Geodesically complete analytic solutions for a cyclic universe

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We present analytic solutions to a class of cosmological models described by a canonical scalar field minimally coupled to gravity and experiencing self interactions through a hyperbolic potential. Using models and methods inspired by 2T-physics, we show how analytic solutions can be obtained in flat/open/closed Friedmann-Robertson-Walker universes. Among the analytic solutions, there are many interesting geodesically complete cyclic solutions in which the universe bounces at either zero or finite sizes. When geodesic completeness is imposed, it restricts models and their parameters to a certain parameter subspace, including some quantization conditions on initial conditions in the case of zero-size bounces, but no conditions on initial conditions for the case of finite-size bounces. We will explain the theoretical origin of our model from the point of view of 2T-gravity as well as from the point of view of the colliding branes scenario in the context of M-theory. We will indicate how to associate solutions of the quantum Wheeler-deWitt equation with our classical analytic solutions, mention some physical aspects of the cyclic solutions, and outline future directions.

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I. INTRODUCTION

In this paper we will analytically express a cyclic universe using exact solutions in a scalar-tensor theory with a scalar field $\sigma(x^\mu)$ minimally coupled to gravity.

The full action of our theory is

$$S = \int d^4x\sqrt{-g}\left\{\frac{1}{2\kappa^2}R(g) - \frac{1}{2}g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma - V(\sigma)\right\}, \quad (1)$$

where the potential is

$$V(\sigma) = \left(\frac{\sqrt{6}}{\kappa}\right)^4\left[b\cosh^4\left(\frac{\kappa\sigma}{\sqrt{6}}\right) + c\sinh^4\left(\frac{\kappa\sigma}{\sqrt{6}}\right)\right]. \quad (2)$$

Here b and c are dimensionless free parameters of the potential, and κ^{-1} is the reduced Planck mass $\kappa^{-1} = \sqrt{\frac{\hbar c}{8\pi G}} = 2.43\times 10^{18}\,\frac{\mathrm{GeV}}{c^2}$. A plot of the potential energy $V(\sigma)$ for various signs and magnitudes of b, c, consistent with stability $(b+c)>0$, show that the profile of this potential is similar to those often used in the study of cosmology. This potential was chosen because we can solve the equations exactly, thus enabling us to perform the type of analysis presented in this paper. We assume that the general features discussed here go beyond the special choice of potential.[1]

[1]In fact, this is not the only potential for which we are able to give a full analysis with the complete set of analytic solutions [1]. In the near future we will present a similar discussion for the potentials $V_1(\sigma) = (\frac{\sqrt{6}}{\kappa})^4(be^{-2\kappa\sigma/\sqrt{6}} + ce^{-4\kappa\sigma/\sqrt{6}})$ and $V_2(\sigma) = (\frac{\sqrt{6}}{\kappa})^4 be^{2p\kappa\sigma/\sqrt{6}}$, where b, c, p are dimensionless real parameters. The profile of $V_1(\sigma)$, with $c>0$ and $b<0$, is similar to the profile of the potential used initially in the cyclic cosmology model in [2].

The model of Eqs. (1) and (2) was initially inspired by 2T-physics [3–5] as described in [6] and Sec. I A below. The same model fits also in the worldbrane scenario [7], as inspired by D-branes in M-theory [8]. The ideas of a cyclic universe [2] modeled in Ref. [9] can be adapted to reproduce the same potential $V(\sigma)$, thus describing a universe that consists of two $3+1$ dimensional orientifolds that periodically collide with each other by oscillating in an extra fifth dimension. It is quite interesting that this connection emerged between 2T-gravity and M-theory. In Secs. I A and I B we will comment on the different origins that converged on this model.

In a previous cosmological application of this model [6] $V(\sigma)$ was an energy density of the order of the grand unification scale $(m_{\mathrm{GUT}})^4$. In that case, $b^{1/4}\sqrt{6}\kappa^{-1}$ or $c^{1/4}\sqrt{6}\kappa^{-1}$ were of order $m_{\mathrm{GUT}}\sim 10^{16}\,\frac{\mathrm{GeV}}{c^2}$, thus leading to dimensionless values for the parameters b or c in the order of 10^{-12}. Exact solutions have a way of finding applications in various fields. For physical applications of our solutions, including cyclic cosmology or other future cases, the value of the parameters b, c, should be chosen appropriately depending on the application.

The complete set of analytic solutions for this model, in a homogeneous, spatially flat, isotropic Friedmann-Robertson-Walker (FRW) universe, was obtained in our earlier paper [6], and some of the solutions' perturbations were studied in [6,10]. In the current paper we will emphasize a subset of these solutions that are geodesically complete and describe a universe smoothly evolving through big bang or big crunch singularities at which the universe shrinks to *zero size*, but then continues to perform periodic expansions and contractions that describe a cyclic universe, all without violating unitarity or the null energy condition in a flat universe. We will also include the effect of spatial curvature for the FRW universes ($k = 0, \pm 1$) in our new exact solutions, and we will exhibit cyclic solutions with *finite-size bounces* as well.

Perturbations such as radiation are easily included in the exact solutions, while anisotropy can be discussed with analytic approximations; but those aspects, as well as the quantum treatment through the Wheeler-deWitt equation, which require more detailed discussions, will appear in a separate paper [11].

The complete set of homogeneous, isotropic classical solutions presented in [6] shows that, the *generic* solutions for the field $\sigma(\tau)$ and the scale factor $a(\tau)$ describe a geodesically *incomplete* geometry. The geodesic incompleteness can be exhibited in terms of conformal time τ as defined by the line element $ds^2 = a^2(\tau)(-d\tau^2 + ds_3^2)$, where ds_3^2 is the line element of the three-dimensional space. As an illustration consider the spatially flat case $ds_3^2 = d\vec{x} \cdot d\vec{x}$. The geodesic $x^\mu(\tau)$ of a massive particle in this flat geometry is described by its velocity

$$\frac{d\vec{x}(\tau)}{d\tau} = \frac{\vec{p}}{\sqrt{\vec{p}^2 + m^2 a^2(\tau)}}, \tag{3}$$

where $\vec{p}$ is the conserved 3-momentum. In terms of this conformal time τ, the complete set of solutions in [6] shows that, for the generic solution, the scale factor $a(\tau)$ starts out at zero size at some time $a(\tau_1) = 0$ and grows to maximum size $a(\tau_2) = a_{\max}$ in a finite amount of conformal time $(\tau_2 - \tau_1) =$ finite. It turns out that $a_{\max}$ is infinite in the case of $b \geq 0$ or finite in the case of $b < 0$. Furthermore $a(\tau)$ has this same behavior in an infinite number of different disconnected intervals of conformal time τ. Each such separate interval describes a universe that starts out with a big bang and expands to maximum size. Moreover there are other disconnected intervals in which the universe contracts from maximum size to zero size. Evidently such generic solutions of the Friedmann equations are geodesically incomplete.

If expressed in terms of cosmic time t defined by the line element $ds^2 = (-dt^2 + a^2(t)ds_3^2)$, the geodesic equation reads

$$\frac{d\vec{x}(t)}{dt} = \frac{\vec{p}}{a(t)\sqrt{\vec{p}^2 + m^2 a^2(t)}}, \tag{4}$$

where $a(t)$ is expressed in terms of cosmic time t. The relation to conformal time is $dt = a(\tau)d\tau$ or $t(\tau) = \int_{\tau_1}^{\tau} a(\tau)d\tau$, where $t(\tau_1) = 0$ defines the big bang at $a(t(\tau_1)) = 0$. Hence $t(\tau)$ is given by the area under the curve in a plot of $a(\tau)$ versus τ, for some interval $\tau_1 \leq \tau \leq \tau_2$, starting with the big bang. An example of a geodesically incomplete curve $a(\tau)$ is Fig. 1 in [6], while examples of geodesically complete ones are given in many figures in the current paper. Since $a(\tau)$ in the generic solution is given in disconnected τ intervals, the cosmic time $t(\tau)$ cannot be defined for negative values before the big bang. Hence the geodesic equation above is artificially stopped at the big bang at the finite value of time $t(\tau_1) = 0$. This is one of the signs of geodesic incompleteness of this geometry. In addition, when the area under the curve is finite (the solutions for $b < 0$), the total cosmic time $t(\tau_2)$ is also finite, and geodesics are again artificially stopped at a finite value of cosmic time $t(\tau_2)$, providing another sign of geodesic incompleteness. In this way, each interval that is geodesically incomplete in conformal time appears again as geodesically incomplete in cosmic time. This type of geometry bounded by singularities, and classical solutions in them, occur often in general relativity and are commonly used in its applications, as in our own paper [6]. But in view of the geodesic incompleteness of the generic solutions of the Friedmann equations displayed in the conformal frame, it feels that this must be an incomplete story.

We think that a more satisfactory approach, especially for cosmology, is to find those models and solutions that describe a geodesically complete geometry. This type of solution is what we will describe in the current paper. It turns out that among the classical solutions presented in [6] there are some unique solutions that are geodesically complete in the *Einstein frame*. In this solution the patches of conformal time in which $a(\tau)$ is real in the Einstein frame[2] are smoothly connected from $\tau = -\infty$ to $\tau = \infty$. Then the universe sails smoothly through singularities, while geodesics of both massless and massive particles smoothly continue through singularities to the next cycle of the cyclic universe.

The requirements for such solutions depend on whether the bounce is at zero size $a(\tau_{\text{bang}}) = 0$ or finite size $a(\tau_{\text{bang}}) > 0$. In the case of zero-size bounce, that occurs when the spatial curvature is zero, $k = 0$, initial conditions of the two fields need to be synchronized and periods of oscillation need to be relatively quantized, as we will describe in detail. These requirements result in some quantization conditions among the available parameters consisting of integration constants of the differential equations for $\sigma(\tau)$, $a(\tau)$ as well as the parameters b, c in the potential energy $V(\sigma)$, and also radiation and anisotropy parameters when they are included. Because of these requirements these geodesically complete zero-size bounce solutions are associated with a countable set of boundary conditions

[2]All solutions, including those that are geodesically incomplete in the Einstein frame, are geodesically complete in other frames. Indeed, in the γ-gauge that we will discuss in Sec. II, all solutions are geodesically complete. However, as viewed from the point of view of the Einstein frame, $a(\tau)$ for such solutions becomes imaginary in the patches that complete the geodesics in the γ-frame, and hence the physical meaning in the Einstein frame becomes obscure. We intend to study this phenomenon in a future paper, but for the current paper we concentrate on only the geodesically complete solutions in which $a(\tau)$ is real for all τ.

(but still an infinite set, in the sense described in Sec. I D). In the case of finite-size bounce none of these constraints occur on boundary conditions, but in this case there is spatial curvature, $k/r_0^2 \neq 0$, which needs to be large enough to compete with the potential energy. In this case, as long as the parameters that define the model are within a certain continuous region, the generic solution is the finite-size bounce solution without any further requirements on boundary conditions.

When perturbations such as radiation and anisotropy are included or when quantum effects in the form of the Wheeler-deWitt equation are taken into account, there still are geodesically complete solutions that have a similar character to what we will present in this paper. They still form a countable set for zero-size bounces, while they are the generic solutions for the finite bounces. Either way, the distinguishing character of geodesic completeness has an appeal that seems important for physical applications, as we will discuss in a future paper [11].

A. 2T-physics origin

We would like to briefly summarize here the main points of how the model in Eq. (1) relates to 2T-gravity [4,5] in 4-space and 2-time dimensions as the conformal shadow in 3-space and 1-time. More generally, according to 2T-physics, a theory in 1T-physics in $(d-1)+1$ dimensions is one of the many shadows that comes from $d+2$ dimensions [3]. A useful shadow that appeals to the intuition of physicists accustomed to relativistic field theory is the one called the *conformal shadow*. In the conformal shadow there is a local scale symmetry (Weyl symmetry). The original 2T-gravity in $4+2$ dimensions does not have a Weyl symmetry in $4+2$ dimensions, instead this crucial gauge symmetry in $3+1$ dimensions is dictated in the conformal shadow as a remnant of general coordinate transformations in the extra $1+1$ dimensions [5]. Other less familiar shadows provide other descriptions of the physics that are related by duality transformations to the conformal shadow, and often they can provide hidden physical information that is systematically missed in the conventional formulation of 1T-physics [3,12,13].

Besides this duality aspect, 2T-physics may also provide additional constraints on the interactions of fields in 1T-physics even within the conformal shadow. The constraints in theories of interest, in the conformal shadow, are mainly on scalar fields and their interactions. These constraints have been determined generally in [14] in the presence of gravity or supergravity. Most of the emergent constraints can be rephrased as being consequences of various symmetries in 1T-field theory, but not all of them. Some of those additional constraints are not motivated by fundamental principles in 1T-physics, as discussed in [14], so they can be taken as signatures of 2T-physics. Here we will deal only with the simplest version of scalar fields that obey the constraints. This is the case when *all scalar fields* are conformally coupled to gravity in the 1T version. This is a familiar form in 1T field theory, but 1T-gravity does not require that *all* the scalars be conformal scalars; by contrast 2T-gravity has this as an outcome for the conformal shadow in one of its allowed versions (more general form of constrained scalars in [14]).

Moreover, in the conformal shadow there is no Einstein-Hilbert term in the action, so there is no dimensionful gravitational constant that would break the local Weyl symmetry explicitly. Instead, the Einstein-Hilbert term emerges from gauge fixing the Weyl symmetry within the conformal shadow (see below). This is a mechanism called ‘‘compensating fields’’ which is familiar in conventional field theory. Such structures of 2T-physics are compatible with the construction of satisfactory models of a complete theory of nature directly in $4+2$ dimensions, including the standard model [15], its generalizations with supersymmetry [16,17] or grand unification, gravity [4,5], supergravity [14], all of which lead to applications in LHC physics and cosmology [6].

The derivation of the *conformal shadow* in $3+1$ dimensions from the $4+2$-dimensional theory is described in detail in [4,5]. For the purpose of the current paper it is possible to skip this detail and start out directly in $3+1$ dimensions by requiring a local scale symmetry (Weyl symmetry). Then, the ordinary looking model in Eqs. (1) and (2) can be presented as a gauge fixed version of the following gauge invariant field theory in $3+1$ dimensions, which is a conformal shadow that contains one scalar field $s(x)$ in addition to a dilaton $\phi(x)$, both conformally coupled to gravity as follows:

$$S = \int d^4x\sqrt{-g}\Big(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g^{\mu\nu}\partial_\mu s\partial_\nu s + \frac{1}{12}(\phi^2 - s^2)R(g) - \phi^4 f\Big(\frac{s}{\phi}\Big)\Big). \quad (5)$$

The field $\phi(x)$ has the wrong sign in the kinetic term, so it is a ghost (negative norm[3]). This sign of the kinetic term is required by the Weyl symmetry if the sign in front of the curvature term $\frac{1}{12}\phi^2 R(g)$ is positive. However, due to the local scale symmetry the ghost can be gauge fixed away, so this theory is unitary. The gauge symmetry is preserved for any potential of the form $\phi^4 f(\frac{s}{\phi})$, where $f(z)$ is an arbitrary function of its argument $z = \frac{s}{\phi}$. In this action there is no Einstein-Hilbert term with a dimensionful gravitational constant but, instead, the factor $(\phi^2 - s^2)^{-1}$ plays the

[3]There are models of cosmology based on the notion of ‘‘quintom matter’’ [18], which also introduce a negative norm ghost field. We should emphasize that those models have actual ghosts and therefore are nonunitary and fundamentally flawed. Despite some similarity, our model is fundamentally different because of the Weyl symmetry that eliminates the ghosts, thus having fewer degrees of freedom. Our action, our solutions which do not violate the null energy conditions, and the discussion of the physics are also different.

role of a spacetime dependent effective "gravitational parameter."

B. Braneworld origin

A cyclic model, inspired by D-branes in M-theory [8], was developed in [2], where it was discussed for a very different potential than Eq. (2). However, it is possible to recover precisely the current model of Eqs. (1) and (2) in the colliding worldbrane scenario as follows. One should compare Eq. (27) in Ref. [9] to the model in Eq. (5) before gauge fixing the Weyl symmetry. Both models have a Weyl symmetry that is a remnant of general coordinate transformations in extra dimensions (although the extra dimensions in the two cases do not have identical signatures). One should compare our fields here $s(x)$, $\phi(x)$ to the fields $\psi^{\pm}(x)$ in Ref. [9], since they are both conformally coupled scalars and have precisely the same kinetic energy terms. Furthermore, the potential $V(\sigma)$ of Eq. (2) is also recovered, if one replaces the unknown terms in Eq. (27) of Ref. [9], $2W_{\rm CFT}[g^+] - 2W_{\rm CFT}[g^-] + S_m[g^+] + S_m[g^-]$, by just a cosmological term on each brane. Namely, replacing the unknown expression by, $b_+\sqrt{g^+} + b_-\sqrt{g^-}$, where $b_\pm$ are constants, and using their definition of $g^\pm$, gives $\sum_\pm b_\pm\sqrt{g^\pm} = \sum_\pm b_\pm\sqrt{-g}(\psi^\pm)^4$. This is indeed the potential $b\phi^4 + cs^4$ in Eq. (5), which in turn leads to the potential $V(\sigma)$ after the Weyl gauge symmetry is fixed to obtain the Einstein frame, as described in [6] and below.

C. Fixing the Weyl symmetry

The Weyl symmetry can be gauge fixed in several forms. In the Einstein gauge denoted by a label E, such as ϕ_E, s_E, $g_E^{\mu\nu}$, the gauge is fixed such that the curvature term $\frac{1}{12}\times(\phi^2 - s^2)R(g)$ becomes precisely the Einstein-Hilbert term $\frac{1}{2\kappa^2}R(g_E)$, so that in the Einstein gauge we have

$$\frac{1}{12}(\phi_E^2 - s_E^2) = \frac{1}{2\kappa^2}. \tag{6}$$

In this gauge it is convenient to parametrize ϕ_E, s_E in terms of a single scalar field σ,

$$\phi_E(x) = \pm\frac{\sqrt{6}}{\kappa}\cosh\left(\frac{\kappa\sigma(x)}{\sqrt{6}}\right),$$
$$s_E(x) = \frac{\sqrt{6}}{\kappa}\sinh\left(\frac{\kappa\sigma(x)}{\sqrt{6}}\right). \tag{7}$$

Then the gauge fixed form of the action in Eq. (5) takes precisely the form of Eq. (1), where the potential $V(\sigma)$ is arbitrary as long as the function $f(z)$ is arbitrary.

The FRW metric in this gauge takes the form

$$ds_E^2 = -dt^2 + a_E^2(t)ds_3^2 = a^2(\tau)(-d\tau^2 + ds_3^2), \tag{8}$$

$$ds_3^2 = \frac{dr^2}{1 - kr^2/r_0^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2); \qquad k = 0, \pm 1, \tag{9}$$

where ds_3^2 is the metric of the three-dimensional space, τ is the conformal time, and $a(\tau) \equiv a_E(t(\tau))$ is the cosmological scale whose dynamics we wish to study in this paper. The relation between ordinary comoving time t and the conformal time is[4]

$$dt = a(\tau)d\tau, \quad \text{or} \quad t(\tau) = \int_0^\tau a(\tau')d\tau'. \tag{11}$$

The scalar curvature of the metric in Eq. (8) is given by

$$R(g_E) = \frac{6}{a^2}\left(\frac{\ddot a}{a} + \frac{k}{r_0^2}\right), \tag{12}$$

where r_0 is a constant radius that sets the scale of the curvature of 3-space[5] when the dimensionless scale factor is $a = 1$.

Thus, in this gauge, for the purpose of homogeneous solutions of the equations of motion, the dynamical variables are $a(\tau)$ and $\sigma(\tau)$ which interact with each other as prescribed by the action (1). Their equations of motion reduce to the Friedmann equations [19] as follows:

$$\frac{\dot a^2}{a^4} = \frac{\kappa^2}{3}\left[\frac{\dot\sigma^2}{2a^2} + V(\sigma)\right] - \frac{k}{r_0^2 a^2}, \tag{13}$$

$$\frac{\ddot a}{a^3} - \frac{\dot a^2}{a^4} = -\frac{\kappa^2}{3}\left[\frac{\dot\sigma^2}{a^2} - V(\sigma)\right], \tag{14}$$

$$\frac{\ddot\sigma}{a^2} + 2\frac{\dot a\,\dot\sigma}{a^3} + V'(\sigma) = 0. \tag{15}$$

[4]In this paper the overdot denotes derivative with respect to conformal time $\dot a(\tau) \equiv da/d\tau$ and $\ddot a(\tau) = d^2a/(d\tau)^2$. The derivative with respect to comoving time t can be rewritten by using the chain rule as $\frac{d}{dt} = \frac{1}{a(\tau)}\frac{d}{d\tau}$. For example, the Hubble parameter $H \equiv \frac{1}{a_E(t)}\frac{da_E}{dt}$ and its derivative $\frac{dH}{dt} + H^2 = \frac{1}{a_E}\frac{d^2a_E}{(dt)^2}$, are expressed as

$$H = \frac{\dot a(\tau)}{a^2(\tau)}, \qquad \frac{dH}{dt} + H^2 = \frac{\ddot a}{a^3} - \frac{\dot a^2}{a^4}. \tag{10}$$

[5]The parameter r_0 sets the scale for the curvature. If normalized to today's curvature, with r_0 representing today's size of the Universe, then $K = k/r_0^2$ is extremely small even when $k \neq 0$. In that case we can completely forget the effect of spatial curvature. However, there are cosmological models that play with the curvature parameter as applied in the early stages of the Universe. In that case r_0 may be within a few factors of 10 of the Planck scale, in which case the curvature is enormous. In order not to miss possible interesting solutions we will not prejudge the size of this term and investigate the solutions that emerge. Then in various physical applications we may or may not neglect the parameter $K = k/r_0^2$.

We had previously found all the exact solutions of these equations for the potential $V(\sigma)$ given in Eq. (2) and a flat universe $k = 0$. These were tabulated in [6]. In this paper we will emphasize the subset of those solutions that are geodesically complete and in addition we will generalize them by including nonzero spatial curvature $k = \pm 1$. Further generalizations including radiation and anisotropic metrics will be given in [11]. To explain what we mean by a geodesically complete solution we need the following discussion.

D. Geodesic completeness

Note that the Einstein gauge in Eq. (6) can be chosen only in patches of spacetime x^μ when the *gauge invariant* quantity $[1 - s^2(x^\mu)/\phi^2(x^\mu)]$ is positive. This quantity may be expressed in the Einstein gauge (i.e. when it is positive only) as $(1 - s_E^2/\phi_E^2) = (\cosh(\kappa\sigma/\sqrt{6}))^{-2}$. We must note that this gauge invariant quantity could vanish at various times τ. We did in fact find that it does vanish at various values of τ in *generic* analytic solutions for $\sigma(\tau)$, $a(\tau)$ given in [6]. However, when $(1 - s_E^2/\phi_E^2)$ vanishes ϕ_E diverges so as to maintain the gauge choice for the gauge dependent quantity $(\phi^2 - s^2)$ as given in Eq. (6). But the Einstein gauge was fixed under the assumption that $(\phi^2 - s^2)$ was positive; if it can vanish can it also change sign? This is the question that initially motivated two of us [6] to study this model in the ϕ, s version, rather than the a, σ version. Will the dynamics require the *gauge invariant* quantity $(1 - s^2/\phi^2)$ to change sign in some patches of spacetime, thus creating patches with antigravity? If yes, what would that mean cosmologically for the Universe we live in?

In our previous study in [6] our exact solutions for ϕ, s showed that generically the dynamics does require the *gauge invariant* $(1 - s^2/\phi^2)$ to change sign. However, the point at which $(1 - s^2/\phi^2)$ vanishes corresponds to a big bang or a big crunch singularity where the scale factor in the Einstein gauge vanishes $a^2(\tau) = a_E(t(\tau)) = 0$ [recall $a(\tau)$ is *gauge dependent*], so the physical interpretation for our own Universe may be stopped exactly where $(1 - s^2/\phi^2)$ vanishes, and therefore the solution could be stopped artificially at that moment in conformal time τ. This is geodesically incomplete, as well as gauge dependent from the point of view of $a(\tau)$ as defined in the Einstein frame. But nevertheless, if one insists that the theory is defined only in the Einstein frame, one could accept a geodesically incomplete patch for a physical interpretation in the usual interpretation of gravity. This type of geodesically incomplete solution, which is very common in applications of gravity, was used in the application to an inflating universe we discussed in our previous paper [6].

In the current paper we take the point of view that geodesically complete solutions are more satisfactory. To discover and better understand the solution, we examine the factor $(\phi^2 - s^2)$ that multiplies $R(g)$ in the action. To overcome the gauge dependent description of the Einstein or other frames, we focus on the gauge invariant quantity $(1 - s^2/\phi^2)$. The point at which it vanishes corresponds to the big bang or big crunch. When it is positive we can choose the Einstein gauge to describe ordinary gravity, but in patches when it is negative there is antigravity. Only the geodesically complete solutions have no antigravity by having $(1 - s^2/\phi^2) \geq 0$ as a function of τ. Of course, quantum corrections near singularities may smooth out the behavior of solutions. Notwithstanding the cloudiness of our understanding of quantum gravity at this time, it still seems to us attractive to identify the geodesically complete geometries and solutions in the applications to cosmology, expecting that this feature survives the quantum effects, as seems to be the case at the level of the Wheeler-deWitt equation [11]. Hence we will identify the circumstances in which there are geodesically complete solutions in which the quantity $(1 - s^2/\phi^2)$ never changes sign.[6] We found that this is indeed possible, and we explicitly obtained those unique geodesically complete solutions that are presented in the present paper and in [11].

As we will see in the explicit solutions given below, it turns out geodesic completeness, for the *bounce at zero size*, requires two ingredients. First, the parameters in the model have to be in a certain range and satisfy certain quantization conditions. In other words not every model can yield geodesically complete cosmological solutions with zero-size bounces. For example, in the case of the flat FRW universe and in the absence of any perturbations, the ratio of the parameters b/c in the potential $V(\sigma)$ above must be in the range $-\frac{1}{4} \leq (b/c) \leq 4$ and must be quantized as in Eq. (27). This condition on b/c is relaxed in the presence of more parameters, such as curvature $(k = \pm 1)$ or radiation, but there is always one combination of parameters and initial conditions that is quantized. Second, even with the quantized parameters, initial conditions for $\phi(\tau)$, $s(\tau)$ must be synchronized with each other in order to generate geodesically complete solutions in which $(1 - s^2/\phi^2)$ never changes sign. If initial conditions are not synchronized, then $(1 - s^2/\phi^2)$ will change sign and all solutions will be geodesically incomplete; but this is what we want to avoid, and on this basis we consider the solutions with synchronized boundary conditions, namely, only the geodesically complete ones, as being those that provide a fuller story of cosmology.

We have also found exact analytic solutions that obey $\phi^2 - s^2 > 0$ at all times, in which the initial conditions *need not be synchronized or the parameters of the model need not be quantized*. Such solutions occur in the presence of spatial curvature in the closed universe $k = 1$, and provide a cyclic cosmology where the universe bounces

[6] I. Bars thanks Paul Steinhardt for stimulating discussions that led us to focus on this question.

at a *finite size*. For this to be possible the curvature needs to be large enough to compete with the potential energy $V(\sigma)$, as will be discussed in Secs. IV C and IV D.

Since $(1 - s^2/\phi^2)$ never changes sign for such solutions the physics at all times is compatible with Einstein's gravity, since then one can indeed choose the Einstein gauge, Eq. (6), at all times in such a universe.

If one takes the point of view that the theory is *defined* directly in the Einstein frame in terms of a, σ, as in Eq. (1), then the ϕ, s configuration in which $(1 - s^2/\phi^2)$ changes sign is a spurious outcome of the parametrization in terms of ϕ, s in Eq. (5). In that case all field configurations in which $(1 - s^2/\phi^2)$ is negative are excluded by definition. If one also requires geodesic completeness then the solutions we present below are the only ones that satisfy the criteria.

This avoids the question of what happens to the physics for those solutions that *are geodesically complete* in a more general sense than the Einstein frame, by allowing $(1 - s^2/\phi^2)$ to change sign. If the theory is defined at a more fundamental level (as in 2T-physics, or as in the colliding branes scenario) in which one would accept all the consequences of the action in the ϕ, s version of Eq. (5), then one must investigate the properties of those solutions as well. What we do know from our explicit solutions [6], in the cases in which initial conditions are not synchronized or parameters are not quantized, is that the quantity $(\phi^2 - s^2)$ does not remain negative after switching sign, but oscillates back to positive. So, it appears that the universe recovers from antigravity and comes back to a period of time with ordinary gravity. However, if allowed to continue its motion in complete geodesics, the sign changes back and forth again and again. Perhaps the physics appears to be all wrong during the time periods (or more generally spacetime patches) where $(\phi^2 - s^2)$ is negative, but we do not really know the physical cosmological consequences of such solutions for our own Universe. We think that it would be interesting to find out eventually the physical viability and meaning in cosmology of the more general geodesically complete solutions allowed by the action Eq. (5). So we will not throw away yet the generic solutions which were included in the list in [6], nor will we settle the associated physics questions in this paper. So, at a less ambitious level, for the moment we concentrate only on the *geodesically complete* cases that *also* satisfy $\phi^2 - s^2 \geq 0$, as required by the action Eq. (1).

II. ANALYTIC SOLUTIONS

To analyze the model in the ϕ, s version we find it useful to gauge fix the Weyl symmetry in other forms. A very useful gauge is to choose the conformal factor of the metric to be 1. We will call this the γ-gauge. In this gauge we will denote the fields with a label γ, such as ϕ_γ, s_γ, $g_\gamma^{\mu\nu}$. Then, the Robertson-Walker metric in this gauge loses the scale factor since $a_\gamma = 1$,

$$ds_\gamma^2 = -d\tau^2 + \frac{dr^2}{1 - kr^2/r_0^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (16)$$

and its curvature is a constant given by

$$R(g_\gamma) = 6K, \quad \text{with} \quad K \equiv \frac{k}{r_0^2}, \qquad k = 0, \pm 1. \quad (17)$$

In this γ-gauge there is no scale factor for the universe, but now both $\phi_\gamma(x)$, $s_\gamma(x)$ are dynamical variables, with ϕ_γ having the wrong sign in the kinetic term. The advantage of this gauge is that the dynamics of ϕ_γ, s_γ become much simpler and can be solved exactly in certain cases. After obtaining the solution one can transform back to the Einstein gauge to find $a(\tau)$, $\sigma(\tau)$. For the case of only time dependent fields the gauge fixed form of the action (5) is

$$L = \frac{1}{2}(-\dot{\phi}_\gamma^2 + \dot{s}_\gamma^2) - \frac{K}{2}(-\phi_\gamma^2 + s_\gamma^2) - \phi^4 f\left(\frac{s}{\phi}\right). \quad (18)$$

But one should also remember that τ reparametrization symmetry of general relativity requires the vanishing Hamiltonian constraint (this is the $G_{00} = T_{00}$ Einstein equation)

$$H = \frac{1}{2}(-p_\phi^2 + p_s^2) + \frac{K}{2}(-\phi_\gamma^2 + s_\gamma^2) + \phi^4 f\left(\frac{s}{\phi}\right) = 0, \quad (19)$$

where the canonical momenta are $p_\phi = -\dot{\phi}_\gamma$ and $p_s = \dot{s}_\gamma$. The negative norm ghost is eliminated because of this constraint on the phase space $(\phi_\gamma, s_\gamma, p_\phi, p_s)$. The Wheeler-deWitt (WDW) equation of our theory $H\Psi(\phi, s) = 0$ takes an interesting form in the ϕ, s basis,

$$\left(\frac{1}{2}(\partial_\phi^2 - \partial_s^2) + \frac{K}{2}(-\phi_\gamma^2 + s_\gamma^2) + \phi^4 f\left(\frac{s}{\phi}\right)\right)\Psi(\phi, s) = 0. \quad (20)$$

As a side remark, note that for $K > 0$ (closed universe) and in the absence of the potential, $\phi^4 f(\frac{s}{\phi}) = 0$ the system in Eqs. (18)–(20) describes the SO(1,1) Lorentz symmetric relativistic harmonic oscillator in $1 + 1$ dimensions, with (ϕ, s) representing the ("time," "space") directions, respectively. As in other cases of the harmonic oscillator in several dimensions, this system has a larger hidden symmetry, which is $SU(1, 1) \supset SO(1, 1)$ in this case. The quantum version of the relativistic harmonic oscillator (i.e. the WDW equation for $f(s/\phi) = 0$) was studied and solved exactly in Secs. VI, VII, and the Appendix of a recent paper [20] by using unitary representations of SU(1,1). This may be taken as a toy model to begin a study of the WDW equation for our case.[7] Of course, we are interested in the full WDW equation, including the potential energy $\phi^4 f(\frac{s}{\phi})$, radiation, and anisotropy, as will be discussed elsewhere [11].

We now turn to the classical equations of motion satisfied by ϕ_γ, s_γ, including the potential energy. Such classical solutions provide a semiclassical approximation to the WDW equation. We are interested in an exactly solvable case so that we can study the issues we raised with certainty. One of those exactly solvable cases corresponds to a special form of the potential, namely $\phi^4 f(s/\phi) = b\phi^4 + cs^4$ that in turn corresponds to the hyperbolic potential $V(\sigma)$ given in Eq. (2). The equations of motion for $\phi_\gamma(\tau)$ and $s_\gamma(\tau)$ are directly obtained from the Lagrangian or Hamiltonian given above. But it is also instructive to derive the equations directly from the Friedmann equations in Eqs. (13)–(15) by using the following connection between the γ-gauge and the Einstein gauge (derived in [6]):

$$a^2 = \frac{\kappa^2}{6}(\phi_\gamma^2 - s_\gamma^2), \qquad \sigma = \frac{\sqrt{6}}{\kappa}\frac{1}{2}\ln\left(\frac{\phi_\gamma + s_\gamma}{\phi_\gamma - s_\gamma}\right), \quad (21)$$

[7] The WDW equation $H\Psi(\phi, s) = 0$ is satisfied by an infinite set of solutions of the relativistic harmonic oscillator [20]. These are $\Psi(\phi, s) = \sum_{n=0}^{\infty} c_n \psi_n(\phi)\psi_n(s)$, where the c_n are arbitrary and $\psi_n(\phi)$, $\psi_n(s)$ are the standard one-dimensional harmonic oscillator wave functions that satisfy the eigenvalue equations $\frac{1}{2}(-\partial_\phi^2 + K\phi^2)\psi_n(\phi) = E_n\psi_n(\phi)$ and $\frac{1}{2}(-\partial_s^2 + Ks^2)\psi_n(s) = E_n\psi_n(s)$, where $E_n = \sqrt{K}(n + \frac{1}{2})$. For all these solutions $\Psi(\phi, s)$ has an overall Gaussian factor $\exp[-\frac{\sqrt{K}}{2}(\phi^2 + s^2)]$ times a polynomial. This shows that the probability distribution $|\Psi(\phi, s)|^2$ for these generic solutions is *not purely "timelike"* (not $\phi^2 > s^2$), but rather it covers both "timelike" and "spacelike" regions in (ϕ, s) space. This is not surprising since for generic boundary conditions the classical equations also do not obey $\phi^2(\tau) - s^2(\tau) \geq 0$ at all times. Only special boundary conditions with synchronized phases for $\phi(\tau)$, $s(\tau)$ at $\tau = 0$ can yield classical solutions that have this property, as we have explained in the text. Similarly, the relativistic harmonic oscillator has just one quantum state whose probability distribution is concentrated in the timelike region $\phi^2 \geq s^2$, has a damping factor of the form $\exp(-(\phi^2 - s^2))$, and vanishes on the light cone $\phi^2 = s^2$. This was given in the Appendix of [20] (interchange spacelike with timelike in that appendix). This solution is the timelike singlet of SU(1,1) [20]. There is also a separate "spacelike singlet of SU(1,1)," while the other generic solutions correspond to a superposition of other nonsinglet unitary representations of SU(1,1) [20]. Referring to the comments of the last two paragraphs in Sec. I D, if the fundamental action is in the Einstein frame [as in Eq. (1)], then only the timelike SU(1,1) singlet is acceptable as a solution of the WDW equation. If, on the other hand, the starting point is more general [as in action (5)], then in order to favor only the solutions that are consistent with $\phi^2 - s^2 > 0$, the model should have an additional ingredient. This could be an appropriate potential energy term, effects of radiation, curvature, etc., or some appropriate constraint that is natural in the model. This would then characterize the "right" model.

which gives

$$V(\sigma) = \frac{6^2}{\kappa^4}\frac{b\phi_\gamma^4 + cs_\gamma^4}{(\phi_\gamma^2 - s_\gamma^2)^2}. \quad (22)$$

Inserting these in the Friedmann equations we obtain the equations for ϕ, s as follows:

$$0 = \ddot{\phi}_\gamma - 4b\phi_\gamma^3 + K\phi_\gamma, \quad (23)$$

$$0 = \ddot{s}_\gamma + 4cs_\gamma^3 + Ks_\gamma, \quad (24)$$

$$0 = \left(\frac{1}{2}\dot{\phi}_\gamma^2 - b\phi_\gamma^4 + \frac{1}{2}K\phi_\gamma^2\right) - \left(\frac{1}{2}\dot{s}_\gamma^2 + cs_\gamma^4 + \frac{1}{2}Ks_\gamma^2\right). \quad (25)$$

The important observation is that the second order equations for ϕ, s decouple from each other, so they are exactly solvable. The third equation simply states that the energy of the ϕ solution must be matched to the energy of the s solution $E_\phi = E_s$. Once the solution is obtained it is transformed back to the Einstein frame by using Eqs. (21), thus providing the desired solutions for $a(\tau)$, $\sigma(\tau)$ in the Einstein frame. The general generic solutions of these equations, for all possible ranges of the parameters and boundary conditions are listed in [6] for the $K = 0$ case. The solutions are expressed in terms of Jacobi elliptic functions[8] as given in [6] and below.

Now we focus on the subset of solutions that satisfies the criteria we laid out. When the spatial curvature of the FRW universe is zero (i.e. $k = 0$), we found that geodesically complete solutions, with $(\phi^2 - s^2) \geq 0$ at all times, can occur only when the ratio b/c takes on the following quantized values in the range $-\frac{1}{4} \leq \frac{b}{c} \leq 4$, with c positive, $c > 0$, and

[8] The Jacobi elliptic functions that we need for our solutions are denoted as $sn(z|m)$, $cn(z|m)$, $dn(z|m)$. These are periodic functions that have properties similar to trigonometric functions. The following formulas [21] are directly useful to verify our solutions explicitly. The derivatives of Jacobi elliptic functions are given in terms of expressions somewhat similar to those for trigonometric functions,

$$\frac{d}{dz}sn(z|m) = cn(z|m) \times dn(z|m), \quad (26)$$

$$\frac{d}{dz}cn(z|m) = -sn(z|m) \times dn(z|m), \quad (27)$$

$$\frac{d}{dz}dn(z|m) = -m \times sn(z|m) \times cn(z|m). \quad (28)$$

They also satisfy quadratic relations, such as

$$(sn(z|m))^2 + (cn(z|m))^2 = 1; \qquad m(sn(z|m))^2 + (dn(z|m))^2 = 1. \quad (29)$$

$$b = \begin{cases} \frac{4c}{n^4} \\ 0 \\ -\frac{c}{(n+1)^4} \end{cases}, \quad \text{with } n = 1, 2, 3, \ldots. \tag{30}$$

Note that each value of n defines a given model. The explicit solutions are given in Eqs. (32), (41), and (49) for $b > 0$, $b < 0$, and $b = 0$, respectively. When additional parameters, such as curvature, radiation, etc., are included in the model, then this quantization condition on the model is relaxed, but still some combination of parameters and integration constants must be quantized as we will discuss. Furthermore, if the spatial curvature is sufficiently large, so that the curvature term in the action can compete with the potential, then we find, geodesically complete, finite, bouncing solutions of a cyclic universe where the bounce occurs at a minimum finite size of the universe. For such cases there is no quantization condition on the parameters of the model but, instead, the initial conditions on the fields must be within a certain range defined by those parameters.

Before we give the mathematical details, we first explain how the physics is easily captured by interpreting these decoupled equations in terms of an analog mechanical problem of a particle moving in a potential. In the case of ϕ the Hamiltonian is $H(\phi) = \frac{1}{2}\dot{\phi}^2 + V(\phi)$, with $V(\phi) = \frac{1}{2}K\phi^2 - b\phi^4$, while in the case of s the Hamiltonian is $H(s) = \frac{1}{2}\dot{s}^2 + V(s)$, with $V(s) = \frac{1}{2}K\phi^2 + cs^4$. According to Eq. (25) the only acceptable solutions for ϕ, s are the ones that satisfy

$$H(\phi(\tau)) = H(s(\tau)) = E. \tag{31}$$

The corresponding potentials are depicted in Figs. 1, 22, and 27 for the cases of $k = 0$, ± 1 We have included the cases of positive b [heavy solid curve $V(\phi)$] and negative b [dashed curve $V(\phi)$]. We have drawn the pictures for only positive c [solid thin curve $V(s)$] while for negative c the $V(s)$ curve is reflected from the horizontal axis in each figure.

These figures, combined with the physical intuition of a particle in potential, capture the physical aspects of our solutions. We approach the mathematical analysis systematically for each figure and investigate the various ranges of the parameters b, c, K and the integration parameter E. We will start with the simplest case of zero curvature and analyze it thoroughly in Sec. III. We will then discuss the positive and negative curvature cases separately in Secs. IV and V, respectively.

III. THE FLAT ($k = 0$) FRW UNIVERSE

We first discuss the flat case ($k = 0$) that has the fewest parameters. For $k = 0$ the only possible solutions are for $E_s = E_\phi > 0$ as shown by the horizontal dashed line in Fig. 1. In the case of s, the potential is an infinite positive well [$V(s) = cs^4$ with $c > 0$], therefore the particle is trapped in the well, and $s(\tau)$ oscillates back and forth

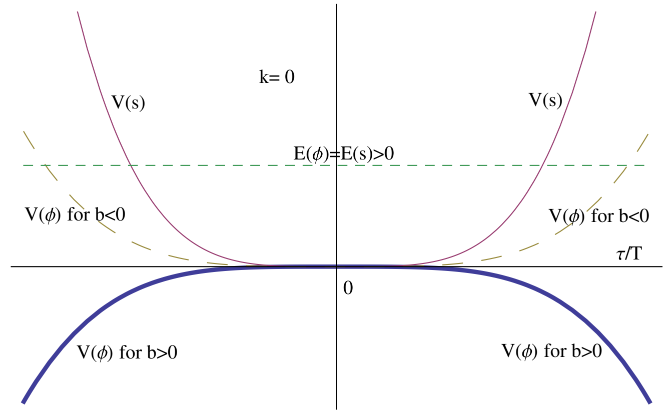

FIG. 1 (color online). The flat FRW universe, $k = 0$.

between turning points $-s_0(E_s) < s(\tau) < s_0(E_s)$ given by $cs_0^4 = E_s$. In the case of ϕ, if b is negative [dashed curve $V(\phi) = -b\phi^4$ with $b < 0$] its behavior is similar to the one just described for s, so $\phi(\tau)$ also oscillates back and forth $-\phi_0(E_\phi) < \phi(\tau) < \phi_0(E_\phi)$ at the energy level $E_\phi = E_s$. But if b is positive, then the particle is in an inverted well [heavy curve $V(\phi) = -b\phi^4$ with $b > 0$]. So, at the energy level $E_\phi = E_s$, which is higher than the peak of the hill, the particle will come up the hill from $\phi = -\infty$, go over the top of the hill, and slide down the hill to infinitely large positive values of ϕ. The trip may also happen in reverse direction depending on initial conditions. It turns out that the trip from $\phi = -\infty$ to $\phi = \infty$ is completed in a finite amount of conformal time τ, so allowing all possible values of conformal time, $\phi(\tau)$ repeats the trip periodically again and again by jumping from $\phi = \infty$ to $\phi = -\infty$. Such solutions (given analytically in [6]) solve all the equations but do not yet address the sign of $\phi^2(\tau) - s^2(\tau)$.

In order to have the oscillation amplitude of ϕ to be larger than the amplitude of s it is necessary to have $-\frac{c}{4} < b < 4c$ (consistent with the curves as drawn in the figure). The lower bound $-\frac{c}{4} < b$ for negative b, is partially understood from the figure which shows that the turning point for ϕ should be at a greater distance from the origin as compared to the turning point for s. However the $\frac{1}{4}$ factor in $-\frac{c}{4} < b$ and the upper bound $b < 4c$ for positive b emerge from the details of the solutions in Eqs. (32), (41), and (49). This is a constraint on the model. If the potential energy $V(\sigma)$ does not satisfy this property it will not be possible to maintain $\phi^2(\tau) \geq s^2(\tau)$ at all times. In addition, to insure $\phi^2(\tau) \geq s^2(\tau)$, we must (i) synchronize the initial conditions of the ϕ, s particles at the origin at $\tau = 0$, namely $\phi(0) = s(0) = 0$; and (ii) also require that their periods are commensurate, so that at the time τ when ϕ returns back to zero s also returns to zero at the same time (although s could make several returns to zero in the meantime). Commensurate periods can be arranged only by quantizing the parameter b/c. This too is a condition on the model. If the quantization is not satisfied in the model $\phi^2(\tau) - s^2(\tau)$ will change sign periodically as a function of time. But when the potential $V(\sigma)$ satisfies the required conditions the model yields geodesically complete solutions in which $\phi^2(\tau) - s^2(\tau)$ never changes sign, but periodically touches zero, which corresponds to a big crunch smoothly followed by a big bang. This is just the solution we sought as given in Eqs. (32), (41), and (49). We see that the model has to be right to be able to yield such a solution.

A. $b > 0$ case

The solutions that satisfy this description are a subset of those in [6] and explicitly given by the following expressions. For positive b, c, the only geodesically complete solution occurs for the quantized values of $b = 4c/n^4$, with $n = 1, 2, 3, \ldots$, as follows:

$$\phi_\gamma(\tau) = \frac{\kappa n}{\sqrt{48c}T} \frac{sn(\frac{2\tau}{nT}|\frac{1}{2})}{1 + cn(\frac{2\tau}{nT}|\frac{1}{2})},$$
$$s_\gamma(\tau) = \frac{\kappa}{\sqrt{48c}T} \frac{sn(\frac{\tau}{T}|\frac{1}{2})}{dn(\frac{\tau}{T}|\frac{1}{2})}. \tag{32}$$

Here the Jacobi elliptic functions, $sn(z|m)$ etc. (see footnote 8), appear only for the case of the parameter $m = 1/2$. The energy level $E_\phi = E_s$ is parametrized in terms of the parameter T which provides a scale for conformal time, as

$$E_\phi = E_s = \frac{1}{16cT^4}, \tag{33}$$

where T (or $E_s = E_\phi$) is one of the integration parameters that appears in integrating the differential equations. Note that this T is related to the overall factor in Eq. (32) that determines the amplitude of oscillations of $s_\gamma(\tau)$.

It is easy to verify that these are solutions of Eqs. (23)–(25) by using the properties of Jacobi elliptic functions given in footnote 8. The plot of these functions in Fig. 2 conveys their periodic properties and shows how $\phi_\gamma^2(\tau) \geq s_\gamma^2(\tau)$ at all times. The quantum $n = 5$ chosen for this figure corresponds to the ratio of the periods of ϕ versus s. The times at which ϕ and s vanish together $\phi_\gamma(\tau) = s_\gamma(\tau) = 0$ are the only times when $\phi_\gamma^2(\tau) = s_\gamma^2(\tau)$, at which point the universe goes through smoothly from a big crunch to a big bang. At an intermediate time $\tau = \tau_{\rm turn}$ the quantity $\phi_\gamma^2(\tau) - s_\gamma^2(\tau)$ attains a maximum; this is the turnaround point at which the universe stops expanding and begins contracting. These features are seen in Fig. 3 for the scale factor $a(\tau)$, and the scalar $\sigma(\tau)$ which are given by Eqs. (21).

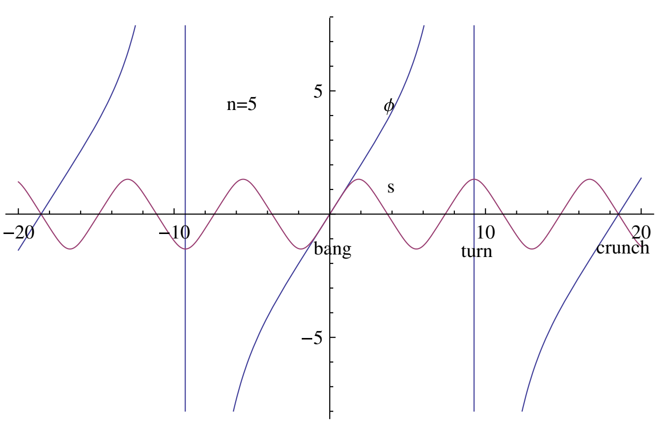

FIG. 2 (color online). $\phi_\gamma(\tau)$ and $s_\gamma(\tau)$ plotted for $n = 5$, $\kappa = \sqrt{6}$, $T = 1$, $c = 1/8$.

A parametric plot for $\phi_\gamma(\tau)$, $s_\gamma(\tau)$ is given in Fig. 4, with ϕ on the horizontal and s on the vertical. This captures similar information to Fig. 2. It is for the model $b/c = 4/n^4$ with $n = 5$, which leads to the five nodes in the figure. The time after the first node is a fast inflation period, as seen also from Fig. 3. In a semiclassical approach to the Wheeler-deWitt equation, the curve shows the region in (ϕ, s) space where the WDW wave function $\Psi(\phi, s)$ is expected to have the largest probability. This is the unique curve (for the $n = 5$ model) purely in the timelike region $\phi^2(\tau) - s^2(\tau) > 0$ in (ϕ, s) space. The corresponding WDW wave function is the analog of the "timelike SU(1,1) singlet" in footnote 7.

Other quantities of interest to convey the properties of the solution include the Hubble parameter $H = \frac{\dot{a}}{a^2}$ (see footnote 4), the kinetic energy of the σ field, $K(\tau) = \frac{\dot{\sigma}^2}{2a^2}$, its potential energy $V(\sigma(\tau))$, and the equation of state parameter given by $w(\tau) = (K(\tau) - V(\tau))/(K(\tau) + V(\tau))$. Their plots appear in Figs. 5–7.

The Hubble parameter decreases from infinity at the big bang, quickly approaching a constant at the turnaround (with a few small ripples depending on n), switches to

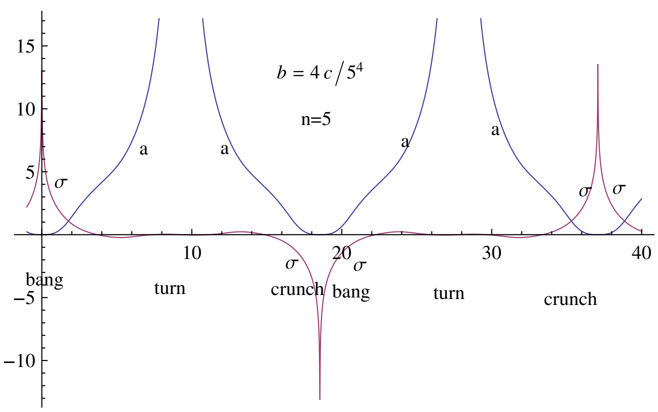

FIG. 3 (color online). $a(\tau)$ and $\sigma(\tau)$ plotted for $n = 5$, $\kappa = \sqrt{6}$, $T = 1$, $c = 1/8$.

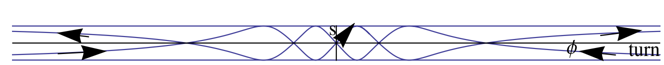

FIG. 4 (color online). The arrow at the origin marks the crunch/bang moments and the arrows at the ends mark the turnarounds.

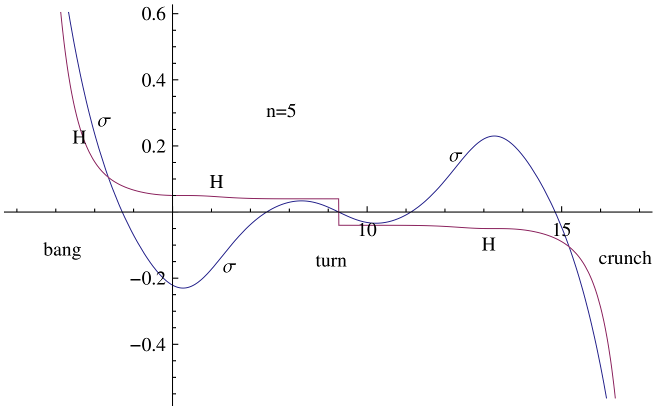

FIG. 5 (color online). Hubble parameter $H(\tau)$ and $\sigma(\tau)$ for $n = 5$.

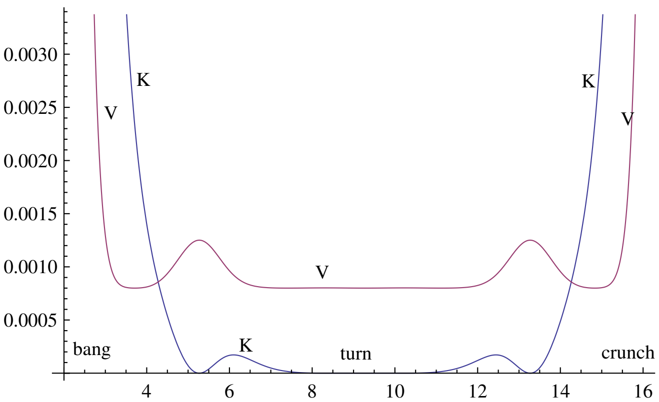

FIG. 6 (color online). Kinetic $K(\tau) = \dot{\sigma}^2/2a^2$ and potential energies $V(\sigma(\tau))$ of the σ field, for $n = 5$.

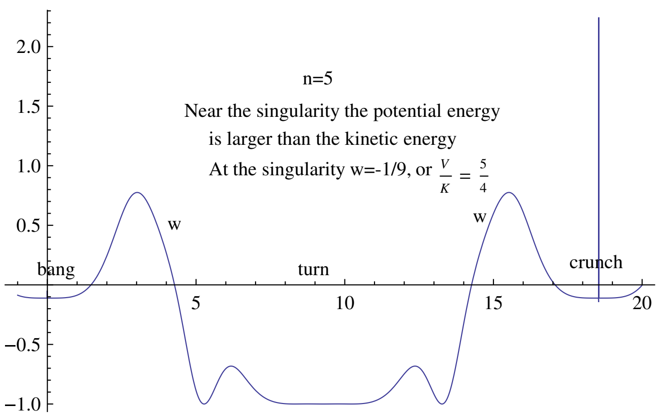

FIG. 7 (color online). The equation of state $w = (K - V)/(K + V)$, for $n = 5$.

negative at turnaround, and then slowly reaches negative infinity at the big crunch.

The potential and kinetic energies of the σ field are fairly close to each other in magnitude most of the time. At the turnaround the kinetic energy vanishes $K(\tau_{\rm turn}) = 0$ while the potential energy is a constant $V(\tau_{\rm turn}) = 6^2 b/\kappa^4$ (since $\sigma = 0$ at turnaround). Both $K(0)$ and $V(0)$ are infinite at the bang or crunch, but $V(0)$ is larger at the singularity since $w(0) = -1/9$ as seen in Fig. 7.

The behavior of various quantities near the bang/crunch singularity is better understood by studying the Taylor expansion near $\tau \to 0$ for any value of n as follows:

$$a(\tau) \to \frac{\kappa}{T\sqrt{48c}}\left(\frac{\tau}{T}\right)^3 \frac{\sqrt{4+n^4}}{2\sqrt{5}n^2}\left[1 + \frac{(n^2-4)}{60n^2}\left(\frac{\tau}{T}\right)^4 + O\left(\frac{\tau}{T}\right)^8\right], \tag{34}$$

$$\frac{\dot{a}(\tau)}{a(\tau)} \to \frac{1}{T}\left(\frac{\tau}{T}\right)^{-1}\left[3 - \frac{n^4-4}{15n^4}\left(\frac{\tau}{T}\right)^4 + O\left(\frac{\tau}{T}\right)^8\right], \tag{35}$$

$$H(\tau) \to \frac{\sqrt{48c}}{\kappa}\left(\frac{\tau}{T}\right)^{-4}\left[\frac{6\sqrt{5}n^2}{\sqrt{4+n^4}} - \frac{\sqrt{5}(n^4-4)}{30n^2\sqrt{4+n^4}}\left(\frac{\tau}{T}\right)^4 + O\left(\frac{\tau}{T}\right)^8\right], \tag{36}$$

$$\sigma(\tau) \to \frac{\sqrt{6}}{\kappa}\left[-\ln\left(\frac{\tau}{T}\right)^2 + \frac{1}{2}\ln\frac{80n^4}{4+n^4} + \frac{n^4-4}{240n^4}\left(\frac{\tau}{T}\right)^4 + O\left(\frac{\tau}{T}\right)^8\right], \tag{37}$$

$$\dot{\sigma}(\tau) \to \frac{\sqrt{6}}{\kappa T}\left(\frac{\tau}{T}\right)^{-1}\left[-2 + \frac{n^4-4}{60n^4}\left(\frac{\tau}{T}\right)^4 + O\left(\frac{\tau}{T}\right)^8\right], \tag{38}$$

$$V(\sigma(\tau)) \to \frac{288c}{\kappa^4}\left(\frac{\tau}{T}\right)^{-8}\left[\frac{50n^4}{(4+n^4)} - \frac{5(n^4-4)}{3(n^4+4)}\left(\frac{\tau}{T}\right)^4 + O\left(\frac{\tau}{T}\right)^8\right], \tag{39}$$

$$K(\sigma(\tau)) \to \frac{288c}{\kappa^4}\left(\frac{\tau}{T}\right)^{-8}\left[\frac{40n^4}{(4+n^4)} + \frac{2(n^4-4)}{3(n^4+4)}\left(\frac{\tau}{T}\right)^4 + O\left(\frac{\tau}{T}\right)^8\right]$$

$$w(\tau) \to -\frac{1}{9} + \frac{2(n^4-4)}{81n^4}\left(\frac{\tau}{T}\right)^4 + O\left(\frac{\tau}{T}\right)^8. \tag{40}$$

The last expression shows that $w = -1/9$ at the singularity for all values of n. This behavior seems to be surprising according to common lore.

We emphasize that this behavior near the singularity is only for our geodesically complete analytic solutions that satisfy both the relative quantization of their periods as well as the synchronization of the initial conditions. If either of these is not satisfied (i.e. for nongeodesically complete solutions in only the Einstein frame) the behavior near the singularity is radically different. This behavior seems to be of measure zero in the space of all solutions. In our next paper [11] we will further analyze this issue including the effects of anisotropy and the quantum effects via the Wheeler-deWitt equation.

B. $b < 0$ case

We repeat the same type of analysis for $b < 0$ which refers to Fig. 1 with the dashed curve representing $V(\phi)$. Geodesically complete solutions occur only if b/c has one of the quantized values $b/c = -1/(n+1)^4$, with $n = 1, 2, 3, \ldots$. Then the unique solution is

$$\phi_\gamma(\tau) = \frac{\kappa(n+1)}{\sqrt{48c}T}\frac{sn(\frac{\tau}{(n+1)T}|\frac{1}{2})}{dn(\frac{\tau}{(n+1)T}|\frac{1}{2})},$$
$$s_\gamma(\tau) = \frac{\kappa}{\sqrt{48c}T}\frac{sn(\frac{\tau}{T}|\frac{1}{2})}{dn(\frac{\tau}{T}|\frac{1}{2})}. \quad (41)$$

We provide a few plots similar to the ones in the previous subsection for the model $n = 5$. Despite some similarities, there are notable differences in the behavior as compared to the $b > 0$ case of the previous section as indicated in the following comments.

From Figs. 8 and 9 we see that the universe grows up to a maximum finite size before it turns around. Another way of plotting the information in Fig. 8 is the parametric plot for $\phi(\tau)$, $s(\tau)$ in Fig. 8; note the five nodes corresponding to $n = 5$. As in the previous case, this figure is associated with the semiclassical probability distribution of the Wheeler-deWitt wave function in (ϕ, s) space.

From Figs. 11 and 12 we see that there are temporary inflation periods during which $H(\tau)$ is temporarily almost a constant, and the acceleration[9] is positive, as seen in Fig. 27. The number of such temporary acceleration periods is determined by n.

It is interesting to speculate on whether this could be a mechanism to explain the current accelerated inflation state of our Universe; namely, could it be that we currently are in such a period which is inflationary only temporarily on the scale of the lifetime of the Universe?

The energy of the σ field is small except near the bang/crunch where it is infinite. The equation of state $w(\tau)$ grows to infinity at turnaround $w(\tau_{\rm turn}) = \infty$, while it takes the value $w(0) = -1/9$ at the bang/crunch where $V(0)/K(0) = 5/4$, while $V(0)$, $K(0)$ are both infinite.

The behavior of various quantities near the bang/crunch singularity is given by the Taylor expansion near $\tau \to 0$ for any value of n as follows:

$$a(\tau) \to \frac{\kappa}{T\sqrt{48c}}\left(\frac{\tau}{T}\right)^3 \frac{\sqrt{(n+1)^4 - 1}}{2\sqrt{5}(n+1)^2} \times \left[1 - \left(\frac{\tau}{T}\right)^4 \frac{(n+1)^4 + 1}{60(n+1)^4} + O\left(\frac{\tau}{T}\right)^8\right], \quad (42)$$

[9]The acceleration is defined in the Einstein frame as $\frac{d^2 a_E(t)}{dt^2}$. This may be written in terms of conformal time as $\frac{1}{a}\partial_\tau(\frac{1}{a}\partial_\tau a)$. For the purpose of the plot we have defined the quantity "acc" as the acceleration divided by an extra factor of a.

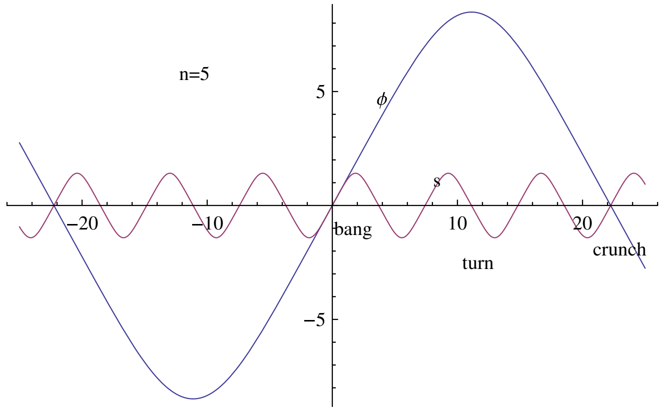

FIG. 8 (color online). $\phi(\tau)$, $s(\tau)$ have finite amplitudes.

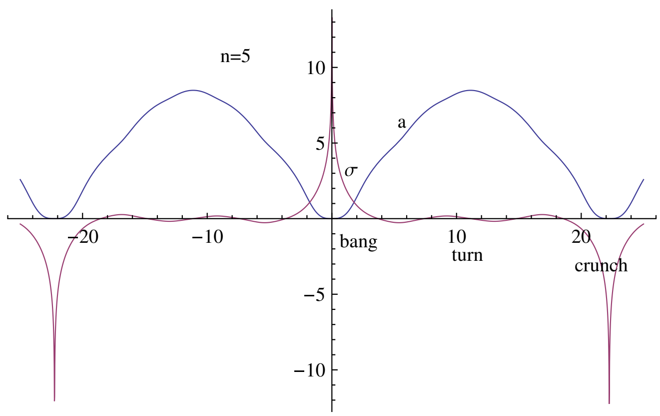

FIG. 9 (color online). $a(\tau)$ has a finite maximum.

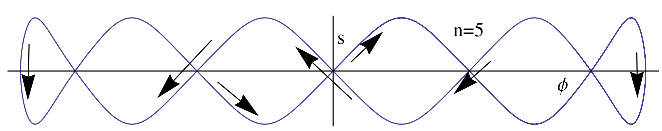

FIG. 10 (color online). Crunch/bang is at the origin and turn-around at the edges.

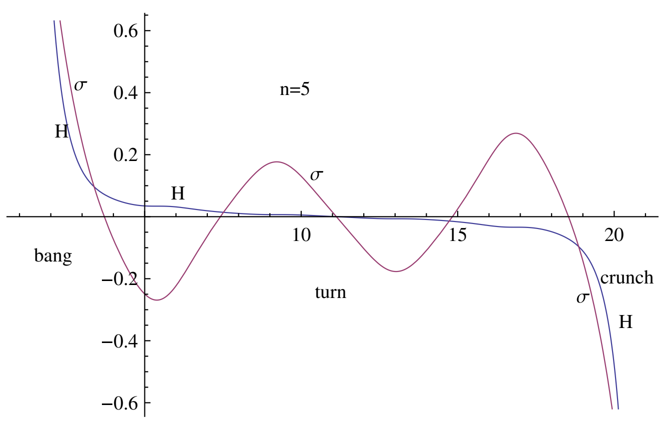

FIG. 11 (color online). Behavior of $\sigma(\tau)$, $H(\tau)$.

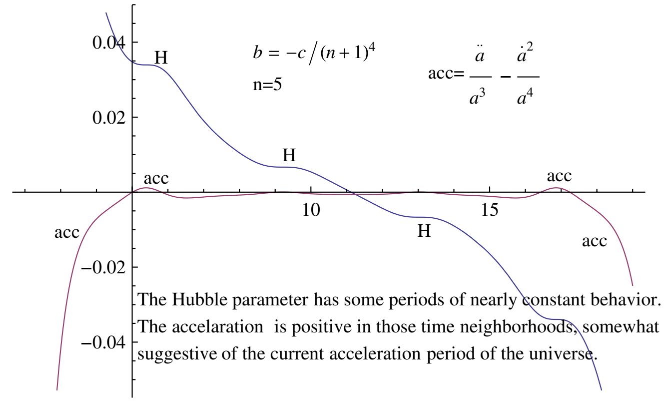

FIG. 12 (color online). Temporary inflation periods.

$$\frac{\dot{a}(\tau)}{a(\tau)} \rightarrow \frac{1}{T}\left(\frac{\tau}{T}\right)^{-1}\left[3-\frac{(n+1)^4+1}{15(n+1)^4}\left(\frac{\tau}{T}\right)^4+O\left(\frac{\tau}{T}\right)^8\right], \tag{43}$$

$$H(\tau)\rightarrow\frac{\sqrt{48c}}{\kappa}\left(\frac{\tau}{T}\right)^{-4}\frac{6\sqrt{5}(n+1)^2}{\sqrt{(n+1)^4-1}} \times\left[1-\frac{(n+1)^4+1}{180(n+1)^4}\left(\frac{\tau}{T}\right)^4+O\left(\frac{\tau}{T}\right)^8\right], \tag{44}$$

$$\sigma(\tau)\rightarrow\frac{\sqrt{6}}{\kappa}\left[-\ln\left(\frac{\tau}{T}\right)^2+\frac{1}{2}\ln\frac{80(n+1)^4}{(n+1)^4-1} +\frac{(n+1)^4+1}{240(n+1)^4}\left(\frac{\tau}{T}\right)^4+O\left(\frac{\tau}{T}\right)^8\right], \tag{45}$$

$$\dot{\sigma}(\tau)\rightarrow\frac{\sqrt{6}}{\kappa T}\left(\frac{\tau}{T}\right)^{-1}\left[-2+\frac{(n+1)^4+1}{60(n+1)^4}\left(\frac{\tau}{T}\right)^4+O\left(\frac{\tau}{T}\right)^8\right], \tag{46}$$

$$V(\sigma(\tau))\rightarrow\frac{288c}{\kappa^4}\left(\frac{\tau}{T}\right)^{-8}\left[\frac{50(n+1)^4}{((n+1)^4-1)} -\frac{5}{3}\frac{(n+1)^4+1}{((n+1)^4-1)}\left(\frac{\tau}{T}\right)^4+O\left(\frac{\tau}{T}\right)^8\right], \tag{47}$$

$$K(\sigma(\tau))\rightarrow\frac{288c}{\kappa^4}\left(\frac{\tau}{T}\right)^{-8}\left[\frac{40(n+1)^4}{((n+1)^4-1)} +\frac{2}{3}\frac{(n+1)^4+1}{((n+1)^4-1)}\left(\frac{\tau}{T}\right)^4+O\left(\frac{\tau}{T}\right)^8\right]$$
$$w(\tau)\rightarrow-\frac{1}{9}+\frac{2}{81}\frac{(n+1)^4+1}{60(n+1)^4}\left(\frac{\tau}{T}\right)^4+O\left(\frac{\tau}{T}\right)^8. \tag{48}$$

C. $b = 0$ case

Finally, for vanishing $b = 0$, the solution corresponds to the $n \rightarrow \infty$ limit of either the positive or negative b branches, and is given by

$$\phi_\gamma(\tau)=\frac{\kappa}{\sqrt{48c}T}\frac{\tau}{T}, \quad s_\gamma(\tau)=\frac{\kappa}{\sqrt{48c}T}\frac{sn(\frac{\tau}{T}|\frac{1}{2})}{dn(\frac{\tau}{T}|\frac{1}{2})}. \tag{49}$$

We provide a few plots similar to the ones in the previous subsections.

These plots correspond to the $n = \infty$ limit of the previous plots for either $b > 0$ or $b < 0$. Therefore their interpretation is similar to the discussion above.

The temporary acceleration periods persist.

The behavior of the energy, pressure and the equation of state are indicated on the figures.

The behavior of various quantities near the bang/crunch singularity is given by the Taylor expansion near $\tau \rightarrow 0$ as follows. These agree with the $n = \infty$ limit of the previous cases,

$$a(\tau)\rightarrow\frac{\kappa}{T\sqrt{48c}}\left(\frac{\tau}{T}\right)^3\frac{1}{2\sqrt{5}}\left[1-\frac{1}{60}\left(\frac{\tau}{T}\right)^4+O\left(\frac{\tau}{T}\right)^8\right], \tag{50}$$

$$\frac{\dot{a}(\tau)}{a(\tau)}\rightarrow\frac{1}{T}\left(\frac{\tau}{T}\right)^{-1}\left[3-\frac{1}{15}\left(\frac{\tau}{T}\right)^4+O\left(\frac{\tau}{T}\right)^8\right], \tag{51}$$

$$H(\tau)\rightarrow\frac{\sqrt{48c}}{\kappa}\left(\frac{\tau}{T}\right)^{-4}\left[6\sqrt{5}1-\frac{1}{6\sqrt{5}}\left(\frac{\tau}{T}\right)^4+O\left(\frac{\tau}{T}\right)^8\right], \tag{52}$$

$$\sigma(\tau)\rightarrow\frac{\sqrt{6}}{\kappa}\left[-\ln\left(\frac{\tau}{T}\right)^2+\frac{1}{2}\ln 80+\frac{1}{240}\left(\frac{\tau}{T}\right)^4+O\left(\frac{\tau}{T}\right)^8\right], \tag{53}$$

$$\dot{\sigma}(\tau)\rightarrow\frac{\sqrt{6}}{\kappa T}\left(\frac{\tau}{T}\right)^{-1}\left[-2+\frac{1}{60}\left(\frac{\tau}{T}\right)^4+O\left(\frac{\tau}{T}\right)^8\right], \tag{54}$$

$$V(\sigma(\tau))\rightarrow\frac{288c}{\kappa^4}\left(\frac{\tau}{T}\right)^{-8}\left[50-\frac{5}{3}\left(\frac{\tau}{T}\right)^4+O\left(\frac{\tau}{T}\right)^8\right], \tag{55}$$

$$K(\sigma(\tau))\rightarrow\frac{288c}{\kappa^4}\left(\frac{\tau}{T}\right)^{-8}\left[40+\frac{2}{3}\left(\frac{\tau}{T}\right)^4+O\left(\frac{\tau}{T}\right)^8\right]$$
$$w(\tau)\rightarrow-\frac{1}{9}+\frac{2}{81}\left(\frac{\tau}{T}\right)^4+O\left(\frac{\tau}{T}\right)^8. \tag{56}$$

IV. THE CLOSED ($k = +1$) FRW UNIVERSE

For $k = +1$ the system of equations is Eqs. (23)–(25). This amounts to the motion of two particles ϕ, s satisfying the equations of motion derived from Hamiltonians

$$H(\phi)=\frac{1}{2}\dot{\phi}^2+V_b(\phi), \qquad H(s)=\frac{1}{2}\dot{s}^2+V_c(s), \tag{57}$$

with

$$V_c(s)=\frac{1}{2}Ks^2+cs^4, \quad \text{and} \quad V_b(\phi)=\frac{1}{2}K\phi^2-b\phi^4, \tag{58}$$

as plotted in Fig. 22 for $K = +1/r_0^2$, and whose energies are constrained by

$$H_\phi = H_s. \tag{59}$$

The motion changes character depending on whether $E_\phi = E_s$ is larger or smaller than the peak of the inverted double well in Fig. 22, i.e. the maximum of $V_{b>0}(\phi)$. This critical value is given by

$$E^* = \frac{K^2}{16b}. \tag{60}$$

Therefore we need to discuss separately the high and low energy levels $E_\phi = E_s$ above and below this critical value as shown in Fig. 22.

A. Higher level $E > E^*$, and $b > 0$ or $b < 0$

For the *higher* level of $E_s = E_\phi > E^*$, the intuitive physics discussion works in exactly the same way as the discussion for the $k = 0$ case at the beginning of Sec. III, for both $b \geq 0$ or $b \leq 0$. In the case of $b > 0$, the particle $s_\gamma(\tau)$ is trapped in an infinite well and oscillates between turning points $\pm s_0(E)$, while $\phi_\gamma(\tau)$ oscillates from minus infinity to plus infinity. In the case of $b < 0$ both particles are trapped in infinite wells, so they oscillate between turning points $\pm\phi_0(E)$ and $\pm s_0(E)$, respectively. The turning points $\pm\phi_0(E)$, $\pm s_0(E)$ are the points where the curves $V_c(s)$, $V_{b<0}(\phi)$ intersect the horizontal curve $E_\phi = E_s = E$ as seen in the figure.

Hence for all cases $b \geq 0$ or $b \leq 0$ at the *higher* energy level of $E_s = E_\phi$, the *geodesically complete* motion is described by plots of $\phi_\gamma(\tau)$, $s_\gamma(\tau)$ that are similar in character to the $k = 0$ case given in Figs. 1–21). In particular for very small curvature K (large values of the curvature radius r_0), the $K \neq 0$ plots should approach the $K = 0$ plots. Therefore, we will not include the $K \neq 0$ plots here.

The exact solutions for the *higher* energy level are denoted as $s_\gamma^+(\tau)$, $\phi_\gamma^+(\tau)$, where the superscript $+$ refers to $E > E^*$ with $E_s = E_\phi = E$. The solution $s_\gamma^+(\tau)$ is given by

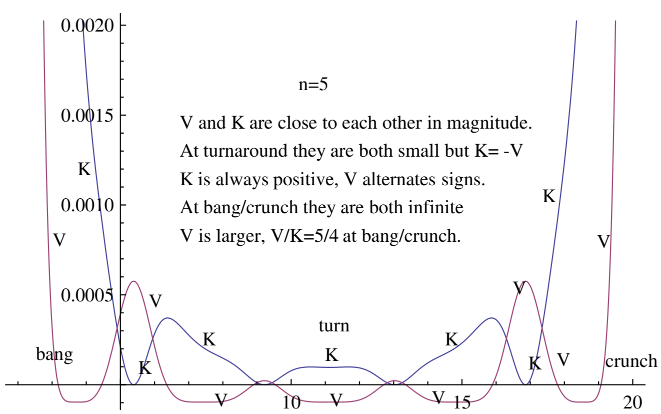

FIG. 13 (color online). Energy components of σ field.

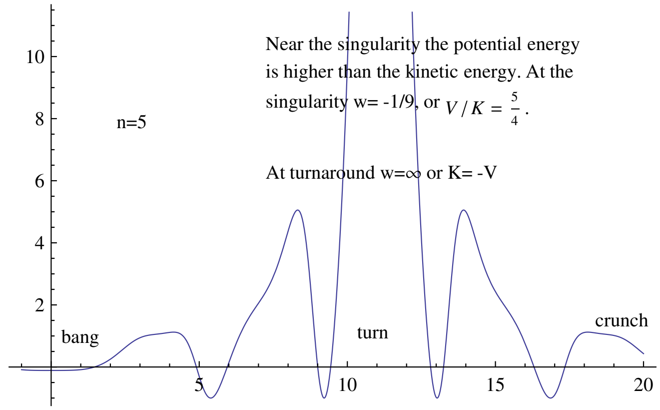

FIG. 14 (color online). Equation of state $w(\tau)$.

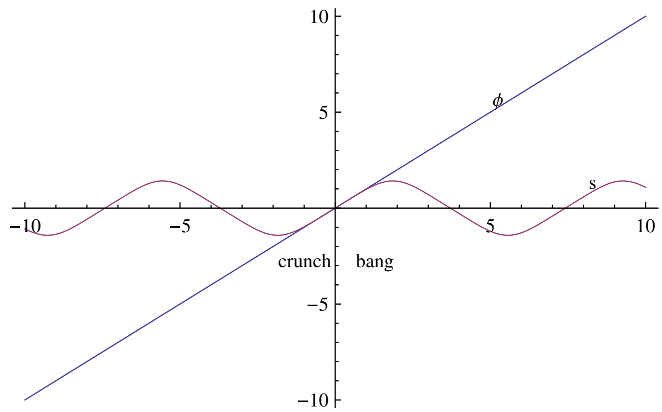

FIG. 15 (color online). $\phi(\tau)$ grows linearly with τ.

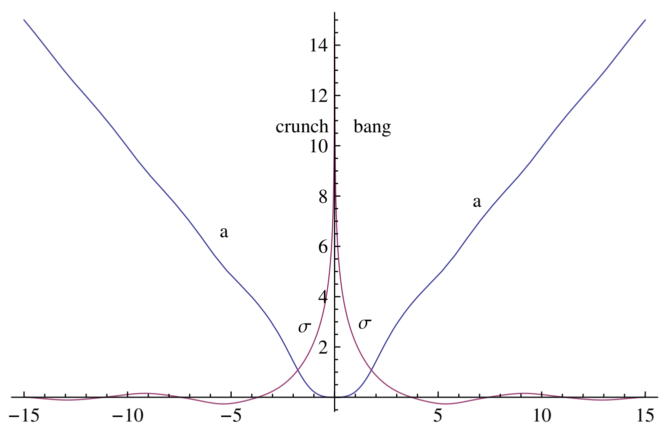

FIG. 16 (color online). There is a single crunch/bang.

$$s_\gamma^+(\tau) = \sqrt{\frac{1 - K^2 T_s^4}{8cT_s^2}} \frac{sn(\frac{\tau}{T_s}|m_s)}{dn(\frac{\tau}{T_s}|m_s)}, \qquad \begin{aligned} m_s(E) &\equiv \tfrac{1}{2}(1 - KT_s^2(E)) \\ T_s(E) &\equiv (16cE + K^2)^{-1/4}, \end{aligned} \tag{61}$$

while $\phi_\gamma^+(\tau)$ has the following expressions for $b > 0$:

$$\phi^{+}_{\gamma,b\geq 0}(\tau) = \sqrt{\frac{1}{8bT_{+}^{2}}}\frac{sn(\frac{\tau}{T_{+}}|m_{+})}{1+cn(\frac{\tau}{T_{+}}|m_{+})}, \qquad \begin{array}{c} m_{+}(E) \equiv \frac{1}{2} + KT_{+}^{2}(E) \\ T_{+}(E) \equiv (64bE)^{-1/4} \end{array}, \tag{62}$$

or $b < 0$

$$\phi^{+}_{\gamma,b\leq 0}(\tau) = \sqrt{\frac{1-K^{2}T_{-}^{4}}{8|b|T_{-}^{2}}}\frac{sn(\frac{\tau}{T_{-}}|m_{-})}{dn(\frac{\tau}{T_{-}}|m_{-})}, \qquad \begin{array}{c} m_{-}(E) \equiv \frac{1}{2}(1 - KT_{-}^{2}(E)) \\ T_{-}(E) \equiv (16|b|E + K^{2})^{-1/4} \end{array}. \tag{63}$$

Note that all symbols T, m that appear in the Jacobi elliptic functions are determined in terms of the energy level $E = E_s = E_\phi$ as given above. The $T_\pm(E)$ are determined in terms of E, b, c, K by the energy condition $E_\phi = E_s = E$, by using the following expressions computed from the Hamiltonians $H(\phi)$, $H(s)$ given in Eqs. (57)–(59) for the solutions $s^{+}_{\gamma}(\tau)$, $\phi^{+}_{\gamma}(\tau)$ above:

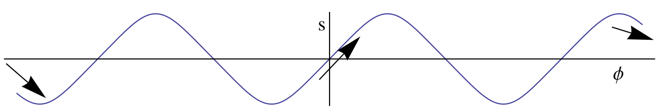

FIG. 17 (color online). Parametric plot equivalent to Fig. 15.

$$E_s = \frac{1}{16cT_s^4}(1 - K^2T_s^4), \qquad E_\phi^{+} = \frac{1}{64bT_+^4},$$
$$E_\phi^{-} = \frac{1}{16|b|T_-^4}(1 - K^2T_-^4). \tag{64}$$

Note also we have assumed that $E_s = E_\phi^{\pm} = E$ is higher than the critical value E^* in Eq. (60). This yields the expressions for $T_{s,\pm}(E)$, $m_{s,\pm}(E)$ given above as well as the ranges for these parameters as a function of the energy level as follows:

$$KT_s^2(E) < \sqrt{\frac{|b|}{|b|+c}}, \qquad KT_-^2(E) < \frac{1}{\sqrt{2}}, \qquad KT_+^2(E) < \frac{1}{2}, \tag{65}$$

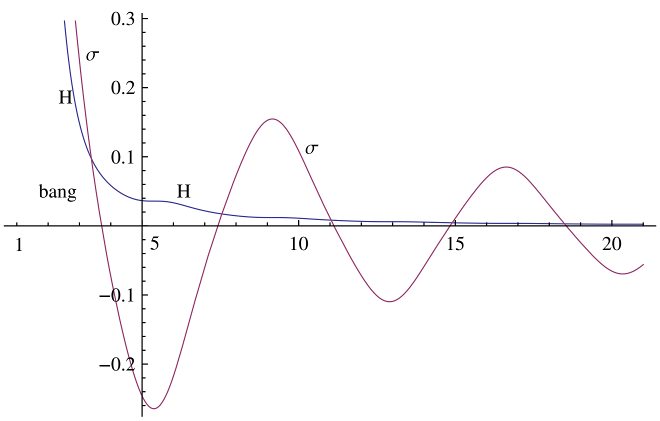

FIG. 18 (color online). $H(\tau)$, $\sigma(\tau)$ decrease.

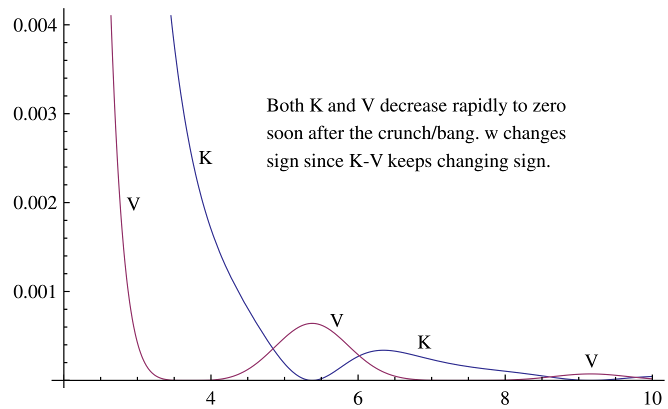

FIG. 20 (color online). The energy of σ field decreases.

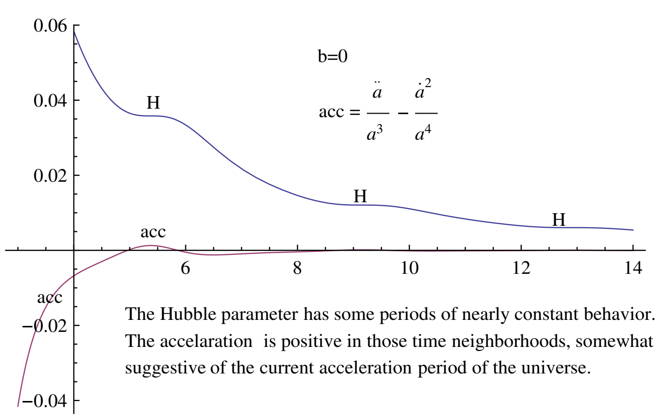

FIG. 19 (color online). Temporary acceleration periods.

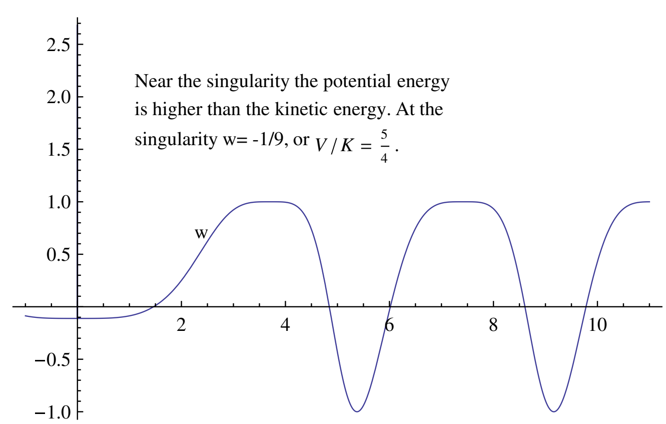

FIG. 21 (color online). Equation of state $w(\tau)$.

which determines the possible range of values for m_s, $m_\pm$ as E changes in the range $E \geq E^*$,

$$\frac{1}{2}(1-(1+c/|b|)^{-1/2}) < m_s(E) \leq \frac{1}{2};$$
$$\left(\frac{1}{2}-\frac{\sqrt{2}}{4}\right) < m_-(E) \leq \frac{1}{2}; \qquad \frac{1}{2} \leq m_+(E) < 1. \tag{66}$$

It is easy to see that in the zero curvature limit $K \to 0$ these solutions reduce to Eqs. (32) for $b \geq 0$ or Eqs. (41) for $b \leq 0$.

As they stand these solutions do not yet satisfy the requirement $\phi^2(\tau) \geq s^2(\tau)$ at all times. This can be satisfied only by requiring the period of ϕ to be a multiple integer of the period of s. The analytic expression for this conditions is

$$b \geq 0: T_+ Q(m_+) = 2nT_s Q(m_s), \quad n = 1, 2, 3, \ldots, \tag{67}$$

$$b \leq 0: T_- Q(m_-) = nT_s Q(m_s), \quad n = 1, 2, 3, \ldots, \tag{68}$$

where the quantity $Q(z)$ is a well-known special function, namely, the quarter period of the corresponding Jacobi elliptic function, and is given by the following integral representation of the EllipticK integral:

$$Q(z) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - z\sin^2\theta}} = \text{EllipticK}(z). \tag{69}$$

The consequence of this is to require a certain combination of parameters (b, c, K, E) to be quantized. The range of parameters in the model (b, c, K, E) that can satisfy the constraint can be determined numerically[10] by using the expressions above. Therefore only a model that can satisfy this condition can give the corresponding geodesically complete solutions.

B. Lower level $E < E^*$, and $b < 0$

When the energy is less than the critical value, the exact solutions are denoted as $s_\gamma^-(\tau)$, $\phi_\gamma^-(\tau)$ where the superscript $-$ refers to the energy interval $0 < E_s = E_\phi = E < E^*$. We consider at first the case of $b < 0$, for which $V_{b<0}(\phi)$ is represented by the dashed curve. There is no difference in this case with the $b < 0$ case above, so the formulas above apply, namely $s_\gamma^-(\tau)$ is the same as $s_\gamma^+(\tau)$, and $\phi_{\gamma,b<0}^-(\tau)$ is the same as $\phi_{\gamma,b<0}^+(\tau)$, except for the fact that now the energy is in the range $0 < E < E^*$,

$$s_{\gamma,b<0}^-(\tau) = \sqrt{\frac{1-K^2T_s^4}{8cT_s^2}\frac{sn(\frac{\tau}{T_s}|m_s)}{dn(\frac{\tau}{T_s}|m_s)}}, \qquad \begin{aligned} m_s(E) &\equiv \tfrac{1}{2}(1-KT_s^2(E)) \\ T_s(E) &\equiv (16cE+K^2)^{-1/4} \end{aligned}, \tag{70}$$

$$\phi_{\gamma,b<0}^-(\tau) = \sqrt{\frac{1-K^2T_-^4}{8|b|T_-^2}\frac{sn(\frac{\tau}{T_-}|m_-)}{dn(\frac{\tau}{T_-}|m_-)}}, \qquad \begin{aligned} m_-(E) &\equiv \tfrac{1}{2}(1-KT_-^2(E)) \\ T_-(E) &\equiv (16|b|E+K^2)^{-1/4} \end{aligned}. \tag{71}$$

Since the energy is less than the critical value, $0 \leq E_s = E_\phi \leq E^*$, we must now restrict the range of the parameters m, T to

$$\sqrt{\frac{|b|}{|b|+c}} \leq KT_s^2(E) \leq 1, \qquad \frac{1}{\sqrt{2}} \leq KT_-^2(E) \leq 1, \tag{72}$$

which implies

$$0 \leq m_s(E) \leq \frac{1}{2}\left(1-\sqrt{\frac{|b|}{|b|+c}}\right),$$
$$m_-(E) \leq \frac{1}{2}\left(1-\frac{1}{\sqrt{2}}\right). \tag{73}$$

To obtain geodesically complete solutions the analog of the quantization condition in Eq. (68) must be further imposed, $T_- Q(m_-) = nTQ(m)$.

C. Lower level $E < E^*$, and $b > 0$, finite bounce

For the case of $b > 0$ [$V_{b>0}(\phi)$ represented by the inverted double well in Fig. 22] there are new features. For ϕ now there is the possibility to either be trapped inside the false vacuum, or be outside of it, depending on initial conditions. When ϕ is trapped in the false vacuum it should oscillate between two turning points; when it is outside it should oscillate between a finite value and infinity. So the solution has the form

$$\phi_{\gamma,b\geq 0}^{-,\text{in}}(\tau) = \frac{\sqrt{KT_{\text{in}}^2-1}}{\sqrt{2b}T_{\text{in}}} sn\left(\frac{\tau}{T_{\text{in}}}|m_{\text{in}}\right), \qquad \begin{aligned} m_{\text{in}}(E) &\equiv (KT_{\text{in}}^2(E)-1) \\ T_{\text{in}}(E) &\equiv \left(\frac{K}{2}+\sqrt{\frac{K^2}{4}-4bE}\right)^{-1/2} \end{aligned}, \tag{74}$$

[10] For example, using MATHEMATICA, which recognizes the function EllipticK(z), one can plot $T_\pm Q(m_\pm)/T_s Q(m_s)$ as a function of one of the parameters E, b, c, K (example E) while the other three are chosen arbitrarily. When the plot matches an integer n, this fixes the value of the remaining parameter (i.e. E in the example above) in terms of the integer n and the other three parameters.

$$\phi^{-,\mathrm{out}}_{\gamma,b\geq 0}(\tau)=\frac{\sqrt{KT_{\mathrm{out}}^2+1}}{\sqrt{2b}T_{\mathrm{out}}}\frac{1}{cn(\frac{\tau+\tau_0}{T_{\mathrm{out}}}|m_{\mathrm{out}})},\qquad \begin{array}{c} m_{\mathrm{out}}(E)\equiv-\frac{1}{2}(KT_{\mathrm{out}}^2(E)-1)\\ T_{\mathrm{out}}(E)\equiv(K^2-16bE)^{-1/4}\end{array}. \tag{75}$$

Meanwhile $s(\tau)$ continues to oscillate as before between two turning points, so it is still given by the same expression, namely,

$$s^-_{\gamma,b>0}(\tau)=\sqrt{\frac{1-K^2T_s^4}{8cT_s^2}}\frac{sn(\frac{\tau}{T_s}|m_s)}{dn(\frac{\tau}{T_s}|m_s)},\qquad \begin{array}{c} m_s(E)\equiv\frac{1}{2}(1-KT_s^2(E))\\ T_s(E)\equiv(16cE+K^2)^{-1/4}\end{array}. \tag{76}$$

The energies of these solutions are computed in terms of the parameters m, T by using the Hamiltonians in Eqs. (57)–(59) as follows:

$$E_s=\frac{1}{16cT_s^4}(1-K^2T_s^4),\qquad E_\phi^{\mathrm{in}}=\frac{KT_{\mathrm{in}}^2-1}{4bT_{\mathrm{in}}^4},$$
$$E_\phi^{\mathrm{out}}=\frac{K^2T_{\mathrm{out}}^4-1}{16bT_{\mathrm{out}}^4}. \tag{77}$$

All energies must be positive and equal to each other $E_\phi^{\mathrm{out}}=E_s=E$ and $E_\phi^{\mathrm{in}}=E_s=E$, as well as smaller than E^*. This yields the expressions for $T_{s,\mathrm{in,out}}(E)$ and $m_{s,\mathrm{in,out}}(E)$ given above as well as the ranges for these parameters as a function of E, as follows:

$$1\geq KT_s^2(E)\geq\sqrt{\frac{b}{c+b}},\quad 2\geq KT_{\mathrm{in}}^2(E)\geq 1,\quad KT_{\mathrm{out}}^2(E)\geq 1.$$

Similarly, the range of values for the parameters m_s, m_{in}, m_{out} are then as follows:

$$0\leq m_s(E)\leq\frac{1}{2}\left(1-\sqrt{\frac{b}{b+c}}\right),\qquad 0\leq m_{\mathrm{in}}(E)\leq 1,$$
$$m_{\mathrm{out}}(E)\leq 0. \tag{78}$$

To obtain the geodesically complete solution, in the case of the *inside* solution the quantization condition $T_{\mathrm{in}}Q(m_{\mathrm{in}})=nTQ(m)$ is required (see footnote 10). However, in the case of the *outside* solution no quantization condition is needed as explained below.

We now comment on the outside solution given by $\phi^{-,\mathrm{out}}_{\gamma,b>0}(\tau)$, $s^-_{\gamma,b>0}(\tau)$ in Eqs. (75) and (76) because it is different in character compared to all the previous cases. It describes a periodically contracting/expanding universe with bounces that occur at minimum finite values of the scale factor, while the maximum is infinite. $\phi^{-,\mathrm{out}}_{\gamma,b>0}(\tau)$ describes the behavior of ϕ outside of the false vacuum; the amplitude for ϕ is always larger than the amplitude for $s^-_{\gamma,b>0}(\tau)$ at all times and for all boundary conditions, including the additional arbitrary parameter τ_0. This solution represents *cyclic bounces at finite minimum sizes of the universe*. This is shown in Fig. 23. The cyclic bounce occurs for all values of $c>0$, all values of $b>0$, and all values of the relative initial conditions τ_0 at $\tau=0$. This is why we added an additional phase τ_0 in the expression of $\phi^{-,\mathrm{out}}_{\gamma,b>0}(\tau)$. There is no need to synchronize the boundary conditions at $\tau=0$ for $\phi(0)$, $s(0)$ in order to get geodesically complete solutions that satisfy $\phi^2(\tau)\geq s^2(\tau)$ at all times. Moreover, b/c need not be quantized since the periods of ϕ, s can now be arbitrary relative to each other.

The corresponding plots for the scale factor $a(\tau)$ and $\sigma(\tau)$ for the bounce are given in Fig. 24. The minimum size of the scale factor $a(\tau)\sim\sqrt{\phi_\gamma^2(\tau)-s_\gamma^2(\tau)}$ at the bounce instant is not the same each time since the initial values of $\phi(\tau)$, $s(\tau)$ are not synchronized and their periods are not related.

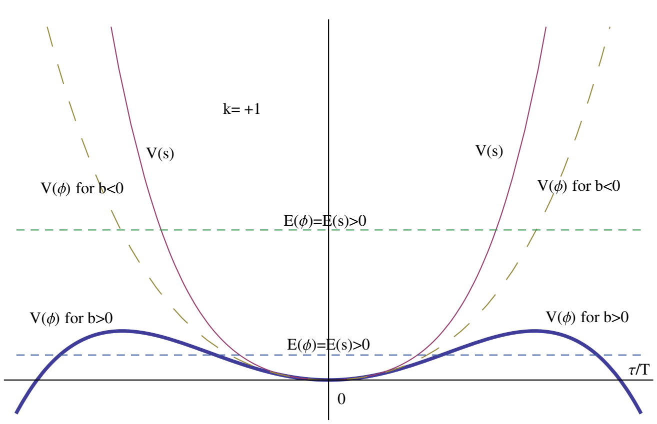

FIG. 22 (color online). The closed FRW universe, $k>0$.

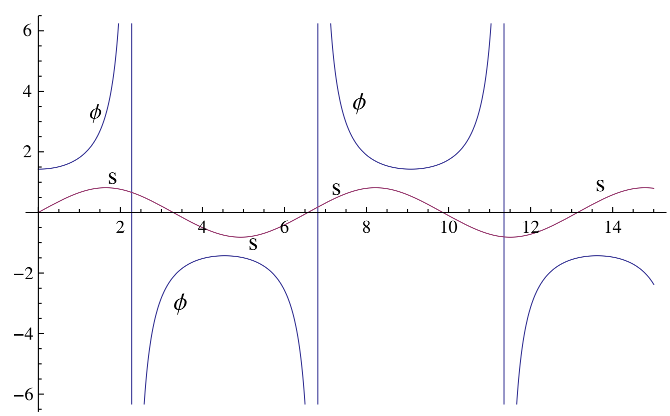

FIG. 23 (color online). $\phi^{-,\mathrm{out}}_{\gamma,b>0}(\tau)$ and $s^-_{\gamma,b>0}(\tau)$ for the bouncing solution.

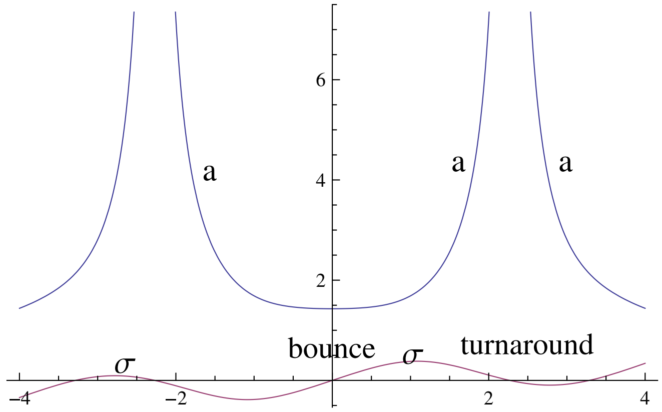

FIG. 24 (color online). The bounce and turnaround.

For this bounce solution to play a physical role in cosmology we need the curvature terms $K\phi^2$, Ks^2 to be able to compete with the potential terms $b\phi^4$, cs^4. For this to be phenomenologically tenable in a complete cosmological model, a period of inflation must follow after the bang so that the universe inflates to its current size and to its almost flat current condition (since r_0 would not be identified with today's size of the Universe).

As a limiting case of the above solutions we point out the special case of the $b > 0$ scenario when $E = E^*$. As seen from Fig. 22, the ϕ field sits still on top of the hill while the s field oscillates in a finite range. The maximum size of the Universe is a finite number determined by the constant value of the ϕ field.

D. The case of $c < 0$

There are also geodesically complete solutions when $c < 0$ and $b > 0$ which we will outline very briefly. The ϕ, s Hamiltonians are the same as before, as in Eqs. (57)–(59), with $K > 0$. There exist two interesting classes of geodesically complete solutions that satisfy $\phi^2 - s^2 \geq 0$, which can occur when $c < 0$ and $b > 0$. This happens when the corresponding potentials $V(\phi)$, $V(s)$ take the form in Figs. 25 and 26.

The first case is depicted in Fig. 25 when $0 \leq E \leq V_{\max}(s) = \frac{K^2}{16|c|}$, and $V_{\max}(\phi) = \frac{K^2}{16b}$ is higher, i.e. with $b < |c|$. Then s oscillates in the region of a false vacuum, while ϕ oscillates outside of the false vacuum all the way to infinity. This is similar to the finite bounce solution we discussed in Sec. IV C and Figs. 23 and 24. For completeness we record the solution,

$$s^{-}_{\gamma,c\leq 0}(\tau) = \frac{\sqrt{KT_c^2 - 1}}{\sqrt{2|c|}T_c}\, sn\left(\frac{\tau}{T_c}\,|m_c\right), \qquad \begin{array}{c} m_c(E) \equiv (KT_c^2(E) - 1) \\ T_c(E) \equiv \left(\frac{K}{2} + \sqrt{\frac{K^2}{4} - 4|c|E}\right)^{-1/2}, \end{array} \tag{79}$$

and

$$\phi^{-,\mathrm{out}}_{\gamma,b\geq 0}(\tau) = \frac{\sqrt{KT_{\mathrm{out}}^2 + 1}}{\sqrt{2b}T_{\mathrm{out}}}\, \frac{1}{cn(\frac{\tau+\tau_0}{T_{\mathrm{out}}}\,|m_{\mathrm{out}})}, \qquad \begin{array}{c} m_{\mathrm{out}}(E) \equiv -\frac{1}{2}(KT_{\mathrm{out}}^2(E) - 1) \\ T_{\mathrm{out}}(E) \equiv (K^2 - 16bE)^{-1/4} \end{array}. \tag{80}$$

As in the previous case of the bounce, this solution also represents *cyclic bounces at finite minimum sizes of the universe*, similar to those in Figs. 23 and 24. It occurs for all values of $b > 0$ and $c < 0$ provided $b < |c|$, and provided K is large enough so that the curvature terms $K\phi^2$, Ks^2 are able to compete with the potential term $b\phi^4$, cs^4. The parameters (c, b, K) do not need to satisfy any quantization conditions. Also, the synchronization of the relative phase is not needed, hence we have allowed the additional integration constant τ_0 in the solution in Eq. (80).

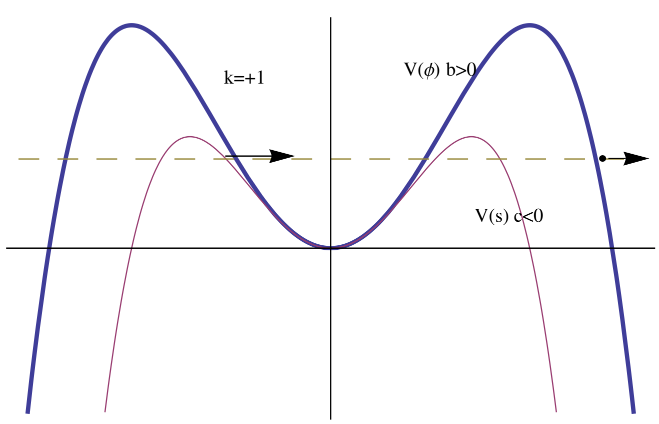

FIG. 25 (color online). s oscillates inside with ϕ outside.

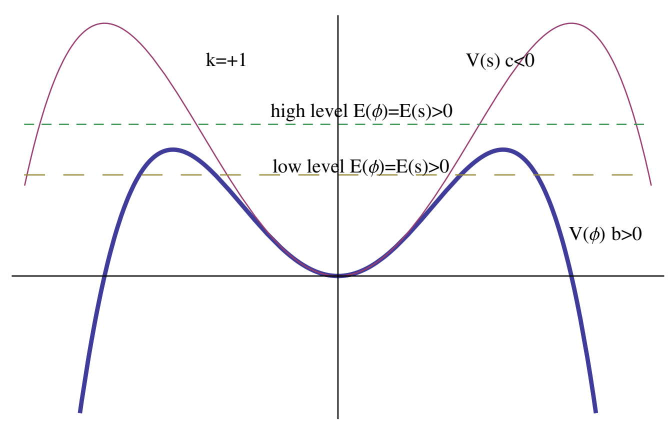

FIG. 26 (color online). s inside; ϕ depends on E.

The second case is depicted in Fig. 26 when $0 \leq E \leq V_{\max}(s)$, where $V_{\max}(s) = \frac{K^2}{16|c|}$ and $V_{\max}(\phi) = \frac{K^2}{16b}$, with $b > |c|$. Then s oscillates in the region of a false vacuum, while the behavior of ϕ depends on whether the energy is low [below $V_{\max}(\phi)$] or high [between $V_{\max}(\phi)$ and $V_{\max}(s)$]. The analytic solutions are similar to the ones discussed above, except that now $c < 0$, and the energy E is limited to the region $0 \leq E \leq V_{\max}(s)$. For completeness we list the geodesically complete solutions that satisfy $\phi^2 - s^2 \geq 0$, together with the quantization condition for their periods

$$\text{high:}\ \begin{cases} s^+_{\gamma,c\leq 0}(\tau) = \frac{\sqrt{KT_c^2-1}}{\sqrt{2|c|}T_c} sn\left(\frac{\tau}{T_c}|m_c\right), & m_c(E) \equiv (KT_c^2(E) - 1) \\ & T_c(E) \equiv \left(\frac{K}{2} + \sqrt{\frac{K^2}{4} - 4|c|E}\right)^{-1/2} \\ \phi^+_{\gamma,b\geq 0}(\tau) = \sqrt{\frac{1}{8bT_+^2}\frac{sn(\frac{\tau}{T_+}|m_+)}{1+cn(\frac{\tau}{T_+}|m_+)}}, & m_+(E) \equiv \frac{1}{2} + KT_+^2(E) \\ & T_+(E) \equiv (64bE)^{-1/4} \\ T_+Q(m_+) = 2nT_cQ(m_c) & \end{cases}, \tag{81}$$

and

$$\text{low:}\ \begin{cases} s^+_{\gamma,c\leq 0}(\tau) = \frac{\sqrt{KT_c^2-1}}{\sqrt{2|c|}T_c} sn\left(\frac{\tau}{T_c}|m_c\right), & m_c(E) \equiv (KT_c^2(E) - 1) \\ & T_c(E) \equiv \left(\frac{K}{2} + \sqrt{\frac{K^2}{4} - 4|c|E}\right)^{-1/2} \\ \phi^{+,\text{in}}_{\gamma,b\geq 0}(\tau) = \frac{\sqrt{KT_b^2-1}}{\sqrt{2|c|}T_b} sn\left(\frac{\tau}{T_b}|m_b\right), & m_b(E) \equiv (KT_b^2(E) - 1) \\ & T_b(E) \equiv \left(\frac{K}{2} + \sqrt{\frac{K^2}{4} - 4bE}\right)^{-1/2} \\ T_bQ(m_b) = nT_cQ(m_c) & \end{cases}. \tag{82}$$

In the low energy level it is also possible for ϕ to oscillate on the outside of the false vacuum. In that case the solution is given by

$$\text{low:}\ \begin{cases} s^+_{\gamma,c\leq 0}(\tau) = \frac{\sqrt{KT_c^2-1}}{\sqrt{2|c|}T_c} sn\left(\frac{\tau}{T_c}|m_c\right), & m_c(E) \equiv (KT_c^2(E) - 1),\quad T_c(E) \equiv \left(\frac{K}{2} + \sqrt{\frac{K^2}{4} - 4|c|E}\right)^{-1/2}, \\ \phi^{+,\text{out}}_{\gamma,b\geq 0}(\tau) = \frac{\sqrt{KT_{\text{out}}^2+1}}{\sqrt{2b}T_{\text{out}}}\frac{1}{cn(\frac{\tau+\tau_0}{T_{\text{out}}}|m_{\text{out}})}, & m_{\text{out}}(E) \equiv -\frac{1}{2}(KT_{\text{out}}^2(E) - 1),\quad T_{\text{out}}(E) \equiv (K^2 - 16bE)^{-1/4}. \end{cases} \tag{83}$$

In this case there is no quantization condition on the parameters, and there is an additional integration constant τ_0 which is arbitrary.

V. THE OPEN ($k = -1$) FRW UNIVERSE

For $k = -1$, the dynamics of ϕ, s is described by the Hamiltonians in Eqs. (57) and (59) with $K < 0$. The corresponding potentials $V(\phi)$, $V(s)$ are plotted in Fig. 27.

For the negative energy level $E_s = E_\phi < 0$, there cannot exist geodesically complete solutions that satisfy $\phi^2(\tau) - s^2(\tau) \geq 0$ at all times because of the following argument. The energy $E_s = E_\phi$ must be above the minimum of $V(s)$ at the double well to have a solution for s. Then the quantity $\phi^2 - s^2$ always changes sign, whether $b > 0$ or $b < 0$. For example, suppose s is at the minimum of $V(s)$, so the solution $s(\tau)$ is just a constant. But $\phi(\tau)$ oscillates between two turning points $\phi_{\min}(E) < \phi(\tau) < \phi_{\max}(E)$, while

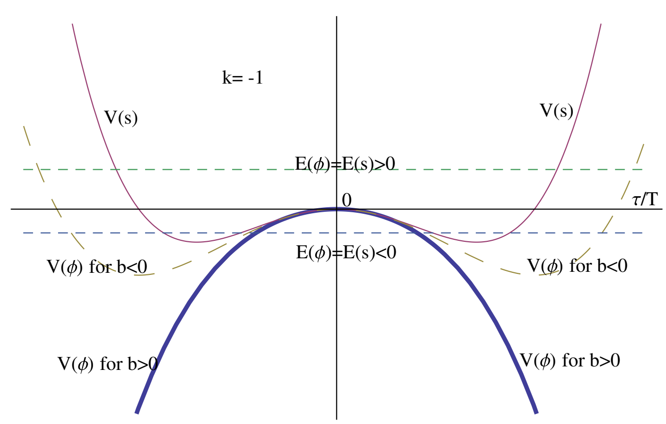

FIG. 27 (color online). The open FRW universe, $k < 0$.

sometimes $\phi^2 - s^2$ is positive and sometimes it is negative. The figure above is drawn for the case $|b| < c$. If we take $|b| > c$ then $V(s)$ will be like the dashed curve and while $V(\phi)$ will be like the solid thin curve (for $b < 0$) or the solid thick curve (for $b > 0$). In these cases again there are no solutions such that $\phi^2 - s^2$ remains positive at all times.

For the positive energy level $E_s = E_\phi > 0$, there are geodesically complete solutions that satisfy $\phi^2 - s^2 > 0$ at all times. In fact this case is formally identical to the case discussed in Sec. IVA. In the present case $K < 0$, $c > 0$, while b can have either sign. The exact solutions are parallel to Eqs. (61)–(63) except for replacing $K = -|K|$ Thus, the solutions are

$$s_\gamma^+(\tau) = \sqrt{\frac{1 - K^2 T_s^4}{8cT_s^2}} \frac{sn(\frac{\tau}{T_s}|m_s)}{dn(\frac{\tau}{T_s}|m_s)}, \qquad \begin{array}{l} m_s(E) \equiv \frac{1}{2}(1 + |K|T_s^2(E)) \\ T_s(E) \equiv (16cE + K^2)^{-1/4} \end{array}, \tag{84}$$

while $\phi_\gamma^+(\tau)$ has the following expressions for $b > 0$:

$$\phi_{\gamma,b\geq 0}^+(\tau) = \sqrt{\frac{1}{8bT_+^2}} \frac{sn(\frac{\tau}{T_+}|m_+)}{1 + cn(\frac{\tau}{T_+}|m_+)}, \qquad \begin{array}{l} m_+(E) \equiv \frac{1}{2} - |K|T_+^2(E) \\ T_+(E) \equiv (64bE)^{-1/4} \end{array}, \tag{85}$$

or $b < 0$

$$\phi_{\gamma,b\leq 0}^+(\tau) = \sqrt{\frac{1 - K^2 T_-^4}{8|b|T_-^2}} \frac{sn(\frac{\tau}{T_-}|m_-)}{dn(\frac{\tau}{T_-}|m_-)}, \qquad \begin{array}{l} m_-(E) \equiv \frac{1}{2}(1 + |K|T_-^2(E)) \\ T_-(E) \equiv (16|b|E + K^2)^{-1/4} \end{array}. \tag{86}$$

As they stand these solutions do not yet satisfy the requirement $\phi^2(\tau) \geq s^2(\tau)$ at all times. This can be satisfied only by requiring the period of ϕ to be a multiple integer of the period of s. The analytic expression for this conditions is, as before,

$$b \geq 0: T_+ Q(m_+) = 2nT_s Q(m_s), \quad n = 1, 2, 3, \ldots, \tag{87}$$

$$b \leq 0: T_- Q(m_-) = nT_s Q(m_s), \quad n = 1, 2, 3, \ldots. \tag{88}$$

Note that in the limit $K \to 0$ these solutions reduce to the solutions for the flat case with $b \geq 0$ and $b \leq 0$. We will not discuss them in any more detail here since this $K < 0$ case is similar to the previous discussion for both $K > 0$ and $K = 0$.

VI. SUMMARY AND OUTLOOK

We have thoroughly analyzed a simple model of a scalar field interacting with gravity in $3 + 1$ dimensions. The model was derived from 2T-gravity in $4 + 2$ dimensions as the "3 + 1 dimensional conformal shadow" [4–6] and can also be constructed in the colliding branes scenario [2] in $4 + 1$ dimensions using the worldbrane notions [7] inspired by M-theory [8].

An essential feature of the model in $3 + 1$ dimensions is an underlying local conformal symmetry (Weyl symmetry) exhibited in the action of Eq. (5). There is no fundamental gravitational constant in this model, but instead there is a gauge dependent dynamical "gravitational parameter" which plays precisely the role of the gravitational constant when the Weyl symmetry is gauge fixed to the Einstein frame, thus agreeing with the standard form of Einstein's gravity and its interactions with matter.

This raised the question of whether the dynamics could force the gravitational parameter $(\phi^2(x^\mu) - s^2(x^\mu))^{-1}$ to change sign in some patches of spacetime where antigravity would emerge, and whether the existence of such patches could have observational consequences in our current Universe in the context of cosmology or otherwise. This is the question that motivated our investigation.

We found out that generically the gravitational parameter does change sign dynamically, and that this change of sign is not a gauge artifact since the gauge invariant quantity $(1 - s^2/\phi^2)$ can be used to monitor the sign change. So our model indicates that patches of antigravity could exist, but we have also found that such patches cannot be reached from our current Universe without going through singularities, such as the big bang or big crunch (perhaps others such as black holes as well).

We have not yet answered what the physical implications of this phenomenon may be for our current Universe, but instead we have limited the current investigation to finding and classifying those classical solutions in the context of cosmology that are geodesically complete for all times. By *all* times we mean that one must go beyond the gauge dependent definition of time and instead seek geodesically complete solutions in all possible choices of time, thus being able to connect information before and after singularities. In this way one is not limited to only some patches while declaring ignorance about other patches.[11]

[11]Of course, we must expect modifications of the classical equations due to quantum gravity especially near singularities. We may need to wait a long time before one is sure about what quantum gravity really is. Given this cloudiness of our knowledge at the present time, we feel that the geodesically complete approach we are pursuing here will still be relevant, and perhaps even provide guidance to clarify the issues in future research.

We found that the conformal time τ is a good evolution parameter for our purpose, so we analyzed the solutions for all values of τ from minus infinity to plus infinity. In this way we learned that the gauge invariant quantity $1 - s^2(\tau)/\phi^2(\tau)$ oscillates back and forth between patches where it is positive and negative (namely gravity/antigravity), and this information is carried smoothly through the singularities. In fact we learned that the point in time where there is a singularity in the Einstein frame (divergent scalar curvature) does not look like a singularity at all in other convenient gauge choices of the Weyl gauge symmetry.

This paper was focused on finding and classifying the complete subset of all classical cosmological solutions for which $1 - s^2(\tau)/\phi^2(\tau)$ never changes sign for all times. Thus, for the classical solutions exhibited in this paper the Universe remains always in the gravity patch, never shifting to antigravity. The Universe starts expanding with a big bang, but eventually it turns around (at a finite or infinite size, depending on the sign of the parameter b) and begins to contract, leading to a big crunch. But this is followed with the same periodic pattern of a big bang, turnaround, big crunch, again and again indefinitely to the future as well as to the past.

There are such cyclic solutions in which the Universe never contracts to zero size, and bounces back after contracting to a finite size. So in these finite bounce solutions the Universe never hits a curvature singularity (as defined in the Einstein frame). These solutions are possible for the closed universe. The finite bounce solutions exist for a large range of the parameters b, c, K that define the model, but one of the integration constants E (which amounts to energy initial conditions for the scalar fields s or ϕ) must lie in a certain range defined by the parameters b, c, K while the other integration constant τ_0 is arbitrary.

There are also cyclic solutions in which the Universe contracts to zero size periodically, thus hitting the curvature singularity (in the Einstein frame) at the big crunches/bangs. These cyclic solutions occur for the flat, open and closed universes, but only if some initial conditions for the ϕ, s fields are synchronized (the integration parameter τ_0 set to $\tau_0 = 0$) and a quantization condition is imposed on a combination of the parameters b, c, K, and the integration parameter E. Thus not every model is capable of yielding geodesically complete cyclic solutions, as illustrated clearly in the case of the flat universe.

Evidently the next stage of this research is to analyze what happens to these solutions under perturbations. This is the topic of our next paper in Ref. [11], where in addition to curvature K, radiation and anisotropy are included both at the classical and quantum (in the sense of the Wheeler-deWitt equation) levels. According to our current understanding, very similar geodesically complete solutions exist in the presence of these perturbations. The question of the physics of antigravity and its effect on our current era of cosmology is an interesting topic that we intend to pursue as a natural evolution of the present discussion. We hope to report on these details in the near future.

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