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Optimized perturbation theory for charged scalar fields at finite temperature and in an external magnetic field

D. C. Duarte,[1,*] R. L. S. Farias,[1,†] and Rudnei O. Ramos[2,‡]

[1]*Departamento de Ciências Naturais, Universidade Federal de São João Del Rei, 36301-000, São João Del Rei, MG, Brazil*
[2]*Departamento de Física Teórica, Universidade do Estado do Rio de Janeiro, 20550-013 Rio de Janeiro, RJ, Brazil*
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Symmetry restoration in a theory of a self-interacting charged scalar field at finite temperature and in the presence of an external magnetic field is examined. The effective potential is evaluated nonperturbatively in the context of the optimized perturbation theory method. It is explicitly shown that in all ranges of the magnetic field, from weak to large fields, the phase transition is second order and that the critical temperature increases with the magnetic field. In addition, we present an efficient way to deal with the sum over the Landau levels, which is of interest especially in the case of working with weak magnetic fields.

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I. INTRODUCTION

Phase transition phenomena in spontaneously broken quantum field theories have long been a subject of importance and interest due to their wide range of possible applications, going from low-energy phenomena in condensed matter systems to high-energy phase transitions in particle physics and cosmology (for reviews, see, for example, [1–3]).

In addition to thermal effects, phase transition phenomena are also known to be triggered by other external effects, for example, by external fields. In particular, those changes caused by external magnetic fields have attracted considerable attention in the past [4] and received reinvigorated interest recently, mostly because of the physics associated with heavy-ion collision experiments. In heavy-ion collisions, it is supposed that large magnetic fields can be generated, and the study of their effects in the hadronic phase transition then became a subject of intense interest (see, e.g., [5] for a recent review). Magnetic fields can lead, in particular, to important changes in the chiral/deconfinement transition in quantum chromodynamics (QCD) [6] and even the possibility of generating new phases [7]

As far the influence of external magnetic fields and thermal effects on phase transformations is concerned, one well-known example that comes to our mind is the physics associated with superconductivity, in particular, in the context of the Ginzburg-Landau theory [8]. Let us recall in that case thermal effects alone tend to produce a phase transition at a critical temperature where superconductivity is destroyed and the system goes to a normal ordered state. The phase transition in this case is second order. However, in the presence of an external magnetic field, but below some critical value, by increasing the temperature the system undergoes a first-order phase transition instead. This simple example already shows that magnetic fields may have influence on the phase transition other than we would expect from thermal effects alone. There are also other examples of more complex systems where external magnetic fields may have a drastic effect on the symmetry behavior. Among these effects, besides the possibility of changing the order of the phase transition, as in the Ginzburg-Landau superconductor, it can in some circumstances strengthen the order of the phase transition, as in the electroweak phase transition in the presence of external fields [9], or there can also be dynamical effects, such as delaying the phase transition [10]. External magnetic fields alone can also lead to dynamical symmetry breaking (magnetic catalysis) [11] (for an earlier account, see Ref. [12]).

Likewise, nonperturbative effects may affect the symmetry properties of a system, once the external parameters are changed, in a way different than seeing through a purely perturbative calculation, or by a mean-field leading-order description. This is because perturbation theory is typically beset by problems, for example, around critical points, due to infrared divergences, or at high temperatures, when powers of coupling constants can become surmounted by powers of the temperature (see, e.g., the textbooks [13,14] for extensive discussions). Thus, high-temperature field theories and the study of phase transitions in general require the use of nonperturbative methods, through which large classes of terms need to be resummed. Familiar techniques used to perform these resummations include, for example, ring-diagram (or daisy and superdaisy) schemes [15,16], composite operator methods [17] and field propagator dressing methods [18,19]. Other methods used include also numerical lattice studies and expansions in parameters not related to a coupling constant, such as the $1/N$ expansion and the ϵ-expansion [20], the use of two-particle irreducible (2PI) effective actions [21,22], hard-thermal-loop resummation [23], variational methods, such as the

*dyana.c.duarte@gmail.com
†ricardofarias@ufsj.edu.br
‡rudnei@uerj.br

screened perturbation theory [24,25], and the optimized perturbation theory (OPT) [26]. Of course, any resummation technique must be implemented with care to avoid possible overcounting of terms and lack of self-consistency. Failure in not following this basic care can lead to a variety of problems, such as predicting nonexistent phenomena or producing a different order for the phase transition. One classical example of this was the earlier implementations of daisy and superdaisy schemes, that at some point were giving wrong results, e.g., predicting a first-order transition [27] for the $\lambda\phi^4$ theory, an unexpected result since the model belongs to the universality class of the Ising model, which is second order. These methods have also initially predicted a stronger first-order phase transition in the electroweak standard model, a result soon proved to be misleading [28]. These wrong results were all because of the wrong implementation of the ring-diagram summation at the level of the effective potential, as clearly explained in the first reference in [28]

In this work we will analyze the phase transition for a self-interacting complex scalar field model and determine how an external magnetic field, combined with thermal effects, affects the transition. All calculations will be performed in the context of the OPT nonperturbative method. Our reasons for revisiting here the phase transition in this model are twofold. First, because this same model has been studied recently in the context of the ring-diagram resummation method [29], where it was found that the ring diagrams render the phase transition first order and that the effect of magnetic fields was to strengthen the order of the transition and also to lower the critical temperature for the onset of the (first-order) phase transition. So in this work we want to reevaluate these findings in the context of the OPT method. We recall, from the discussion of the previous paragraph, that the ring-diagram method requires special attention in its implementation and that previous works have already concluded erroneously about its effects on the transition. The OPT method has a long history of successful applications (for a far from complete list of previous works and applications see, e.g., Refs. [30,31] and references therein). The OPT method automatically resums large classes of terms in a self-consistent way so as to avoid possible dangerous overcounting of diagrams. In our implementation of the OPT here, we will see that already at the first order in the OPT it is equivalent to the daisy and superdaisy schemes. The OPT then provides a safe comparison with these other nonperturbative schemes. In particular, to our knowledge, this is the first study of the OPT method when considering the inclusion of an external magnetic field. Finally, we also want to properly treat the effects of small magnetic fields (which requires summing over very large Landau levels) in an efficient way, particularly suitable for numerical work. This way we can evaluate in a precise way the effects of external magnetic fields ranging from very small to very large field intensities (in which case, in general, just a few Landau levels suffice to be considered). For this study we will make use of the Euler-Maclaurin formula and fully investigate the validity of its use as an approximation for the Landau level sums for different ranges for the magnetic field.

The remaining of this paper is organized as follows. In Sec. II we introduce the model and explain the application of the OPT method for the problem we study in this paper. It is shown explicitly which terms are resummed by the OPT. We also verify that the Goldstone theorem is fulfilled in the OPT. In Sec. III we study the phase transition in the spontaneously broken self-interacting quartic complex scalar field in the OPT method, first by incorporating only thermal effects and then by including both temperature and an external magnetic field. In Sec. IV we study the advantages of using the Euler-Maclaurin formula for the sum over the Landau levels. The accuracy and convergence of the method is fully examined. We determine that the sum over the Landau levels, in the regime of low magnetic fields, where typically we must sum over very large levels so as to reach good accuracy, can be efficiently performed within the very few first terms in the Euler-Maclaurin formula. As an application, an analytical formula for the critical temperature for phase restoration, as a function of the magnetic field, $T_c(B)$, is derived. Our final conclusions are given in Sec. V. Finally, an appendix is included to show some of the details of the renormalization of the model in the context of the OPT.

II. THE COMPLEX SCALAR FIELD MODEL AND THE OPT IMPLEMENTATION

In our study we will make use of a self-interacting quartic complex scalar field model with a global $U(1)$ symmetry and spontaneously symmetry breaking at tree level in the potential, whose Lagrangian density is of the standard form

$$\mathcal{L} = |\partial_\mu \phi|^2 + m^2|\phi|^2 - \frac{\lambda}{3!}|\phi|^4, \tag{2.1}$$

where $m^2 > 0$ for spontaneously symmetry breaking. It is convenient to write the complex scalar field ϕ in terms of real and imaginary components, $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$. In terms of a (real) vacuum expectation value (VEV) for the field, $\langle\phi\rangle \equiv \varphi/\sqrt{2}$, we can, without loss of generality, shift the field ϕ around its VEV in the ϕ_1 direction

$$\phi_1 \to \varphi + \phi_1 \qquad \phi_2 \to \phi_2. \tag{2.2}$$

The Feynman propagators for ϕ_1 and ϕ_2, in terms of the VEV then read

$$D_{\phi_1}(P) = \frac{i}{P^2 + m^2 - \frac{\lambda}{2}\varphi^2 + i\varepsilon}, \tag{2.3}$$

$$D_{\phi_2}(P) = \frac{i}{P^2 + m^2 - \frac{\lambda}{6}\varphi^2 + i\varepsilon}. \tag{2.4}$$

Using the tree-level VEV for the field, $\varphi_0^2 = 6m^2/\lambda$ in Eqs. (2.3) and (2.4), ϕ_1 is then associated with the massive Higgs mode (with mass squared $m_1^2 = 2m^2$ at the tree level) and ϕ_2 is the Goldstone mode of the field, remaining massless throughout the symmetry-broken phase.

A. Implementing the optimized perturbation theory

The implementation of the OPT in the Lagrangian density is the standard one [30,31], where it is implemented through an interpolation procedure

$$\mathcal{L} \to \mathcal{L}_\delta = \sum_{i=1}^{2}\left\{\frac{1}{2}(\partial_\mu \phi_i)^2 - \frac{1}{2}\Omega^2\phi_i^2 + \frac{\delta}{2}\eta^2\phi_i^2 - \frac{\delta\lambda}{4!}(\phi_i^2)^2\right\} + \Delta\mathcal{L}_{\mathrm{ct},\delta}, \tag{2.5}$$

where $\Omega^2 = -m^2 + \eta^2$ and $\Delta\mathcal{L}_{\mathrm{ct},\delta}$ is the Lagrangian density part with the renormalization counterterms needed to render the theory finite. In $\mathcal{L}_\delta$, the dimensionless parameter δ is a bookkeeping parameter used only to keep track of the order that the OPT is implemented (it is set to one at the end) and η is a (mass) parameter determined variationally at any given finite order of the OPT. A popular variational criterion used to determine η is known as the principle of minimal sensitivity (PMS), defined by the variational relation [32]

$$\left.\frac{d\Phi^{(k)}}{d\eta}\right|_{\bar{\eta},\delta=1} = 0, \tag{2.6}$$

which is applied to some physical quantity $\Phi^{(k)}$, calculated up to some order-k in the OPT. The optimum value $\bar{\eta}$, which satisfies Eq. (2.6), is a function of the original parameters of the theory and it is in general a nontrivial function of the couplings. It is because of the variational principle used that nonperturbative results are generated. Other variational criteria can likewise be defined differently, but they produce the same final result for the quantity $\Phi^{(k)}$, as shown recently [33].

In terms of the interpolated Lagrangian density, Eq. (2.5), the Feynman rules in the OPT method are as follows. The interaction vertex is changed from $-i\lambda$ to $-i\delta\lambda$. The quadratic terms $\delta\eta^2\phi_i^2/2$ in Eq. (2.5) define new vertex insertion terms. The propagators for ϕ_i ($i = 1, 2$) now become

$$D_{\phi_1,\delta}(P) = \frac{i}{P^2 - \Omega^2 - \frac{\delta\lambda}{2}\varphi^2 + i\varepsilon}, \tag{2.7}$$

$$D_{\phi_2,\delta}(P) = \frac{i}{P^2 - \Omega^2 - \frac{\delta\lambda}{6}\varphi^2 + i\varepsilon}. \tag{2.8}$$

B. The effective potential in the OPT method

As in the many other previous applications [31,33], we apply the PMS, Eq. (2.6), directly on the effective potential, which is the most convenient quantity to study the phase structure of the model. To first order in the OPT, using the previous Feynman rules in the OPT method given above, the effective potential $V_{\mathrm{eff}}(\varphi)$ is given by the vacuum diagrams shown in Fig. 1.

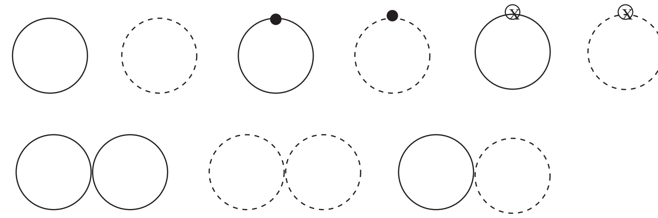

FIG. 1. Feynman diagrams contributing to the effective potential to first order in the OPT. Solid lines stand for the ϕ_1 propagator, while dashed lines stand for the ϕ_2 propagator. A black dot is an insertion of $\delta\eta^2$. A crossed dot is a mass renormalization insertion.

From the diagrams shown in Fig. 1 and by further expanding the propagators Eqs. (2.7) and (2.8) in δ, the explicit expression for the renormalized effective potential at first order in the OPT (some of the details of the renormalization in the OPT method are given in the Appendix) becomes

$$\begin{aligned} V_{\mathrm{eff}}(\varphi) = {} & \frac{\Omega^2}{2}\varphi^2 - \delta\frac{\eta^2}{2}\varphi^2 + \delta\frac{\lambda}{4!}\varphi^4 - i\int_P \ln(P^2 - \Omega^2) \\ & - \delta\eta^2\int_P \frac{i}{P^2 - \Omega^2} + \delta\frac{\lambda}{3}\varphi^2\int_P \frac{i}{P^2 - \Omega^2} \\ & + \delta\frac{\lambda}{24\pi^2}\frac{1}{\epsilon}\Omega^2\int_P \frac{i}{P^2 - \Omega^2} + \delta\frac{\lambda}{3}\left[\int_P \frac{i}{P^2 - \Omega^2}\right]^2, \end{aligned} \tag{2.9}$$

where the momentum integrals are expressed in Euclidean space and the regularization is performed in $\overline{\mathrm{MS}}$ scheme, where, in the finite temperature only case,

$$\int_P \equiv iT\sum_{P_0 = i\omega_n}\left(\frac{e^{\gamma_E}M^2}{4\pi}\right)^{\epsilon}\int\frac{d^d p}{(2\pi)^d}, \tag{2.10}$$

where γ_E is the Euler-Mascheroni constant, M is an arbitrary mass regularization scale, $d = 3 - 2\epsilon$ is the dimension of space and $\omega_n = 2\pi nT$, ($n = 0, \pm 1, \ldots$) are the Matsubara frequencies for bosons at temperature T.

The inclusion of an external magnetic field presents no additional difficulty. For example, without loss of generality, we can consider a constant magnetic field in the z-space direction and use a gauge where the external electromagnetic field is $A_\mu = (0, 0, Bx, 0)$. The Feynman propagator for a charged scalar field with charge e becomes $P^2 - m^2 \to -\omega_n^2 - E_k^2(p_z, B)$, where $E_k(p_z, B)$ is the energy dispersion (for charged scalar bosons) in an external constant magnetic field [34],

$$E_k^2 = p_z^2 + m^2 + (2k + 1)eB, \tag{2.11}$$

where the last term denotes the Landau-levels ($k = 0, 1, 2, \ldots$). Likewise, the momentum integrals, taking into account the degeneracy multiplicity of the Landau levels [35], are now represented by

$$\int_P \equiv i\frac{eB}{2\pi}\sum_{k=0}^{+\infty} T \sum_{P_0=i\omega_n}\left(\frac{e^{\gamma_E}M^2}{4\pi}\right)^{\epsilon}\int \frac{d^{d-2}p_z}{(2\pi)^{d-2}}. \quad (2.12)$$

C. Optimization results and Goldstone theorem

Let us now apply the PMS condition (2.6) on the effective potential Eq. (2.9). We obtain straightforwardly that the optimum $\bar{\eta}$ satisfies the nontrivial (renormalized) gap equation

$$\bar{\eta}^2 = \frac{\lambda}{3}\varphi^2 + \frac{2\lambda}{3}\left[\int_P \frac{i}{P^2-\Omega^2} + \frac{\Omega^2}{16\pi^2}\frac{1}{\epsilon}\right]\Bigg|_{\eta=\bar{\eta}}. \quad (2.13)$$

The extrema of the effective potential are defined as usual, by requiring that the first derivative of the effective potential with respect to φ vanishes. This gives us the trivial solution $\bar{\varphi} = 0$ and the solution for the minimum

$$\bar{\varphi}^2 = 6\frac{m^2}{\lambda} - 4\left[\int_P \frac{i}{P^2-\Omega^2} + \frac{\Omega^2}{16\pi^2}\frac{1}{\epsilon}\right]. \quad (2.14)$$

We can now verify the effective mass for the field, in particular, for the ϕ_2 component of the complex scalar field. The effective mass for ϕ_2 in the OPT is given by

$$m^2_{\rm eff,2} = -m^2 + \bar{\eta}^2 + \frac{\lambda}{6}\bar{\varphi}^2 + \Sigma_2(\bar{\eta}), \quad (2.15)$$

where $\Sigma_2(\bar{\eta})$ is the (renormalized) field's self-energy, which at first order in the OPT is trivially found to be given by

$$\Sigma_2^{(1)}(\bar{\eta}) = -\bar{\eta}^2 + \frac{2\lambda}{3}\left[\int_P \frac{i}{P^2-\Omega^2} + \frac{\Omega^2}{16\pi^2}\frac{1}{\epsilon}\right]\Bigg|_{\eta=\bar{\eta}}. \quad (2.16)$$

Using now Eqs. (2.16), (2.13), and (2.14) in Eq. (2.15), we obtain immediately that $m^2_{\rm eff,2} = 0$, which is nothing but the result expected due to Goldstone theorem for the symmetry-broken complex scalar field model. It can also be verified that this result carries out at higher orders in the OPT method, in which case, at some order-k in the OPT, the self-energy is the one at the respective order, $\Sigma^{(k)}$, entering in the above equations. Previous demonstrations of the Goldstone theorem in the OPT were for the linear sigma model [36] and for the $SU(2)$ Nambu–Jona-Lasinio model [37].

Finally, it is noticed from the PMS equation given above, Eq. (2.13), that the OPT naturally resums the leading-order loop terms of the field's self-energy. So it is quite analogous, at already in the first order in the OPT approximation, to the ring-diagram resummation [15,28]. And that this resummation is also performed automatically in a self-consistent way is clear from the interpolation procedure. While $\bar{\eta}$ enters in all propagators, thus carrying the self-energy corrections, the insertions of $\bar{\eta}$ [the term $\delta\eta^2\phi_i^2/2$ in Eq. (2.5) is treated as an additional interaction vertex] subtract spurious contributions at each order, thus avoiding dangerous overcounting of diagrams (this is also similar to the procedure used in the screened perturbation theory [24,25]). Going to higher orders in the OPT resums higher order classes of diagrams. For our purposes of comparing with the ring-diagram results, it suffices then to keep up to first order in the OPT.

III. PHASE STRUCTURE AND SYMMETRY RESTORATION AT FINITE TEMPERATURE AND IN AN EXTERNAL MAGNETIC FIELD

We are now ready to study the phase structure and the symmetry restoration in the complex scalar field model at finite temperature and in an external magnetic field. It is convenient to first investigate the case of finite temperature only, so as to later compare with the case when an external magnetic field is added.

A. Symmetry restoration at finite temperature

It is convenient to write the explicit expressions for each term entering in the effective potential in Eq. (2.9). At finite temperature and in the absence of an external magnetic field, using (2.10), we see that the first-momentum integral in (2.9) becomes (recalling that $\Omega^2 = -m^2 + \eta^2$)

$$-i\int_P \ln(P^2-\Omega^2) = -\frac{\Omega^4}{2(4\pi)^2}\frac{1}{\epsilon} + Y(T,\eta), \quad (3.1)$$

where we have identified explicitly the divergent term and the finite term, $Y(T,\eta)$, is given by

$$Y(T,\eta) = -\frac{1}{2(4\pi)^2}\left[\frac{3}{2} + \ln\left(\frac{M^2}{\Omega^2}\right)\right]\Omega^4 + \frac{T^4}{\pi^2}\int_0^\infty dz z^2 \ln\left[1-\exp\left(-\sqrt{z^2+\Omega^2/T^2}\right)\right]. \quad (3.2)$$

Likewise, for the remaining momentum integrals in (2.9) we obtain

$$-\delta\eta^2\int_P \frac{i}{P^2-\Omega^2} = -\delta\eta^2\left[-\frac{\Omega^2}{(4\pi)^2}\frac{1}{\epsilon} + X(T,\eta)\right], \quad (3.3)$$

$$\delta\frac{\lambda}{3}\varphi^2\int_P \frac{i}{P^2-\Omega^2} = \delta\frac{\lambda}{3}\varphi^2\left[-\frac{\Omega^2}{(4\pi)^2}\frac{1}{\epsilon} + X(T,\eta)\right], \quad (3.4)$$

where $X(T,\eta)$ is given by

$$X(T,\eta) = \frac{\Omega^2}{16\pi^2}\left[\ln\left(\frac{\Omega^2}{M^2}\right) - 1\right] + \frac{T^2}{2\pi^2}\int_0^\infty dz \frac{z^2}{\sqrt{z^2+\frac{\Omega^2}{T^2}}}\frac{1}{\exp(\sqrt{z^2+\frac{\Omega^2}{T^2}})-1}. \quad (3.5)$$

Next, there is the term coming from the mass counterterm in Eq. (2.9) (see the Appendix),

$$\delta \frac{\lambda}{24\pi^2}\frac{1}{\epsilon}\Omega^2\int_P \frac{i}{P^2-\Omega^2} = -\frac{2\delta\lambda}{3(4\pi)^4}\Omega^4\frac{1}{\epsilon^2}+\frac{\delta\lambda}{24\pi^2}\Omega^2 X(T,\eta)\frac{1}{\epsilon} - \delta\frac{2\lambda}{3}\frac{\Omega^4}{(4\pi)^4}W(\eta), \tag{3.6}$$

where we have also defined

$$W(\eta)=\frac{1}{2}\left[\ln\left(\frac{\Omega^2}{M^2}\right)-1\right]^2+\frac{1}{2}+\frac{\pi^2}{12}. \tag{3.7}$$

Finally, we have the two-loop contributions shown in Fig. 1. They all sum up in the OPT expansion to give

$$\delta\frac{\lambda}{3}\left[\int_P\frac{i}{P^2-\Omega^2}\right]^2=\delta\frac{\lambda}{3}\frac{\Omega^4}{(4\pi)^4}\frac{1}{\epsilon^2}-\delta\lambda\frac{\Omega^2}{24\pi^2}X(T,\eta)\frac{1}{\epsilon} + \delta\frac{2\lambda}{3}\frac{\Omega^4}{(4\pi)^4}W(\eta)+\delta\frac{\lambda}{3}X^2(T,\eta). \tag{3.8}$$

From Eqs. (3.1), (3.3), (3.4), (3.6), and (3.8), we find the complete expression for the renormalized effective potential at first order in the OPT

$$V_{\rm eff}(\varphi,T,\eta)=-\frac{m^2}{2}\varphi^2+(1-\delta)\frac{\eta^2}{2}\varphi^2+\delta\frac{\lambda}{4!}\varphi^4+Y(T,\eta) + \delta\left\{-\eta^2+\frac{\lambda}{3}[\varphi^2+X(T,\eta)]\right\}X(T,\eta). \tag{3.9}$$

The phase structure at finite temperature is completely determined by Eqs. (3.9), (2.13), and (2.14). The optimum $\bar\eta$, determined by the PMS criterion, and the VEV for the scalar field, can now be both, respectively, expressed as

$$\bar\eta^2=\frac{\lambda}{3}\varphi^2+\frac{2\lambda}{3}X(T,\bar\eta), \tag{3.10}$$

and

$$\bar\varphi^2=6\frac{m^2}{\lambda}-4X(T,\bar\eta). \tag{3.11}$$

In the OPT we can exactly compute the critical temperature of phase transition. Using that at the critical point the VEV of the field vanishes, $\bar\varphi(T_c)=0$, then from Eq. (3.10) we obtain that

$$\bar\eta^2(T_c)=\frac{2\lambda}{3}X(T_c,\bar\eta(T_c)), \tag{3.12}$$

which upon using it in Eq. (3.11), we obtain

$$0=6\frac{m^2}{\lambda}-4X(T_c,\bar\eta(T_c))\Rightarrow m^2-\bar\eta^2(T_c)=-\Omega^2(T_c)=0. \tag{3.13}$$

From the definition of $X(T,\bar\eta)$, Eq. (3.5), we then obtain

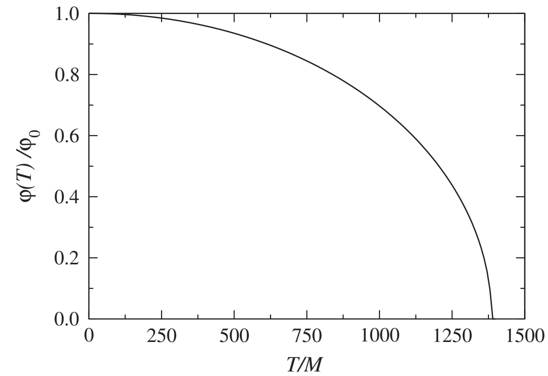

FIG. 2. Temperature dependence of the minimum of the effective potential $\varphi(T)$ in the OPT with the PMS optimization criterion. φ_0 is the minimum of the tree-level potential. The parameters used are $m/M=20$ and $\lambda=0.00375$.

$$X(T_c,\bar\eta(T_c))=\frac{T_c^2}{2\pi^2}\int_0^\infty dz\frac{z}{\exp(z)-1}=\frac{T_c^2}{12}, \tag{3.14}$$

and when using the above result back in Eq. (3.13), we obtain the exact result for the critical temperature at this first order in the OPT

$$T_c^2=\frac{18m^2}{\lambda}. \tag{3.15}$$

This result for T_c is the same predicted before for a scalar field model with two real components [38], obtained in the high-temperature one-loop approximation. The result (3.15) is exact in our OPT approximation and obtained independent of any high-temperature approximation, as usually assumed in any previous calculations.

From Eqs. (3.10) and (3.11), we obtain directly the behavior of the VEV $\bar\varphi$ as a function of the temperature. For comparison purposes, we use analogous parameters as adopted in Ref. [29],[1] where $m/M=20$ and $\lambda=0.00375$. For these parameters, the authors in Ref. [29] find a first-order phase transition. In Fig. 2 we show the result for the minimum of the effective potential, $\bar\varphi\equiv\varphi(T)$ as a function of the temperature. The effective potential, for different temperatures and the same parameters as in Fig. 2, is shown in Fig. 3. Note that the effective potential has an imaginary part for temperatures below T_c, Eq. (3.15), since there can be values of φ for which $-m^2+\bar\eta^2$ becomes negative. This happens for values of φ in between the inflection points of the potential, which defines the spinodal region of instability in between the (degenerate) VEV of the field, determined by Eq. (3.11). In Fig. 3, for the case of $T<T_c$, we show the real part of the effective potential,

[1] Note that in [29] the authors used a different numerical factor for the quartic term in the tree-level potential, which gives rise to the extra factor 3/2 in the numerical value for λ in our notation. In the approximations used in [29], the terms involving the regularization scale were not included. Here we express all quantities in terms of the regularization scale M.

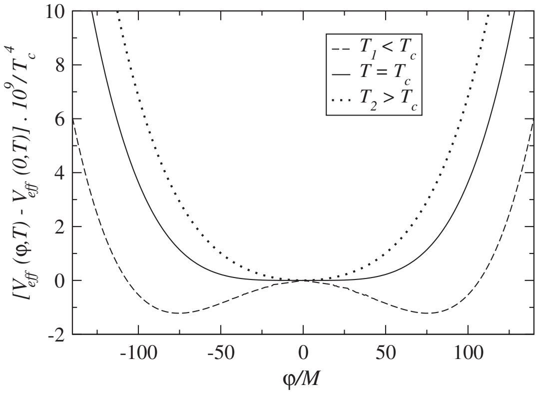

FIG. 3. The effective potential (subtracting the vacuum energy at $\varphi = 0$), for the same parameters of Fig. 2 and for temperatures: $T_1/M = 1380$, $T_2/M = 1390$ and at the critical temperature, $T_c/M \simeq 1385.64$.

so to be able to show all the potential, including the spinodal region.

It should be noted that the minimum of the effective potential, determined by the coupled Eqs. (3.10) and (3.11), is well-defined and unique, where the only solutions for Eq. (3.11) are the two degenerate minima for $T < T_c$, while for $T \geq T_c$, the only solution is $\bar{\varphi} = 0$. Therefore, the phase transition can readily be inferred to be second order for any set of parameters. It is also clear from the results shown by Figs. 2 and 3 that the transition due to thermal effects only is of second order. The effective potential does not develop local minima and the VEV of the field varies continuously from $T = 0$ till $T = T_c$, where it vanishes. This is the expected behavior, since the complex scalar field model belongs to the same universality class of a $O(2)$ Heisenberg model, for which the phase transition is second order [1].

Thus, we see that the OPT method correctly describes the phase transition in the model, capturing the correct physics. Furthermore, as an added bonus, we can exactly compute the critical temperature of transition.

B. The phase structure in a constant external magnetic field

Let us now investigate the phase structure when an external magnetic field is also applied to the system. In the presence of a constant magnetic field, the Feynman rules change as shown in Sec. II B. The momentum integrals in Eq. (2.9) are given by (2.12) and the dispersion relation for charged particles is given by Eq. (2.11), which takes into account the Landau energy levels of a charged particle in a magnetic field.[2]

[2]Note that the dispersion relation Eq. (2.11) only applies for the fields in the complex base (ϕ, ϕ^*), since (ϕ_1, ϕ_2) are not appropriate eigenstates of charge. However, when reexpressing the effective potential in the complex scalar field base, the final result turns out to be the same as in Eq. (2.9) or (3.9), but with the functions X and Y depending now also on the magnetic field.

At finite B, the first-momentum integral term in (2.9) becomes

$$-i\int_P \ln(P^2 - \Omega^2) = \frac{eB}{2\pi}\sum_{k=0}^{+\infty}\left(\frac{e^{\gamma_E}M^2}{4\pi}\right)^{\epsilon}\int\frac{d^{1-2\epsilon}p_z}{(2\pi)^{1-2\epsilon}}T \times \sum_{n=-\infty}^{+\infty}\ln[\omega_n^2 + p_z^2 + \Omega^2 + (2k+1)eB]. \quad (3.16)$$

By performing the sum over the Matsubara frequencies in (3.16), we obtain two terms. One is independent of the temperature and the other is for $T \neq 0$. The $T = 0$ term is

$$\begin{aligned}\frac{eB}{2\pi}\sum_{k=0}^{+\infty}\left(\frac{e^{\gamma_E}M^2}{4\pi}\right)^{\epsilon}\int\frac{d^{1-2\epsilon}p_z}{(2\pi)^{1-2\epsilon}}\sqrt{p_z^2 + \Omega^2 + (2k+1)eB} &= -\frac{(eB)^2}{4\pi^2}\left(\frac{e^{\gamma_E}M^2}{8\pi eB}\right)^{\epsilon}\frac{\Gamma(-1+\epsilon)}{(4\pi)^{-\epsilon}}\zeta\left(-1+\epsilon, \frac{\Omega^2 + eB}{2eB}\right)\\ &= -\frac{\Omega^4}{32\pi^2}\left[\frac{1}{\epsilon} + 1 + \ln\left(\frac{M^2}{2eB}\right) + \mathcal{O}(\epsilon)\right] + \frac{(eB)^2}{4\pi^2}\zeta'\left(-1, \frac{\Omega^2 + eB}{2eB}\right), \quad (3.17)\end{aligned}$$

where to write the second line in (3.17), we used the analytic continuation of the Hurwitz zeta function [39]

$$\zeta(s, a) = \sum_{k=0}^{\infty}\frac{1}{(k+a)^s}, \quad (3.18)$$

and $\zeta'(s, a)$ is its s-derivative. In writing the final terms in Eq. (3.17) we have also dropped constant terms since they do not affect the phase structure.

The $T \neq 0$ part of (3.16) is

$$\frac{eB}{\pi}T^2\sum_{k=0}^{+\infty}\int_{-\infty}^{+\infty}\frac{dz}{2\pi}\ln\left\{1 - \exp\left[-\sqrt{z^2 + \frac{\Omega^2}{T^2} + (2k+1)\frac{eB}{T^2}}\right]\right\}. \quad (3.19)$$

We can now write Eq. (3.16) in a form analogous to Eq. (3.1)

$$-i\int_P \ln(P^2 - \Omega^2) = -\frac{\Omega^4}{2(4\pi)^2}\frac{1}{\epsilon} + \tilde{Y}(T, B, \eta), \quad (3.20)$$

where $\tilde{Y}(T, B, \eta)$ is given by

$$\tilde{Y}(T,B,\eta) = -\frac{\Omega^4}{32\pi^2}\left[1+\ln\left(\frac{M^2}{2eB}\right)\right] + \frac{(eB)^2}{4\pi^2}\zeta'\left(-1,\frac{\Omega^2+eB}{2eB}\right) + \frac{eB}{\pi}T^2\sum_{k=0}^{+\infty}\int_{-\infty}^{+\infty}\frac{dz}{2\pi} \times \ln\left\{1-\exp\left[-\sqrt{z^2+\frac{\Omega^2}{T^2}+(2k+1)\frac{eB}{T^2}}\right]\right\}. \quad (3.21)$$

Analogous manipulations leading to Eq. (3.20) allow us to write the remaining momentum integral terms in Eq. (2.9) as

$$\int_P \frac{i}{P^2-\Omega^2} = -\frac{\Omega^2}{(4\pi)^2}\frac{1}{\epsilon} + \tilde{X}(T,B,\eta), \quad (3.22)$$

where

$$\tilde{X}(T,B,\eta) = \frac{eB}{8\pi^2}\ln\left[\Gamma\left(\frac{\Omega^2+eB}{2eB}\right)\right] - \frac{eB}{16\pi^2}\ln(2\pi) - \frac{\Omega^2}{16\pi^2}\ln\left(\frac{M^2}{2eB}\right) + \frac{eB}{2\pi}\sum_{k=0}^{+\infty}\int_{-\infty}^{+\infty}\frac{dz}{2\pi} \times \frac{1}{\sqrt{z^2+\frac{\Omega^2}{T^2}+(2k+1)\frac{eB}{T^2}}} \times \frac{1}{\exp[\sqrt{z^2+\frac{\Omega^2}{T^2}+(2k+1)\frac{eB}{T^2}}]-1}. \quad (3.23)$$

Actually, Eq. (3.22) is just the derivative of Eq. (3.20) with respect to Ω^2, as can be easily checked.

The final expression for the renormalized effective potential at finite temperature and in a constant external magnetic field has an analogous form as Eq. (3.9), where by using Eqs. (3.20) and (3.22) in (2.9), we obtain

$$V_{\rm eff}(\varphi,T,B,\eta) = -\frac{m^2}{2}\varphi^2 + (1-\delta)\frac{\eta^2}{2}\varphi^2 + \delta\frac{\lambda}{4!}\varphi^4 + \tilde{Y}(T,B,\eta) + \delta\left\{-\eta^2 + \frac{\lambda}{3}[\varphi^2+\tilde{X}(T,B,\eta)]\right\}\tilde{X}(T,B,\eta). \quad (3.24)$$

Finally, the PMS criterion and the minimum of the effective potential are again also of the form as Eqs. (3.10) and (3.11)

$$\bar{\eta}^2 = \frac{\lambda}{3}\varphi^2 + \frac{2\lambda}{3}\tilde{X}(T,B,\bar{\eta}), \quad (3.25)$$

and

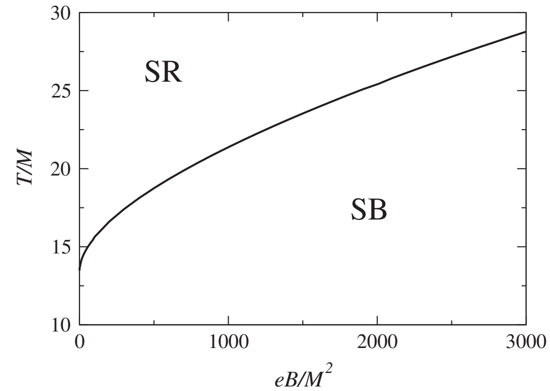

FIG. 4. The phase diagram of the system in the (B,T) plane. The parameters considered here are $\lambda = 0.1$, $m/M = 1$. The solid line separates the regions of symmetry-broken (SB) and symmetry-restored (SR) phases.

$$\bar{\varphi}^2 = 6\frac{m^2}{\lambda} - 4\tilde{X}(T,B,\bar{\eta}). \quad (3.26)$$

With Eqs. (3.24), (3.25), and (3.26), we are now in position to study the effects of an external magnetic field in the phase structure of the model, in addition to the thermal effects. In Fig. 4 we show the phase diagram for the symmetry-broken complex scalar field model in the (B,T) plane. It shows that the magnetic field effect enlarges the symmetry-broken phase, increasing the critical temperature for phase transition. We must note that this result for the phase diagram is very different from that seen in the superconductor or in the scalar quantum electrodynamics (QED) case [40]. In contrast to our case, where we are only studying the effects of B and T on the charged scalars, in the scalar QED there are also the interactions with the gauge field. In the scalar QED case with a local $U(1)$ gauge symmetry, in the symmetry-broken phase, because of the Higgs mechanism the gauge field becomes massive and the external magnetic field gets screened (the Meissner effect). As the magnetic field is increased, eventually above a critical field the phase is restored through a first-order phase transition. In the study made in this paper, on the other hand, we consider only the case of a broken *global* $U(1)$ symmetry.[3]

The effect of the magnetic field on the global $U(1)$ symmetry can also be seen in the next two figures. In Fig. 5 we have plotted the VEV of the field at fixed $T > T_c$, thus starting from a symmetry-restored phase, $\bar{\varphi} = 0$, as a function of B. The critical temperature, for the parameters used, is $T_c/M \simeq 13.42$, which fully agrees with Eq. (3.15), and the symmetry returns to be broken,

[3]One of the authors (ROR) would like to thank K. G. Klimenko for discussions on this topic and for also pointing out to him the possible similarity with the symmetry behavior found in this paper due to the magnetic field, with pion condensation seen in the Nambu–Jona-Lasinio model with two flavor quarks plus baryon and isospin chemical potentials [41].

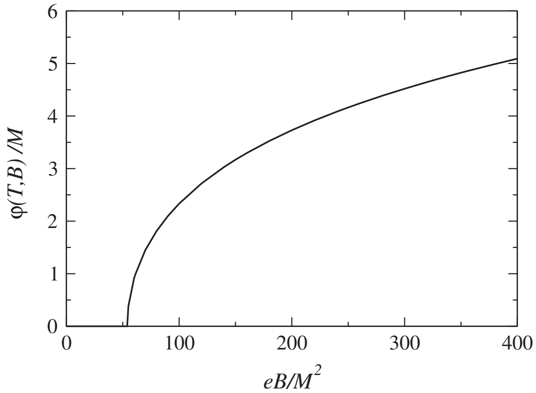

FIG. 5. Magnetic field dependence of the VEV of the field, $\bar{\varphi}(T, B)$, for a fixed value of T above T_c. The parameters considered here are: $\lambda = 0.1$, $m/M = 1$ and $T/M = 15$.

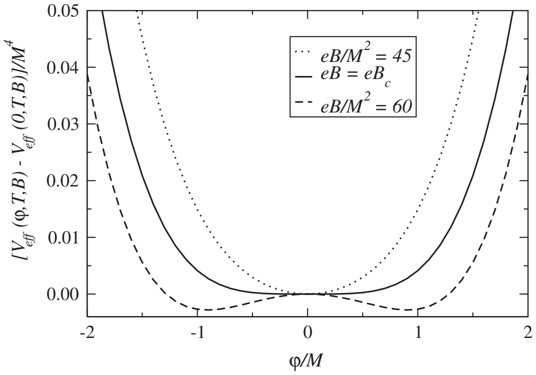

FIG. 6. The effective potential for the same parameters of Fig. 5 and for three different values of the magnetic field: $eB/M^2 = 45$, $eB_c/M^2 = 53.93$ and $eB/M^2 = 60$.

$\bar{\varphi} \neq 0$, for a magnetic field above a critical value given by $eB_c/M^2 \simeq 53.93$. In Fig. 6 the effective potential is plotted for the same fixed temperature of Fig. 5 and for values of B below, at and above the critical value B_c, above which the symmetry becomes broken again (as before, to show the spinodal region of the potential, we have plotted only the real part of it). It should be noticed from these results that the phase transition is once again second order. The effect of the magnetic field is to enlarge the symmetry-broken region by making the critical temperature larger, but it does not change the order of the transition.

Finally, we have again used for comparison a case with analogous parameters as considered in Ref. [29], and contrary to the results found in that reference, we have found again here only a second-order phase transition. For the analogous parameters used in [29] (see observations made in previous subsection), $m/M = 20$, $\lambda = 0.00375$ and $eB/M^2 = 30$, the change of the critical temperature in relation to the $B = 0$ case shown in Fig. 3 is only to produce a slight shift of the critical temperature to a value 0.08% higher. In Fig. 7, we have plotted the VEV of the field for the case of zero magnetic field and for a much higher value of the magnetic field, $eB/M^2 = 3000$, so as to be able to visualize the difference. Even so, the difference even for such a value of magnetic field is only marginal, changing the critical temperature obtained at $B = 0$, $T_c(B = 0)/M \simeq 1385.64$, to $T_c(B)/M \simeq 1396.87$ for the value of B used. In all cases we have tested for a nonvanishing magnetic field and other different parameters of the potential, the transition is again verified to be second order, with the VEV varying always continuously from its value at $T = 0$ to zero at $T = T_c(B)$.

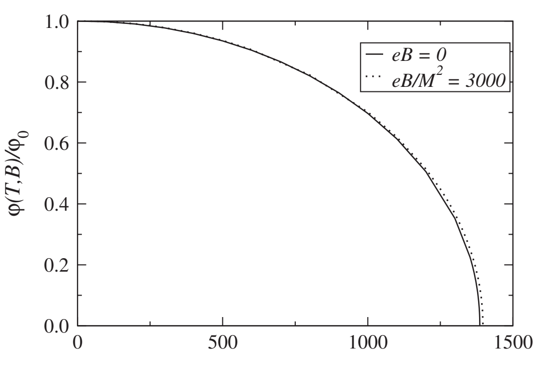

FIG. 7. Temperature dependence of the normalized VEV $\bar{\varphi}(T, B)/\varphi_0$. The critical temperature for $B = 0$ is $T_c^{B=0}/M \simeq 1385.64$. Including the effects of the magnetic field, with $eB/M^2 = 3000$, T_c increase to $T_c/M \simeq 1396.87$. The other parameters used here are: $\lambda = 0.00375$ and $m/M = 20$.

IV. USING THE EULER-MACLAURIN FORMULA FOR THE SUM OVER LANDAU LEVELS

When working with quantum field theories in a magnetic field, we need to deal with the sum over the Landau levels. At $T = 0$ this does not present a problem in general, since we can express the expressions in terms of zeta functions. We have seen this in the previous section, where all terms at $T = 0$ were presented in analytical form. However, at finite temperature this is not possible in general; therefore, we either need to numerically perform the sums over the Landau levels, or find suitable approximations for the expressions. This job becomes easier in the high-magnetic field regime, with $\sqrt{eB} \gg T$, for which most of the time suffices to use only the very first terms of the sum, or even just the leading Landau level term. Higher-order terms are quickly Boltzmann suppressed in this case. But this is not so in the low-magnetic field regime, with $\sqrt{eB} \lesssim T$, and $\Omega/T \lesssim 1$, where most of the time requires to sum up to very large Landau-levels. This has the potential to make any numerical computation to become quickly expensive, especially when there are in addition numerical integrations involved, as in our case here (note that we have not made use of high-temperature approximations, but worked with the complete expressions in terms of the temperature-dependent integrals). Simple approximate analytical results, which are always desirable to obtain, can also become difficult to obtain without a suitable approximation that can be used. One such approximation was discussed in Ref. [29], but it is valid only for very small magnetic fields. Here we discuss a more natural alternative for dealing with the cases of low magnetic fields, based on the Euler-Maclaurin formula.

The Euler-Maclaurin (EM) formula provides a connection between integrals and sums. It can be used to evaluate finite sums and infinite series using integrals, or vice-versa. In its most general form it can be written as [42]

$$\sum_{k=a}^{b} f(k) = \int_a^b f(x)dx + \frac{1}{2}[f(a)+f(b)] + \sum_{i=1}^{n} \frac{b_{2i}}{(2i)!}[f^{(2i-1)}(b) - f^{(2i-1)}(a)] + \int_a^b \frac{B_{2n+1}(\{x\})}{(2n+1)!} f^{(2n+1)}(x)dx, \tag{4.1}$$

where b_i are the Bernoulli numbers, defined by the generating function

$$\frac{x}{\exp(x)-1} = \sum_{n=0}^{\infty} b_n \frac{x^n}{n!}, \tag{4.2}$$

and $B_n(x)$ are the Bernoulli polynomials, with generating function

$$\frac{z\exp(zx)}{\exp(z)-1} = \sum_{n=0}^{\infty} B_n(x)\frac{z^n}{n!}. \tag{4.3}$$

The notation $\{x\}$ in $B_{2k+1}(\{x\})$ in Eq. (4.1) means the fractional part of x and $f^{(k)}(x)$ means the kth derivative of the function. The last term in Eq. (4.1) is known as the *remainder*.

Next, we study the reliability of the use of the EM formula as a suitable approximation for the sums over Landau levels and we estimate the errors involved in doing so. In the following, we will call the zeroth order in the EM formula when only the first term in the right-hand side of Eq. (4.1) is kept, the first order when also the second term is kept and so on so forth. Thus, for example, the Landau sum part of $\tilde{Y}(T, B, \eta)$ in Eq. (3.21), up to second order in the EM formula, becomes

$$L\tilde{Y} = \sum_{k=0}^{\infty}\int_{-\infty}^{+\infty}\frac{dz}{2\pi}\ln[1-e^{-E(z,\Omega,T,B,k)}] \simeq \int_{-\infty}^{+\infty}\frac{dz}{2\pi}\left\{\int_0^{\infty}\ln[1-e^{-E(z,\Omega,T,B,k)}]dk + \frac{1}{2}\ln[1-e^{-E(z,\Omega,T,B,0)}] - \frac{eB}{12T^2}\frac{1}{E(z,\Omega,T,B,0)[e^{E(z,\Omega,T,B,0)}-1]}\right\}, \tag{4.4}$$

where

$$E(z,\Omega,T,B,k) = \sqrt{z^2 + \frac{\Omega^2}{T^2} + (2k+1)\frac{eB}{T^2}}. \tag{4.5}$$

Likewise, the Landau sum part of $\tilde{X}(T, B, \eta)$ in Eq. (3.23), up to second order in the EM formula, becomes

$$L\tilde{X} = \sum_{k=0}^{\infty}\int_{-\infty}^{+\infty}\frac{dz}{2\pi}\frac{1}{E(z,\Omega,T,B,k)}\frac{1}{e^{E(z,\Omega,T,B,k)}-1} \simeq \int_{-\infty}^{+\infty}\frac{dz}{2\pi}\left\{\int_0^{\infty}\frac{1}{E(z,\Omega,T,B,k)}\frac{1}{e^{E(z,\Omega,T,B,k)}-1}dk + \frac{1}{2}\frac{1}{E(z,\Omega,T,B,0)[e^{E(z,\Omega,T,B,0)}-1]} + \frac{eB}{12T^2}\frac{1}{E^2(z,\Omega,T,B,0)[e^{E(z,\Omega,T,B,0)}-1]}\times\left[\frac{1}{E(z,\Omega,T,B,0)} + \frac{e^{E(z,\Omega,T,B,0)}}{e^{E(z,\Omega,T,B,0)}-1}\right]\right\}. \tag{4.6}$$

In Table I we assess the reliability of using the EM at each order (we have studied it up to the fourth order) for the sums in Eqs. (4.4) and (4.6), for different values for the ratio eB/T^2, and we compare the results with the exact ones.

From the results shown in Table I, we see that the EM approximation produces results that already at first order have good accuracy compared to the exact values coming from the full Landau sums. It performs particularly very well for values of $eB \ll T^2$, reaching convergence within just the very first few terms in the expansion. However, for values $eB \gtrsim T^2$, increasing the order in the EM series, it quickly looses accuracy. This can be traced to the fact that as we go to a higher order in the derivatives appearing in the sum on the right-hand side of Eq. (4.1), higher powers of eB/T^2 are produced, eventually spoiling the approximation (though it oscillates around the true value). This apparent runaway behavior is only cured by the introduction of the last term in Eq. (4.1), the remainder.

It should be noted that Eq. (4.1) is not an approximation to the sum, but it is actually an identity. The important term in that context is the remainder, the last term in Eq. (4.1). In all our numerical tests, within the numerical precision for the integrals (for convenience, we have performed all the numerical integrations in *Mathematica* [43]), we have verified that Eq. (4.1) has agreed with the results from the sum over the Landau levels in the left-hand side of Eqs. (4.4) and (4.6), for all values tested and at all orders (when including the remainder), with a precision of less than 0.005%. This is quite impressive, recalling that we can in principle keep only the first few terms in Eqs. (4.4) and (4.6) (the first-order approximation) when including the remainder. For numerical computation this is a tremendous advantage, since Eq. (4.1) can easily be coded, compared to having to perform sums over very large Landau levels (particularly in the cases of low magnetic fields) and the multiple integrations that may be required.

TABLE I. The precision of the EM formula (without the remainder) at each order (up to fourth order), compared with the result from the Landau sum. In all cases we have considered $\Omega = 0$ in Eqs. (4.4) and (4.6), when extending them up to fourth order.

			$L\tilde{Y}$			$L\tilde{X}$	
eB/T^2	order	EM approx.	Landau sum	error (%)	EM approx.	Landau Sum	error (%)
	0	−688.7717		0.04	508.2575		1.67
	1	−689.0258		1.00×10^{-4}	515.7328		0.22
0.001	2	−689.0271	−689.0264	1.00×10^{-4}	517.0372	516.8633	0.03
	3	−689.0270		1.00×10^{-4}	516.7096		0.03
	4	−689.0271		1.00×10^{-4}	517.2016		0.07
	0	−68.6567		0.35	47.7384		5.00
	1	−68.8954		5.10×10^{-3}	49.8997		0.70
0.01	2	−68.8990	−68.8989	1.00×10^{-4}	50.3031	50.2488	0.11
	3	−68.8988		1.00×10^{-4}	50.2007		0.10
	4	−68.8990		1.00×10^{-4}	50.3549		0.21
	0	−6.6735		2.99	3.9418		14.07
	1	−6.8706		0.13	4.4852		2.22
0.1	2	−6.8797	−6.8793	5.80×10^{-3}	4.6037	4.5873	0.36
	3	−6.8791		2.90×10^{-3}	4.5725		0.30
	4	−6.8796		4.40×10^{-3}	4.6199		0.70
	0	−0.5400		18.70	0.2195		34.89
	1	−0.6497		2.18	0.3128		7.21
1	2	−0.6652	−0.6642	0.15	0.3416	0.3371	1.34
	3	−0.6637		0.07	0.3329		1.25
	4	−0.6651		0.14	0.3465		2.79
	0	−0.0159		59.54	0.0034		66.99
	1	−0.0328		16.54	0.0081		21.36
10	2	−0.0407	−0.0393	3.56	0.0112	0.0103	8.74
	3	−0.0383		2.55	0.0095		7.77
	4	−0.0413		5.09	0.0124		20.39
	0	-6.8468×10^{-6}		88.48	5.9362×10^{-7}		89.52
	1	-3.6528×10^{-5}		38.52	3.4235×10^{-6}		39.54
100	2	-8.3692×10^{-5}	-5.9413×10^{-5}	40.87	8.3706×10^{-6}	5.6626×10^{-6}	47.82
	3	-1.1413×10^{-5}		80.80	3.6803×10^{-6}		35.00
	4	-3.8726×10^{-4}		551.81	5.6924×10^{-5}		905.27

Given all the advantages of using the EM formula, it is quite surprising that it is not used frequently in physics, despite being well-known in mathematics. In particular, we see that it is well suitable in problems involving external magnetic fields. To our knowledge, among the few works that we are aware of that have previously made use of the EM formula before were, for example, Refs. [44–46]. In Ref. [44] the author derived the effective potential of the Abelian-Higgs model, including both thermal effects and an external magnetic field, in the one-loop approximation. The EM formula was used at its zeroth order (by just transforming the sum in an integral) to obtain analytical results for the effective potential in the high-magnetic field region. But from the results of Table I, this is exactly the region and order in the approximation that lead to the largest errors. In [45] the EM formula was used to obtain an expression for the effective potential for vector bosons and to study pair production in an external magnetic field, while in [46] it was used to obtain analytical expressions for the internal energies for noninteracting bosons confined within a harmonic oscillator potential. None of these previous works have assessed the reliability of the use of the EM formula.

As an application of the EM formula to our problem, we estimate from it the dependence of the critical temperature with the magnetic field. Recall from Eq. (3.13) the critical temperature can be obtained by setting $\Omega = 0$ and $\bar{\varphi} = 0$ in the equation for the VEV. Thus, from Eq. (3.26), we need to solve the equation

$$6\frac{m^2}{\lambda} - 4\tilde{X}(T_c, B)|_{\Omega=0} = 0. \tag{4.7}$$

As we have seen from the results of Table I, the EM formula for low magnetic fields produce reliable results already at the second order, and for very small magnetic fields, good precision is reached even at first order. Thus, for example, from the expansion in Eq. (4.6), and performing the k integral in the first term, we obtain

$$
\begin{aligned}
L\tilde{X} \simeq & -\frac{1}{\pi}\frac{T^2}{eB}\int_0^\infty dz \ln(1 - e^{-\sqrt{z^2+a^2}}) \\
& + \frac{1}{2\pi}\int_0^\infty dz \frac{1}{\sqrt{z^2+a^2}}\frac{1}{e^{\sqrt{z^2+a^2}}-1} \\
& + \frac{eB}{12\pi T^2}\int_0^\infty dz \frac{1}{(z^2+a^2)(e^{\sqrt{z^2+a^2}}-1)} \\
& \times \left(\frac{1}{\sqrt{z^2+a^2}} + \frac{e^{\sqrt{z^2+a^2}}}{e^{\sqrt{z^2+a^2}}-1}\right),
\end{aligned} \tag{4.8}
$$

where $a^2 = eB/T_c^2$. Results for the integrals in Eq. (4.8) can be found in the low-magnetic field regime, where $a \ll 1$. This regime is equivalent to the high-temperature regime in quantum field theory, from where a series expansion in a can be found. In fact, the integrals in Eq. (4.8) can be easily related to the $h_i(a)$ Bose-Einstein integrals found in that context (see, for example, the App. A in Ref. [14]), from where we obtain the leading-order terms in the expansion in powers of a

$$
L\tilde{X} \simeq \frac{\pi}{6a^2} - \frac{5}{24a} + \mathcal{O}(a^0). \tag{4.9}
$$

Going to higher order in the EM series only leads to $\mathcal{O}(a^n)$ (with $n \geq 0$) terms, for $a = \sqrt{eB}/T_c \ll 1$.

Using the result (4.9) in Eq. (3.23) and then back in Eq. (4.7), after some algebra, we obtain the approximate result for the critical temperature as a function of the magnetic field

$$
T_c^2(B) \simeq 18\frac{m^2}{\lambda} + \frac{25eB}{32\pi^2}\left(1 + \sqrt{1 + \frac{1152\pi^2 m^2}{25\lambda eB}}\right). \tag{4.10}
$$

The result (4.10) shows the growth of the critical temperature with the magnetic field, as seen numerically from the results obtained in the previous section. A comparison with our previous results also shows that Eq. (4.10) provides an excellent fit for $T_c(B)$, leading to results with less than 1% error compared to the full numerical results for T_c, for values of field such that $eB/T_c^2 \lesssim 1$. For example, for the parameters $\lambda = 0.1$ and $eB = 400m^2$, the approximation (4.10) gives $T_c/m \simeq 17.97$, while the full numerical calculation gives $T_c/m = 18.11$.

V. CONCLUSIONS

In this paper we have revisited the phase transition problem for the self-interacting complex scalar field model when thermal effects and an external magnetic field are present. We have studied this problem in the context of the nonperturbative method of the optimized perturbation theory. We have shown that the OPT method preserves the Goldstone theorem and that carrying out the approximation to first order is analogous to the ring-diagram resummation method. The OPT carries out the resummation of self-energy diagrams in a self-consistent way, avoiding overcounting issues, which have previously plagued the ring-diagram method in its very early applications.

By using the OPT method, we have demonstrated that the phase transition in the model is always second order in the presence of thermal effects and by including an external magnetic field, there is no change to the phase transition order. The effect of the external magnetic field is to strengthen the symmetry-broken phase, producing a larger VEV for the field and also a larger critical temperature. The study we made in this paper considered only the effects of the magnetic field and temperature on the charged scalar field [the model is one with a global $U(1)$ symmetry]. Our study, in a sense, is closer to the case of studying the effects of B and T in the linear sigma model [47], where no couplings to gauge fields (except to the external field) are considered in general. The inclusion of other field degrees of freedom, or when promoting the symmetry from a global one to a local symmetry, can of course change both qualitatively and quantitatively how the magnetic field affects the system. In particular, as concerning the possibility of producing a first-order phase transition, either due to thermal effects, or by a magnetic field or from both. This is what we expect in the context of the scalar QED, where due to the screening of the magnetic field in the broken phase, the phase diagram and transition are very different. The phase structure can also be very different when adding other interactions and having other symmetries. For example, in the context of the electroweak phase transition [9,10,48], it has been shown that the effect of the magnetic field is to make the first-order transition stronger. By omitting vacuum energy terms from the effective potential, which may have phenomenological motivations, as in Ref. [47], may also lead to a first-order transition, instead of a second-order one.

Still comparing our results with those obtained in the Abelian-Higgs model, there is a tantalizing question of whether for very strong magnetic fields a new phase could be formed. This could be, for example, a phase with global vortices condensation, thus restoring the symmetry again. This would be in analogy to the local, Nielsen-Olesen vortex condensation that can happen in type II superconductors [40]. Vortex condensation in that case is energetically favorable to happen for fields beyond a critical value and when the mass of the Higgs field becomes larger that the mass of the gauge field. This is an interesting possibility to investigate in the future.

As an aside, but a complementary part of this work, we have verified the reliability of the use of the Euler-Maclaurin formula as an approximation for the sum over Landau energies in a magnetic field. We have verified that it produces results with errors of less than 0.1% already at the first few orders in the EM formula and that it leads to particularly suitable approximations in the low-magnetic field region ($eB/T^2 \ll 1$). The low-magnetic field region is typically the region where we have to face the problem of summing over very larger number of Landau levels, which, in practice, can be overwhelming in terms of CPU time, when working numerically. The EM formula in this case can be a very valuable tool, both for numerical computations, but also for obtaining approximate analytical results, which would be otherwise very difficult through the use of the Landau sums directly. As an application of the EM formula, we have obtained an approximate expression for the dependence of the critical temperature with the magnetic field in our problem. We expect that the EM formula can also be used in many other applications involving the effects of magnetic fields in phase transitions, as in condensed matter problems or in the recent interest in studying the effects of magnetic fields in the QCD phase transition. Furthermore, our results (including the use of the OPT method) can also be of interest in the study of cosmological phase transitions in general. Magnetic fields can be easily generated in the early universe [49]. These fields can then influence subsequent cosmological phase transitions or also be important in particle physics phenomena in the early universe. Works in those contexts are in progress and we will report on them elsewhere.

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APPENDIX A: RENORMALIZATION IN THE OPT METHOD

In this appendix we briefly explain the renormalization of the effective potential Eq. (2.9). First, note that the interpolation procedure in the OPT method, Eq. (2.5), introduces only new quadratic terms, thus, it does not alter the renormalizability of the original theory. Therefore, the counterterms needed to render the theory finite have the same polynomial structure of the original Lagrangian [30].

In obtaining the renormalized effective potential, we first note that from the order δ term Eq. (3.4), we obtain the mass renormalization counterterm entering in $\Delta \mathcal{L}_{\rm ct,\delta}$ in Eq. (2.5),

$$-\frac{1}{2} A_{\delta^1} \phi_i^2, \tag{A1}$$

where

$$A_{\delta^1} = \delta \frac{\lambda \Omega^2}{24\pi^2 \epsilon}. \tag{A2}$$

From Eq. (A1), we obtain the two mass counterterms diagrams contributing to the effective potential at first order in the OPT and shown in Fig. 1.

By collecting all the divergences from the vacuum loop terms in Eq. (2.9), we have that the divergence from the contribution (3.4) is canceled by the counterterm Eq. (A1). A potential temperature- and magnetic-field-dependent divergence, proportional to either X in the two-loop vacuum term Eq. (3.8), or $\tilde{X}$ in the case of a finite external magnetic field, is explicitly canceled against an identical term coming from the mass counterterm diagrams, Eq. (3.6). The remaining divergences in the effective potential are only vacuum ones, independent of the background field, and they can be all canceled by introducing a vacuum counterterm $\Delta V_{\rm ct}$, added to the effective potential. At the first order in the OPT, $\Delta V_{\rm ct}$ is given by

$$\Delta V_{{\rm ct},\delta^1} = \left(\frac{\Omega^4}{2(4\pi)^2} - \frac{\delta \eta^2}{16\pi^2}\Omega^2\right)\frac{1}{\epsilon} + \frac{\delta\lambda\Omega^4}{3(16\pi^2)^2}\frac{1}{\epsilon^2}. \tag{A3}$$

Thus, at first order in the OPT, we only require the two counterterms, Eqs. (A1) and (A3). Going to second order in the OPT it will also require a coupling constant renormalization counterterm, $C_{\delta^2}\lambda\phi_i^4/4!$, from which we can built vertex counterterm vacuum diagram contributions to the effective potential [33] (see also the first two references in [30]). At second order in the OPT we would also obtain δ^2 contributions for Eqs. (A1) and (A3). Going to higher orders in the OPT only produces additional $\mathcal{O}(\delta^n)$, $n > 2$, contributions to the mass, vertex and vacuum counterterms.

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