetic dipole, the static charge (6) being thus a carrier of an equivalent magnetic moment $\mathcal{M}$,

$$\boldsymbol{A} = \frac{[\mathcal{M} \times \mathbf{x}]}{r^3}, \qquad \mathcal{M} = \boldsymbol{\theta}(Ze)^2 \frac{2g}{5a}. \tag{10}$$

Let us study the consequences of this relation for particle physics. Lower bounds on the NC scale based on high-energy experiments have been drastically improved during recent years. The analysis of primordial nucleosynthesis data [12] gives $\Lambda_{\rm NC} \gtrsim 3$ TeV as a conservative estimate, while with other choices of parameters the bound increases to 10^3 TeV. From ultrahigh energy cosmic ray experiments one deduces [13] that $\Lambda_{\rm NC} \gtrsim 200$ TeV. The data for photon-neutrino interaction put the bound for time-space NC into approximately the same range [14]. A much stronger bound, $\Lambda_{\rm NC} \gtrsim 5 \times 10^{11}$ TeV, was obtained [15] by analyzing an atomic magnetometer experiment [16]. Characteristic energy scale of this experiment is below 1 eV, while the typical scale of modern particle physics experiments reaches TeV. Between these scales the value of effective NC parameter may change considerably. One of mechanisms for such a change may be due to the QFT effects which may manifest themselves through the renormalization group variation of couplings with characteristic energy. Therefore, it is important to study independently the high-energy restrictions, which we do below by using (10).

For a particle of unit charge, $Z = 1$, and mass m the NC correction to magnetic moment reads

$$\delta_{\rm NC}|\mathcal{M}| = \alpha|\theta|\mu\frac{4m}{5a}, \tag{11}$$

where α is the fine structure constant and $\mu = e/(2m)$ is the corresponding magneton. From now on, we put $\hbar = c = 1$ and, consequently, $g = e$. For the proton, by taking the charge radius of 0.9 fm for a we conclude that the correction to the magnetic moment is below the experimental error of $2.3 \times 10^{-8}\mu_N$ [17] already for $\Lambda_{\rm NC} \simeq 0.24$ TeV. An estimate of the NC proton magnetic moment contribution into the hyperfine splitting of the energy states in a hydrogen atom, based on a NC theory [9,18] of electron spectrum, does not strengthen the bound on l found in [6] with the use of the nondipole magnetic solution given as Eq. (13) below. In the case of leptons, we require that NC corrections to the magnetic moment anomaly,

$$\delta_{\rm NC}((g_l - 2)/2) = 4\alpha|\theta|m_l/(5a_l) \tag{12}$$

lie within experimental errors, which are 3×10^{-13} for electrons, and 6×10^{-10} for muons [17]. With the estimate $a_e,\ a_\mu < 10^{-3}$ fm that corresponds to the LEP energy scale of 200 GeV we obtain $\Lambda_{\rm NC} \gtrsim 45$ TeV in the case of electrons and $\Lambda_{\rm NC} \gtrsim 14$ TeV in the case of muons. These two bounds are of the order of currently accepted restrictions on the NC scale, but do not improve them.

The situation changes if we accept that the charge radius of leptons is defined solely by the NC effects. That is, $a_e \simeq a_\mu \simeq \sqrt{|\theta|} = \Lambda_{\rm NC}^{-1}$. Then, from the restrictions on the muon anomalous magnetic moment we derive $\Lambda_{\rm NC} \gtrsim 10^3$ TeV, while for the electron we have $\Lambda_{\rm NC} \gtrsim 10^4$ TeV, or $l_{\rm NC} \lesssim 2 \times 10^{-8}$ fm. This is the strongest bound on the NC scale among the ones, which follow from the high-energy data.

In the argumentation above we used the experimental errors only while completely ignoring possible theoretical uncertainties. This can be done for the following simple reason. Any theoretical calculation based on the usual commutative quantum field theory predicts the magnetic dipole moment directed along the spin vector $\boldsymbol{S}$, which characterizes the state of a particle. The NC correction (10) is parallel to the NC vector $\boldsymbol{\theta}$, which is a characteristic of the background space-time. We expect that relative orientation of $\boldsymbol{S}$ and $\boldsymbol{\theta}$ taken for various particles in various experiments is random. The effect of noncommutativity is in widening the range of experimental data rather than in shifting the central value. Therefore, the experimental error does give a bound on the NC effects even without taking into account theoretical uncertainties.

Yet another, also smooth, solution of Eq. (8) for the vector potential is worth discussing:

$$\begin{aligned} A_{\rm I}^{(1)k}(\mathbf{x}) &= -\frac{g}{4}\left(\frac{Ze}{a^2}\right)^2\left(\frac{2}{5}\frac{r^2}{a^2} + \frac{8}{5}\frac{a^3}{r^3} - 1\right)\theta^{ik} x^i, \\ A_{\rm II}^{(1)k}(\mathbf{x}) &= -\frac{g}{4}\left(\frac{Ze}{r^2}\right)^2\theta^{ik} x^i, \end{aligned} \tag{13}$$

that is not regular in the origin, but decreases at large distance from the source faster than (9), in other words it is more localized. Unlike Eq. (9) solution (13) is not the field of a magnetic dipole, since it decreases away from the source faster than that. The second line in (13) does not depend on the size a of the charge and coincides with the magnetic solution found in [6] for the field produced by a pointlike static charge outside of it, i.e. for $r \neq 0$. It is highly singular, $\sim 1/r^3$, in the origin $r = 0$.

Correspondingly, it does not make a solution in a reasonable class of generalized functions, when continued to the point $r = 0$. (In this respect it deeply differs from the standard solution $A_{\rm II}^{(0)0}$ in (7), which, in the limit $a \to 0$, is less singular, $\sim 1/r$, and makes a generalized-function solution to the Laplace equation with $\delta^3(\mathbf{x})$ as its inhomogeneity (see e.g. [19]). That solution is defined in the whole $\mathbb{R}^3$, the point $r = 0$ included.) For this reason our choice is in favor of the nonsingular solution (9).

One can consider the solution which satisfies weaker conditions at infinity, so that an external homogeneous magnetic field is allowed. In such a case, one can find [8] many interesting similarities and differences with the QED effects [20].

To summarize, our main result is the (remarkably simple) formula (10) for NC magnetic moment of a spherical charge. Equation (10) is subject to quantum corrections and classical corrections of higher powers in the noncommutativity parameter θ, which are both small. So, we are confident that this result will remain valid in a more complete approach, as well as the bounds it imposes on the NC scale.

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