Principle of relative locality

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We propose a deepening of the relativity principle according to which the invariant arena for nonquantum physics is a phase space rather than spacetime. Descriptions of particles propagating and interacting in spacetimes are constructed by observers, but different observers, separated from each other by translations, construct different spacetime projections from the invariant phase space. Nonetheless, all observers agree that interactions are local in the spacetime coordinates constructed by observers local to them. This framework, in which absolute locality is replaced by relative locality, results from deforming energy-momentum space, just as the passage from absolute to relative simultaneity results from deforming the linear addition of velocities. Different aspects of energy-momentum space geometry, such as its curvature, torsion and nonmetricity, are reflected in different kinds of deformations of the energy-momentum conservation laws. These are in principle all measurable by appropriate experiments. We also discuss a natural set of physical hypotheses which singles out the cases of energy-momentum space with a metric compatible connection and constant curvature.

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I. INTRODUCTION

How do we know we live in a spacetime? And if so how do we know we all share the same spacetime? These are the fundamental questions we are investigating in this note.

We local observers do not directly observe any events macroscopically displaced from our measuring instruments. As naive observers looking out at the world, and no less as particle physicists and astronomers, we are basically "calorimeters" with clocks. Our most fundamental measurements are the energies and angles of the quanta we emit or absorb, and the times of those events. Judging by what we observe, we live in energy-momentum space, not in spacetime.

The idea that we live in a spacetime is constructed by inference from our measurements of momenta and energy. This was vividly illustrated by Einstein's procedure to give spacetime coordinates to distant events by exchanges of light signals [1]. When we use Einstein's procedure we take into account the time it takes the photons to travel back and forth but we throw away information about their energy, resulting in a projection into spacetime. When we do this we presume that the same spacetime is reconstructed by exchanges of light signals of different frequencies. We are also used to assuming that different local observers, distant from each other, reconstruct the same spacetime by measurements of photons they send and receive.

But why should the information about the energy of the photons we use to probe spacetime be inessential? Might that just be a low energy approximation? And why

should we presume that we construct the same spacetime from our observations as observers a cosmological distance from us?

Special and general relativity are based on an hypothesis, which is that the observations made by different observers can be organized into a common description of events taking place within a Lorentzian spacetime. This implies a notion of "absolute locality," such that *when pairs of spacetime events coincide for one observer they also coincide for all other observers*. We show below that the notion of absolute locality is equivalent to the assumption that energy-momentum space is a linear manifold so that energy-momentum conservation is linear. This corresponds to an idealization in which we throw away information about the energy of the quanta we use in Einstein's localization procedure to assign spacetime coordinates to distant events.

As we show here, this idealization can be transcended in a simple and powerful generalization of special relativity which is motivated by general considerations of the unification of gravity with quantum physics. We propose a new framework in which we can relax in a controlled manner the concept of absolute locality. This is done by curving the geometry of energy-momentum space. This as we will see implies that absolute locality is replaced by a notion of "relative locality" in which different observers see different spacetimes, and the spacetimes they observe are energy and momentum dependent. As a result, coincidences of events are still objective for all observers local to those events, but they are not in general manifest in the spacetime coordinates constructed by distant observers.

One way to motivate this new physical framework is by thinking about the symmetry of the vacuum. The most

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basic question that can be asked of any physical system is what is the symmetry of the ground state that governs its low lying excitations. This is no less true of spacetime itself, moreover in general relativity, and presumably in any description of the quantum dynamics of spacetime, the symmetry of the ground state is dynamically determined. We also expect that the classical spacetime geometry of general relativity is a semiclassical approximation to a more fundamental quantum geometry. In this paper we show how simple physical assumptions about the geometry of energy-momentum space may control the departure of the spacetime description from the classical one.

We will first restrict attention to an approximation in which $\hbar$ and G_{Newton} both may be neglected while their ratio

$$\sqrt{\frac{\hbar}{G_{\text{Newton}}}} = m_p \qquad (1)$$

is held fixed.[1] In this approximation, quantum and gravitational effects may both be neglected, but there may be new phenomena on scales of momentum or energy given by m_p. At the same time, because $l_p = \sqrt{\hbar G_{\text{Newton}}} \to 0$ we expect no features of quantum spacetime geometry to be relevant.

Since our approximation gives us an energy scale, but not a length scale, we will begin by presuming that, within its domain of applicability, energy-momentum space should play a more fundamental role than spacetime. This is in accord with the operational point of view we mentioned in the opening paragraph. So we begin in energy-momentum space by asking how it may be deformed in a way that is measured by a scale m_p. Once that is established we will derive the properties of spacetime from dynamics formulated in energy-momentum space. For convenience we work first in the limit just described, after which we will briefly turn on $\hbar$.

By following this logic below, we will find that physics may be governed by a novel principle, which we call the *Principle of Relative Locality*. This states that,

Physics takes place in phase space and there is no invariant global projection that gives a description of processes in spacetime. From their measurements local observers can construct descriptions of particles moving and interacting in a spacetime, but different observers construct different spacetimes, which are observer-dependent slices of phase space.

In the next section we introduce an operational approach to the geometry of energy-momentum space, which we build on in Sec. III to give a dynamics of particles on a curved energy-momentum space. We see how a modified version of spacetime geometry is emergent from the dynamics which is formulated on energy-momentum space. In these sections we consider a general energy-momentum

space geometry, which illuminates a variety of new phenomena that might be experimentally probed corresponding to the curvature, torsion and nonmetricity of energy-momentum space. However, one advantage of this approach is that with a few reasonable physical principles the geometry of energy-momentum space can be reduced to three choices, depending on the sign of a parameter. As we show in Sec. IV, this gives this framework both great elegance and experimental specificity. In Sec. V we make some preliminary observations as to how the geometry of energy-momentum space may be probed experimentally, after which we conclude.

II. AN OPERATIONAL APPROACH TO THE GEOMETRY OF MOMENTUM SPACE

We take an operational point of view in which we describe physics from the point of view of a local observer who is equipped with devices to measure the energy and momenta of elementary particles in her vicinity. The observer also has a clock that measures local proper time. We construct the geometry of momentum space[2] $\mathcal{P}$ from measurements made of the dynamics of interacting particles. We assume that to each choice of calorimeter and other instruments carried by our observer there is a preferred coordinate on energy-momentum space, k_μ. But we also assume that the dynamics can be expressed covariantly in terms of the geometry of $\mathcal{P}$ and do not depend on the choice of calorimeter's coordinates. We note that the k_μ measure the energy and momenta of excitations above the ground state, hence the origin of momentum space, $k_\mu = 0$, is physically well defined.

Our local observer can make two kinds of measurements. One type of measurement can be done only with a single particle and it defines, as we will see, *a metric* on momentum space. The other type of measurement involve multi particles and defines *a connection*. A key mathematical idea underlying our construction is that a connection on a manifold can be determined by an algebra [2], in the present case this will be an algebra that determines how momenta combine when particles interact.

A. The metric geometry of momentum space

First we describe the metric geometry. Our local observer can measure the rest energy or relativistic mass of a particle which is a function of the four-momenta. She can

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[2]For brevity, in the remainder of this article we often refer to the manifold of relativistic four-momenta (the "energy-momentum space") compactly as "momentum space". For the four-momenta of our "momentum space" k_μ (our Greek indices run 0, 1, 2, 3), we adopt conventions such that the energy component is k_0 while the spatial-momentum components are k_j (our Latin indices run 1, 2, 3). In order to avoid confusion, in the few instances when we refer exclusively to the k_j components we shall characterize them as the spatial momentum.

also measure the kinetic energy K of a particle of mass m moving with respect to her, but local to her. We postulate that these measurements determine the metric geometry of momentum space. To this end we interpret the mass as the geodesic distance from the origin, and this gives the dispersion relation

$$D^2(p) \equiv D^2(p, 0) = m^2. \tag{2}$$

The kinetic energy is defined to be given by the geodesic distance between a particle at rest for a given observer (with momentum $p = (p_0, 0, 0, 0)$) and a particle of identical mass carrying kinetic energy K, having momentum p', as measured by the same observer. In this case $D(p) = D(p') = m$ and the kinetic energy is related to the geodesic distance on the momentum manifold as follows

$$D^2(p, p') = -2mK. \tag{3}$$

The minus sign expresses the fact that the geometry of momentum space is Lorentzian. From these measurements one can reconstruct a metric[3] on $\mathcal{P}$

$$dk^2 = h^{\mu\nu}(k)dk_\mu dk_\nu. \tag{4}$$

B. The algebra of interactions

Now we describe the construction of the connection on momentum space. This is determined by processes in which n particles interact, n_{in} incoming and n_{out} outgoing, with $n = n_{\text{in}} + n_{\text{out}}$. This proceeds by the construction of an algebra, which then determines the connection.

Associated to each interaction there must be a combination rule for momentum, which will in general be nonlinear. We denote this rule for two particles by

$$(p, q) \rightarrow p'_\mu = (p \oplus q)_\mu. \tag{5}$$

Hence the momentum space $\mathcal{P}$ has an algebraic structure defined by the deformed sum rule $\oplus$. We assume that more complicated processes are built up by iterations of this deformed sum rule—but to begin with we assume neither linearity, nor commutativity nor associativity. We stress that we are not arguing that any of these properties should be necessary; on the contrary we propose to treat the structure of momentum space just like any other experimental issue, including the possibility of nonassociativity and/or noncommutativity of the deformed sum rule.

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[3]In the standard case of physics in Minkowski spacetime, $h^{\mu\nu}$ is the dual Minkowski metric and $\mathcal{K}_\mu(k) = \sum_I k_\mu^I$. A scale m_p may be introduced by deforming the geometry of $\mathcal{P}$ so that it is curved. The correspondence principle (to be introduced below) assures that we recover the standard flat geometry of $\mathcal{P}$ in the limit $m_p \rightarrow \infty$.

We will also need an operation that turns outgoing momenta into incoming momenta. This can be called the antipode and is denoted, $p \rightarrow \ominus p$ and it satisfies[4]

$$(\ominus p) \oplus p = p \oplus (\ominus p) = 0. \tag{6}$$

Then we have the conservation law of energy-momentum for any process, giving, for each type of interaction, four functions on $\mathcal{P}^n$, depending on momenta of interacting particles, which vanish

$$\mathcal{K}_\mu(k^I) = 0. \tag{7}$$

For example, for a process with two incoming momenta p_μ, q_μ, and an outgoing momentum k_μ one might have

$$\mathcal{K}_\mu(p, q, k) = (p_\mu \oplus q_\mu) \oplus (\ominus k_\mu) = 0 \tag{8}$$

which, by virtue of (6) can be equivalently written as

$$(p_\mu \oplus q_\mu) = k_\mu. \tag{9}$$

In order to keep the presentation as general as possible in the following we shall not specify which particles our incoming and which are outgoing from a process. We shall mainly focus on the illustrative example of 3-particle interactions, for which we shall take the conservation law

$$\mathcal{K}_\mu(p, q, k) = (p_\mu \oplus q_\mu) \oplus k_\mu = 0, \tag{10}$$

keeping in mind that for an outgoing momentum one would place in (10) the antipode of that momentum. For example, for the case of a particle-decay process, we could have (10) taking the form $[((\ominus p_\mu^{(f)}) \oplus (\ominus p_\mu^{(f)'})) \oplus p^{(i)}]_\mu = 0$.

C. From the algebra of interactions to the connection on momentum space

Corresponding to the algebra of combinations of momenta there is a connection on $\mathcal{P}$. The geometry of momentum space is studied in detail in [2], but the basics are as follows. The algebra of the combination rule determines a connection on $\mathcal{P}$ by

$$\frac{\partial}{\partial p_\mu} \frac{\partial}{\partial q_\nu} (p \oplus q)_\rho |_{q, p=o} = -\Gamma_\rho^{\mu\nu}(0). \tag{11}$$

The torsion of $\Gamma_\rho^{\mu\nu}$ is a measure of the asymmetric part of the combination rule

$$-\frac{\partial}{\partial p_\mu} \frac{\partial}{\partial q_\nu} ((p \oplus q)_\rho - (q \oplus p)_\rho)_{q, p=o} = T_\rho^{\mu\nu}(0). \tag{12}$$

Similarly the curvature of $\mathcal{P}$ is a measure of the lack of associativity of the combination rule

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[4]And more generally $(\ominus p) \oplus (p \oplus k) = k$, where $\ominus$ is a left inverse.

$$2 \frac{\partial}{\partial p_{[\mu}} \frac{\partial}{\partial q_{\nu]}} \frac{\partial}{\partial k_{\rho}} ((p \oplus q) \oplus k - p \oplus (q \oplus k))_{\sigma}|_{q,p,k=o}$$

$$= R_{\sigma}^{\mu\nu\rho}(0) \qquad (13)$$

where the bracket denote the antisymmetrization.

We note that there is no physical reason to expect a combination rule for momentum to be associative, once it is nonlinear. Indeed, the lack of associativity means there is a physical distinction between the two processes of Fig. 1, which is equivalent to saying there is a definite microscopic causal structure. That is, *causal structure of the physics maps to nonassociativity of the combination rule for momentum which in turn maps to curvature of momentum space.* The curvature of momentum space makes microscopic causal orders distinguishable, and hence meaningful. This gives rise to proposals to measure the curvature of momentum space which we will discuss below.

The mathematical framework allows the possibility that the combination rule for momentum be noncommutative. This has a direct physical meaning, which is that physical processes must satisfy conditions on how momentum and energy are distributed among the incoming particles in a process, which hold in addition to the (nonlinear) laws of total energy and momentum conservation. In Fig. 2 we illustrate this possibility for the example of photon absorption by an atom. The photon of momentum p_{μ} excites the atom from the initial state, characterized by an internal energy level ω_I and momentum k_{μ}, so that the final state has momentum q_{μ} and is in the internal energy level ω_{II}. One might find experimentally that q_{μ} is obtained from p_{μ} and k_{μ} through a nonlinear composition law, such that $q_{\mu} = k_{\mu} \oplus p_{\mu}$ (but $q_{\mu} \neq p_{\mu} \oplus k_{\mu}$).

At the level we are working, which is classical particle mechanics, all particles can be presumed to be distinguishable. But looking ahead to quantum physics where particles can be indistinguishable we can say the following. For distinguishable particles the interaction would be allowed only if, say, $p_f = p_i \oplus p_i'$, whereas for indistinguishable
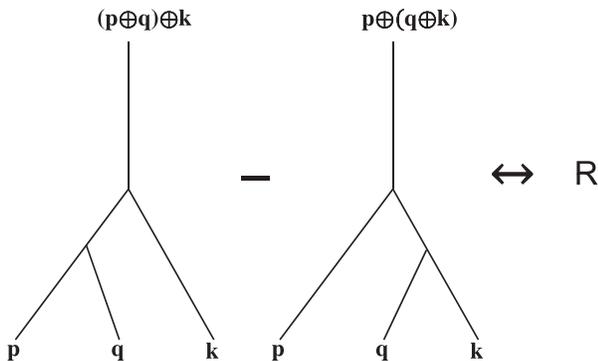
FIG. 1. Curvature of the connection on momentum space produces nonassociativity of composition rule.
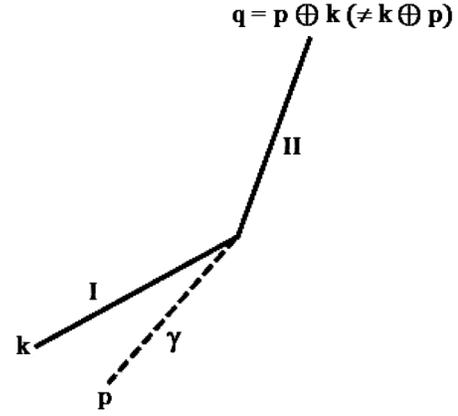
FIG. 2. Torsion implies that only one order of (deformed) summation of momenta actually is allowed for the process. In figure the atom initially in internal state I with momentum k_{μ} could absorb the photon with momentum q_{μ} to go in an another internal state II with momentum q_{μ}, with $q_{\mu} = k_{\mu} \oplus p_{\mu} \neq p_{\mu} \oplus k_{\mu}$.

incoming particles of momenta p_i, p_i' the interaction could be allowed both for $p_f = p_i \oplus p_i'$ and for $p_f = p_i' \oplus p_i$.

To determine the connection, torsion and curvature away from the origin of momentum space we have to consider translating in momentum space, i.e. we can denote

$$p \oplus_k q = k \oplus ((\ominus k \oplus p) \oplus (\ominus k \oplus q)) \qquad (14)$$

the identity for this product is at $0_k = k$. Then

$$\frac{\partial}{\partial p_{\mu}} \frac{\partial}{\partial q_{\nu}} (p \oplus_k q)_{\rho}|_{q,p=k} = -\Gamma_{\rho}^{\mu\nu}(k). \qquad (15)$$

Thus, the action of adding an infinitesimal momentum dq_{μ} from particle J to a finite momentum p_{μ} of particle I defines a parallel transport on $\mathcal{P}$.

$$p_{\mu} \oplus dq_{\mu} = p_{\mu} + dq_{\nu}\tau_{\mu}^{\nu}(p) \qquad (16)$$

where $\tau(p)$ is the parallel transport operation from the identity to p. It can be expanded around $p = 0$

$$\tau_{\mu}^{\nu}(p) = \delta_{\mu}^{\nu} - \Gamma_{\mu}^{\nu\rho}p_{\rho} - \Gamma_{\mu}^{\nu\rho\sigma}p_{\rho}p_{\sigma} + \cdots \qquad (17)$$

with

$$\Gamma_{\mu}^{\nu\rho\sigma} = \partial_p^{\sigma}\Gamma_{\mu}^{\nu\rho} - \Gamma_{\lambda}^{\sigma\nu}\Gamma_{\mu}^{\lambda\rho} - \Gamma_{\lambda}^{\sigma\rho}\Gamma_{\mu}^{\nu\lambda}. \qquad (18)$$

The corresponding conservation law thus has the form to second order

$$\mathcal{K}_{\mu}(k) = \sum_I k_{\mu}^I - \sum_{J \in \mathcal{J}(I)} C_{I,J}\Gamma_{\mu}^{\nu\rho}k_{\nu}^J k_{\rho}^I + \ldots \qquad (19)$$

where $\mathcal{J}(I)$ is the set of particles that interact with the I'th one and $C_{I,J}$ are coefficients that depend on the form of the conservation law.

We will shortly study the consequences of curvature and torsion on momentum space for the dynamics of particles.

We will see that the meaning of the curvature of momentum space is that it implies a limitation of the usefulness of the notion that processes happen in an invariant spacetime, rather than in phase space. The hypothesis of a shared, invariant spacetime, in which all observers agree on the locality of distant interactions, turns out to be a direct consequence of the linearity of the usual conservation laws of energy and momentum. When we deform the conservation laws by making them nonlinear, this gives rise to relative locality.

III. THE EMERGENCE OF SPACETIME FROM THE DYNAMICS OF PARTICLES

We take the point of view that spacetime is an auxiliary concept which emerges when we seek to define dynamics in momentum space. If we take the momenta of elementary particles to be primary, they themselves need momenta, so that a canonical dynamics can be formulated. The momenta of the momenta are quantities x_I^μ that live in the cotangent space of $\mathcal{P}^n$ at a point k_μ^I.

A. Variational principle

Given these we can define the free particle dynamics by

$$S_{\text{free}}^I = \int ds(x_J^\mu \dot{k}_\mu^J + \mathcal{N}_J \mathcal{C}^J(k)) \qquad (20)$$

where s is an arbitrary time parameter and $\mathcal{N}_J$ is the Lagrange multiplier imposing the mass shell condition

$$\mathcal{C}^J(k) \equiv D^2(k) - m_J^2. \qquad (21)$$

We emphasize that the contraction $x_J^\mu \dot{k}_\mu^J$ does not involve a metric, and the dynamics is otherwise given by constraints which are functions only of coordinates on $\mathcal{P}$ and depend only the geometry of $\mathcal{P}$. This leads to the Poisson brackets,

$$\{x_I^\mu, k_\nu^J\} = \delta_\nu^\mu \delta_I^J. \qquad (22)$$

We then have a single-particle phase space, Γ, which is the cotangent bundle of $\mathcal{P}$. This is coordinatized locally by x^μ, k_ν. We note that there is neither an invariant projection from Γ to a spacetime, $\mathcal{M}$, nor is there defined any invariant spacetime metric. Yet this structure is sufficient to describe the dynamics of a free particle. The fact that there is no invariant projection to a spacetime is related to the nonlinearity of momentum space. Indeed under a non-linear redefinition $p \rightarrow F(p)$ the conjugated coordinates is given by $x \rightarrow (\partial p/\partial F)x$, so the new canonical coordinate appears to be momentum space dependent. This is this mixing between "spacetime" and momentum space that is the basis of the relative locality. We can call the x_J^μ *Hamiltonian spacetime coordinates* because they are defined as being canonically conjugate to coordinates on momentum space.

To describe a process involving several particles we employ a multiparticle phase space which is a simple product of the phases spaces for each particle.

Note that we do not need a spacetime or spacetime metric to describe how these particles interact. If we consider the process with n interacting particles we want to impose conservation of the nonlinear quantities, $\mathcal{K}_\mu$. We do this by introducing a Lagrange multiplier to guarantee conservation (7). The action is

$$S^{\text{total}} = \sum_J S_{\text{free}}^J + S^{\text{int}} \qquad (23)$$

where for the incoming particles

$$S_{\text{free}}^J = \int_{-\infty}^0 ds(x_J^\mu \dot{k}_\mu^J + \mathcal{N}_J \mathcal{C}^J(k)), \qquad (24)$$

while for the outgoing particles

$$S_{\text{free}}^J = \int_0^\infty ds(x_J^\mu \dot{k}_\mu^J + \mathcal{N}_J \mathcal{C}^J(k)). \qquad (25)$$

The interaction contribution to the action is simply a Lagrange multiplier times the conservation law (7).

$$S^{\text{int}} = \mathcal{K}(k(0))_\mu z^\mu. \qquad (26)$$

We have set the interaction to take place at affine parameter $s = 0$ for each of the particles. At this point z^μ can be just considered to be a Lagrange multiplier to enforce the conservation of momentum (7) at the interaction where for each particle $s = 0$.

We vary the total action. After an integration by parts in each of the free actions we have

$$\delta S^{\text{total}} = \sum_J \int_{s_1}^{s_2} \left(\delta x_J^\mu \dot{k}_\mu^J - \delta k_\mu^J \left[\dot{x}_J^\mu - \mathcal{N}_J \frac{\delta \mathcal{C}^J}{\delta k_\mu^J} \right] \right.$$
$$\left. + \delta \mathcal{N}_J \mathcal{C}^J(k) \right) + \mathcal{R}. \qquad (27)$$

Here $\mathcal{R}$ contains both the result of varying S^{int} and the boundary terms from the integration by parts. $s_{1,2}$ are $0, \infty, -\infty$ depending on whether the term is incoming or outgoing. Before examining the boundary terms we confirm we have the desired free parts of the equations of motion

$$\dot{k}_\mu^J = 0 \qquad \dot{x}_J^\mu = \mathcal{N}_J \frac{\delta \mathcal{C}^J}{\delta k_\mu^J} \qquad \mathcal{C}^J(k) = 0. \qquad (28)$$

We fix $\delta k_\mu^J = 0$ at $s = \pm\infty$ and examine the remainder of the variation

$$\mathcal{R} = \mathcal{K}(k)_\mu \delta z^\mu - \left(x_J^\mu(0) - z^\nu \frac{\delta \mathcal{K}_\nu}{\delta k_\mu^J} \right) \delta k_\mu^J. \qquad (29)$$

Here x_J^μ and k_μ^J are taken for each particle at the parameter time $s = 0$. This has to vanish if the variational principle is to have solutions. From the vanishing of the coefficient of δz^μ we get the four conservation laws of the

interaction (7). From the vanishing of the coefficient of δk_μ^J we find $4n$ conditions which hold at the interaction

$$x_J^\mu(0) = z^\nu \frac{\delta \mathcal{K}_\nu}{\delta k_\mu^J}. \qquad (30)$$

Using (19) this gives conditions

$$x_J^\mu(0) = z^\mu - z^\nu \sum_{L \in \mathcal{J}(J)} C_{J,L} \Gamma_\nu^{\mu\rho} k_\rho^L + \dots \qquad (31)$$

This tells us that to leading order, in which we ignore the curvature of momentum space, all of the worldlines involved in the interaction meet at a single spacetime event, z^μ. The choice of z^μ is not constrained and cannot be, for its variation gives the conservation laws (7). Thus, we have recovered the usual notion that interactions of particles take place at single events in spacetime from the conservation of energy and momentum. This is good because in quantum field theory conservation implies locality, and it is good to have a formulation of classical interactions where this is also the case.

However when we include terms proportional to z^μ, which is to say when the observer is not at the interaction event, we see that the relationship between conservation of energy and momentum and locality of interactions is realized a bit more subtly. The interaction takes place when the condition (31) is satisfied, that is at n separate events, separated from z^μ by intervals

$$\Delta x_J^\mu(0) = -z^\nu \sum_{L \in \mathcal{J}(J)} C_{J,L} \Gamma_\nu^{\mu\rho} k_\rho^L + \dots \qquad (32)$$

These relations (31) and (32), illustrate concisely the relativity of locality. For some fortunate observers the interaction takes place at the origin of their coordinates, so that $z^\mu = x_J^\mu(0) = 0$ in which case the interaction is observed to be local. Any other observer, translated with respect to these, has a nonvanishing z^μ and hence sees the interaction to take place at a distant set of events. These are centered around z^μ but are not precisely at the same values of the coordinates. That is the coordinates of particles involved in an interaction removed from the origin of the observer by a vector z^μ are spread over a region of order

$$\Delta x \approx |z||\Gamma|k \qquad (33)$$

The relationship (30) possess a very nice mathematical meaning too. Since the momentum space is in general curved the proper way to define the conjugate coordinates is as elements of the cotangent bundle of $\mathcal{P}$. The cotangent space based at p^I and the cotangent space based at 0 are different spaces in the general curved case. This expresses mathematically the relativity of locality. The Hamiltonian particle coordinate x_I represent an element of $T_{p^I}^*\mathcal{P}$ while the interaction coordinate being dual to the conservation law represent an element of $T_0^*\mathcal{P}$. (30) represent a relation between these two spaces and remarkably it can be shown

[2] that indeed the coefficient $\partial\mathcal{K}/\partial k$ evaluated when $\mathcal{K} = 0$ is the parallel transport operator, more precisely

$$\partial_p^\nu(k \oplus p)_\mu|_{p=\ominus k} = (\tau(k)^{-1})_\mu^\nu \qquad (34)$$

where $\tau(k)$ is the parallel transport operator of vectors from 0 to k introduced earlier therefore $\tau(k)^{-1}$ is the parallel transport operator of covectors from 0 to k.

B. The physical meaning of relative locality

Is this a real, physical nonlocality or a new kind of coordinate artifact? It is straightforward to see that it is the latter, because the $\Delta x_J^\mu(0)$ can be made to vanish by making a translation to the coordinates of another observer. In a canonical formulation, translations are generated by the laws of conservation of energy and momentum, (7). Given any local observable in phase space $\mathcal{O}$ observed by a local observer, Alice, we can construct the observable as seen in coordinates constructed by another observer, Bob, distant from Alice, by a translation labeled by parameters b^μ.

$$\delta_b \mathcal{O} = b^\nu \{\mathcal{K}_\nu, \mathcal{O}\}. \qquad (35)$$

Since momentum space is curved, and $\mathcal{K}_\nu$ is nonlinear, it follows that the "spacetime coordinates" x_J^μ of a particle translate in a way that is dependent on the energy and momenta of the particles it interacts with, $x_J^\mu \to x_J'^\mu(0) = x_J^\mu(0) + \delta_b x_J^\mu(0)$ where

$$\begin{aligned} \delta_b x_J^\mu(0) &= b^\nu \{\mathcal{K}_\nu, x_J^\mu\} \\ &= -b^\mu + b^\nu \sum_{L \in \mathcal{J}(J)} C_{J,L} \Gamma_\nu^{\mu\rho} k_\rho^L + \dots \end{aligned} \qquad (36)$$

This is a manifestation of the relativity of locality, i.e. local spacetime coordinates for one observer mix-up with energy and momenta on translation to the coordinates of a distant observer[5]

This mixing under translations effect also entirely accounts for the separation of an interaction into apparently distinct events, because with $b^\nu = -z^\nu$, we see that Δx_J^μ of (30) is equal to $\delta_b x_J^\mu(0)$ of (36). Thus, the observer whose new coordinates we have translated to observes a single interaction taking place at $x_J^\mu \to x_J'^\mu(0) = 0$.

Thus, if I am a local observer and see an interaction to take place via a collision at my origin of coordinates, a distant observer will generally see it in their coordinates as spread out in spacetime by (30). And vice versa. There is not a physical nonlocality, as all momentum conserving interactions are seen as happening at a single spacetime event by some family of observers, who are local to the interaction. But it becomes impossible to localize distant

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[5]The discussion above concerns one interaction vertex only. The issue of translational invariance in the multiparticle case has been discussed in depth in the recent papers in [3,4].

interactions in an absolute manner. Furthermore, all observers related by a translation agree about the momenta of the particles in the interaction, because under translations (35) $\delta_\nu k_\mu^I = 0$.

Note that if the curvature and torsion vanish there is no mixing of spacetime coordinates with momenta under translations, so there is an invariant definition of spacetime. Thus, the flatness of momentum space is responsible for the notion of an absolute spacetime, just as the linearity of the velocity additivity allows Newtonian physics to have an absolute time.

Note also that the translations of spacetime coordinates define sections $\mathcal{S}$ on Γ, which extend from the origin of local coordinates. These tell us how to translate events at the origin of the coordinates of an observer to coordinates measured by a distant observer. These sections provide local and energy-dependent definitions of spacetime, relative to observers and energy scales. These sections are defined by Hamiltonian vector fields on Γ which are defined acting on functions f on Γ by

$$V_b f = \{b^\mu \mathcal{K}_\mu, f\}$$
$$= b^\mu \sum_J \left(1 - \sum_{K \in \mathcal{J}(J)} C_{J,K} \Gamma_\mu^{\nu\rho} k_\rho^K + \dots\right) \frac{\partial f}{\partial x_J^\mu}. \quad (37)$$

We can check that these commute and hence define submanifolds of Γ. We can define an inverse metric on the sections $\mathcal{S}$ defined by

$$g^{\mu\nu}(x, k) = g(dx^\mu, dx^\nu) = h^{\rho\sigma}(k)\{\mathcal{K}_\rho, x^\mu\}\{\mathcal{K}_\sigma, x^\nu\}. \quad (38)$$

We note that this metric on the sections $\mathcal{S}$ is momentum dependent. Thus, we arrive at a description of the geometry of spacetime which is energy dependent. This metric is in fact just the fiber metric at the point k where the fiber is the cotangent plane at this point.[6]

But note that we can take all the $k_\mu^I = 0$ in which case $g^{\mu\nu}(x, k = 0) = \eta^{\mu\nu}$ is the Minkowski metric. So observers who probe spacetime with zero momentum probes will see Minkowski spacetime. However, the coordinates of this invariant zero momentum section are noncommutative.

$$\{z^\mu, z^\nu\} \neq 0. \quad (39)$$

Indeed, if one wants to describe the spacetime as probed by the zero momentum probes this means that we desire to model the spacetime as the cotangent space of the origin. In order to achieve this we need to parallel transport the event at p back to events at 0. This means that we interpret the coordinates z as being covariant covector fields defined by

[6]We are usually familiar with such a phenomenon in the dual picture where gravity is turned on and spacetime is curved, in which case momentum space is represented by covector fields and the metric induced on each fiber is dependent on the spacetime point. It can be said that relative locality is a dual gravity.

$$z^\mu = \tau(p)^\mu_\nu x_p^\nu \quad (40)$$

where x_p is the coordinate dual to p with respect to the Poisson bracket and living in the cotangent space at p. The Poisson commutator can now be evaluated, it is related to the Lie bracket of the covariantly constant vector field on $\mathcal{P}$ and its expansion is given by

$$\{z^\mu, z^\nu\} = (\tau_{\bar\mu}^\mu \partial^{\bar\mu} \tau_{\bar\nu}^\nu - \tau_{\bar\mu}^\nu \partial^{\bar\mu} \tau_{\bar\nu}^\mu) x^{\bar\nu}$$
$$= T_\sigma^{\mu\nu} z^\sigma + R_\sigma^{\mu\nu\rho} p_\rho z^\sigma + \cdots \quad (41)$$

where we have expanded around $p = 0$ in the second equality.

However, when we go to zero momentum we can no longer neglect the limitations on local measurements coming from the uncertainty principle of quantum mechanics. Quantum mechanically a particle of energy p_0 can only be localized with accuracy not greater than $\hbar/p_0$, and this combines with the relativity of locality expressed by (33) to give uncertainty relation of the form

$$\Delta x \geq \frac{\hbar}{p_0} + |x||\Gamma|p_0, \quad (42)$$

characterizing the limitation on the sharpness of coordinates of particles with energy p_0. Taking small p_0 helps reduce the relative-locality features but increases the quantum mechanical uncertainty.

To understand the implications of this in more detail we will next specialize from the general case by imposing physical conditions which restrict the geometry of momentum space.

IV. SPECIALIZING THE GEOMETRY

As we have seen, the geometry of momentum space can code several kinds of deformations of the energy-momentum conservation laws, which take advantage of the flexibility to choose the metric, torsion, curvature and nonmetricity of the connection. This gives us an arena within which we can formulate and test new physical principles. These impose constraints on the choice of the geometry of $\mathcal{P}$. To illustrate this we next turn from the general case to show how a set of simple principles restricts us to a one parameter set of momentum space geometries, and consequently, an almost unique set of experimental predictions.

Consider the following four increasingly strong physical principles:

(1) *The correspondence principle*: Special relativity describes accurately all processes involving momenta small compared to some mass scale m_d. While it is natural to presume that $m_d \approx m_p$, the scale m_d should be determined experimentally.

(2) *The weak dual equivalence principle*: The algebra of combination of momenta, and hence the geometry

of $\mathcal{P}$ are universal; they do not depend on which kinds of particles are involved in interactions.

(3) *The strong dual equivalence principle or $E = mc^2$:* There is equivalence between the rest mass energy defined by the metric and the inertial mass which involves the connection.

(4) *Maximal symmetry* The geometry of momentum space is isotropic and homogeneous: invariant under Lorentz transformations and invariant under "translations".

The geometry of momentum space is discussed in more detail in [2] where it is precisely shown how these principles lead to a unique geometry. This unique geometry is characterized by a constant: the dual cosmological constant which as the dimension of an inverse mass square. When this dual cosmological constant is zero we recover usual special relativity, when it is nonzero, momentum space is genuinely curved.

The first principle implies first that the metric of momentum space is a Lorentzian metric (which we already have implicitly assumed). It also implies that the torsion and nonmetricity of $\Gamma_\mu^{\nu\rho}$ must be at least of order $1/m_p$, while curvature must be of order of $1/m_p^2$.

The weak equivalence principle implies that the combination of momenta do not depend on the colors or charges of particles and is the same as the composition for identical particles. For identical particles there is no operational way to give an order of the combination rule if we have Bose statistics, therefore taken strongly, that is if we do not allow for any modification of the statistics of identical particles, this principle also implies that the product is symmetric and hence the connection is torsionless.

The strong equivalence principle relates the metric and the connection of $\mathcal{P}$ by imposing that the connection is metric compatible. The metric determines distance between two points in momentum space, and hence governs the mass shell relations of single particles, while the connection determine what is the straightest path between two points, and hence is determined by interactions which combine momenta. Since they are given by different physics, they are in principle independent. However, there are indications, to be discussed in [5] that, at least in some cases, the nonmetricity of the connection is related to violations of the equivalence between relativistic energy and mass. It is intriguing to envisage that a relationship between the metric and the connection of momentum space might codify Einstein's observation that in a relativistic theory $E = mc^2$.

The first three principles impose therefore that the geometry of momentum space is entirely fixed by a Lorentzian metric. The connection is then the unique connection which is torsionless and compatible with the metric.

The fourth principle of maximal symmetry is the most restrictive. This could be called the principle of "special

relative locality" since it essentially implies a unique fixed dual geometry on momentum space. What it means in a nutshell is that the space of Killing vector fields of the metric form a ten dimensional Lie algebra. This symmetry algebra also preserve the connection. This implies that there exists Lorentz transformations Λ acting on $\mathcal{P}$ fixing the identity 0 such that

$$\Lambda(p \oplus q) = \Lambda(p) \oplus \Lambda(q). \tag{43}$$

This implies that the conservation law transform covariantly under Lorentz transformations

$$\mathcal{K}(\Lambda k) = \Lambda \mathcal{K}(k). \tag{44}$$

Hence, if we impose that the geometry of $\mathcal{P}$ is invariant under Lorentz transformations then we gain an action of the Lorentz group on the phase space Γ. From this we can conclude that there are for each interaction event, a family of local observers which see the interaction to take place at $z^\mu = 0$ and hence be local.

There also exist a notion of translations[7] T_r such that $T_r(0) = r$ and

$$T_r(p \oplus_k q) = T_r(p) \oplus_{T_r k} T_r(q) \tag{45}$$

where $\oplus_k$ is the translated combination rule (14).

Here we expand more on how Lorentz invariance can be made compatible with curved momentum space. First it is well known that around the identity 0 we can set up a special set of coordinates: The Riemannian normal coordinates. In this coordinates the distance from 0 is given by the usual flat space formula hence the mass shell condition in this coordinates is simply[8]

$$\mathcal{C}(k) = k_0^2 - k_i^2 - m^2 = 0. \tag{46}$$

The Lorentz transformations that preserves the zero momenta and the metric therefore acts in the usual manner in this coordinate systems.

Moreover under the hypothesis of homogeneity these coordinates can be extended to cover almost all the manifold $\mathcal{P}$. The Lorentz generators therefore satisfy the usual algebra. If we assume in turn that the Lorentz transformations are canonical transformations preserving the Poisson bracket, they also satisfy the usual Poisson algebra. That is given the boost and rotation generators N_i and M_j, $i, j, k = 1, 2, 3$, we have

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[7]The algebra of translations on momentum space does not have to be commutative. It can be defined to be the left translation $T_r(p) \equiv r \oplus p$.

[8]The distance $D^2(k, k')$ of two points away from the origin do not assume this simple form (46), which applies only to measurements of distances from the origin. Consequently, the action of Lorentz boosts on the geometry can be nontrivial even if the action on the coordinates k_μ in which (46) holds is linear.

$$\{M_i, M_j\} = \epsilon_{ijk} M_k,$$
$$\{M_i, N_j\} = \epsilon_{ijk} N_k, \tag{47}$$
$$\{N_i, N_j\} = -\epsilon_{ijk} N_k$$

and the generators act on the momentum space $\mathcal{P}$ through these brackets. Moreover, as in special relativity, we assume that the momentum space $\mathcal{P}$ splits into collections of orbits of the Lorentz group: the zero momentum point, which is left invariant by Lorentz transformations; the positive and negative energy mass shells of massive particles; the light cone corresponding to massless particles; and mass shells of tachyons of imaginary mass. It is a direct consequence of this assumption that the function $\mathcal{C}(k)$ in the mass shell condition (21) must be a Lorentz scalar, so that all Lorentz observers agree what the value of the invariant mass m^2 is. It follows then that vectors corresponding to the infinitesimal Lorentz transformations are Killing vectors of the metric (4), so that the surfaces of constant distance from the origin (the point in $\mathcal{P}$ corresponding to zero momentum) are orbits of action of Lorentz group.

Combining the assumptions of Lorentz symmetry with translation invariance or, equivalently, homogeneity then completely determines the geometry of $\mathcal{P}$, up to an overall scale. Indeed, according to the latter the geometry of a fixed mass orbit $|k|^2 + m^2 = 0$ is the same of all the masses. Thus the geometry of $\mathcal{P}$ is not only invariant under Lorentz transformations acting along the orbits, but also under translations, mapping one orbit to another. All together we have therefore the 10-parameter (in 4 dimensions) group of transformations that leave the geometry of the momentum space invariant, and therefore this space is a maximally symmetric manifold. It is well known that there are only three such manifolds: the flat space and the (anti-) de Sitter spaces. On all three of these spaces the Lorentz group action is naturally defined. Notice that the $\mathcal{P}$ may be a submanifold of one of these spaces, satisfying the requirements that all the Lorentz orbit belong to this submanifold.

Two points are worth noting as we now transition from discussing the theoretical frameworks to a first sketch of their phenomenological implications. The two cases of positive and negative constant curvature on $\mathcal{P}$ are expected to be rather different in their phenomenological implications. Also, while action of the Lorentz transformations will be induced on the spacetime coordinates, this action may be deformed and will depend on which coordinates are used to label spacetime events and processes. As we have seen the induced spacetime coordinates are observer dependent, and are also either noncommutative or energy-momentum dependent. The details of how these affect the action of Lorentz transformations on spacetime reserved for discussion in a future publication.

V. DETERMINING EXPERIMENTALLY THE GEOMETRY OF MOMENTUM SPACE

In the 19th century, Gauss proposed that the geometry of space should be empirically determined, and there is a legend he set about measuring the curvature of space by determining the angles between three mountain peaks. Similarly, we here propose that the geometry of momentum space is to be measured rather than assumed. In this section we establish that the geometry of momentum space can produce observable effects. We do this by means of simple sketches of idealized experiments. We postpone for further work the consideration of observations that could actually be done with present and near future technology.

There are two classes of experiments that can be contemplated. There are experiments which aim to measure features of the geometry of momentum space; these can be distinguished from tests of the specific hypotheses described in the previous section. We describe two examples from the first class, which aim to measure the curvature of momentum space.

The first example we have chosen is from atomic physics, and shows that the curvature of a connection on momentum space, defined by (13) corresponds to a measurable quantity. We assume the availability of a beam of atoms of same type prepared all in the ground state with the same initial momentum. The idealized measurement procedure we consider assumes the availability of four energy levels: in addition to the ground state it involves excited levels *I*, *II* and *III*. Ideally we would want these energy levels to be such that there is a small step in energy from ground to I excited, a large step in energy from *I* to *II* and another small step in energy from *II* to *III*. We excite the atoms with lasers tuned to the transition between these states as shown in Fig. 3.

By the scheme in Fig. 3 we can bring atoms from the ground state to the level III excited following two routes:

$$\text{route A} \equiv \text{ground} \rightarrow \text{I} \rightarrow \text{III} \tag{48}$$

and

$$\text{route B} \equiv \text{ground} \rightarrow \text{II} \rightarrow \text{III} \tag{49}$$

The resulting momenta for the two processes are $(p \oplus k_{0,I}) \oplus k_{I,III}$ and $(p \oplus k_{0,II}) \oplus k_{II,III}$, and can be compared by measuring the laser frequency needed to bring the *route A* atoms back to the ground state and the laser frequency needed to bring the *route B* atoms back to the ground state. In the idealized situation of $k_{0,I} \simeq k_{II,III} \equiv q$, $k_{0,II} \simeq k_{I,III} \equiv Q$, this procedure would give a clean comparison between $(p \oplus q) \oplus Q$ and $(p \oplus Q) \oplus q$, which following the analysis we offered in the previous sections is indeed a sensitive indicator of the nontriviality of the connection on momentum space. We expect that other more practical and sensitive measurement procedures for the geometry of momentum space will be gradually found,
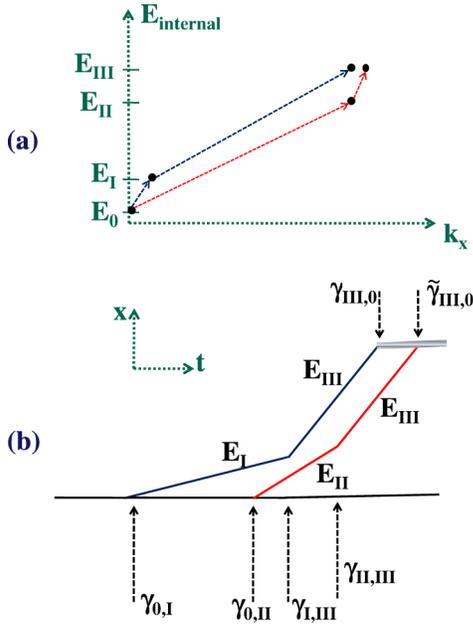
FIG. 3 (color online). Schematic description of a measurement procedure aimed at exposing the implications of a nontrivial connection on momentum space. The *route A* and *route B* described in the text are here shown both in momentum space [panel (a)] and as spacetime trajectories of the atoms [panel (b)].

if a dedicated research effort is inspired by our proposal. The simplicity of the scheme described in Fig. 3 serves the purpose of showing very clearly that the geometry of momentum space can manifest itself in measurable quantities. The procedure sketched in Fig. 3 is also representative of a whole class of strategies for measuring nonassociativity and/or noncommutativity of the law of composition of momenta, which, as we have shown, are expressed, respectively, by the curvature and torsion of a connection on $\mathcal{P}$.

It is interesting to note also that there is a simple analogy between the nonlinear composition of momenta we have discussed here and the nonlinear law of composition of velocities, in Special Relativity. It is not always stressed that the composition law of velocities in special relativity is nonassociative [6]. This nonassociativity is absent for addition of colinear velocities but is measured in Thomas procession [7]. This suggest an experiment inspired by the Thomas precession experiment [8]. The idea is to follow a system in orbit (an electron in an atom or a particle circling in the LHC). Such a system is enclosing a loop in momentum space at each period of revolution, which enclose the curvature in momentum space. At each period the localization of the orbiting particle will be shifted compared to the localization of a particle at rest. Effectively, the particle will experience an infinitesimal boost N_i at each period given by

$$N_\lambda = \frac{\Delta A_{\rho\sigma}}{m_P^2} R_\lambda^{\rho\sigma\mu} p_\mu \approx \frac{\Delta A_{\rho\sigma}}{m_P^2} m R_\lambda^{\rho\sigma 0} \qquad (50)$$

where ΔA_{cd} is the area of the loop in momentum space and m the mass of the particle. One should be able to observe then a spacetime displacement due to relative locality effect. Even if this effect is tiny this type of observation possess a huge potential since it is a cumulative effect and we can use the large number of orbits that develop over time.

These experiments represent ways to measure the curvature of momentum space, which is the main effect that needs to be probed. We can also test more broadly the solidity of the strong and weak dual equivalence principle. The metricity of the connection could be tested by looking for violations of the equivalence between mass and rest energy. There are numerous nuclear physics experiments that rely on the equivalence between mass and energy, the accuracy that is achieved in these experiments is not Planckian, but they clearly deserve a closer scrutiny in light of the new principle we propose.

It will be also important to have a direct experimental bound on the momentum space torsion. One way to probe it is to put Bose statistics under experimental scrutiny, since we have argued that a nonvanishing torsion will in effect correspond to a modification of the statistics[9] This issue deserves a deeper analysis in order to propose specific effects to look for. Also one could imagine a momentum space Eötvos experiment showing that all type of matter add momentum in the same way.

We can add that a nontrivial geometry of momentum space can in the most general form produce deformations of well tested symmetries of quantum field theory including CPT and crossing symmetry[9]. This is because they imply nonlinearities in conservation laws, which would show up as violations of the linear form of those laws. For example, we expect that standard arguments on crossing symmetry in which an incoming particle with four momentum k_μ is replaced by an outgoing antiparticle with four momentum $-k_\mu$ will be deformed so the antiparticle has instead momentum $\ominus k_\mu$. This will introduce nonlinearities which will show up as violations of the usual crossing symmetry. Thus the tight experimental constraints on these symmetries constrains the geometry of momentum space. This is expressed by the *correspondence principle* we discussed in the previous section. Thus, a first task for phenomenologists will be to understand the bounds on the mass scale m_ρ from existing tests of fundamental symmetries in quantum field theory [9].

These examples illustrates that the principles proposed here open a new type of investigation, both experimental and theoretical, into the geometry of momentum space.

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[9]The usual argument in favor of the standard statistics uses crucially the existence of an absolute spacetime, and the independence of the state of a system on its momentum space history. Logically, this derivation should be revisited in light of the relative locality principle.

VI. CONCLUSIONS

The passage from special relativistic locality to relative locality reminds us of the passage we navigated a century ago from absolute space to spacetime. The additivity of velocity implies there is an absolute time by which velocity is measured. If we hypothesize that the combinations of velocity might become nonlinear, without weakening the principle of the relativity of inertial frames, we need an invariant scale, to measure the scale of the nonlinearities, which must be a velocity itself. Hence there is an invariant velocity we call c. This then allows us to interchange distances and times, which makes possible the existence of an absolute spacetime, which replaces the notion of absolute space. Space itself remains, but as an observer-dependent concept, because of the relativity of the simultaneity of distant events.

Similarly, as we observed above, the additivity of momenta and energy implies the existence of an absolute spacetime. When we contemplate weakening that to a nonlinear combination rule for momenta in physical interactions, we need an invariant momentum scale. We have taken this scale to be m_p but of course from a phenomenological point of view it should be taken as having a free value to be constrained by experiment. This, together with $\hbar$ makes it possible to interchange distances and momenta, which makes possible the mixing of spacetime coordinates with energy and momenta, so that the only invariant structure is the phase space. We saw above explicitly how nonlinearity in conservation of energy and momentum directly forces translations of spacetime coordinates to depend on momenta and energies. Local spacetime remains, but as an observer-dependent concept, because of the relativity of the locality of distant events.

Relative locality suggests novel points of departure for attempts to discover the right quantum theory of gravity. Here, we have discussed its implications in a semiphenomenological perspective framed by the approximation (1) but it is possible it goes deeper. For example, one can suggest that the fundamental description is one in which there is dynamical curvature in phase space, so that the fundamental constants are G_{Newton} and m_p. This implies that Planck's constant could be a derived quantity, $\hbar = G_{\text{Newton}} m_p^2$, suggesting that quantum mechanics is emergent from a dynamics of phase space geometry.

The idea that momentum space might be curved has a long history. To our knowledge, it was first put forward more than 70 years ago in a paper by Born [10]. The idea seems to have been next discussed independently in the celebrated paper by Snyder [11], which was followed by series of papers of Russian physicists from the 1950s to the 1980s [12]. There the main motivation was the ultraviolet divergencies in quantum field theories; it was hoped that curving momentum space may help taming these divergencies.

As Snyder first pointed out, the curvature of momentum space implies the noncommutativity of spacetime coordinates which generate translations on momentum space. A picture of quantum geometry in which a curved momentum space is dual to a noncommutative spacetime was introduced by Majid, who stressed the relation to quantum groups [13]. This idea was realized in the construction of κ-Minkowski spacetime which is a noncommutative geometry invariant under a deformation of Poincaré symmetry known as the κ-Poincaré algebra [14,15].[10]

Still more generally, Connes has stressed [16] that the fundamental observables are spectra of energy and momentum, tightly cementing the relationship between the geometry of momentum space and noncommutative geometry.

Another motivation comes from lower dimensional physics. It is by now well established by different methods [17–24] that in $2 + 1$ dimensions, where gravity is described by a topological field theory [25], the effective momentum space of particles is deformed and becomes a (curved) three dimensional group manifold.

So the world we have described here is realized in at least one well developed example, which is quantum gravity coupled to matter in $2 + 1$ dimensions.

In the physical $3 + 1$ dimensions the situation is less clear. It has been suggested that, as in the 3D case, the "no gravity" limit is governed by topological field theory, effectively leading to curved momentum space [26]. The curved momentum space was also employed in the context of group field theory, which generates amplitudes for spin foam models of quantum gravity [27]. Quantum field theory with curved momentum space has been recently discussed in detail in [28]. The dynamical momentum space in the context of the cosmological constant problem has been recently discussed [29].

Last but not least, momentum spaces of constant curvature find its natural application as a model of doubly (or deformed) special relativity (DSR) [30–33], whose second observer-independent scale is naturally associated with the curvature of the momentum space [32][11] (see also [37]). These formulations were closely related to noncommutative geometry. A second formulation of DSR expressed the same idea as an energy dependence of spacetime [33]. For a long time it has been suspected that these were different ways of formulating the same theories, the developments of this paper show how an energy dependent metric and noncommutative spacetime coordinates are different ways of expressing a deeper idea, which is relative locality.

The nonassociativity of momentum addition in noncommutative geometry was also explored in [6], which stressed

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[10]The geometry of κ–Poincaré theory fits the general scheme presented here with a connection with nonzero torsion but zero curvature.

[11]The existence of a regime defined by Eq. (1) was discussed by several authors including [34–36].

the analogy to the nonassociativity of velocity composition in special relativity. The nonassociativity of momentum conservation was also further explored in [38].[12]

The issue of possible macroscopic nonlocal effects in DSR was raised several times [40–44]. Attempts to address these issues led to some partial anticipations of the ideas proposed here [45–47]. From the present perspective we can appreciate that the concern was well justified and there is a pot of gold under the rainbow of apparent nonlocality. At the same time, the principle of relative locality tells us that the theories discussed here do not contain physical nonlocalities of the kind that were suggested in [41–43]. Instead, energy-momentum conserving interactions are always local in spacetime when observed by observers who are local to them. The fact that a distant observer is only able to localize an event involving several particles with different momenta to within a region whose scale is proportional to its distance is inevitable in theories with curved momentum spaces, and should not be misconstrued as entailing a violation of the physical principle that physical interactions are local.

Nonetheless, if theories are still local, in the restricted sense of relative locality, there are, as we discussed above, new phenomena that can be studied experimentally. In addition, as will be explained elsewhere, the problem of whether photons emitted simultaneously are detected simultaneously, after traveling for very long distances, can be cleanly addressed. This is relevant for the timing of arrival time measurements in gamma ray bursts [48–50].

We have seen here that the notion of curved momentum space has generic and vivid consequences for our understanding of basic physics. We do not live in spacetime. We live in Hilbert space, and the classical approximation to that is that we live in phase space. If we think that we can filter out colors and frequencies of particles to arrive at a picture of particles moving in spacetime which is independent of the momenta those particles carry, that is only an illusion that has been possible because of the smallness of elementary particle scales in Planck units. Similarly, if we

think that observers distant from each other see the same spacetime that is only because we are only beginning to make precise measurements of quantities where the combination $|z||\Gamma|k|/\hbar$ is not negligible.

The crucial idea underlying and unifying all these developments turns out to be one that is a direct consequence of the curvature of momentum space: the relativity of locality.

Even apart from fundamental physics, there are situations in condensed matter physics, where it is convenient to understand excitations as living in a curved momentum space [51]. The considerations of this paper may be relevant for those cases. Or, to put it the other way, just as some condensed matter or fluid systems provide analogues for relativity and gravity, it may be that condensed matter systems with curved momentum spaces may give us analogues to the physics of relative locality.

So look around. You see colors and angles, i.e. you are seeing into phase space. The idea that underlying it is an energy independent, invariant spacetime geometry could be an approximation, reliable only to the extent that we measure the geometry with quanta small compared to the Planck energy and we neglect phenomena of order of $|z||\Gamma|k|/\hbar$. Whether this is correct or not is for experimental physics to decide. If it turns out to be correct, then a new arena opens up for experimental physics and astronomy, which is the measurement of the geometry of momentum space.

ACKNOWLEDGMENTS

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