Scalar field localization on 3-branes placed at a warped resolved conifold

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We have studied the localization of a scalar field on a 3-brane embedded in a six-dimensional warped bulk of the form $M_4 \times C_2$, where M_4 is a 3-brane and C_2 is a 2-cycle of a six-dimensional resolved conifold $\mathcal{C}_6$ over a $T^{1,1}$ space. Since the resolved conifold is singularity-free in $r = 0$ depending on a resolution parameter a, we have analyzed the behavior of the localization of a scalar field when we vary the resolution parameter. On one hand, this enables us to study the effects that a singularity has on the field. On the other hand we can use the resolution parameter as a fine-tuning between the bulk Planck mass and 3-brane Planck mass and so it opens a new perspective to extend the hierarchy problem. Using a linear and a nonlinear warp factor, we have found that the massive and massless modes are trapped to the brane even in the singular cone ($a \neq 0$). We have also compared the results obtained in this geometry and those obtained in other six-dimensional models, such as stringlike geometry and cigarlike universe geometry.

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I. INTRODUCTION

Since the original Randall-Sundrum model [1,2] many works have intended to extend the localization of various fields on a 3-brane embedded in the higher dimensional bulk. Besides the localization of gravity and other fields, many models have been suggested to explain other physical problems, for instance, the small value of the cosmological constant [3]. In order to explain the geometry used to localize the fields in noncompact extra dimensions, some authors have assumed that the 3-brane is generated by a topological defect. In a six-dimensional bulk, Cohen and Kaplan [4] have found such a geometry generated by a global string. In this context the geometry has cylindrical symmetry and a naked singularity at $r = 0$, where the 3-brane is placed, and also another singularity far from the origin. Gregory [5] has found a nonsingular stringlike solution by adding a cosmological constant to the bulk and splitting the metric inside and outside the core of the string defect. For the continuity boundary condition on the core, Gregory has found the phase space of solutions describing the stable and unstable points. For a geometry generated by a local stringlike defect, Gherghetta and Shaposhnikov [6] have found a solution with negative cosmological constant on bulk that has trapped gravity. Oda [7] has extended this solution for a bulk built from a warped product of a $(p-1)$-brane and a S^n sphere and he has studied the localization of many kinds of fields. All the solutions above assume that the transverse space has spherical symmetry and the whole bulk has cylindrical symmetry. We have studied an extension of this approach for the localization of a scalar field where the transverse space has a conifold geometry whose singularity depends on a resolution parameter.

The conifold here is a conical manifold $\mathcal{C}^n$ over a X^{n-1} called a base space. X^{n-1} is a topological equivalent to S^{n-1} defined by the coset $X^{n-1} = SU(n-1)/SU(n)$ [8]. It has a naked singularity that arises as an orbifold fixed point of the group Z_n, i.e., $\mathcal{C}^n = R^n/Z_n$. The conifold is an example of a Calabi-Yau space, a Ricci-flat manifold that is a candidate to an internal space in compactification of string theories. The conifold is a generator of all Calabi-Yau spaces through a process that generates singularities and is called conifold transitions [9]. In this process, some fields become massless and then the spectrum of the fields is changed [10]. Despite these interesting properties, the general relativity is not well-defined on singularities and sometimes it is necessary take off the conical singularity. There are two main processes to smooth out the singularity: the first one is called deformation because it deforms the quadric that defines the conifold; the second is called resolution because it introduces a resolution parameter that controls the blowup of singularity [8]. These processes are used to study the extensions of anti-de Sitter–conformal field theory (AdS-CFT) correspondence [11,12].

The change of spectrum on conifold spaces and the symmetry properties of their smoothed versions have motivated the study of those spaces in brane world scenarios. Firouzjahi and Tye [13] have studied the behavior of the gravitational and Kaluza-Klein (KK) modes on a deformed conifold and they have shown that the graviton has a rather uniform probability distribution everywhere while a KK mode is peaked in the region near $r = 0$. This region is called the throat because it has a big curvature and interpolates between asymptotically flat regions. Furthermore, Noguchi *et al.* [14] have used the Klebanov-Strassler throat of a deformed conifold in order to obtain localized gravitational KK modes. Since the supergravity solution of a 3-brane converges to $\mathrm{AdS}_5 \times S^5$ for $r \to 0$, Brummer, Hebecker, and Trincherini [15] have used a throat of a conifold to deduce and extend the original Randall-Sundrum geometry. Further, Vázquez-Poritz [16] has shown that the Z_2 symmetry of the Randall-Sundrun model can be deduced from a dimensional reduction from a six-dimensional Eguchi-Hanson resolved conifold. Since this symmetry is natural in Eguchi-Hanson spaces, Vázquez-Poritz has shown that the metric used for the localization of the gravity can de obtained from a particular conifold. Furthermore, Pontón and Poppitz [17] have studied the relation between gravity localization on stringlike defects and an AdS-CFT correspondence on the so-called hidden brane. Since the stringlike geometry has a conical singularity far from the origin the authors have found that the singularity could be resolved using the AdS-CFT duality. On the other hand, Kehagias [18] has used a compact conical transverse space to explain the small value of the cosmological constant. All of these points have motivated us to study geometries where the transverse space is a smoothed conifold and ask whether that geometry could localize some kind of field in a 3-brane.

In this work we replaced the usual spherically symmetric transverse space by a 2-cycle of the resolved conifold. Since the resolved conifold has spherical symmetry for a fixed r and the radial metric component approaches to one asymptotically, this geometry converges to a stringlike one if we put the 3-brane far from the tip of the cone. Another feature of the resolved conifold that has great importance in the Randall-Sundrum-like model is its $\mathbb{Z}_2$ symmetry as pointed out in Ref. [12]. We have studied here the effects that variations of the resolution parameter, or in other terms the singularity, have on the localization of a scalar field in a 3-brane placed in the origin of the resolved conifold. The study of the effects of geometrical singularities on the localization problems has already done by Cvetic, Lu, and Pope [19], where the geometry was generated by a singular domain wall as well as by Gregory and Santos [20] in the global vortex geometry. In the present work, however, we have chosen a transverse space whose singularity depends continually on a parameter.

The resolved conifold geometry also generalizes the so-called cigarlike geometries. Indeed, in cigar manifolds the curvature is great but not infinity around the origin and flat asymptotically [21]. In resolved conifold geometry the value of the curvature in the origin is parametrized and asymptotically the curvature converges to zero or another constant. Using a cigarlike geometry without cosmological constant de Carlos and Moreno have found a supersymmetric solution that has trapped gravity [22]. On the other hand, the so-called Ricci flow is given by a parameter evolution of the metric through a heat-type equation called the Ricci equation [21,23,24]. This flow provides information about the stability of the manifold like the formation or blowup of singularities. Therefore, it is interesting to use the smoothed conifold geometries to study the stability of the bulk geometry in brane worlds using methods of geometric analysis like the Ricci equation.

This work is organized as follows. In Sec. II we have defined the metric for the resolved conifold and we have studied its principal properties as well as its dependence on the resolution parameter. We also have defined the conifold 2-cycle that we have worked out. In Sec. III we have proposed a warped metric ansatz, studied its Einstein equations, and compared it with the well-studied stringlike solutions. Still in this section, we have studied this geometry for a linear and a nonlinear warp factor. In Sec. IV we have studied the localization of a real scalar field using a linear and a nonlinear warp factor for both massive and massless modes. Finally we present our conclusions in Sec. V, summarizing our results and presenting some perspectives.

II. CONIFOLD GEOMETRY

The 6-conifold is a conical manifold $C_6 \subset \mathbb{C}^4$ defined by the quadric [8]:

$$z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0. \tag{1}$$

The metric of a 6-conifold over a X^5 compact space is

$$ds_6^2 = dr^2 + r^2 ds^2(X^5). \tag{2}$$

This space has a naked singularity in $r = 0$. For $X^5 = T^{1,1} = SU(2) \times SU(2)/U(1)$ the metric is [10,11]

$$ds_6^2 = dr^2 + \frac{r^2}{9}(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)^2 + \frac{r^2}{6}(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2 + d\theta_2^2 + \sin^2\theta_2 d\phi_2^2). \tag{3}$$

A smooth version of this conifold, called the resolved conifold, has the metric [12,25]

$$ds_6^2 = \left(\frac{r^2 + 6a^2}{r^2 + 9a^2}\right)dr^2 + \frac{r^2}{9}\left(\frac{r^2 + 9a^2}{r^2 + 6a^2}\right)(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)^2 + \frac{1}{6}r^2(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2) + \frac{1}{6}(r^2 + 6a^2)(d\theta_2^2 + \sin^2\theta_2 d\phi_2^2). \tag{4}$$

We have plotted the scalar curvature in Fig. 1 where we have shifted the origin to point $r = 5$. Note that the curvature is smooth for $a = 1$ in $r = 0$ and the curvature diverges in the origin for $a = 0$. Furthermore, the resolved conifold has a positive curvature and is asymptotically flat. These issues have motivated us to use this manifold as a prototype of the extension of transverse spaces in the brane worlds. As a matter of fact, many authors have studied the localization of fields in spherical backgrounds whose transverse space has positive, constant, and nonsingular curvature [4–7]. Since the resolved conifold is parametrized by the resolution parameter that controls the singularity at $r = 0$, we can study the effects the singularity has on the localization of fields in this particular space.

Note that in the limit $r \rightarrow 0$ the metric converges to a spherical one of radius a,

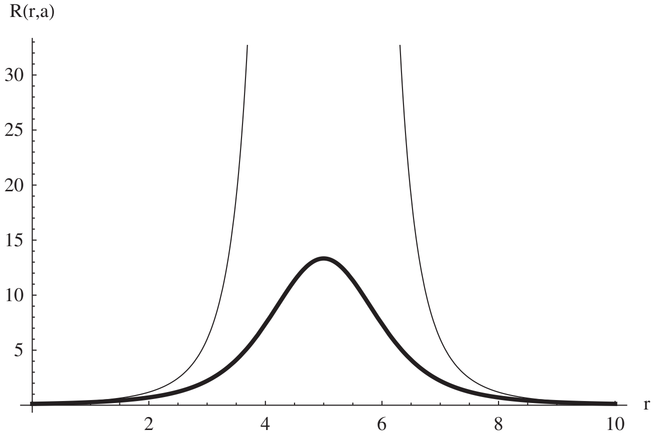

FIG. 1. The scalar curvature of the resolved conifold. The origin was shifted to the point $r = 5$. For $a = 1$ (thick line), the function is regular in $r = 0$ while for $a = 0$ (thin line) the scalar curvature diverges in the origin.

$$\lim_{r\to 0} ds_6^2 = a^2(d\theta_2^2 + \sin^2\theta_2 d\phi_2^2), \tag{5}$$

that has no singularity. Topologically this can be seen as a result of taking out a small neighborhood around $r = 0$ and replacing it by a S^2 of radius a. Since in the limit $a \to 0$ we reobtain the singular conifold again, the radius a can be used to measure how smooth the conifold is and then it is called the resolution parameter.

Now if we take as constants our angular coordinates ψ, ϕ_1, θ_2, ϕ_2, the cone 2-cycle can be written as the 2-resolved cone, namely,

$$ds_2^2 = \left(\frac{r^2 + 6a^2}{r^2 + 9a^2}\right)dr^2 + \frac{1}{6}(r^2 + 6a^2)d\theta^2. \tag{6}$$

This cone has a radial metric component $g_{rr} = \alpha(r) = (\frac{r^2+6a^2}{r^2+9a^2})$ whose graphic is plotted in Fig. 2. Note that $\lim_{r\to\infty} g_{rr} = 1$ and therefore asymptotically the cone approaches the plane $\mathbb{R}^2$ with cylindrical metric of an effective radius $r_{\rm eff} = \sqrt{\frac{(r^2+6a^2)}{6}}$ which is the transverse metric used in stringlike geometries. Near $r = 0$ we have a hyperbolic behavior with high curvature and this region is called the throat. The angular resolved conifold metric component $\beta(r, a) = \frac{(r^2+6a^2)}{6}$ has a conical singularity dependent on the resolution parameter. It is worthwhile to mention that as $a \to 0$ the width of the throat approaches zero.

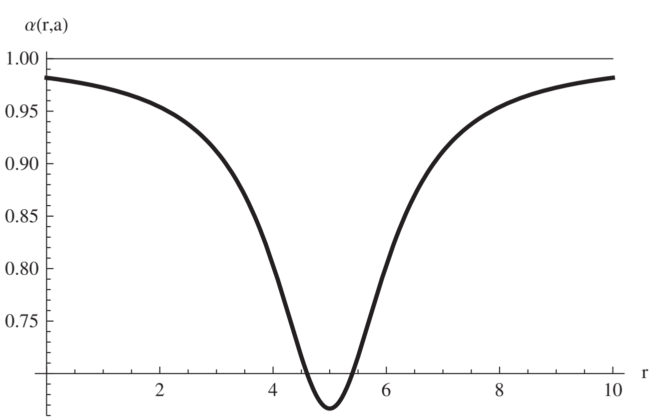

FIG. 2. Radial metric component of a resolved conifold. For $a = 1$ (thick line), the factor has a large variation and asymptotically converges to one. For $a = 0$ (thin line) the radial factor is equal to one as in the stringlike defects geometry.

Since the angular metric components diverge, the effective radius of the base sphere grows without limit. The scalar curvature of this 2-manifold is

$$R = R(r, a) = -\frac{6a^2(r^2 + 18a^2)}{(r^2 + 6a^2)^3}. \tag{7}$$

For the sake of comparison we cite here the scalar curvature for the Hamilton cigar geometry [21], namely,

$$R_H = \frac{4}{(1 + r^2)}. \tag{8}$$

Thus, this 2-cycle of the resolved conifold is a space of varying negative scalar curvature that converges asymptotically to zero. This behavior is similar to the Hamilton cigar that is a Ricci soliton used in the study of the stability of manifolds [24]. The Hamilton cigar is an example of a solution of the Ricci flow equation:

$$\frac{\partial g_{ab}}{\partial t} = -2R_{ab}, \tag{9}$$

where $t \in [0, 1]$ is a parameter that describes the evolution of the geometry of the manifold.

III. BULK GEOMETRY

Once described the metric and the 2-cycle of the resolved conifold which we will use as a transverse space, we want to study the localization of a scalar field in a 3-brane embedded in a six-dimensional bulk of form $M_6 = M_4 \times \mathcal{C}_2$, where $\mathcal{C}_2$ is a 2-cycle of the resolved conifold described above.

The action for the gravitational is

$$S_g = \frac{1}{2K_6^2}\int_{\mathcal{M}_6} d^6x\sqrt{-g}(R - 2\Lambda) + \int_{\mathcal{M}_6} dx^6\sqrt{-g}L_m \tag{10}$$

where $K_6^2 = \frac{8\pi}{M_6^4}$ and M_6^4 is the six-dimensional bulk Planck mass.

Further, let us assume the following ansatz for the energy-momentum tensor:

$$T^\mu_\nu = t_0(r)\delta^\mu_\nu, \tag{11}$$

$$T^r_r = t_r(r), \tag{12}$$

$$T^{\theta}_{\theta} = t_{\theta}(r), \tag{13}$$

where

$$T_{ab} = -\frac{2}{\sqrt{-g}}\frac{\partial L_m}{\partial g^{ab}}. \tag{14}$$

Now, let us choose the metric components in such a way that we can obtain the stringlike defect geometry in some limits. From now on, the metric ansatz will be

$$ds_6^2 = e^{-A(r)}\hat{g}_{\mu\nu}dx^{\mu}dx^{\nu} + \alpha(r,a)dr^2 + \beta(r,a)e^{-B(r)}d\theta^2, \tag{15}$$

where $a \geq 0$ is the resolution parameter. This ansatz extends the solution for stringlike defects by the inclusion of the resolved conifold metric factors $\alpha(r,a)$, $\beta(r,a)$. Since the radial component approaches one at infinity, if we put the 3-brane in a point far from the origin of the resolved conifold, this ansatz goes to the Oda ansatz one [7]. Furthermore, the geometry of the bulk is parameter-dependent which enables us to control the singularity.

The Einstein equations for the metric ansatz in Eq. (15) are

$$\begin{aligned}3A'' &+ B'' - \frac{3}{2}A'B' - B'^2 - 3A'^2 + \frac{3}{2}\left(\frac{\beta'}{\beta} - \frac{\alpha'}{\alpha}\right)A' \\ &+ \left(\frac{\beta'}{\beta} - \frac{1}{2}\frac{\alpha'}{\alpha}\right)B' - \frac{\beta''}{\beta} + \frac{1}{2}\frac{\alpha'}{\alpha}\frac{\beta'}{\beta} \\ &+ \alpha\left(\frac{1}{2}e^{A}\hat{R} - 2\Lambda + 2K_6^2 t_0\right) = 0,\end{aligned} \tag{16}$$

$$-3A'^2 - 2A'B' + 2\frac{\beta'}{\beta}A' + \alpha(e^{A}\hat{R} - 2\Lambda + 2K_6^2 t_r) = 0, \tag{17}$$

$$4A'' - 5A'^2 - 2\frac{\alpha'}{\alpha}A' + \alpha(e^{A}\hat{R} - 2\Lambda + 2K_6^2 t_{\theta}) = 0. \tag{18}$$

The continuity equation for the energy-momentum tensor is

$$\nabla^{a}T_{ab} = 0. \tag{19}$$

This equation yields a constraint on the components of the energy-momentum tensor

$$t_r' = 2A'(t_r - t_0) + \frac{B'}{2}(t_r - t_{\theta}) + \frac{\beta'}{2\beta}(t_{\theta} - t_r). \tag{20}$$

Equations (16)–(18) and the continuity equation (20) differ from the solution of the stringlike defects by the addition of the angular factor β of the resolved conifold metric.

Let us now sum the radial and angular Einstein equations and assume that $A(r) = B(r)$. This yields a linear differential equation for $A(r)$ in the form

$$2A''(r) - \left(\frac{\alpha'}{\alpha} + \frac{\beta'}{\beta}\right)A'(r) + K_6^2\alpha(t_{\theta} - t_r) = 0. \tag{21}$$

Defining

$$\delta(r,a) = -\left(\frac{\alpha'}{\alpha} + \frac{\beta'}{\beta}\right), \tag{22}$$

$$\chi(r,a) = K_6^2\alpha(t_{\theta} - t_r), \tag{23}$$

Eq. (21) can be rewritten as

$$2A''(r) + \delta(r,a)A'(r) + \chi(r,a) = 0. \tag{24}$$

The solution of Eq. (24) is

$$A(r) = A(0) - \int_0^r\left(\frac{\int_0^{r'}\eta(r'',a)\chi(r'',a)dr''}{\int_0^{r'}\eta(r'',a)dr''}\right)dr'. \tag{25}$$

Let us suppose the boundary conditions

$$A(0) = q, \tag{26}$$

$$\lim_{r\to\infty}A(r) = \infty, \tag{27}$$

where q is a constant.

Equation (24) with boundary conditions gives the warp factor. In the point $r = 0$, the metric defined by Eq. (15) becomes

$$ds_6^2 = \hat{g}_{\mu\nu}dx^{\mu}dx^{\nu} + \frac{a^2}{6}d\theta^2. \tag{28}$$

This is a factorizable metric of the space $M_4 \times S^1$ where S^1 has radius $\frac{a}{\sqrt{6}}$. Therefore, the 3-brane can be realized as a normal fiber bundle of strings in $r = 0$.

In this geometry, the relationship between the four-dimensional Planck mass (M_4) and the bulk Planck mass (M_6) is given by

$$M_4^2 = 2\pi M_6^4\int_0^{\infty}e^{-A(r)-(B(r)/2)}\sqrt{\alpha(r,a)\beta(r,a)}dr. \tag{29}$$

Therefore, we can use the resolution parameter in order to tune the ratio between the Planck masses and so explain the hierarchy between them. This is an extension of the stringlike tuning of the Planck masses: in the stringlike geometry, the adjustment is made by the six-dimensional cosmological constant Λ and the tension of the string μ [6,7]. Here, we have added a dependence on a geometrical parameter a. Note that the hierarchy is well-defined even for the singular cone ($a = 0$). Therefore, using parameter-dependent transverse spaces we could obtain a parameter-dependent hierarchy. Since there are many parameter-dependent spaces these manifolds could be used to solve the hierarchy problem. We argue that this dependence could be related to possible transformations in transverse space, for instance, the conical transitions.

A. Linear warp factor

Now let us choose a specific warp factor $A(r)$ and study its geometrical consequences. Let us choose the linear warp factor, i.e., $A(r) = kr$, where k is a real constant. This warp factor was widely used both in Randall-Sundrum models [1,2] and in stringlike geometries [5–7]. This warp factor was the first used to solve the hierarchy problem. Further, in Randall-Sundrum and stringlike geometries, it provides a AdS_6 geometry to the bulk, i.e., a maximally symmetric space with negative cosmological constant.

With that choice the Einstein equations become

$$\begin{aligned} B'' - \frac{3k}{2}B' - B'^2 - 3k^2 + \frac{3k}{2}\left(\frac{\beta'}{\beta} - \frac{\alpha'}{\alpha}\right) \\ + \left(\frac{\beta'}{\beta} - \frac{1}{2}\frac{\alpha'}{\alpha}\right)B' - \frac{\beta''}{\beta} + \frac{1}{2}\frac{\alpha'}{\alpha}\frac{\beta'}{\beta} \\ + \alpha\left(\frac{1}{2}e^A\hat{R} - 2\Lambda + 2K_6^2 t_0\right) = 0, \end{aligned} \tag{30}$$

$$-3k^2 - 2kB' + 2\frac{\beta'}{\beta}k + \alpha(e^A\hat{R} - 2\Lambda + 2K_6^2 t_r) = 0, \tag{31}$$

$$-5k^2 - 2\frac{\alpha'}{\alpha}k + \alpha(e^A\hat{R} - 2\Lambda + 2K_6^2 t_\theta) = 0. \tag{32}$$

Summing the radial and angular equations above, we obtain the solution for the warp factor $B(r)$:

$$\begin{aligned} B(r) = kr + \ln(\alpha(r,a)\beta(r,a)) \\ + \frac{K_6^2}{k}\int_0^r \alpha(r',a)(t_r(r') - t_\theta(r'))dr'. \end{aligned} \tag{33}$$

Therefore, we can get $B(r) = A(r)$ if

$$\ln(\alpha(r,a)\beta(r,a)) + \frac{K_6^2}{k}\int_0^r \alpha(r',a)(t_r(r') - t_\theta(r'))dr' = 0. \tag{34}$$

Equation (34) provides a constraint between the resolved conical geometry and the content of matter. In a vacuum, the warp factor $B(r)$ is given by

$$B(r) = kr + \ln(\alpha(r,a)\beta(r,a)). \tag{35}$$

The solution for the function $B(r)$ above differs from the stringlike defect one by the conifold metric components α, β [7]. In addition, note that in general $B(r)$ depends on the resolution parameter. Therefore, we can make $\beta(r,a) = 1$ and still detect the effects of the resolution on the conifold.

Let us suppose now an angular energy-momentum tensor of the form

$$t_\theta(r) = \zeta e^{kr} + \lambda(r) + \rho, \tag{36}$$

where ζ, ρ are constants and

$$\lambda(r) = \frac{1}{4K_6^2\alpha(r)}\left(5P^2 + 2P\frac{\alpha'}{\alpha} - 3\left(\frac{\alpha'}{\alpha}\right)^2\right). \tag{37}$$

Now let us suppose the 3-brane M_4 is a maximally symmetric space. Therefore, we can define a 3-cosmological constant Λ_3, satisfying

$$\hat{R}_{\mu\nu} - \frac{\hat{R}}{2}\hat{g}_{\mu\nu} = -\Lambda_3\hat{g}_{\mu\nu}. \tag{38}$$

Thus, its scalar curvature $\hat{R}$ must be constant. Therefore, from Eq. (32) we conclude that

$$\zeta = \frac{\hat{R}}{2K_6^2}, \tag{39}$$

$$\rho = \frac{\Lambda}{K_6^2}, \tag{40}$$

$$k = P. \tag{41}$$

It is worthwhile to mention that the solution above for t_θ differs from the stringlike type by the terms α and $\frac{\alpha'}{\alpha}$ and so we obtain the stringlike defect as a special case of the resolved conifold one.

For $B(r) = A(r)$ the components of the energy-momentum tensor are

$$t_r(r) = \frac{\Lambda}{K_6^2} - \frac{e^{kr}\hat{R}}{K_6^2} + \frac{1}{\alpha K_6^2}\left(5k^2 - 2\frac{\beta'}{\beta}k\right), \tag{42}$$

$$\begin{aligned} t_0(r) = \frac{\Lambda}{K_6^2} - \frac{e^{kr}\hat{R}}{2K_6^2} + \frac{1}{2\alpha K_6^2}\left(\frac{11}{2}k^2, +2\left(\frac{\beta'}{\beta} - \frac{\alpha'}{\alpha}\right)k \right. \\ \left. - \frac{\beta''}{\beta} + \frac{1}{2}\frac{\alpha'}{\alpha}\frac{\beta'}{\beta}\right). \end{aligned} \tag{43}$$

Since β' diverges, then for $\hat{R} \leq 0$ the component t_0 satisfies the energy dominant condition, $t_0 \geq 0$.

This linear warp factor satisfies the boundary conditions (26) and (27). Since the warp factor diverges asymptotically we can impose the following condition:

$$A'(r) > 0. \tag{44}$$

Now let us analyze the asymptotic behavior of the linear warp factor. The angular Einstein equation is

$$-5k^2 - 2\frac{\alpha'}{\alpha}k + \alpha(e^A\hat{R} - 2\Lambda + 2K_6^2 t_\theta) = 0. \tag{45}$$

Asymptotically $\lim_{r\to\infty}\frac{\alpha'}{\alpha} = 0$ and $\lim_{r\to\infty}\alpha = 1$. Thus, for $t_\theta = -\frac{e^{kr}\hat{R}}{2K_6^2}$, the angular equation becomes

$$5k^2 + 2\Lambda = 0 \Rightarrow \Lambda < 0. \tag{46}$$

Therefore, the bulk is asymptotically AdS_6 for the linear warp factor. Further, since

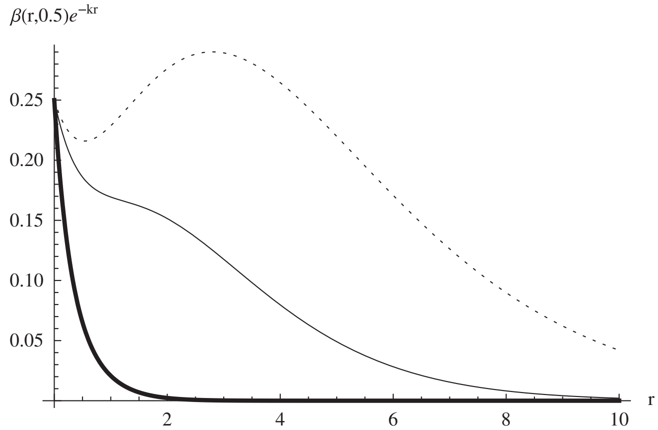

FIG. 3. Angular metric factor of bulk for $a = 0.5$. For $k = 3$, we have a monotonic exponential behavior characteristic of the stringlike defects and cigar geometries (thick line). For $k = 0.9$ (thin line) and $k = 0.6$ (dotted line) the angular factor grows until a maximum and then decreases exponentially. Note that the former behavior makes the angular component vanish more slowly.

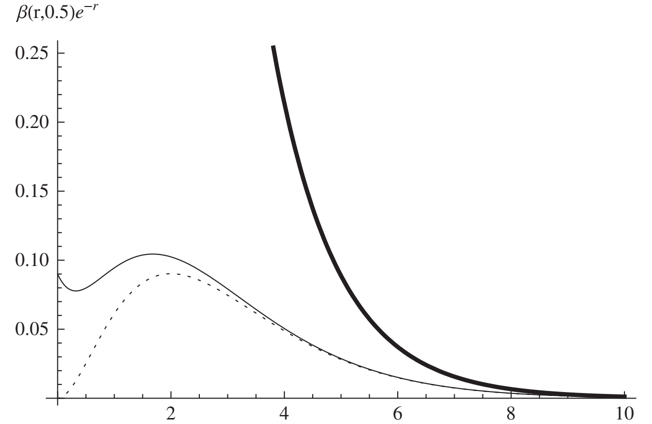

FIG. 4. Angular metric component for $k = 1$. For $a = 3$ the component decreases exponentially like for stringlike defects and cigar geometries (thick line). For $a \simeq 0.3$ (thin line) there is no conical singularity and the component reaches a maximum before decay. For $a = 0$ (dotted line) the component begins with a conical singularity at the origin, increases until a maximum, and then vanishes slowly.

$$R = 3\Lambda - \frac{K_6^2}{2} T, \tag{47}$$

the energy-momentum tensor has a core around the brane which is very similar to the stringlike geometry [6].

Another important feature in this approach is the freedom of the factor k since the constant P must only to be positive. We have plotted the angular component for some different values of k and for $a = 0.5$ in Fig. 3. For $k = 3$, this component has an exponential decreasing behavior and then approaches the configuration of stringlike defects and cigar geometries [5–7,22]. However, for $0.5 \leq k \leq 1$ the angular component increases until it reaches a maximum and then decreases exponentially. Hence, the decreasing of the k parameter has the feature of damping the exponential decreasing of the angular component. The same features appear if we fix k and vary the resolution parameter. Indeed, as $a \rightarrow 0$ a peak arises making the angular component vanish more slowly as shown in Fig. 4. In addition, the angular metric component has the same asymptotic behavior for any value of a but it changes its behavior close to the brane.

For the linear warp factor the scalar curvature is given by

$$R = -\frac{3}{2} \frac{(1620a^6k^2 + kr^5(5kr - 4) + 12a^4(60k^2r^2 - 13kr + 6) + a^2r^2(105k^2r^2 - 50kr + 4))}{(r^2 + 6a^2)^3} \tag{48}$$

whose graphic is plotted in Fig. 5. For better viewing, let us make the change of variable

$$r \rightarrow r - 5. \tag{49}$$

Therefore, the bulk has a varying negative scalar curvature.

B. Nonlinear warp factor

In addition to the configuration above, we have used another warp factor slightly different from the warp factor previously studied by Fu, Liu, and Guo [26]. Our proposed warp factor is given by

$$A(r) = B(r) = \cosh(r) + \tanh(r)^2. \tag{50}$$

Note that, like in the Randall-Sundrum model where the warp factor is a modulus function, this nonlinear warp factor is symmetric with respect to reflection on the brane; i.e., it has Z_2 symmetry, as shown in Fig. 6. Furthermore, it gives a localized angular component as seen in Fig. 7. We have plotted the scalar curvature for this warp factor for $a = 0$ and $a = 1$. Note from Fig. 8 that the behavior of the scalar curvature is opposite of the linear case because for the nonlinear case the scalar curvature is regular for $r = 0$ but diverges at infinity. Moreover, the curvature is positive around the origin and is negative for large distances.

Therefore, the geometry for the warp factor $A(r) = \cosh(r) + \tanh(r)^2$ has a well behavior in the origin but diverges asymptotically. Furthermore, asymptotically the warp factor satisfies

$$\lim_{r\rightarrow\infty} A'(r) = 0, \tag{51}$$

$$\lim_{r\rightarrow\infty} A''(r) > 0. \tag{52}$$

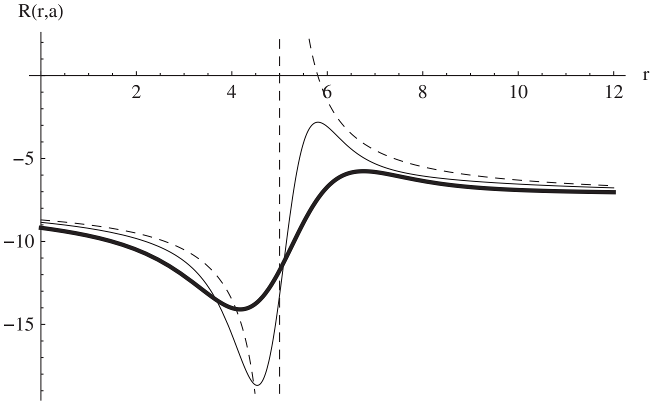

FIG. 5. Bulk scalar curvature for the linear warp factor ($k = 1$). We have put the brane in $r = 5$. For $a = 1$ (thick line) the curvature increases until an asymptotic value around -7.5. For $a = 0.5$ (thin line) the curvature grows until reaching a maximum and then decreases to the same asymptotic value. For $a = 0$ (dashed line) the curvature diverges on the brane.

Then, for $t_\theta = 0$ and $\hat{R} = 0$ the angular Einstein equation satisfies

$$4A''(r) - 2\Lambda = 0 \Rightarrow \Lambda > 0. \tag{53}$$

However, since the scalar curvature diverges far from the brane, we cannot conclude that the bulk converges to the dS_6 space. Indeed, since the scalar curvature diverges at infinity and

$$R = K_6^2 T + 6\Lambda \Rightarrow \lim_{r\to\infty} T = -\infty, \tag{54}$$

whatever produces this geometry, the scalar curvature has a radial energy-momentum tensor component that diverges asymptotically. This feature contrasts a lot with the string-like geometry.

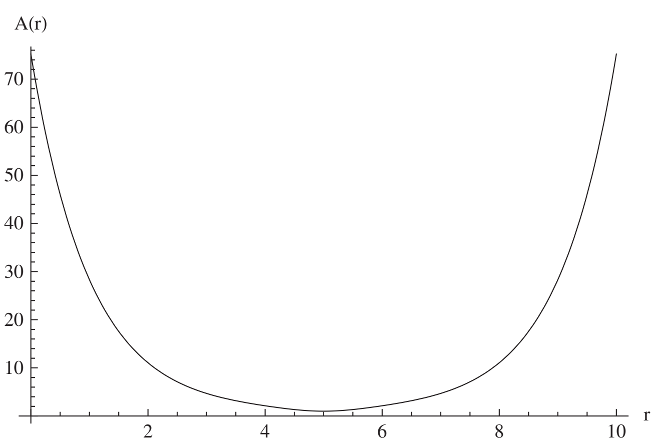

FIG. 6. Nonlinear warp factor. This function is symmetric around the brane and diverges asymptotically.

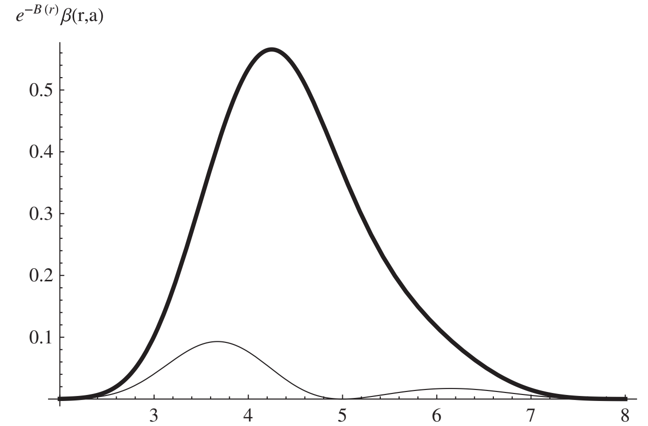

FIG. 7. Bulk angular metric factor for the nonlinear warp factor. For $a = 1$ (thick line) the component is similar to a Gaussian function and is not zero on the brane—there is no conical singularity. For $a = 0$ the component has two maxima and it has a conical singularity.

IV. SCALAR FIELD IN MINIMAL COUPLING

Now let us study the localization of a scalar field in the geometry analyzed so far. The action for a scalar field minimally coupled to the gravity is

$$S_s = \int_{\mathcal{M}_6} dx^6 \sqrt{-g} g^{ab} \partial_a \phi \partial_b \phi. \tag{55}$$

The equation of motion for the scalar field is given by

$$\partial_A(\sqrt{-g} g^{AB} \partial_B \Phi) = 0. \tag{56}$$

Let us assume that this scalar field is a product of a 4-component field with Poincaré symmetry and another scalar field living only in the 2-cycle of a conifold, i.e.,

$$\Phi(x^\mu, r, \theta) = \hat{\phi}(x^\mu)\tilde{\phi}(r, \theta). \tag{57}$$

With Poincaré symmetry the 3-brane scalar field must satisfy the mass condition

$$\partial_\mu \partial^\mu (\hat{\phi}(x^\mu)) = m^2 \hat{\phi}(x^\mu). \tag{58}$$

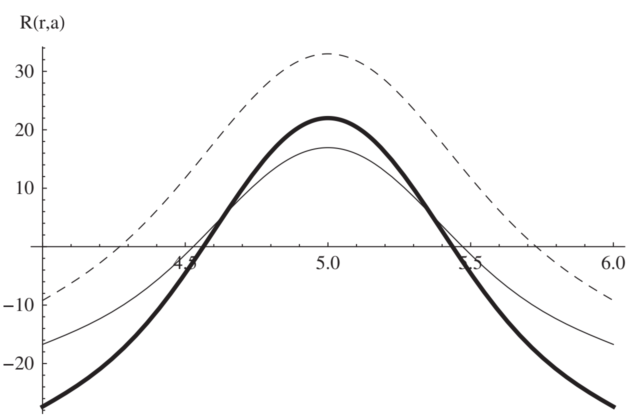

FIG. 8. Bulk scalar curvature for the nonlinear warp factor.

Since $0 \leq \theta \leq 2\pi$, let us assume that $\tilde{\phi}(r, \theta)$ can be expanded in the Fourier series as

$$\tilde{\phi}(r, \theta) = \chi(r) \sum_{l=0}^{\infty} e^{il\theta}. \quad (59)$$

Using the ansatz (59) yields

$$\left(\frac{\sqrt{-g}}{\alpha(r, a)} \chi'(r)\right)' - \frac{l^2 \sqrt{-g} e^B}{\beta(r, a)} \chi(r) + m^2 \sqrt{-g} e^A \chi(r) = 0. \quad (60)$$

Equation (60) is a Sturm-Liouville-like equation. Further, let us look for solutions that satisfy the boundary conditions

$$\chi'(0) = \lim_{r \to \infty} \chi'(r) = 0. \quad (61)$$

If we have two solutions for Eq. (60), namely, $\chi_i(r)$ and $\chi_j(r)$, the orthogonality relations between them are

$$\int_0^{\infty} \sqrt{\alpha(r, a)\beta(r, a)} e^{(-A(r)-(B(r)/2))} \chi_i * \chi_j dr = \delta_{ij}. \quad (62)$$

We can rewrite Eq. (60) as

$$\chi''(r) - \frac{1}{2}\left(4A' + B' + \frac{\alpha'}{\alpha} - \frac{\beta'}{\beta}\right)\chi'(r) + \alpha e^A\left(m^2 - l^2 \frac{e^{B-A}}{\beta}\right)\chi(r) = 0. \quad (63)$$

Note that Eq. (63) is similar to that found in stringlike geometries [6,7], regardless of the conifold terms $\alpha(r, a)$, $\beta(r, a)$. Further, we can see from Eq. (63) that it is possible to add the conifold terms to the warp factors. Here we can choose two distinct paths. On the one hand it is possible to study the resolution behavior directly in the warp factors. On other hand, we can study the localization in a factorized geometry, i.e., without the exponential warp factors $e^{A(r)}$, $e^{B(r)}$. In addition, we could define an effective angular number $l_{\rm eff} = \frac{l^2}{\beta(r,a)}$ which would depend on the point and on the resolution parameter.

A. Massive modes

Let us simplify Eq. (63) making the following change of variable

$$z = z(r) = \int_0^r \alpha(r')^{1/2} e^{A(r')/2} dr'. \quad (64)$$

Since the radial metric component $\alpha(r, a)$ is a non-negative smooth function of r, i.e., $\forall\ r \in [0, \infty)$, $\alpha(r, a) > 0$, so for a fixed a, $\alpha(r_1) \neq \alpha(r_2) \Leftrightarrow r_1 \neq r_2$. Thus,

$$\frac{dz}{dr} > 0, \quad (65)$$

and so $z(r)$ is a smooth, monotonic increasing function of r.

Using the change of variable in Eq. (64), Eq. (63) turns to be

$$\ddot{\chi}(z) - \frac{1}{2}\left(3\dot{A} + \dot{B} - \frac{\dot{\beta}}{\beta}\right)\dot{\chi}(z) + (m^2 - l^2 \beta^{-1} e^{B-A})\chi(z) = 0. \quad (66)$$

In order to simplify further, let us write $\chi(z)$ in the form

$$\chi(z) = e^{(3A+B-\ln(\beta))/4} \Psi(z). \quad (67)$$

From Eq. (66), the $\Psi(z)$ function must obey

$$-\ddot{\Psi}(z) + V(z)\Psi(z) = m^2 \Psi(z), \quad (68)$$

where

$$V(z) = \left(\frac{3\dot{A} + \dot{B} - \beta^{-1}\dot{\beta}}{4}\right)^2 - \frac{(3\ddot{A} + \ddot{B} + \beta^{-2}(\dot{\beta})^2 - \beta^{-1}\ddot{\beta})}{4} + l^2 \beta^{-1} e^{B-A}. \quad (69)$$

This is a time-independent Schrödinger-like equation. We can study the localization of the scalar field by analyzing the behavior of the potential around a potential well. Returning to the r coordinate the potential can be written as

$$V(r, a, l) = \frac{e^{-A}}{\alpha}\left\{\frac{1}{16}(15A'^2 + B'^2 + 8A'B') - \frac{1}{4}(3A'' + B'') + (3A' + B')\frac{\alpha'}{8\alpha} - \frac{\alpha'\beta'}{8\alpha\beta} - \frac{\beta'}{8\beta}A' + \left(\frac{\beta'}{4\beta}\right)'\right\} + l^2\beta^{-1}e^{B-A}. \quad (70)$$

The study of this potential provides graphic information about the possible stable solutions around a minimum.

1. Linear warp factor

Now, let us analyze the case for the linear warp factor keeping the condition $A(r, k) = B(r, k)$. The potential is given by

$$V(u, a, l, k) = \frac{e^{-ku}}{\alpha}\left\{\frac{3}{2}k^2 + \frac{k}{2}\frac{\alpha'}{\alpha} - \frac{1}{8}k\frac{\beta'}{\beta} - \frac{1}{8}\frac{\alpha'}{\alpha}\frac{\beta'}{\beta} + \frac{1}{4}\left(\frac{\beta'}{\beta}\right)'\right\} + l^2\beta^{-1}. \quad (71)$$

We have plotted the potential for $l = 0$, $k = 1$ in Fig. 9. For large values of a the potential well decays exponentially. As $a \to 0$ two asymmetric minima arise far from the brane. However, for $a = 0$, the point $r = 5$ turns out to be an infinite potential well. Therefore, it is possible to find a localized solution of Eq. (66). Indeed, the eigenfunction $\chi(r)$ must satisfy the differential equation

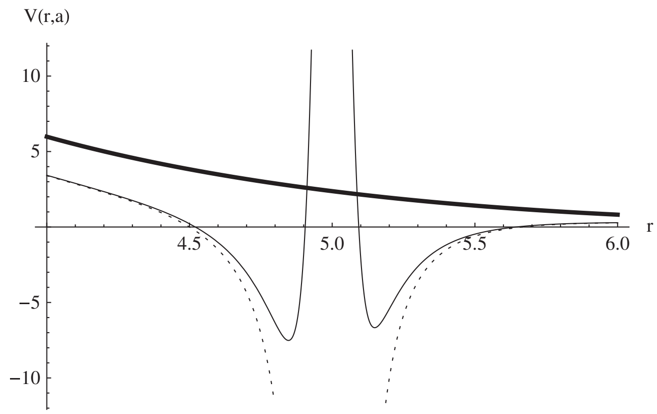

FIG. 9. Potential for linear warp factor and $l = 0$. For $a = 1$ (thick line) the potential decays exponentially. For $0.1 > a > 0$ (thin line) it appears as two asymmetric potential wells beyond the point $r = 5$ but at this point the potential is a maximum. For $a = 0$ a potential well arises on the brane (dotted line).

$$\chi''(r) + \left(\frac{(r-5)}{(r-5)^2 + 9a^2} - \frac{5}{2}k\right)\chi'(r) + \left(\frac{(r-5)^2 + 6a^2}{(r-5)^2 + 9a^2}\right)e^{k(r-5)}m^2\chi(r) = 0. \quad (72)$$

For $(\frac{a}{r-5}) \to 0$, which is valid for the singular cone ($a = 0$) and for distant points $[(r-5) \to \infty]$, the radial function satisfies

$$\chi''(r) + \left(\frac{1}{(r-5)} - \frac{5}{2}k\right)\chi'(r) + e^{k(r-5)}m^2\chi(r) = 0. \quad (73)$$

Note that Eq. (73) differs from the stringlike [6] equation for massive modes by the factor $\frac{1}{r-5}$. For $(r-5) \gg \max(a, \frac{2}{2k})$, Eq. (73) converges to a well-known equation of stringlike defects [6]

$$\chi''(r) - \frac{5}{2}k\chi'(r) + e^{kr}m^2\chi(r) = 0, \quad (74)$$

and so asymptotically the field has the same features as in stringlike geometry. For instance, the spectrum of mass has the same asymptotic behavior.

In order to study the behavior of massive modes near the brane let us take the limit $(\frac{r-5}{a}) \to 0$, $a \neq 0$, and therefore the equation for the radial component becomes

$$\chi''(r) + \left(\frac{r-5}{9a^2} - \frac{5}{2}k\right)\chi'(r) + \frac{2}{3}e^{k(r-5)}\chi(r) = 0. \quad (75)$$

We can also use Eq. (75) to study the asymptotic behavior of the eingenfunction for the singular conifold $a = 0$, whereas $(\frac{r-5}{a}) \to 0$.

For $|r-5| \ll \frac{45a^2k}{2}$, Eq. (75) turns to be the stringlike equation for massive modes [6]. Hence, the resolved conifold geometry resembles asymptotically and close to the brane the stringlike geometry. Therefore, for $(r-5) \to 0$ or $(r-5) \to \infty$ we have the well-known stringlike solution

$$\chi(r) \to e^{5/4cr}\left(C_1 J_{5/2}\left(\frac{2m}{c}e^{c/2}r\right) + C_2 Y_{5/2}\left(\frac{2m}{c}e^{c/2}r\right)\right). \quad (76)$$

However, this solution in Eq. (76) is not normalizable for $[0, \infty)$. In order to normalize the field we can use a cutoff distance r_c and apply the boundary condition at the point $r = r_c$ instead of at infinity. Then we take the limit $r_c \to \infty$ and analyze the behavior of the mass spectrum. Since the solution of Eq. (72) behaves near and far from the brane like the Bessel functions, for $r_c > \max(a, \frac{2}{2k})$, the asymptotic mass spectrum must be of the form

$$m_n = c\left(n - \frac{1}{2}\right)\frac{\pi}{2}e^{-(cr_c/2)}. \quad (77)$$

On the other hand, expanding the exponential until first order yields

$$\chi''(r) + \left(\frac{r-5}{9a^2} - \frac{5}{2}k\right)\chi'(r) + \frac{2}{3}(1 + k(r-5))\chi(r) = 0. \quad (78)$$

Therefore, this equation describes either the behavior of the field in the neighborhood of the brane or the asymptotic behavior of the eigenfunction for $a \neq 0$. The solution is the product between an exponential function and the confluent hypergeometric function of the second kind, namely,

$$\chi(r, a) = E(r, a)M\left(\frac{(1 - 6a^2 - 135a^4 - 324a^6)}{2}, \frac{1}{2}, -\frac{10 + 45a^2 + 216a^4 - 2r}{6\sqrt{2}a}\right), \quad (79)$$

where $E(r, a) = e^{(10+45a^2+108a^4-r)/18a^2}$. The graphic of this function was plotted in Fig. 10.

2. Nonlinear warp factor

Now let us study the solutions for the nonlinear warp factor. First, we have plotted the potential using several values of the resolution parameter a and $l = 0$ in Fig. 11. For $a \geq 0.5$ there is a potential well on the brane ($r = 5$) and then there are massive modes trapped to the brane. Nevertheless, for $0.5 > a > 0$, there is a potential barrier on the brane and there is only a potential well beside the brane. However, for $a = 0$ a potential well appears again on the brane and thus there are localized states on the brane in the singular conifold.

The eigenfunction must satisfy

$$\chi''(r) + \left(\frac{(r-5)}{(r-5)^2 + 9a^2} - \frac{5}{2}A'(r)\right)\chi'(r) + \left(\frac{(r-5)^2 + 6a^2}{(r-5)^2 + 9a^2}\right)e^{A(r)}m^2\chi(r) = 0. \quad (80)$$

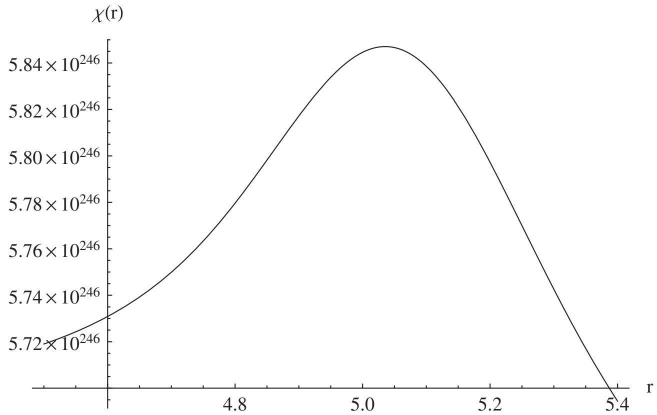

FIG. 10. Eigenfunction for linear warp factor and $a = 1$, $k = 1$. The solution is defined for points close to the brane. The field has compact support and is well-defined on the brane ($r = 5$).

For $(\frac{(r-5)}{a}) \longrightarrow 0$ we can expand the warp factor and its exponential around $r = 5$. This yields the equation

$$\chi''(r) + \left(\frac{(r-5)}{9a^2} - \frac{5}{2}(r-5)\right)\chi'(r) + \frac{2}{3}(r-5)^2 m^2 \chi(r) = 0. \tag{81}$$

The solution for this equation is again a product between a exponential function and the hypergeometric confluent function. Its plot is shown in Fig. 12.

For $(\frac{a}{r-5}) \longrightarrow 0$ the eigenfunction satisfies

$$\chi''(r) + \left(\frac{5(r-5)}{2}\right)\chi'(r) + (r-5)^2\chi(r) = 0 \tag{82}$$

whose solution is

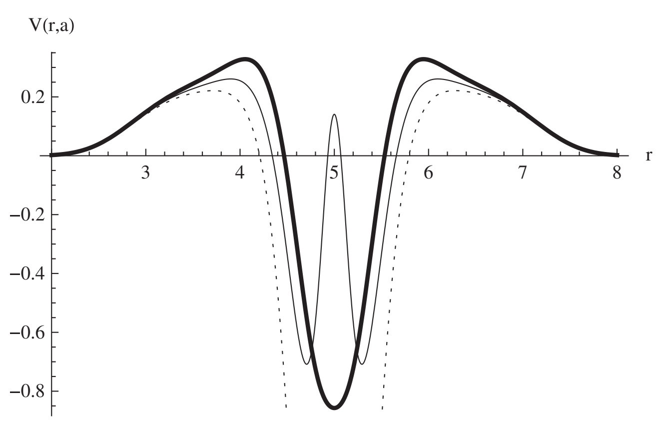

FIG. 11. Potential for the nonlinear warp factor. For $a = 0.24$ (thick line) there is only one potential well around the brane (the usual volcano potential); for $a = 0.16$, there is a maximum in the brane and the formation of two minima besides the brane (thin line); as $a \rightarrow 0$ the width of the maximum decreases and for $a = 0$ the potential has a infinite minimum on the brane (dashed line).

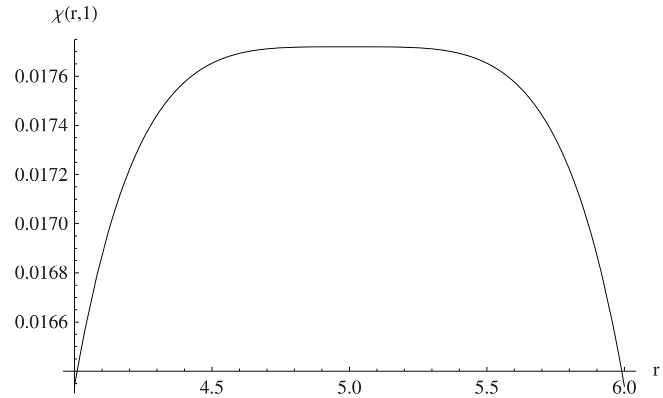

FIG. 12. Eigenfunction for the nonlinear warp factor close to the brane.

$$\chi(r) = e^{10r-r^2}\left(H\left(-\frac{4}{3}, \frac{\sqrt{3}}{2}(r-5)\right) + M\left(\frac{2}{3}, \frac{1}{2}, \frac{\sqrt{3}}{2}(r-5)\right)\right). \tag{83}$$

We have plotted the solution above in Fig. 13. Note that the function is well-defined on the brane but asymmetric in relation to the brane. Therefore, the eigenfunction for the singular conifold ($a = 0$) is localized on the brane. In addition, since the eigenfunction vanishes at infinity, the eigenfunction for $a \neq 0$ is also localized.

B. Massless modes

Now let us turn our attention to the massless modes (or Kaluza-Klein modes). Considering $m = 0$ and $l = 0$ (which is called *s-wave*) [7], the radial equation (63) becomes

$$\chi_k''(r) - \frac{1}{2}\left(4A' + B' + \frac{\alpha'}{\alpha} - \frac{\beta'}{\beta}\right)\chi_k'(r) = 0. \tag{84}$$

The constant function $\chi(r) = \chi_0$ satisfies the equation above. Thus, this solution is said to be localized if its

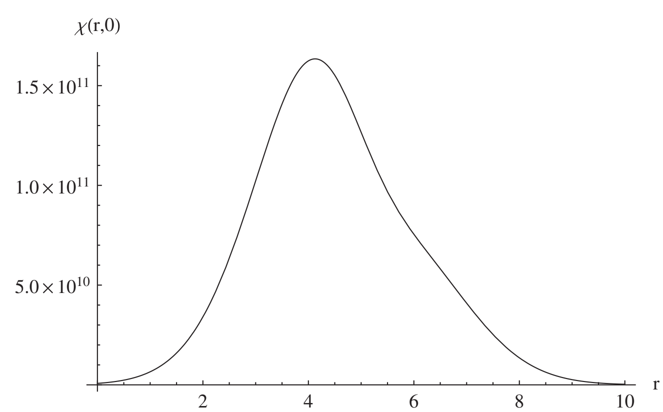

FIG. 13. Eigenfunction for the nonlinear warp factor far from the brane and/or for $a = 0$.

action is localized around the 3-brane, i.e., if its action has compact support. Since Eq. (84) is a Sturm-Liouville equation and we are seeking localized functions that satisfy the asymptotic condition

$$\lim_{|r-5|\to\infty} \chi'(r) = 0, \tag{85}$$

we can find a spectra of eigenfunctions $(\phi(r))_n$ satisfying the orthonormal condition

$$\int_0^\infty e^{-(3A(r)/2)}\sqrt{\alpha\beta} * \phi_n(r)\phi_m(r)dr = \delta_{nm}. \tag{86}$$

Therefore, we can define the eigenfunction in flat space as

$$\chi_n(r, a) = e^{-(3A(r)/4)}(\alpha(r, a)\beta(r, a))^{1/4}\phi_n(r). \tag{87}$$

On the other hand, since

$$\chi_0^2 \int_0^\infty e^{-(3A(r)/2)}\sqrt{\alpha\beta}dr = 1, \tag{88}$$

the zero-mode eigenfunction is

$$\chi_0(r, a) = le^{-(3A(r)/2)}\sqrt{\alpha(r, a)\beta(r, a)}, \tag{89}$$

where

$$l = \frac{1}{\int_0^\infty e^{-(3A(r)/2)}\sqrt{\alpha\beta}dr}. \tag{90}$$

Again, the eigenfunction is quite similar to that found in Refs. [6,7] regardless of the factor $(\alpha(r, a)\beta(r, a))^{1/2}$. We have plotted the zero-mode $\psi_0(r, a)$ for some values of a using the linear warp factor in Fig. 14 and for the nonlinear warp factor in Fig. 15. Since for every a the function is integrable we can say that the massless field is localized on the brane even for the singular conifold case.

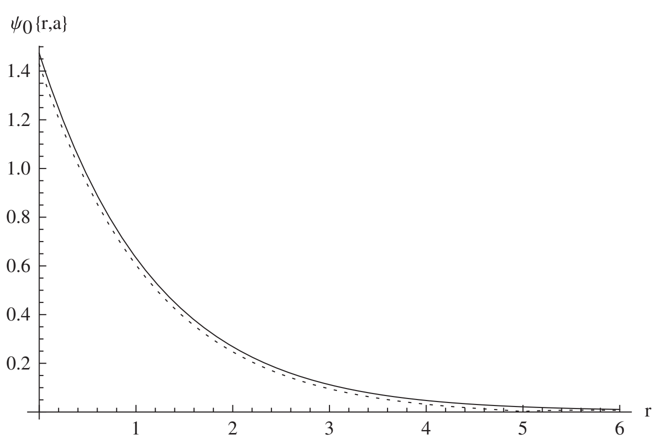

FIG. 14. Plot of the zero-mode eigenfunction for the linear warp factor. For any a it has an exponential decreasing behavior and thus it is localized (but not symmetric) around the brane.

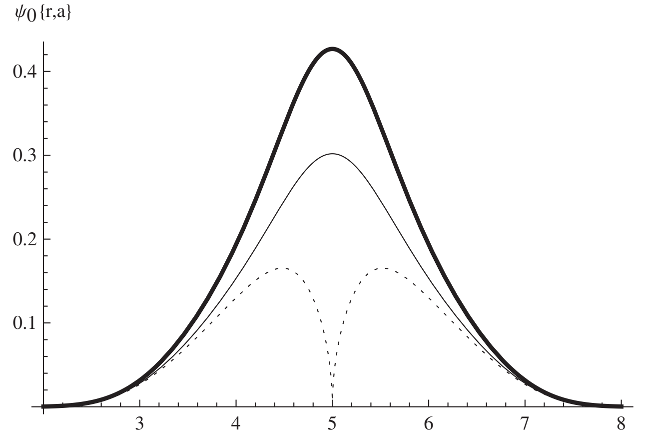

FIG. 15. Plot of the zero-mode eigenfunction for the nonlinear warp factor. The function has compact support even for the singular conifold $a = 0$ (dotted line). Note that this solution is localized and symmetric around the brane.

V. CONCLUSION AND PERSPECTIVES

We have studied the localization of a scalar field on a 3-brane in a resolved conifold background of six dimensions, built from a warped product between a 2-cycle of a resolved conifold and a 3-brane placed in the tip of the cone. We have chosen a geometry such that when the resolution parameter goes to zero or when we put the 3-brane far from the origin, we reobtain the well-studied stringlike geometries.

The use of the resolved conifold as a transverse space brought a very nice feature: an extension to the solution of the hierarchy problem using the resolution parameter a. Indeed, in the Randall-Sundrum and stringlike models, the ratio between the six-dimensional Planck mass M_6 and the four-dimensional Planck mass M_4 depends only on the cosmological constant Λ and on the tension of the brane μ. Since this parameter has a purely geometric origin, it has opened the way to use other parameter-dependent manifolds like Eguchi-Hanson spaces, Taub-Nut, etc., as transverse spaces. Furthermore, using parameter-dependent transverse spaces we could study the evolution and stability of the hierarchy between M_6 and M_4 through some mechanisms, for example, the conical transitions or the geometrical flux.

For a linear warp factor, the bulk geometry asymptotically approaches the AdS_6 space ($\Lambda < 0$) which is similar to the stringlike geometry. Further, the massive modes have a family of potentials parametrized by the resolution parameter. For $a \geq 1$, the potential decreases exponentially and then it is not possible to have stable trapped states in the brane. As $a \to 0$, two asymmetric minima appear around the 3-brane put at the point $r = 5$, but the potential reaches its maximum value on the brane. The respective eigenfunction is localized around the deepest minimum close to the brane. In the limit $a = 0$, the potential well turns out to be an infinite potential well in the brane and thus the eigenfunction is localized in the brane. Indeed, both for $\frac{a}{(r-5)} \rightarrow 0$ and for $\frac{(r-5)}{a} \rightarrow 0$ we obtain the well-known solution for stringlike geometry [6]. In addition, the massless modes were localized even for the singular case. Indeed, the action for a constant solution in this geometry has compact support and therefore is normalizable. The main difference between this conifold geometry and the standard stringlike one lies in the parametrization of the potentials by a geometric factor. This parametrization provides the asymmetric minima potentials far from the brane. Therefore, the resolution parameter could be used as a filter of fields that can or cannot be localized on the brane.

Using the nonlinear warp factor $A(r) = \cosh(r) + \tanh(r)^2$ the scalar curvature diverges asymptotically and so the energy-momentum tensor is not restricted to points close to the brane, which is in contrast to the stringlike geometry. For the scalar field, the massive modes are trapped on the 3-brane even for the singular conifold. For $a \geq 1$ the potential behaves like the standard volcano potential with a minimum on the brane. As $a \rightarrow 0$ two more local minima appear inside the global minimum and the minimum on the brane turns out to be a maximum. Therefore, in the range $0 < a \leq 1$ it is not possible to localize the scalar field on the brane. In the same way as for the linear warp factor, for $a = 0$ the potential on the brane returns to be a potential well but now infinite. At last, the massless modes are localized for all values of a. A difference between this geometry and the stringlike one lies in the radial field solutions. Indeed, for stringlike geometries the usual field solutions are Bessel functions and in this work we have found confluent hypergeometric functions that depend on two parameters and which are very sensible to variations on these parameters. This was expected since the field equation depends on the mass m and angular number l. Also, the field equation, through the geometry, depends on the resolution parameter a. We argue that by using other parameter-dependent geometries we could find more general solutions and study their stability.

We also argue that the successful localization of massless modes in the singular conifold for both the linear and nonlinear warp factor might be related to the fact that in the conifold transitions, where we make singularities in the manifold, some fields become massless. Thus, this close relationship between conical singularity and the spectrum of massless fields provides the localization of the massless modes in the singular geometry.

This work suggests many perspectives. Using the same geometry studied so far, we could study the localization of other fields in this scenario, like the vector, gravitational, and spinor fields. Since these fields have more degrees of freedom than the scalar field, the effects of asymmetry of the potential could be more relevant, for example, for the resonance of fermionic modes. Another way it could be used is in parameter-dependent geometries like the deformed conifold or even orbifold instead of the resolved conifold. Since these smooth conifolds are related to supergravity solutions that near the horizon behave like $AdS_5 \times S^5$, we could study the localization of fields where the transverse spaces are well-known solutions like the Eguchi-Hanson spaces or the Klebanov-Strassler throat. Furthermore, since we have parametrized the geometry, we could study the stability of this geometry using some analytical method like the Ricci equation, where the variable of the flux would be the resolution parameter. Since there is a relation between the resolved conifold geometry and the cigarlike geometry, we could study the flux through a parametrized cigar geometry. On the other hand, we could use the geometry bulk used here to study other problems such as the small value of the cosmological constant in the 3-brane and the supersymmetry breakdown.

It is worthwhile to mention that even though we have studied the behavior of the field in that geometry we have not said how this geometry was generated. Hence, a next step could be deduce this geometry from some field and so give a physical meaning to the resolution parameter.

ACKNOWLEDGMENTS

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