Static and spherically symmetric black holes in $f(R)$ theories

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(Received 9 August 2011; published 3 October 2011)

We study some features of static and spherically symmetric solutions (SSS) with a horizon in $f(R)$ theories of gravitation by means of a near-horizon analysis. A necessary condition for an $f(R)$ theory to have this type of solution is obtained. General features of the effective potential are deduced, and it is shown that there exists a limit on the curvature at the horizon, in both cases for any $f(R)$. Finally, we calculate the expression for the energy of the collision of two massive particles in the center of mass frame.

DOI: 10.1103/PhysRevD.84.084006 PACS numbers: 04.50.Kd, 04.70.Bw

I. INTRODUCTION

Although gravity has been shown to be with high accuracy in accordance with General Relativity (GR) in a number of situations in which the curvature is small [1], there is no observational evidence of the behavior of the gravitational field for very large values of the curvature. In this regard, objects such as black holes and neutron stars are the ideal places to look for deviations from GR in the strong regime [2]. The task of understanding what kind of deviations can be expected, and their relation to observable quantities is of relevance both theoretically and from the observational point of view. The latter is important in view of developments that offer the prospect of surveying phenomena occurring in the vicinity of the horizon in the near future, such as the possibility of obtaining "black hole images" [3,4].

As a first step in discussing deviations from GR in strong-field gravity, we will examine here some features of static and spherically symmetric (SSS) black hole solutions in theories with a Lagrangian that is a function of the Ricci scalar R [6]. Different aspects of this type of solutions in $f(R)$ theories have been previously in discussed in [8], mostly resorting to exact solutions and/or phase space analysis. We will take here a complementary path which consists in extracting information about relevant quantities from the behavior of the geometry near the horizon. For an arbitrary $f(R)$, it will be assumed that a SSS black hole solution exists, being described by a general metric adapted to these symmetries. As a result of expanding the metric functions in series of the distance to the horizon (whose radius r_0 is given by $g_{00}(r_0) = 0$), and using the equations of motion for the metric, we shall obtain a necessary condition that the $f(R)$ must satisfy for the existence of the SSS black hole solution. It will also be shown that the near-horizon geometry is constrained by the equation-of-motion method (EOM), and the consequences of these constraints in the redshift, the curvature, and the energy of a collision of particles in the center of mass frame will be analyzed. Let us begin by presenting the relevant equations in the next section.

II. EQUATIONS OF MOTION AT THE HORIZON

The vacuum equations of motion for an $f(R)$ theory are given by

$$\frac{df}{dR}R_{\mu\nu} - \frac{f}{2}g_{\mu\nu} - (\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)\frac{df}{dR} = 0, \quad (1)$$

along with the trace of Eq. (1):

$$3\Box\frac{df}{dR} + \frac{df}{dR}R - 2f = 0. \quad (2)$$

In the case of a SSS metric in Schwarzschild's coordinates, the nonzero equations following from Eq. (1) are

$$\frac{df}{dR}R_{00} + g_{00}A = 0, \quad (3)$$

$$\frac{df}{dR}R_{11} + g_{11}A - \frac{d^3f}{dR^3}(\partial_1 R)^2 - \frac{d^2f}{dR^2}\nabla_1(\partial_1 R) = 0, \quad (4)$$

$$\frac{df}{dR}R_{22} + g_{22}A = 0, \quad (5)$$

where

$$A \equiv -\frac{f}{2} + \frac{d^3f}{dR^3}g^{11}(\partial_1 R)(\partial_1 R) + \frac{d^2f}{dR^2}\Box R. \quad (6)$$

To describe the SSS spacetime, the metric

$$ds^2 = -e^{-2\phi(r)}\left(1 - \frac{b(r)}{r}\right)dt^2 + \frac{dr^2}{1 - \frac{b(r)}{r}} + r^2 d\Omega^2 \quad (7)$$

will be adopted, where the function ϕ is known as the anomalous redshift. We shall assume that there is a horizon at $r = r_0$ [9], where r_0 is given implicitly by $b(r_0) = r_0$. From this expression, we see that the dependence of r_0 with the mass will likely be different from the linear relation $r_0 = 2M$ in Schwarzschild's solution. We assume in the following that all the functions in the metric, as well as the derivatives of $f(R)$, can be developed in series around the horizon, so that

$$b(r) = b_0 + b_0'(r - r_0) + \tfrac{1}{2}b_0''(r - r_0)^2 + \ldots$$
$$\phi(r) = \phi_0 + \phi_0'(r - r_0) + \tfrac{1}{2}\phi_0''(r - r_0)^2 + \ldots,$$
$$\frac{df}{dR} = \frac{df}{dR}\bigg|_0 + \left[\frac{d}{dr}\left(\frac{df}{dR}\right)\right]_0 (r - r_0) + \ldots,$$

where the subindex zero indicates that the corresponding quantity is evaluated at the horizon, and the prime denotes derivative with respect to the coordinate r. Replacing these expressions in the EOM and taking the limit $r \to r_0$ is a rather long and straightforward calculation that requires care, because there will be both finite terms and terms that diverge as $(r - r_0)^{-1}$ in this limit. The latter arise from the "11" component of the Ricci tensor, given by

$$R_{11} = -\frac{1}{2r^2(1 - \frac{b}{r})}\left\{2r^2\left(1 - \frac{b}{r}\right)(\phi'' - \phi'^2) + 3\phi'(b - rb') + b''r\right\},$$

and also from the term $\Box R = g^{11}\nabla_1\partial_1 R$. For the EOM to be satisfied at the horizon, we must impose that both the finite and the divergent terms be zero. Doing so leads to the following relations: from Eq. (3), we obtain

$$3\phi_0' r_0(1 - b_0') + b_0'' r_0 - 2b_0' = 0. \tag{8}$$

The finite part of Eq. (4) yields

$$\frac{df}{dR}\bigg|_0\left[3\phi_0' b_0' - b_0'' + 2\phi_0' + 2b_0'(r_0\phi_0'' - r_0\phi_0'^2 - \phi_0') + 4\frac{b_0'}{r_0}\right] + 2b_0' R_0' \frac{d^2 f}{dR^2}\bigg|_0 = 0, \tag{9}$$

with

$$R_0' = -\frac{1}{r_0^2}\left\{b_0''(1 - 3\phi_0' r_0) + (1 - b_0')(5r_0\phi_0'' - 2r_0\phi_0^2 - 2\phi_0') + r_0 b_0''' - 4\frac{b_0'}{r_0}\right\},$$

while the divergent part yields

$$2\frac{df}{dR}\bigg|_0 b_0' + r_0^2 f_0 = 0. \tag{10}$$

From the finite part of Eq. (5), we get

$$-2\frac{df}{dR}\bigg|_0 \phi_0' + r_0 R_0'^2 \frac{d^3 f}{dR^3}\bigg|_0 + r_0 B_0 \frac{d^2 f}{dR^2}\bigg|_0 = 0, \tag{11}$$

where $B_0 = B_0(b_0', b_0'', b_0''', b_0^{iv}, \phi_0', \phi_0'', \phi_0''')$ is a rather long expression which we shall not use in this paper. Finally, the divergent part of Eq. (5) gives

$$-5r_0^2\phi_0''(b_0' - 1) + r_0\phi_0'[-3r_0 b_0'' + 2(b_0' - 1)] + 2r_0^2\phi_0'^2(b_0' - 1) + r_0(b_0''' r_0 + b_0'') - 4b_0' = 0, \tag{12}$$

which is equivalent to $R_0' = 0$. Assuming $\frac{df}{dR}|_0 \neq 0$ (we shall see below that this is a reasonable asssumption), it follows from Eq. (9) that

$$3\phi_0' b_0' - b_0'' + 2\phi_0' + 2b_0'(r_0\phi_0'' - r_0\phi_0'^2 - \phi_0') + 4\frac{b_0'}{r_0} = 0. \tag{13}$$

It can also be shown from these relations that the trace equation at $r = r_0$, given by

$$\frac{df}{dR}\bigg|_0 R_0 - 2f_0 + \frac{3}{2r_0}(1 - b_0')\frac{d^2 f}{dR^2}\bigg|_0 R_0' = 0,$$

is identically zero. These equations and some of their consequences will be analyzed in the following. Before closing this section, let us remark that although these relations were obtained using a series development around $r = r_0$, they are exact, in the sense that the higher order terms go to zero when $r \to r_0$.

III. NEAR-HORIZON BEHAVIOR

It will be shown in this section that information about the near-horizon geometry and the $f(R)$ can be extracted from Eqs. (8), (10), (12), and (13), and used to study relevant quantities. To begin with, taking into account that $R_0 = -4b_0'/r_0^2$, Eq. (10) can be rewritten as

$$\frac{f_0}{\frac{df}{dR}|_0} = \frac{R_0}{2}, \tag{14}$$

which furnishes an easy-to-use, coordinate-independent necessary condition for a given $f(R)$ to have SSS black hole solutions. In particular, it follows that Schwarzschild's metric (for which $b(r) = \text{constant}$) is not a solution of those theories for which

$$\frac{f_0}{\frac{df}{dR}|_0} \neq 0,$$

a result which agrees with the conclusions obtained in [10,11]. This expression can be used to test whether a given theory has SSS black hole solutions. For instance, it follows from Eq. (14) that the theory defined by $f(R) = \alpha R^n$ may only have SSS black hole solutions for $n = 2$.

This condition may be strengthened by the use of the inequality $\frac{df}{dR} > 0$, which must be satisfied in order to avoid ghostlike behavior of cosmological perturbations [12]. Using the latter condition along with Eq. (14), we conclude that for a given $f(R)$ theory to be free of ghostlike cosmological perturbations and to have a SSS solution with $R_0 > 0$ ($R_0 < 0$), f_0 must be positive (negative). In particular, in the case of Schwarzschild's spacetime, the condition $f_0 = 0$ must be met.

Going back to the set of equations obtained in the previous section, notice that Eqs. (10) and (11) involve the function f and its derivatives at the horizon, while Eqs. (8), (12), and (13) are constraints on the geometry in the neighborhood of r_0. In particular, from Eqs. (10) and (13), the first and second derivatives of ϕ at the horizon can be expressed as functions of r_0, b_0', and b_0''. Hence, due to the equations of motion, the near-horizon geometry up to second order in the distance to the horizon is determined by the function b. This will be exemplified below in the case of the effective potential for photons.

Another condition on the near-horizon metric comes from the redshift, given by

$$1 + z = \frac{\nu_R}{\nu_E} = \left[\frac{g_{00}(E)}{g_{00}(R)}\right]^{1/2}.$$

We shall assume here that the reception point R is at infinity, and the emission point E is near the horizon. It follows that

$$1 + z \approx \left[\frac{e^{-2\phi_0}}{r_0}(1 - b_0')(r - r_0)\right]^{-1/2}$$

sufficiently near the horizon. Hence, the condition

$$1 - b_0' > 0 \qquad (15)$$

must be satisfied if the redshift is to be well-defined near the horizon. In fact, this is the condition for the metric to have the right sign near the horizon, and also for the tidal forces at the horizon be coincident in sign (and finite) with those of Schwarzschild's black hole (see the Appendix). Using the relation $R_0 = -4b_0'/r_0^2$, this condition entails the existence of a limit for the curvature at the horizon:

$$R_0 > -\frac{4}{r_0^2}.$$

Inequality (15) is also important for the motion of particles near the horizon. In the case of massless particles, the effective potential is defined by

$$V_{\rm eff} = \frac{L^2}{r^3} e^{-2\phi}(r - b).$$

With the definition $v = V/L^2$, it follows that the first derivative at the horizon is given by

$$\left.\frac{dv}{dr}\right|_0 = \frac{1}{r_0^3} e^{-2\phi_0}(1 - b_0'). \qquad (16)$$

Hence, due to Eq. (15), the first derivative of the effective potential is positive, as in the case of Schwarzschild's solution. Since at infinity the effective potential must go to zero, the solution must have at least one unstable circular orbit for photons. Qualitative differences in the effective potential appear only to second order in the distance to the horizon, with the second derivative given by

$$\left.\frac{d^2v}{dr^2}\right|_0 = -\frac{1}{r_0^4} e^{-2\phi_0}[2(1 - b_0')(3 + 2\phi_0' r_0) + r_0 b_0''].$$

Using Eq. (8), we can eliminate ϕ_0', yielding

$$\left.\frac{d^2v}{dr^2}\right|_0 = -\frac{1}{r_0^4} e^{-2\phi_0}\left[6 - \frac{10}{3}b_0' - \frac{1}{3}b_0'' r_0\right], \qquad (17)$$

in such a way that $\frac{d^2v}{dr^2}|_0$ depends of b_0'', on which we have no constraints [13]. Notice also that ϕ_0 acts as a scale in the series development near the horizon.

Collisions

Another phenomenon for which there may be differences between the type of black hole under study here and Schwarzschild's is the collision of two particles. As shown in [14], the maximal collision energy in the center of mass system for two particles of mass m moving in Schwarszchild's geometry, given by $E_{\rm CM}(r_0) = 2\sqrt{5}m$, is reached at the horizon, and is attained when the two particles have angular momentum equal in magnitude and opposite in sign. Let us see how this result changes for the case at hand.

The energy in the center of mass system is given by [15]

$$E_{\rm CM} = \sqrt{2}m\sqrt{1 - g_{\mu\nu}v^\mu_{(1)}v^\nu_{(2)}},$$

where $v^\mu_{(i)}$ is the 4-velocity of each particle. In turn, the 4-velocities are furnished by the geodesic equations, which in the case of the metric (7) are:

$$\frac{dr}{d\tau} = -\sqrt{e^{2\phi} - \left(1 - \frac{b}{r}\right)\left(1 + \frac{\tilde{L}^2}{r^2}\right)},$$
$$\frac{d\phi}{d\tau} = \frac{\tilde{L}}{r^2},$$
$$\frac{dt}{d\tau} = \frac{e^{2\phi}}{1 - b/r},$$

where $\tilde{L}$ is the angular momentum per unit mass, we are taking $\theta = \pi/2$ and assuming that the particles start from rest at infinity. Using these equations and the metric given in Eq. (7) in the expression for $E_{\rm CM}$, a straightforward calculation shows that at the horizon,

$$E_{\rm CM}(r_0) = m\sqrt{4 + \frac{(\tilde{L}_1 - \tilde{L}_2)^2}{b_0^2}},$$

where $\tilde{L}_i$ is the angular momentum of each particle. This expression reduces to the one quoted above in the case of Schwarszchild, $b_0 = r_0 = 2$ (with $M = 1$), and $\tilde{L}_1 = -\tilde{L}_2 = 4$ [16]. We see that in general $E_{\rm CM}$ will be different $E_{\rm CM}(r_0) = 2\sqrt{5}m$, due to the fact that the relation that determines the radius of the horizon ($b(r_0) = r_0$) is different from $r_0 = 2$.

IV. DISCUSSION

As a first step in the investigation of strong gravity effects in $f(R)$ theories, we have studied the near-horizon behavior of a static and spherically symmetric black hole solution. We have obtained a necessary condition for a given $f(R)$ to have such a solution, and showed that the equations of motion constrain the near-horizon geometry. These constraints entail that there is a maximum allowed value for the curvature at the horizon, and also that the effective potential differs qualitatively from that of Schwarzschild's only at second order in the distance to the horizon. In particular, it was concluded that there must be an unstable orbit for photons. We have also obtained the expression for the center of mass energy for the collision of two particles of mass m at the horizon, which can depends on the function $b(r)$ of the metric, evaluated at r_0. Hence, this energy is different from the analog expression in GR. All these results were obtained by means of a local analysis, without any constraint on the curvature scalar or the behavior at infinity, and are suitable for application in other problems, such as the Kerr solution and the no-hair theorems. These issues will be discussed in a future publication.

APPENDIX

The tidal forces at the horizon in Schwarszchild's black hole are such that [17]

$$R^{\hat{1}}_{\hat{0}\hat{0}\hat{1}}\Big|_0 > 0, \qquad R^{\hat{2}}_{\hat{0}\hat{0}\hat{2}}\Big|_0 < 0.$$

In the geometry given by Eq. (7),

$$R^{\hat{1}}_{\hat{0}\hat{0}\hat{1}}\Big|_0 = \frac{b_0'' r_0 - (2 + 3\phi_0' r_0)(b_0' - 1)}{2r_0^2},$$

$$R^{\hat{2}}_{\hat{0}\hat{0}\hat{2}}\Big|_0 = R^{\hat{3}}_{\hat{0}\hat{0}\hat{3}}\Big|_0 = \frac{b_0' - 1}{2r_0^2}.$$

[1] See for instance C. M. Will, Living Rev. Relativity **9**, 3 (2005).
[2] See for instance D. Psaltis, arXiv:0806.1531.
[3] See for instance A. E. Broderick and A. Loeb, J. Phys. Conf. Ser. **54**, 448 (2006).
[4] Another phenomenon which probes the strong-field regime, but which is not observable with present-day technology, is the strong gravitational lensing [5].
[5] See K. S. Virbhadra, Phys. Rev. D **79**, 083004 (2009), and references therein.
[6] These theories have been analyzed in detail lately, due to the fact that they can describe the observed accelerated expansion of the universe (which takes place in the low-curvature regime) without the introduction of dark energy [7].
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[9] In case of multiple horizons, r_0 designates the radius of the most external one.
[10] D. Psaltis, *et al.*, Phys. Rev. Lett. **100**, 091101 (2008).
[11] A. M. Nzioki, *et al.*, Phys. Rev. D **81**, 084028 (2010).
[12] See A. De Felice and S. Tsujikawa, Living Rev. Relativity **13**, 3 (2010).
[13] Going to third order, the second derivative of ϕ evaluated at the horizon that would appear can be expressed in terms of the derivatives of b at the horizon using Eq. (12).
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[16] These are the maximum allowed values such that particles reach the horizon with maximum tangential velocity [14].
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