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Electrodynamics on κ-Minkowski space-time

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In this paper, we derive Lorentz force and Maxwell's equations on kappa-Minkowski space-time up to the first order in the deformation parameter. This is done by elevating the principle of minimal coupling to noncommutative space-time. We also show the equivalence of minimal coupling prescription and Feynman's approach. It is shown that the motion in kappa space-time can be interpreted as motion in a background gravitational field, which is induced by this noncommutativity. In the static limit, the effect of kappa deformation is to scale the electric charge. We also show that the laws of electrodynamics depend on the mass of the charged particle, in kappa space-time.

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I. INTRODUCTION

Quantum gravity effects are expected to lead to space-time uncertainties at Planck scale. Noncommutative geometry provides a natural way to incorporate this microscopic structure of space-time. κ-deformed space-time [2,3] is a prototype of the Lie algebraic type noncommutative space-time, fuzzy sphere being the well-known example of this family [4,5]. The κ-deformed space-time is known to emerge naturally in the low-energy limit of certain quantum gravity models. It is also the space-time associated with doubly special relativity [6–8]. In recent years, algebraic structure and symmetries of κ-space-time have been investigated in detail [9–11].

Generically, field theory models on noncommutative space-time do have highly nonlocal and nonlinear interactions, and are characterized by an interdependence of high- and low-energy behavior, known as UV/IR mixing, which had been studied in detail in field theory models on Moyal space-time as well as on κ space-time [12]. In noncommutative space-times, Lorentz symmetry in the usual sense is broken, but it is shown that this symmetry can be retained by a Hopf algebra approach [13]. Thus, the conventional notions of field quanta can be generalized to noncommutative field theories. Following these developments, field theory models on κ-deformed space-time have been constructed and studied [14,15]. Investigations, trying to obtain bounds on κ deformation parameter using experimental and observational results are being carried out in last couple of years [16–19].

Construction and study of $U(1)$ gauge theory on κ-space-time using star product approach was taken up in [20]. Using Feynman's approach, Maxwell's equations on κ-space-time were obtained in [21].

In Feynman's approach, starting with Newton's equation of motion and assumed (quantum) commutators between coordinates and velocities, one derives the homogeneous Maxwell's equations by the repeated application of Jacobi identities [22]. This method has been generalized to relativistic case in [23]. It has also been shown that the quantum mechanical particles consistently interact with scalar, gauge, and gravitational fields only. In the commutative space-time, it is known that the results obtained by Feynman's approach and minimal prescription are equivalent [22,24]. Feynman's approach has been generalized to obtain inhomogeneous Maxwell's equation in [25] and various other aspects of this method has been studied in [26]. In recent times, this method has been used to obtain Maxwell's equation in Moyal space-time also [27]. In [24], it was shown that by assuming the minimal coupling of gauge field and the ensuing relation between kinetic and conjugate momenta, one can derive Lorentz force equation and Maxwell's equations. In this way, one can work with Poisson brackets rather than (quantum) commutators in Feynman's approach. Thus, this approach of [24] allows one to take a classical limit of the obtained equations, in a proper fashion.

In this paper, we generalize a variant of Feynman's approach [24] to κ-space-time and derive the deformed Maxwell's equations and force equation valid up to first order in deformation parameter. In our approach, we do power series expansion of the noncommutative coordinates, momenta, as well as functions of noncommutative coordinates and momenta in terms of commutative coordinates, momenta and deformation parameter (keeping terms up to the first order in deformation parameter). We also exploit the generalization of minimal coupling prescription to κ-space-time in our calculations.

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This paper is organized as follows. In the next section, we briefly recall Feynman's approach using minimal prescription [24]. Here we show how the force equation as well as all the Maxwell's equations can be derived. Our main results are presented in Sec. III. In Sec. III A, we discuss the derivation of force equation on κ space-time for a electrically neutral particle. The force equation we obtain here is valid up to first order in the deformation parameter a. In Sec. III B, we derive the Lorentz force equation and in Sec. III C, Maxwell's equation for a charged particle in κ-space-time is obtained. Here also, all equations derived are valid up to first order in the deformation parameter a. In sSec. III D, we discuss the natural realization of the coordinates of κ-space-time and obtain the Maxwell's equation in this realization. Finally we conclude with discussion in Sec. IV.

We work with $\eta_{\mu\nu} = (+, -, -, -)$.

II. MINIMAL COUPLING AND FEYNMAN'S APPROACH TO ELECTRODYNAMICS

We start with the same basic assumptions as those in Feynman's approach [22–24]. The coordinates of a relativistic particle in 4-D Minkowski space-time is described by $x_\mu(\tau)$, $(\mu = 0, 1, 2, 3)$, where τ is a parameter, and they satisfy the following commutation relations

$$[x_\mu(\tau), x_\nu(\tau)] = 0, \qquad [x_\mu(\tau), \dot{x}_\nu(\tau)] = -\frac{i}{m}\eta_{\mu\nu}, \tag{1}$$

where $\dot{x}_\mu = \frac{dx_\mu}{d\tau}$. Newton's equation is also assumed

$$F_\mu(x, \dot{x}) = m\ddot{x}_\mu. \tag{2}$$

We introduce kinetic momentum $\pi_\mu = m\dot{x}_\mu$, so that we have

$$[x_\mu, \pi_\nu] = -i\eta_{\mu\nu}, \tag{3}$$

and we can write $\pi_\mu(\tau)$ explicitly as

$$\pi_\mu = p_\mu - eA_\mu(x), \tag{4}$$

where $eA_\mu(x)$ is, for now, an arbitrary function of x, and $p_\mu(\tau)$ is canonical momentum satisfying

$$[p_\mu, p_\nu] = 0, \qquad [x_\mu, p_\nu] = -i\eta_{\mu\nu}. \tag{5}$$

This is the principal of minimal coupling [24]. It is obvious that with $F_\mu = \frac{d\pi_\mu}{d\tau}$, and taking the derivative with respect to τ of Eq. (3) one gets the following relations

$$[x_\mu, F_\nu] = -\frac{1}{m}[\pi_\mu, \pi_\nu], \qquad [\pi_\mu, \pi_\nu] = -ieF_{\mu\nu}(x), \tag{6}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Because of (5) we have useful identities

$$[x_\mu, f(x, p)] = -i\frac{\partial f}{\partial p^\mu}, \qquad [p_\mu, f(x, p)] = i\frac{\partial f}{\partial x^\mu}. \tag{7}$$

Here (and from now on) we take the ordering prescription where x is put always on the left, and p on the right. Force $F_\mu(x, \dot{x})$ can be understood as a function of x and p, that is $F_\mu(x, p)$, and this is important because we only know how to integrate over commuting variables. So, using (7) we can now integrate (6) over p^μ and get

$$F_\nu = \frac{e}{m}F_{\nu\mu}p^\mu + \tilde{G}_\nu(x), \tag{8}$$

where $\tilde{G}_\nu(x)$ is a function of x and we have used the prescription that p goes to the right (for constructing Hermitian operators we can symmetrize $xp \to \frac{1}{2}(xp + px)$). Now, using the definition of kinetic momentum $\pi_\mu = m\dot{x}_\mu$, relation (4), and defining $G_\mu(x) = \tilde{G}_\mu(x) + \frac{e^2}{m}F_{\mu\nu}A^\nu$ we get the Lorentz force

$$F_\mu = G_\mu(x) + eF_{\mu\nu}\dot{x}^\mu. \tag{9}$$

In the minimal coupling approach all the Jacobi identities are satisfied by construction, i.e.,

$$\begin{aligned}
&[x_\mu, [x_\nu, x_\rho]] + [x_\nu, [x_\rho, x_\mu]] + [x_\rho, [x_\mu, x_\nu]] = 0,\\
&[x_\mu, [x_\nu, \pi_\rho]] + [x_\nu, [\pi_\rho, x_\mu]] + [\pi_\rho, [x_\mu, x_\nu]] = 0,\\
&[x_\mu, [\pi_\nu, \pi_\rho]] + [\pi_\nu, [\pi_\rho, x_\mu]] + [\pi_\rho, [x_\mu, \pi_\nu]] = 0,\\
&[\pi_\mu, [\pi_\nu, \pi_\rho]] + [\pi_\nu, [\pi_\rho, \pi_\mu]] + [\pi_\rho, [\pi_\mu, \pi_\nu]] = 0.
\end{aligned} \tag{10}$$

The first two equations in (10) are trivially satisfied, the third is in Feynman's approach equal to the statement that $F_{\mu\nu}$ is a function of x, and the fourth yields homogeneous Maxwell equation

$$\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0. \tag{11}$$

If we take (9) as a definition of $G_\mu(x)$, it is straightforward to see

$$[\pi_\mu, G_\nu] - [\pi_\nu, G_\mu] = 0, \qquad \partial_\mu G_\nu - \partial_\nu G_\mu = 0, \tag{12}$$

which means that $G_\mu = \partial_\mu\phi(x)$. We can conclude that minimal coupling and Feynman's approach are in complete correspondence. From the definition of the commutator we can show that

$$[\pi_\nu, [\pi_\mu, [\pi^\mu, \pi^\nu]]] = 0, \tag{13}$$

and by defining

$$[\pi_\mu, [\pi^\mu, \pi^\nu]] = ej^\nu, \tag{14}$$

we get

$$[\pi_\nu, j^\nu] = 0, \qquad \partial_\mu j^\mu = 0. \tag{15}$$

Thus we see that $j_\mu(x)$ is the conserved current, and by definition Eq. (14) gives the inhomogeneous Maxwell's equations.

$$\partial_\mu F^{\mu\nu} = j^\nu. \tag{16}$$

Now we have the complete set of Maxwell equations which are covariant and we recognize $A_\mu(x)$ as a gauge field, and e as an electric charge of a particle.

III. κ-DEFORMED ELECTRODYNAMICS

A. $e = 0$ case

Minimal coupling approach seems natural for exploring noncommutative spaces, all it takes is to substitute $x_\mu \to \hat{x}_\mu$, where $[\hat{x}_\mu, \hat{x}_\nu] \neq 0$. We will consider a class of noncommutating spaces, the so-called κ-Minkowski space-time, which are defined by

$$[\hat{x}_\mu, \hat{x}_\nu] = i(a_\mu \hat{x}_\nu - a_\nu \hat{x}_\mu), \tag{17}$$

where a_μ is the deformation parameter and $\hat{x}_\mu$ is a noncommutating coordinate operator. In the case when $a \to 0$, we have $[\hat{x}_\mu, \hat{x}_\nu] \to 0$, that is $\hat{x}_\mu \to x_\mu$, so we take a perturbative approach to find the realization of $\hat{x}_\mu$ in terms of operators x_μ and p_μ from the commutating space, up to the first order in the deformation parameter a_μ. So we write

$$\hat{x}_\mu = x_\mu + \delta\hat{x}_\mu(a), \tag{18}$$

where $\delta\hat{x}_\mu(a)$ can be constructed from x_μ, p_μ and a_μ as

$$\delta\hat{x}_\mu(a) = \alpha x_\mu(a \cdot p) + \beta(x \cdot a)p_\mu + \gamma(x \cdot p)a_\mu, \qquad \alpha, \beta, \gamma \in \mathbb{R} \tag{19}$$

Taking into account that (17) must be satisfied up to the first order in deformation parameter a_μ we get the constraint on the real parameters α, β, and γ

$$\gamma - \alpha = 1, \qquad \beta \in \mathbb{R}. \tag{20}$$

Now we have to construct the noncommutative momentum operator $\hat{p}$, but we are missing the relation $[\hat{p}_\mu, \hat{x}_\nu] = ?$; all we know is that in the zeroth order in a_μ Eq. (5) holds. First let us consider the $e = 0$ case and postulate $\dot{\hat{x}}_\mu(e = 0) \equiv \frac{1}{m}\hat{p}_\mu$, then taking the derivative of (17) with respect to τ gives

$$[\hat{p}_\mu, \hat{x}_\nu] + [\hat{x}_\mu, \hat{p}_\nu] = i(a_\mu \hat{p}_\nu - a_\nu \hat{p}_\mu), \tag{21}$$

which only fixes the antisymmetric part of $[\hat{p}_\mu, \hat{x}_\nu]$. We can take $\hat{p}_\mu$ to be

$$\hat{p}_\mu = p_\mu + \delta\hat{p}_\mu(a), \tag{22}$$

and demand that (21) and Jacobi identities between $\hat{p}_\mu$, $\hat{x}_\nu$ and $\hat{x}_\rho$ must be satisfied up to the first order in a and get the explicit form of $\delta\hat{p}_\mu(a)$. This construction is equivalent to just taking the form of $\delta\hat{x}_\mu(a)$ given in (19) and substitute x with p (which is also equivalent with $\dot{p}_\mu = 0$) and then we get

$$\hat{p}_\mu = p_\mu + (\alpha + \beta)(a \cdot p)p_\mu + \gamma a_\mu p^2. \tag{23}$$

Now we have

$$[\hat{p}_\mu, \hat{p}_\nu] = 0. \tag{24}$$

Using (23) we have

$$[\hat{p}_\mu, \hat{x}_\nu] = i\eta_{\mu\nu}(1 + s(a \cdot p)) + i(s + 2)a_\mu p_\nu + i(s + 1)a_\nu p_\mu, \qquad s = 2\alpha + \beta. \tag{25}$$

We are considering $e = 0$ case, so there is no difference between canonical and kinetic momentum. Analogous to commutative space we have the Newton-like equation

$$\hat{F}_\mu(e = 0) \equiv \hat{G}_\mu = \frac{d\hat{p}_\mu}{d\tau}. \tag{26}$$

Taking the derivative with respect to τ of Eq. (25) we get

$$[\hat{G}_\mu, \hat{x}_\nu] = i\eta_{\mu\nu}s(a \cdot G) + i(s + 2)a_\mu G_\nu + i(s + 1)a_\nu G_\mu, \tag{27}$$

where we used (24), and the fact that all equations from Sec. II. hold up to the zeroth order in a. We want to find $\hat{G}_\mu$, but we can not simply integrate (27). The force $\hat{G}_\mu$ can be written as

$$\hat{G}_\mu = G_\mu(x) + \delta\hat{G}_\mu(a), \tag{28}$$

and combining (27) and (28) we can get an equation for $\delta\hat{G}_\mu(a)$

$$[\delta\hat{G}_\mu(a), \hat{x}_\nu] = i\frac{\partial\delta\hat{G}_\mu(a)}{\partial p^\nu} = -[G_\mu, \hat{x}_\nu] + i\eta_{\mu\nu}s(a \cdot G) + i(s + 2)a_\mu G_\nu + i(s + 1)a_\nu G_\mu, \tag{29}$$

which can be easily integrated over p^ν. Before writing $\hat{G}_\mu$ explicitly, it is convenient to find an operator that commutes with $\hat{x}_\mu$. We find an operator $\hat{y}_\mu$ such that

$$[\hat{x}_\mu, \hat{y}_\nu] = 0, \quad \hat{y}_\mu = x_\mu + \gamma x_\mu(a \cdot p) + (\gamma - 1)(x \cdot p)a_\mu + \beta(x \cdot a)p_\mu \tag{30}$$

and define $f(\hat{y})$ as

$$f(\hat{y}) = f(x) + \gamma\left(x \cdot \frac{\partial f}{\partial x}\right)(a \cdot p) + (\gamma - 1)\left(a \cdot \frac{\partial f}{\partial x}\right)(x \cdot p) + \beta(a \cdot x)\left(\frac{\partial f}{\partial x} \cdot p\right), \tag{31}$$

so that

$$[f(\hat{y}), \hat{x}_\mu] = 0. \tag{32}$$

Finally we can write the operator for the force in the noncommutating space when $e = 0$ as

$$\hat{F}_\mu(e = 0) = G_\mu(\hat{y}) + s(a \cdot G)p_\mu + (s + 2)a_\mu(G \cdot p) + (s + 1)G_\mu(a \cdot p), \tag{33}$$

that is,

$$\hat{F}_\mu(e=0) = G_\mu(\hat{y}) + ms(a\cdot G)\dot{x}_\mu + m(s+2)a_\mu(G\cdot\dot{x}) + m(s+1)G_\mu(a\cdot\dot{x}). \tag{34}$$

From the above, we see that a neutral particle with mass m, moving in a κ-deformed Minkowski space-time can be interpreted as moving in an "electromagnetic"-like background that couples proportionally to the deformation parameter a_μ, because these corrections are linear in $\dot{x}$.

B. $e \neq 0$ case and corrections to the Lorentz force

In Sec. II we have shown that the minimal coupling principle leads to well known Lorentz force, so we want to generalize the minimal coupling principle in a consistent way. If we postulate $\hat{\pi}_\mu = m\dot{\hat{x}}_\mu$ and the simplest way to introduce gauge field A_μ, as $\hat{\pi}_\mu = \hat{p}_\mu - eA_\mu(\hat{x}\,\text{or}\,\hat{y})$, then from the Jacobi identities and

$$[\hat{\pi}_\mu, \hat{x}_\nu] + [\hat{x}_\mu, \hat{\pi}_\nu] = i(a_\mu\hat{\pi}_\nu - a_\nu\hat{\pi}_\mu), \tag{35}$$

we get very restrictive conditions on A_μ. It is better to understand minimal coupling principle as a way to introduce a connection between canonical and kinetic momentum through a gauge field in a way that the commutation relations $[\hat{\pi}_\mu, \hat{x}_\nu]$ and $[\hat{p}_\mu, \hat{x}_\nu]$ are the same by form, which is also true in the commutating case. In correspondence with (25) we write

$$[\hat{\pi}_\mu, \hat{x}_\nu] = i\eta_{\mu\nu}(1 + s(a\cdot\pi)) + i(s+2)a_\mu\pi_\nu + i(s+1)a_\nu\pi_\mu. \tag{36}$$

We also write $\hat{\pi}_\mu$ as $\hat{\pi}_\mu = \pi_\mu + \delta\hat{\pi}_\mu(a)$ and from (36) we get explicitly

$$\hat{\pi}_\mu = \hat{p}_\mu - eA_\mu(\hat{y}) - e[(s+2)(A\cdot p)a_\mu + s(A\cdot a)p_\mu + (s+1)A_\mu(a\cdot p)]. \tag{37}$$

Taking the derivative with respect to τ of (36) and using $\hat{F}_\mu = \frac{d\hat{\pi}_\mu}{d\tau}$, we get

$$[\hat{F}_\mu, \hat{x}_\nu] + \frac{1}{m}[\hat{\pi}_\mu, \hat{\pi}_\mu] = i\eta_{\mu\nu}s(a\cdot F) + i(s+2)a_\mu F_\nu + i(s+1)a_\nu F_\mu. \tag{38}$$

Writing the force as $\hat{F}_\mu = F_\mu + \delta\hat{F}_\mu(a)$, we can get a equation for $\delta\hat{F}_\mu(a)$ as

$$[\delta\hat{F}_\mu(a), x_\nu] = [\hat{x}_\nu, F_\mu] - \frac{1}{m}[\hat{\pi}_\mu, \hat{\pi}_\nu] + i\eta_{\mu\nu}s(a\cdot F) + i(s+2)a_\mu F_\nu + i(s+1)a_\nu F_\mu, \tag{39}$$

where F_μ is the force from the commutative space and

$$\begin{aligned}[\hat{\pi}_\mu, \hat{\pi}_\nu] = -ie\bigg[&F_{\mu\nu} + 2(s+1)F_{\mu\nu}a\cdot p + i(\alpha+\beta)a\cdot\frac{\partial F_{\mu\nu}}{\partial x} + \gamma\left(x\cdot\frac{\partial F_{\mu\nu}}{\partial x}\right)(a\cdot p) + (\gamma-1)\left(a\cdot\frac{\partial F_{\mu\nu}}{\partial x}\right)(x\cdot p)\\ &+ \beta(a\cdot x)\left(\frac{\partial F_{\mu\nu}}{\partial x}\cdot p\right) + sa^\alpha(F_{\alpha\nu}p_\mu - F_{\alpha\mu}p_\nu) + (s+2)(a_\mu F_{\alpha\nu} - a_\nu F_{\alpha\mu})p^\alpha\\ &+ i\gamma\left(a_\mu\frac{\partial^2 A_\nu}{\partial x_\alpha\partial x^\alpha} - a_\nu\frac{\partial^2 A_\mu}{\partial x_\alpha\partial x^\alpha}\right)\bigg] + ie^2\bigg[(s+2)A^\alpha\left(a_\mu\frac{\partial A_\nu}{\partial x^\alpha} - a_\nu\frac{\partial A_\mu}{\partial x^\alpha}\right)\\ &+ s(a\cdot A)F_{\mu\nu} + (s+1)a^\alpha\left(A_\mu\frac{\partial A_\nu}{\partial x^\alpha} - A_\nu\frac{\partial A_\mu}{\partial x^\alpha}\right) - \frac{\partial A_\nu}{\partial x^\alpha}\frac{\partial A_\mu}{\partial x^\beta}(x^\alpha a^\beta - a^\alpha x^\beta)\bigg].\end{aligned} \tag{40}$$

The right-hand side (rhs) of Eq. (39) can be explicitly calculated and the left-hand side is

$$[\delta\hat{F}_\mu(a), x_\nu] = i\frac{\partial(\delta\hat{F}_\mu(a))}{\partial p^\nu}, \tag{41}$$

so we can integrate (39) over p^ν. After tedious calculation and expressing everything in terms of $\dot{\hat{x}}_\mu$, we get

$$\hat{F}_\mu = \hat{G}_\mu + eF_{\mu\nu}(\hat{y})\dot{\hat{x}}^\nu + e\tilde{F}_{\mu\nu}\dot{\hat{x}}^\nu - m\Gamma_{\mu\nu\lambda}\dot{\hat{x}}^\nu\dot{\hat{x}}^\lambda + O(a\cdot e^2) + O(a^2), \tag{42}$$

where $\hat{G}_\mu = \hat{F}_\mu(e=0)$ is defined in (33), $F_{\mu\nu}(\hat{y})$ is defined like (31), that is,

$$F_{\mu\nu}(\hat{y}) = F_{\mu\nu}(x) + \gamma\left(x\cdot\frac{\partial F_{\mu\nu}}{\partial x}\right)(a\cdot p) + (\gamma-1)\left(a\cdot\frac{\partial F_{\mu\nu}}{\partial x}\right)(x\cdot p) + \beta(a\cdot x)\left(\frac{\partial F_{\mu\nu}}{\partial x}\cdot p\right), \tag{43}$$

and the remaining two terms are

$$\tilde{F}_{\mu\nu} = i\bigg[(\alpha+\beta)a\cdot\frac{\partial F_{\mu\nu}}{\partial x} - \gamma\left(a_\mu\frac{\partial^2 A_\nu}{\partial x_\alpha\partial x^\alpha} - a_\nu\frac{\partial^2 A_\mu}{\partial x_\alpha\partial x^\alpha}\right)\bigg],$$

$$\Gamma_{\mu\nu\lambda} = e[(\alpha+\beta)F_{\mu\nu}a_\lambda + (\gamma+\beta)F_{\mu\rho}a^\rho\eta_{\lambda\nu} - sF_{\rho\nu}a^\rho\eta_{\mu\lambda} - (3\gamma - 2\beta - 1)F_{\mu\nu}a_\lambda]. \tag{44}$$

Terms proportional to $\dot{x}$ can be interpreted as a correction to the Lorentz force due to the background electromagnetic field, and those proportional to $\dot{x}^2$ as quasigravitational effects caused by the background curvature induced by the noncommutativity of space-time. Both effects are proportional to ea_μ.

C. κ-deformed Maxwell equations

In our approach, all the Jacobi identities using $\hat{\pi}$, $\hat{p}$ and $\hat{x}$ are satisfied by construction up to the first order in the deformation parameter a. Formally the Jacobi identity

$$[\hat{\pi}_\mu,[\hat{\pi}_\nu,\hat{\pi}_\rho]]+[\hat{\pi}_\nu,[\hat{\pi}_\rho,\hat{\pi}_\mu]]+[\hat{\pi}_\rho,[\hat{\pi}_\mu,\hat{\pi}_\nu]]=0 \tag{45}$$

leads to

$$\partial_\mu\hat{F}_{\nu\rho}+\partial_\nu\hat{F}_{\rho\mu}+\partial_\rho\hat{F}_{\mu\nu} = i([\delta\hat{\pi}_\mu(a),F_{\nu\rho}]-e[A_\mu,\delta\hat{F}_{\nu\rho}(a)]+\text{cyclic}(\mu,\nu,\rho)), \tag{46}$$

where

$$[\hat{\pi}_\mu,\hat{\pi}_\nu]\equiv -ie\hat{F}_{\mu\nu}=-ieF_{\mu\nu}(x)-ie\delta\hat{F}_{\mu\nu}(a) \tag{47}$$

is given in (40), so we see that $\hat{F}_{\mu\nu}$ is expressed in terms of operators from the commutative space. This is the κ-deformed analogue of the homogeneous Maxwell equation. The rhs of (46) can be explicitly calculated in terms of commutative variables and fields $\vec{E}$ and $\vec{B}$, that satisfy the usual Maxwell equations. From

$$[\hat{\pi}_\nu,[\hat{\pi}_\mu,[\hat{\pi}^\mu,\hat{\pi}^\nu]]]=0, \tag{48}$$

and by defining

$$[\hat{\pi}_\mu,[\hat{\pi}^\mu,\hat{\pi}^\nu]]=e\hat{j}^\nu, \tag{49}$$

we have

$$[\hat{\pi}_\mu,\hat{j}^\mu]=0, \tag{50}$$

so that $\hat{j}$ is a conserved current and we formally have

$$\partial_\mu\hat{F}^{\mu\nu}=\hat{j}^\nu+i[\delta\hat{\pi}_\mu(a),F^{\mu\nu}]-ie[A_\mu,\delta\hat{F}^{\mu\nu}(a)]+O(a^2). \tag{51}$$

This is the κ-deformed analogue of the inhomogeneous Maxwell equation. The rhs in (51) can also be explicitly calculated.

Now we study the κ-deformed space-time where $a_\mu=(a_0,\vec{0})$, that is, $a_0\equiv a=\kappa^{-1}$ and $a_i=0$. We define

$$\begin{aligned}\hat{F}_{0i}&=-\hat{E}_i, & F_{0i}&=-E_i,\\ \hat{F}_{ij}&=-\epsilon_{ijk}\hat{B}_k, & F_{ij}&=-\epsilon_{ijk}B_k.\end{aligned} \tag{52}$$

Note that $\hat{F}_{\mu\nu}$, $\hat{E}_i$ and $\hat{B}_i$ are functions of commutative operators x and p [see (40)]. Now we can rewrite the rhs of (46) and (51) in terms of commutative electric and magnetic field and get

$$\vec{\nabla}\cdot\hat{\vec{B}}=a(\alpha+\beta)\dot{\vec{B}}\cdot\vec{p}-ae(\vec{D}_B\cdot\vec{B}+s\vec{E}\cdot\vec{B}), \tag{53}$$

$$\begin{aligned}\vec{\nabla}\times\hat{\vec{E}}+\frac{\partial\hat{\vec{B}}}{\partial t}&=-a\Big[(\alpha+\beta)(\dot{\vec{B}}p_0-\dot{\vec{E}}\times\vec{p})\\&\quad+\gamma\Big(p\cdot\frac{\partial\vec{B}}{\partial x}+\frac{\partial\vec{B}}{\partial x}\cdot p\Big)\Big]\\&\quad+ae[\vec{\Box}_E\times\vec{E}+D_B\vec{B}-(s+2)B_i\vec{\nabla}A_i],\end{aligned} \tag{54}$$

$$\begin{aligned}\vec{\nabla}\cdot\hat{\vec{E}}&=\hat{\rho}-a(\alpha+\beta)(p_0\vec{\nabla}\cdot\vec{E}-\dot{\vec{E}}\cdot\vec{p})\\&\quad+ae(\vec{D}_E\cdot\vec{E}-(s+2)\vec{B}^2),\end{aligned} \tag{55}$$

$$\begin{aligned}\vec{\nabla}\times\hat{\vec{B}}-\frac{\partial\hat{\vec{E}}}{\partial t}&=\hat{\vec{j}}+a\Big[(\alpha+\beta)(p_0\dot{\vec{E}}+\dot{\vec{E}}p_0-p_0\vec{\nabla}\times\vec{B}-\dot{\vec{B}}\times\vec{p})\\&\quad+\gamma\Big(p\cdot\frac{\partial\vec{E}}{\partial x}+\frac{\partial\vec{E}}{\partial x}\cdot p\Big)\Big]\\&\quad-ae(\vec{\Box}_B\times\vec{B}+D_E\vec{E}+(s+2)\vec{B}\times\vec{E}+sE_i\vec{\nabla}A_i).\end{aligned} \tag{56}$$

These equations represent the κ-deformed set of Maxwell equations. The operators $\vec{D}_B$, $\vec{D}_E$, D_B, D_E, $\vec{\Box}_B\times$ and $\vec{\Box}_E\times$ are given as follows:

$$\begin{aligned}D_B&=(\vec{r}\cdot\vec{\nabla})\phi\frac{\partial}{\partial t}-\dot{\phi}(\vec{r}\cdot\vec{\nabla})+(2s+3)\phi\frac{\partial}{\partial t}\\&\quad-2(s+1)\dot{\phi}+(s+2)(\vec{A}\cdot\vec{\nabla})+(s+2)(\vec{\nabla}\cdot\vec{A}),\\D_E&=D_B+s\phi\frac{\partial}{\partial t}-2(s+1)\dot{\phi}-2(s+1)(\vec{\nabla}\cdot\vec{A}),\\\vec{D}_B&=(\vec{r}\cdot\vec{\nabla})\vec{A}\frac{\partial}{\partial t}-\dot{\vec{A}}(\vec{r}\cdot\vec{\nabla})+(s+1)\vec{A}\frac{\partial}{\partial t}-2(s+1)\dot{\vec{A}},\\\vec{D}_E&=-\vec{D}_B+s\phi\vec{\nabla}+2(s+1)\dot{\vec{A}},\\\vec{\Box}_B\times&=(\vec{r}\cdot\vec{\nabla})\vec{A}\times\frac{\partial}{\partial t}-\dot{\vec{A}}\times(\vec{r}\cdot\vec{\nabla})\\&\quad-s\phi\vec{\nabla}\times+(s+1)\vec{A}\times\frac{\partial}{\partial t}-(3s+4)\dot{\vec{A}}\times,\\\vec{\Box}_E\times&=-\vec{\Box}_B\times-s\phi\vec{\nabla}\times.\end{aligned} \tag{57}$$

D. *Natural* realization

We see that all the corrections to commutative electrodynamics depend on the realization of operator $\hat{x}_\mu$

$$\hat{x}_\mu=x_\mu+\alpha x_\mu(a\cdot p)+\beta(x\cdot a)p_\mu+\gamma(x\cdot p)a_\mu, \tag{58}$$

that is, on the parameters α, β and γ. We are going to investigate the so-called *natural* realization [28]. The easiest way to get the *natural* realization is to demand that (58) is Hermitian, that is $\hat{x}^\dagger = \hat{x}$ and put $\gamma = 0$, then we get $\alpha = -1$, and $\beta = 1$, and for operator $\hat{x}$ in *natural* realization we have

$$\hat{x}_\mu^{\text{nat}} = x_\mu[1 - (a \cdot p)] + (x \cdot a)p_\mu. \tag{59}$$

The parameter s becomes $s^{\text{nat}} = -1$. For the complex operators given in (57) in *natural* realization we get

$$\begin{aligned}
\vec{D}_B &= (\vec{r}\cdot\vec{\nabla})\vec{A}\frac{\partial}{\partial t} - \dot{\vec{A}}(\vec{r}\cdot\vec{\nabla}),\\
\vec{D}_E &= -\vec{D}_B - \phi\vec{\nabla},\\
D_B &= (\vec{r}\cdot\vec{\nabla})\phi\frac{\partial}{\partial t} - \dot{\phi}(\vec{r}\cdot\vec{\nabla}) + \phi\frac{\partial}{\partial t} + (\vec{A}\cdot\vec{\nabla}) + (\vec{\nabla}\cdot\vec{A}),\\
D_E &= D_B - \phi\frac{\partial}{\partial t},\\
\vec{\Box}_E\times &= \dot{\vec{A}}(\vec{r}\cdot\vec{\nabla}) - (\vec{r}\cdot\vec{\nabla})\vec{A}\times\frac{\partial}{\partial t} + \dot{\vec{A}}\times,\\
\vec{\Box}_B\times &= -\vec{\Box}_E\times + \phi\vec{\nabla}\times.
\end{aligned} \tag{60}$$

And for the κ-deformed Maxwell equations in the *natural* realization we finally have

$$\begin{aligned}
&\vec{\nabla}\cdot\hat{\vec{B}} = -ae(\vec{D}_B\cdot\vec{B} - \vec{E}\cdot\vec{B}),\\
&\vec{\nabla}\times\hat{\vec{E}} + \frac{\partial\hat{\vec{B}}}{\partial t} = ae(\vec{\Box}_E\times\vec{E} + D_B\vec{B} - B_i\vec{\nabla}A_i), \quad \vec{\nabla}\cdot\hat{\vec{E}}\\
&\qquad = \hat{\rho} + ae(\vec{D}_E\cdot\vec{E} - \vec{B}^2),\\
&\vec{\nabla}\times\hat{\vec{B}} - \frac{\partial\hat{\vec{E}}}{\partial t} = \hat{\vec{j}} - ae(\vec{\Box}_B\times\vec{B} + D_E\vec{E} + \vec{B}\times\vec{E} - E_i\vec{\nabla}A_i).
\end{aligned} \tag{61}$$

For the the force operator we have

$$\hat{F}_\mu^{\text{nat}} = \hat{G}_\mu^{\text{nat}} + eF_{\mu\nu}^{\text{nat}}(\hat{y})\dot{\hat{x}}^\nu - m\Gamma_{\mu\nu\lambda}^{\text{nat}}\dot{x}^\nu\dot{x}^\lambda, \tag{62}$$

where

$$\begin{aligned}
\tilde{F}_{\mu\nu}^{\text{nat}} &= 0,\\
\Gamma_{\mu\nu\lambda}^{\text{nat}} &= ae(-F_{\mu 0}\eta_{\lambda\nu} + F_{0\nu}\eta_{\mu\lambda} + 3F_{\mu\nu}\delta_\lambda^0),\\
F_{\mu\nu}^{\text{nat}}(\hat{y}) &= F_{\mu\nu} + at\left(\frac{\partial F_{\mu\nu}}{\partial x}\cdot p\right) - a\dot{F}_{\mu\nu}(x\cdot p),\\
\hat{G}_\mu^{\text{nat}} &= G_\mu(\hat{y}) - amG_0\dot{x}_\mu + am(G\cdot\dot{x})\delta_\mu^0.
\end{aligned} \tag{63}$$

Note [see Eq. (62)] that the force depends not only on the charge of the particle, but on its mass also. This mass dependence vanishes in the limit of $a \rightarrow 0$.

IV. CONCLUSION

In this paper, we have constructed a force equation and Maxwell's equation on κ-deformed space-time. For this construction, we have generalized a variation of Feynman's approach [24] to κ-deformed noncommutative space-time. This approach also starts with the same assumptions as in Feynman's approach [22,23], and uses the notion of canonical conjugate momenta and their commutators (or Poisson brackets) with coordinates. Then, as in the Feynman's approach, by repeated use of Jacobi identity, the force equation and Maxwell's equations are derived. The main differences in this approach are the use of minimal coupling prescription for the gauge field, and the existence of a classical limit. This approach [22–24] allows us to take the classical limit which is obtained by replacing $(i\hbar)^{-1}[]$ with $\{\}_{PB}$.

We have obtained the κ-dependent modification to the Newtons force equation in Sec. III A. Then we introduce the gauge field in Sec. III B, and derive the Lorentz force equation, as well as Maxwell's equations, in the κ-space-time. The additional contributions due to κ-deformation of space-time to the force equation that are linear in $\dot{x}$ can be interpreted as due to a background electromagnetic field and those proportional to $\dot{x}^2$ as a induced curvature of space-time. This is in similar spirit as the induced gravity in Moyal space-time considered in [29]. Here, these corrections are obtained up to first order in the deformation parameter. This change in the Lorentz force equation will affect the trajectories of charged particles in external electromagnetic fields. This can lead to possible observable effects in the beams of high-energy accelerators. It is clear that these effects would violate Lorentz symmetry and modify the dispersion relations. These aspects are investigated in detail in [17].

The κ-dependent corrections to the force equation and Maxwell's equations change with the choice of realization of noncommutative coordinates we use. We have investigated this modification for natural realization [28] in Sec. III D. This realization is Hermitian and has the property that the corresponding momentum transforms as a 4-vector under Lorentz algebra.

The κ-deformed Maxwell equations obtained here are complicated, even in the natural realization. With further simplifying assumptions, they can be compared easily with what we know in the commutative case. For the static case, with $\hat{\rho}$, E, and ϕ set to zero we get the equations for κ-deformed magnetostatic. They are

$$\vec{\nabla}\cdot\hat{\vec{B}} = 0 \qquad \vec{\nabla}\times\hat{\vec{B}} = \hat{\vec{j}}. \tag{64}$$

In the electrostatic limit, there are additional terms. The framework employed here to derive the force equation and Maxwell's equations allows us to replace the (quantum) commutators with corresponding Poisson brackets to get the classical result [22–24]. Thus, by substituting commutator with Poisson bracket, that is $\frac{1}{i\hbar}[,] \rightarrow \{,\}$ and all the operators go to c-number functions. Now the parameter τ becomes the proper time of a particle.

In this classical limit we calculate the corrections to the classical Coulomb force between two test particles of charge e at rest separated at a distance r in the

κ-deformed noncommutative space-time. Since we consider nonrelativistic case, we get $\frac{d\tau}{dt}=1$. By setting $\dot{x}_i=0$ and with vanishing G_μ, B, A and $E=\frac{e}{4\pi r^2}$ in (62), we get

$$\hat{F}=\frac{e^2}{4\pi r^2}(1-2am). \tag{65}$$

Thus we see that the Coulomb law does not change the form. The effect of noncommutativity can be interpreted as change in the charge of the particle $e\rightarrow e(1-2am)^{1/2}$. This shows that the electrodynamics depends on the mass of the particle as well as its charge. The same feature was shown in [21] also.

Next, we consider the case of a particle of mass m and charge e, moving in a constant external electric field E, with a velocity $\vec{v}$. From Eq. (62), we find

$$\begin{aligned}\hat{\vec{F}}&=e\vec{E}(\gamma-am(2\gamma^2+\vec{v}^2))-aem(\vec{E}\cdot\vec{v})\vec{v}\\&=\gamma e\vec{E}-ame\vec{E}(2\gamma^2+\vec{v}^2)-aem(\vec{E}\cdot\vec{v})\vec{v},\end{aligned} \tag{66}$$

where $\gamma=(1-\vec{v}^2)^{1/2}$. With further choice $\vec{E}=(E,0,0)$, we get

$$\begin{aligned}\hat{F}^x&=eE(\gamma-am(2\gamma^2+\vec{v}^2))-aem(Ev^x)v^x\\\hat{F}^y&=-aem(Ev^x)v^y\\\hat{F}^z&=-aem(Ev^x)v^z.\end{aligned} \tag{67}$$

These can be solved to get

$$\dot{\hat{y}}(\tau)=\dot{y}(0)e^{-aeEx(\tau)} \tag{68}$$

$$\hat{y}(\tau)=\dot{y}(0)\tau-\dot{y}(0)aeE\int d\tau x(\tau)+y(0).$$

It is easy to see that $\hat{z}(\tau)$ also obeys the same equation as $\hat{y}(\tau)$, showing the a-dependent modification of $\hat{y}(\tau)$ and $\hat{z}(\tau)$ as deviations from classical trajectories. This pure noncomutative effect may also put some bounds on the parameter a. With initial conditions, $\dot{y}(0)=\dot{z}(0)=0$, we get the a-dependent modified equation

$$\ddot{\hat{x}}=eE/m\frac{1}{\sqrt{1-\dot{x}^2}}-\frac{2aeE}{1-\dot{x}^2}-2aeE\dot{x}^2. \tag{69}$$

Eq. (67) shows that the a-dependent modification to force equation also depends on the mass of the particle apart from its charge.

We also note that, to the first order in the deformation parameter, there are no corrections to Newton's law of gravity. This can be seen by setting $\dot{x}_i, G_0=0$ and $G_i=-G\frac{m^2}{r^2}$ in Eq. (34). This is different from what was shown in [19]. In [19], using the Hamiltonian framework, a modification to Newton's second law due to κ-deformation was derived. The (nonrelativistic) Hamiltonian for a free particle was obtained by taking the appropriate limit of the energy-momentum relation valid in κ-space-time. Here, up to first order in the deformation parameter a, the effect of deformation was to modify the mass $m\rightarrow m(1+am)$. Though the $\frac{1}{r}$ potential was also modified (up to first order in a), this term did not contribute to the force equation. Here also we do see the same feature.

Also, the fact that the force equation was obtained in [19] using the Hamiltonian framework different from what we get here raises the question whether the equations (of motion) obtained here are derivable from a Hamiltonian or a Lagrangian. In the commutative case, the condition for existence of Lagrangian/ Hamiltonian from which equations of motion can be derived had been studied [30,31].This problem, in the Moyal space was investigated in [32]. We plan to study this in the case of κ-space-time.

In Feynman's approach, due to the nonvanishing commutators between the coordinates and velocities, the rotation symmetry is broken and it was shown that by including magnetic angular momentum, this symmetry can be restored [26]. Inclusion of magnetic monopoles in Feynman's approach was also considered. We plan to address these issues separately. Generalizing the method adopted here for general relativistic case, where the metric will depend on the space-time coordinate, is of immense interest. This work is in progress and will be reported elsewhere.

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