Equality between gravitational and electromagnetic absorption cross sections of extreme Reissner-Nordström black holes

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The absorption cross section of Reissner-Nordström black holes for the gravitational field is computed numerically, taking into account the coupling of the electromagnetic and gravitational perturbations. Our results are in excellent agreement with low- and high-frequency approximations. We find equality between gravitational and electromagnetic absorption cross sections of extreme Reissner-Nordström black holes for all frequencies, which we explain analytically. This gives the first example of objects in general relativity in four dimensions that absorb the electromagnetic and gravitational waves in exactly the same way.

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There is a very interesting phenomenon that is caused by electrically charged black holes: the conversion of gravitational to electromagnetic radiation and vice versa. This conversion of radiation is a consequence of the electrostatic field that surrounds charged black holes, and it has been studied by several authors [1–5].

Recently, the present authors showed that the conversion of electromagnetic to gravitational radiation plays an important role in the absorption of pure electromagnetic wave impinging upon a charged black hole [6]. There, we computed numerically the electromagnetic absorption cross section, as well as the conversion coefficients for both axial (odd) and polar (even) modes.

We present here the analysis of the absorption cross section of Reissner-Nordström (RN) black holes when the incident wave is purely gravitational. The transmission and reflection coefficients are different from those for electromagnetic waves, leading to a different absorption cross section except in the extreme RN case. In the case of extreme RN black holes, we find exact equality between the absorption cross sections for electromagnetic and gravitational waves for all frequencies. We also prove this equality analytically and conjecture its relation to the supersymmetry of these black holes. We use natural units with $G = c = 1$.

RN black holes are static charged black holes. The mass M and the electric charge Q satisfy the inequality $|Q| \leq M$. The equations for the electromagnetic and gravitational perturbations are coupled in RN spacetime, and they cannot be considered separately. However, it is possible to write down decoupled equations for two different functions of each polarization [3], namely:

$$\frac{d^2}{dr_*^2}\varphi_\pm^\lambda + (\omega^2 - V_\pm^\lambda)\varphi_\pm^\lambda = 0, \tag{1}$$

where $\lambda =$ axial, polar and r_* is the Wheeler tortoise coordinate defined by $dr/dr^* = f$. The $\pm$ signs in Eq. (1) are related to the different expressions of the effective potentials $V_\pm^\lambda$, which are given for axial modes by

$$V_\pm^{\text{axial}} = f\left[\frac{l(l+1)}{r^2} - \frac{3M}{r^3} + \frac{4Q^2}{r^4} \pm \frac{\Omega}{r^3}\right], \tag{2}$$

where

$$\Omega = [9M^2 + 4Q^2(l-1)(l+2)]^{1/2}, \tag{3}$$

and for polar modes by

$$V_\pm^{\text{polar}} = f(X \pm \Omega S), \tag{4}$$

where

$$X = \frac{f}{(r\Lambda)^2}\left(\frac{8Q^2}{r^2} - \frac{6M}{r}\right)^2 + \frac{8Q^2 f}{r^4\Lambda} + \frac{l(l+1)(l-1)(l+2)}{r^2\Lambda} + \frac{3M}{r^3} + \frac{4Q^2}{r^4\Lambda}\left(2 - \frac{6M}{r} + \frac{4Q^2}{r^2}\right), \tag{5}$$

$$S = \frac{l(l+1)}{r^3\Lambda} + \frac{2f}{r^3\Lambda^2}\left[(l-1)(l+2) + \frac{4Q^2}{r^2}\right] - \frac{1}{r^3\Lambda}\left(\frac{2M}{r} - \frac{2Q^2}{r^2}\right), \tag{6}$$

$$\Lambda = (l-1)(l+2) + \frac{6M}{r} - \frac{4Q^2}{r^2}. \tag{7}$$

We note that the modes φ_+^λ exist for $l \geq 1$ and that the modes φ_-^λ exist for $l \geq 2$.

From the radial equations, it is possible to obtain the asymptotic forms of the functions $\varphi_\pm^\lambda$ and, consequently,

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the asymptotic forms of the electromagnetic and gravitational waves. The asymptotic forms of the decoupled radial functions are

$$\varphi_{\pm}^{\lambda} \propto \begin{cases} e^{-i\omega r_*} + A_{\pm}^{\lambda} e^{i\omega r_*}, & (r_* \to +\infty); \\ B_{\pm}^{\lambda} e^{-i\omega r_*}, & (r_* \to -\infty). \end{cases} \tag{8}$$

The relations between the radial functions $\varphi_{\pm}^{\lambda}$ and the radial part of electromagnetic (F^{λ}) and gravitational (G^{λ}) perturbations at infinity are

$$F^{\lambda} = \varphi_{+}^{\lambda} \cos\psi - \varphi_{-}^{\lambda} \sin\psi, \tag{9}$$

$$G^{\lambda} = \varphi_{+}^{\lambda} \sin\psi + \varphi_{-}^{\lambda} \cos\psi, \tag{10}$$

where

$$\sin(2\psi) = -2PQ\frac{[(l-1)(l+2)]^{1/2}}{\Omega}, \quad |\psi| < \frac{\pi}{4}. \tag{11}$$

Here, $P = +1$ (-1) for the polar (axial) modes.

For the case in which the incident wave is purely gravitational, we have at infinity:

$$G^{\lambda} \sim G_{\text{in}}^{\lambda} e^{-i\omega r_*} + G_{\text{out}}^{\lambda} e^{i\omega r_*}, \tag{12}$$

$$F^{\lambda} \sim F_{\text{out}}^{\lambda} e^{i\omega r_*}. \tag{13}$$

Through Eqs. (8)–(10), (12), and (13), we find

$$C_{\omega l}^{\lambda} = \frac{F_{\text{out}}^{\lambda}}{G_{\text{in}}^{\lambda}} = \frac{\sin(2\psi)}{2}(A_{+}^{\lambda} - A_{-}^{\lambda}), \tag{14}$$

$$R_{\omega l}^{\lambda} = \frac{G_{\text{out}}^{\lambda}}{G_{\text{in}}^{\lambda}} = A_{+}^{\lambda} \sin^2\psi + A_{-}^{\lambda} \cos^2\psi. \tag{15}$$

The conversion, reflection, and transmission coefficients are given, respectively, by $|C_{\omega l}^{\lambda}|^2$, $|R_{\omega l}^{\lambda}|^2$, and $|T_{\omega l}^{\lambda}|^2 = 1 - |C_{\omega l}^{\lambda}|^2 - |R_{\omega l}^{\lambda}|^2$. Unlike in the Schwarzschild case, for which we have only one radial function to be determined (see, for instance, Ref. [7]), here we have to solve two decoupled equations for each polarization. The coefficients A_{+}^{λ} and A_{-}^{λ} in Eq. (8) can be directly found through the solutions of Eq. (1).

The absorbed energy is a mixture of electromagnetic and gravitational radiation. Close to the event horizon we have

$$F(r_* \to -\infty) \approx F_{\text{tr}} e^{-i\omega r_*}, \tag{16}$$

$$G(r_* \to -\infty) \approx G_{\text{tr}} e^{-i\omega r_*}. \tag{17}$$

Using these two equations, together with Eqs. (8)–(10), we find that the transmission coefficients can be written in terms of $B_{\pm}^{\lambda}$ as

$$|T_{\omega l}^{\lambda}|^2 = |B_{+}^{\lambda}|^2 \sin^2\psi + |B_{-}^{\lambda}|^2 \cos^2\psi. \tag{18}$$

The gravitational absorption cross section for RN black holes is

$$\sigma_{\text{abs}} = \sum_{l=2}^{\infty} \sigma_{\text{abs}}^{(l)}, \tag{19}$$

$$\sigma_{\text{abs}}^{(l)} = \frac{\pi}{2\omega^2}(2l+1)\sum_{\lambda} |T_{\omega l}^{\lambda}|^2, \tag{20}$$

where $\sigma_{\text{abs}}^{(l)}$ is the absorption cross section of the l-th partial wave, with the sum over λ including axial and polar

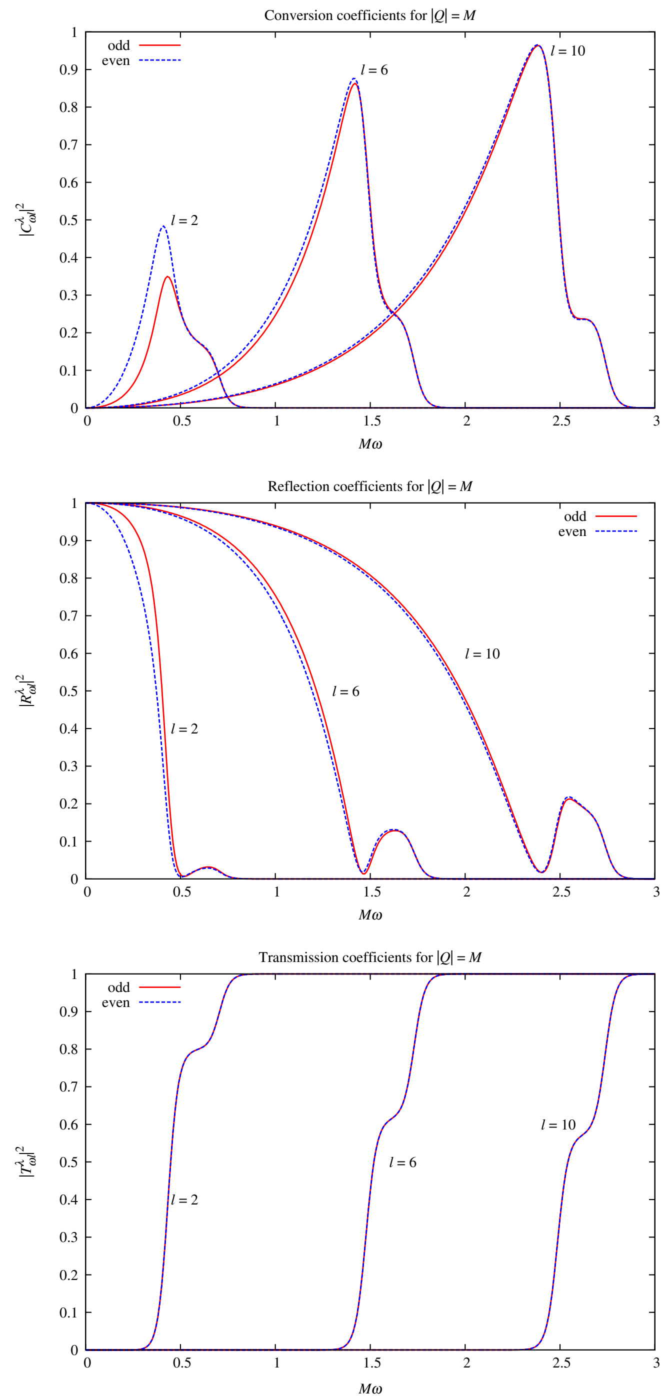

FIG. 1 (color online). Conversion (*top*), reflection (*middle*), and transmission (*bottom*) coefficients for a gravitational wave scattered by an extreme RN black hole.

modes. We note that the transmission coefficients are equal for axial and polar modes [8].

The results in the low-frequency limit for the absorption cross section can be obtained in Schwarzschild and extreme RN black hole cases. The low-frequency gravitational absorption cross section for Schwarzschild black holes can be found in Ref. [9], and it reads $\sigma_{\rm abs} \approx \frac{256}{45}\pi M^2 (M\omega)^4$. For the extreme RN case ($|Q| = M$), one can use Eq. (1) to find the low-frequency absorption cross section in the gravitational case. The result is $\sigma_{\rm abs} \approx \frac{16}{9}\pi M^2 (M\omega)^6$. This is equal to the low-frequency limit of the electromagnetic absorption cross section of extreme RN black holes, presented in Ref. [6].

In fact, it can be shown that the gravitational and electromagnetic absorption cross sections are equal for extreme RN black holes for all ω. As noted in Ref. [10], the transmission coefficients defined by Eq. (8) satisfy $|B^{\lambda}_{+,l}|^2 = |B^{\lambda}_{-,l+1}|^2$ (with the l dependence emphasized here). From this equation, Eqs. (11) and (18)–(20), and the corresponding equations for the electromagnetic waves, we find that the total absorption cross sections for these waves are equal for all ω and given by

$$\sigma_{\rm em, grav} = \frac{2\pi}{\omega^2} \sum_{l=2}^{\infty} l |B^{\rm axial}_{-,l}|^2 \quad (|Q| = M). \tag{21}$$

Now, we apply numerical methods to solve the radial Eq. (1) for arbitrary values of the incident wave frequency. The solutions are obtained by evolving them from $r \gtrsim r_+$ to a region away from the black hole.

The numerical results for the conversion, reflection, and transmission coefficients are presented in Fig. 1 for the extreme RN black hole case. They show some similarities with the ones for the electromagnetic perturbation [6]. We emphasize that there is no $l = 1$ mode in the gravitational case. We can observe that the reflection and conversion coefficients depend on the mode polarization, but the amount of reflected energy (energy that goes back to infinity as gravitational and electromagnetic radiation) and transmitted energy (the energy that is absorbed by the black hole) do not depend on the wave polarization [8]. This can clearly be seen in the graphs of the transmission coefficients [cf. Fig. 1 (bottom)].

We call attention to some interesting features of the conversion, reflection, and transmission coefficients shown in Fig. 1: the inflection points in the conversion and transmission coefficients, and the local maxima of the reflection coefficients. These features are related to the critical orbit of massless particles. We can associate an impact parameter to a partial wave as $b = (l + 1/2)/\omega$. These features can be found to occur when $b \approx b_c$, where b_c is the critical impact parameter, for which a null geodesic circulates the black hole an infinite number of times. For a fixed l, as ω

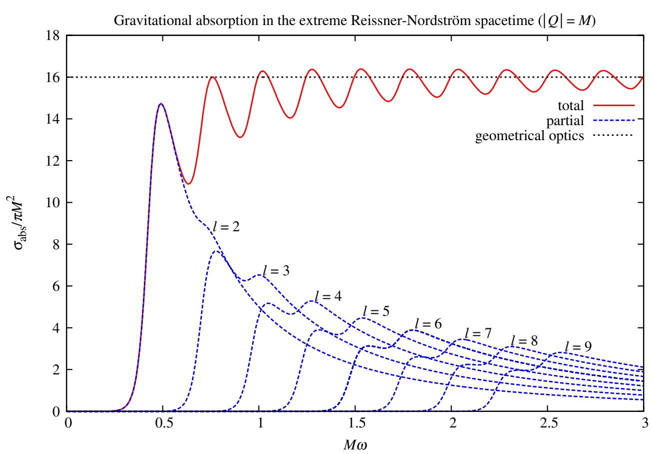

FIG. 2 (color online). Partial and total gravitational absorption cross section for extreme RN black holes. We see that the partial absorption cross sections inherit the points of inflection from the transmission coefficients, but this does not manifest distinctively in the total absorption cross section, which tends to oscillate regularly around the high-frequency limit value.

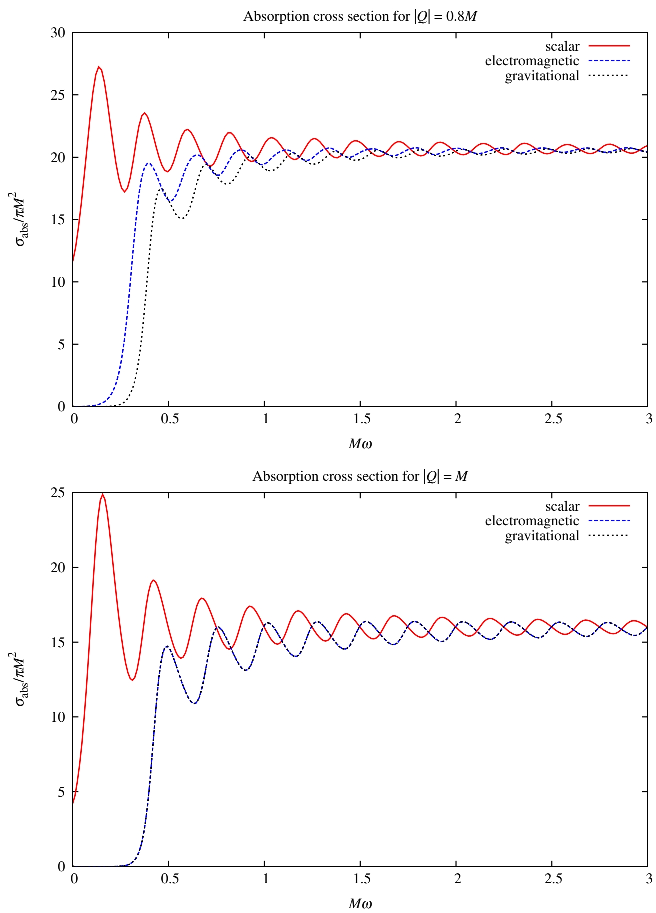

FIG. 3 (color online). Total absorption cross sections of charged black holes for massless particles of spin 0, 1, and 2. *Top*: RN black holes with $|Q| = 0.8M$; *bottom*: extreme RN black holes. As the black hole charge approaches extremality, the total absorption cross section for the electromagnetic and gravitational radiations get closer and are equal for extremely charged black holes.

increases, the impact parameter b decreases. Therefore, the conversion becomes more effective as ω increases from zero because the wave spends more time in the vicinity of the black hole. As the impact parameter becomes closer to its critical value, the wave starts to be absorbed and the transmission coefficient increases from zero. As this happens, part of the wave that would be converted gets absorbed, and then the conversion coefficient starts decreasing rapidly. When the impact parameter gets close to the critical value, the conversion and transmission factors have points of inflection, and the reflection coefficients exhibit local maxima.

Once the transmission coefficients are determined, the absorption cross section follows directly from Eqs. (19) and (20). In Fig. 2, we show the partial and total gravitational absorption cross sections for extremely charged black holes. We see that each partial absorption cross section has an inflection point inherited from the behavior of the transmission coefficients [cf. Fig. 1 (bottom)].

The total absorption cross sections of RN black holes for massless bosonic particles are presented in Fig. 3. There, we have $|Q| = 0.8M$ (top), and extreme (bottom) cases. For $|Q| = 0.8M$, the results for the electromagnetic and gravitational waves oscillate in opposite phase to the scalar case [11] in the high-frequency limit. The equality of the total absorption cross sections of gravitational and electromagnetic waves for extreme RN black holes is confirmed numerically in this figure. This equality only holds for the total absorption cross section. For the partial absorption cross sections, the results are obviously different for low values of l; for instance, the partial wave with $l = 1$ present in the electromagnetic case is absent in the gravitational wave. As compared with the scalar case [12] for high frequencies, the result for the electromagnetic and gravitational absorption oscillate in opposite phase to it. Such phase difference in the absorption cross section is observed in the absorption by Schwarzschild black holes when the comparison is made between bosonic and fermionic cases [13].

In summary, we have computed the gravitational absorption cross section for RN black holes. We have verified numerically that the absorption of gravitational waves by charged black holes does not depend on the wave polarization and tends to the geometrical optics absorption cross section. We also showed and numerically confirmed that the total absorption cross sections for electromagnetic and gravitational waves are the same for extreme RN black holes in the entire range of the wave frequency. This does not happen for nonextremal RN black holes. For instance, the results for Schwarzschild black holes are very different for the electromagnetic and gravitational cases, mainly in the low-frequency regime [13]. Although the relation $|B^{\lambda}_{+,l}|^2 = |B^{\lambda}_{-,l+1}|^2$ has been explained [14] using the supersymmetry of extreme RN black holes [15] in the context of $N = 2$ supergravity [16], the special value of the mixing angle in Eq. (11) for $|Q| = M$ is also necessary for the equality between the gravitational and electromagnetic absorption cross sections. It will be interesting to see whether this mixing angle can also be explained using supersymmetry.

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