Exact solution for the Casimir stress in a spherically symmetric medium

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We calculated the stress of the quantum vacuum, the Casimir stress, in a spherically symmetric medium, Maxwell's fish eye, surrounded by a perfect mirror and derived an exact analytic solution. Our solution questions the idea that the Casimir force of a spherical mirror is repulsive—we found an attractive stress in the medium that diverges at the mirror.

Casimir suggested an intriguing model that could explain the stability of charged particles and the value of the fine-structure constant [1]. The argument goes as follows: Imagine the particle as an electrically charged hollow sphere. Two forces are acting upon it: the electrostatic repulsion and the force of the quantum vacuum, the Casimir force—presumed to be attractive [2–8]. The stress σ of the quantum vacuum on a spherical shell of radius a must be given by a dimensionless constant times $\hbar c/a^4$ on purely dimensional grounds—the quantum stress is an energy density proportional to $\hbar$, and $\hbar c/a^4$ carries indeed the units of an energy density. Now, the electrostatic energy of the sphere is proportional to the square e^2 of its charge and is also inversely proportional to a^4 [9]. Therefore, an attractive Casimir force balances the electrostatic repulsion regardless of how small a is, provided $e^2/(\hbar c)$ assumes a certain value given by the strength of the Casimir force. This strength depends on the internal structure of the particle—the fact that it is a spherical shell—but not on its size, which could be imperceptibly small. Casimir's model, however crude, could simultaneously explain the fine-structure constant $e^2/(\hbar c)$ and the stability of charged elementary particles.

All one needs to do is calculate the Casimir force on a spherical shell, but such calculations are notoriously difficult. After a marathon struggle with special functions, Boyer succeeded in numerically computing the force for an infinitely conducting, infinitely thin shell and found a surprising result [10] that shattered Casimir's idea: the vacuum force is repulsive and so cannot possibly balance the electrostatic repulsion. Boyer's heroic calculation was confirmed in a sophisticated and elegant paper by Milton, DeRaad, and Schwinger [11] and by others [12]. The spherical shell has become the archetype for a shape that causes Casimir repulsion [13]. However, doubts have been lingering about whether the repulsive force of the shell may be an artifact of the simple model used [14,15], for the following reason: the bare stress of the quantum vacuum is always infinite and this infinity is removed by regularization procedures [2–8]. The most plausible regularization involves considering the relative stress between or inside macroscopic bodies. But, an infinitely thin sphere does not represent an extended macroscopic body, nor multiple bodies. Suppose the physically relevant vacuum stress of an extended spherical shell tends to infinity in the limit when the shell becomes infinitely thin and infinitely conducting. In this case the regularization would remove this physically significant infinity, producing a finite result that may very well have the wrong sign. Our paper supports the contention that the Casimir repulsion of the spherical shell could be an artifact of regularization.

Consider a minor modification of Casimir's model (Fig. 1). Imagine that the spherical shell (though still infinitely conducting and infinitely thin) is no longer hollow, but filled with a medium of gradually varying electric permittivity ε and magnetic permeability μ. In this way, the shell has become part of a macroscopic body where the Casimir stress gradually builds up. We calculate the regularized stress tensor σ in the material and see how σ behaves near the spherical mirror. For preserving the spherical symmetry, we assume that ε and μ depend only on the distance r from the center of the sphere. The stress tensor σ will be radially symmetric and may change with increasing r. Considering how σ varies, we obtain a physically well-defined Casimir-force density $\nabla \cdot \sigma$. We assume a toy model for ε and μ where σ turns out to have an exact solution. Our model is (Fig. 2)

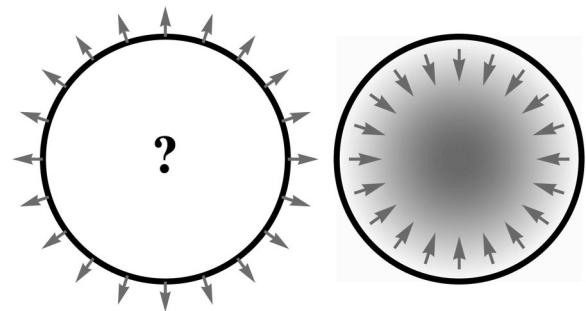$$\varepsilon = \mu = \frac{2n_1}{1 + (r/a)^2} \qquad (1)$$

FIG. 1. The Casimir force on a spherical shell (left) is repulsive [10], or is it? We assumed the shell to be filled with a medium (right) and found an attractive Casimir stress in the material. The shades of grey indicate the profile of the medium (plotted in Fig. 2).
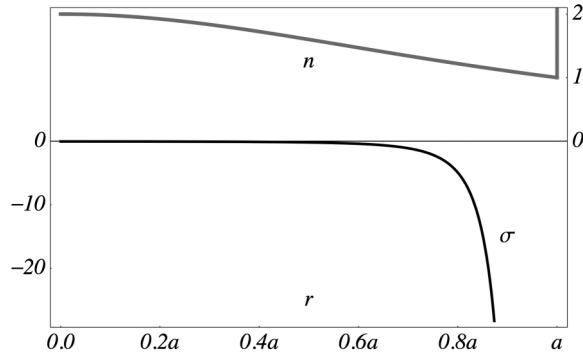
FIG. 2. Index profile $n(r)$ (grey curve) of the medium inside the shell and the resulting vacuum stress $\sigma(r)$ (black curve, in units of $\hbar c/a^4$). As $r \to a$, the stress σ tends to $-\infty$.

where n_1 is a constant. Equation (1) is valid for $r \leq a$, and at $r = a$ we place a perfect spherical mirror where $\varepsilon \to \infty$. The mirror is part of the dielectric material, but, as it is infinitely thin, we can model it by the boundary conditions it imposes: the transversal components of the electric field strength are zero. Equation (1) describes Maxwell's fish eye [16–19]. We have chosen this model because, in our calculation of the Casimir force, we take advantage of the mathematical fact that Maxwell's fish eye implements the geometry of a simple curved space: it represents the three-dimensional surface of the four-dimensional hypersphere in stereographic projection (Fig. 3) [19,20]. We have found a simple, exact expression for the vacuum-stress tensor:

$$\sigma = -\frac{\hbar c \mathbb{1}}{\pi^2 a^4 n (1 - r^2/a^2)^4} \qquad (2)$$

where $\mathbb{1}$ is the unity matrix. The stress is isotropic, negative, and falls monotonically, so the Casimir-force density $\nabla \cdot \sigma$ is always attractive in our model (Fig. 2). Close to the mirror, the stress and the force in the material tend to infinity. Since the material of our toy model [Eq. (1)] represents only a modest modification of homogenous space with refractive index n_1, it is likely that the
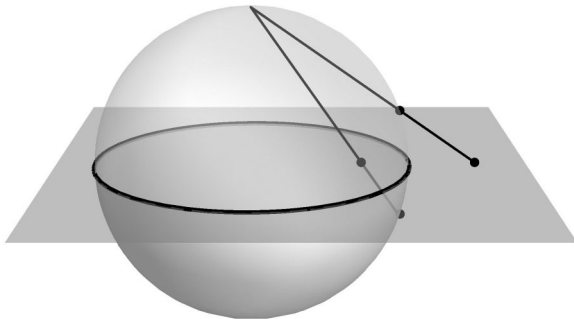

FIG. 3. The medium (plotted in Fig. 2) represents the geometry of the hypersphere (shown here as a sphere) in stereographic projection (lines). The figure illustrates the reflection (points) at the mirror (black circle) on the hypersphere and in physical space.

Casimir stress close to a perfect spherical mirror around a ball of index n_1 is infinite as well, even when $n_1 = 1$, i.e. for a hollow perfect mirror. An imperfect mirror, on the other hand, may lead to a finite and possibly attractive vacuum force. In any case, our present calculation shows that the perfect spherical mirror produces an artifact, a diverging Casimir-force density, in a relatively normal material, which casts new doubts on the claim of Casimir repulsion for the hollow sphere [10–12] and gives new hope for Casimir's fascinating explanation of the fine-structure constant and the stability of elementary charged particles [1]. Of course, in a more realistic theory the charged particle should not be regarded as being a classical object interacting with the quantum vacuum, as in Casimir's case [1], but rather as a self-consistent quantum structure. Speculation aside, we found a nontrivial exact solution for the Casimir force. Analytic solutions for Casimir forces are extremely rare—the attraction between two plates with infinite ε [2], the repulsion between plates with infinite ε and μ [21], and the attractive force on a homogeneous spherical ball with infinite ε [22] have been solved; we believe we have discovered the first exact solution in a gradually varying medium.

We use Lifshitz theory [3,8] for our calculation because this is the best physically motivated and tested theory of Casimir forces [7]. Lifshitz theory relates the vacuum stress to the electromagnetic Green function (as in Schwinger's source theory [23]). The physical stress σ of the quantum vacuum is expressed as

$$\sigma = \lim_{r_0 \to r}[\sigma(r, r_0) - \sigma_0(r, r_0)],$$
$$\sigma(r, r_0) = \tau(r, r_0) - \frac{\mathbb{1}}{2} \operatorname{Tr}\tau(r, r_0) \qquad (3)$$

where the τ are the correlation functions of the fields in the vacuum state between the points r and r_0 at equal times; the τ are finite for $r \neq r_0$. The stress is regularized by subtracting a bare vacuum stress from $\sigma(r, r_0)$ in the limit $r_0 \to r$. Our calculation turns out to be independent of the actual regularizer σ_0 as long as σ_0 depends only on local properties of the medium. The total correlation function τ consists of the sum of the electric and the magnetic field correlation functions given by [8]

$$\tau_{\text{el}} = -\frac{\hbar c \varepsilon(r)}{\pi} \int_0^\infty \kappa^2 G_s(r, r_0; i\kappa) d\kappa,$$
$$\tau_{\text{mag}} = \frac{\hbar c}{\pi \mu(r_0)} \int_0^\infty \nabla \times G_s(r, r_0; i\kappa) \times \overleftarrow{\nabla}_0 d\kappa \qquad (4)$$

where the arrow indicates differentiation from the right; G_s denotes the symmetrized electromagnetic Green function $(G + G^{\text{T}})/2$ for purely imaginary wave numbers $i\kappa$ (i.e. for imaginary frequencies $ic\kappa$). The Green function G describes the electric field at the spectator point r generated by a point source at r_0 pointing in all possible spatial

directions. The Green function is thus a bitensor that obeys the wave equation

$$\nabla \times \tfrac{1}{\mu}\nabla \times G + \varepsilon\kappa^2 G = \delta(\boldsymbol{r} - \boldsymbol{r}_0)\mathbb{1}. \qquad (5)$$

In our case [Eq. (1)], the medium is impedance-matched,

$$\varepsilon(\boldsymbol{r}) = \mu(\boldsymbol{r}) = n(\boldsymbol{r}). \qquad (6)$$

We show that for impedance matching the electric correlation function equals the magnetic one. For this, we represent G as

$$G = \mathrm{G} + \frac{\delta(\boldsymbol{r} - \boldsymbol{r}_0)\mathbb{1}}{n\kappa^2} \qquad (7)$$

and obtain from the wave equation [Eq. (5)]

$$\nabla \times \frac{1}{n}\nabla \times \mathrm{G} + n\kappa^2 \mathrm{G} = -\frac{\nabla \times \delta(\boldsymbol{r} - \boldsymbol{r}_0)\mathbb{1} \times \overset{\leftarrow}{\nabla}_0}{n(\boldsymbol{r})n(\boldsymbol{r}_0)\kappa^2} \qquad (8)$$

where we expressed the double curl of the delta-function term [Eq. (7)] in terms of derivatives with respect to $\boldsymbol{r}$ and $\boldsymbol{r}_0$. Notice that the magnetic Green function, defined as

$$G_{\mathrm{mag}} = -\frac{\nabla \times G \times \overset{\leftarrow}{\nabla}_0}{n(\boldsymbol{r})n(\boldsymbol{r}_0)\kappa^2}, \qquad (9)$$

obeys the same wave equation [Eq. (8)]. Consequently, G_{mag} agrees with G apart from a delta-function term, but such a term does not matter in the correlation functions [Eq. (4)] where we regard $\boldsymbol{r} \neq \boldsymbol{r}_0$ before we take the limit $\boldsymbol{r}_0 \rightarrow \boldsymbol{r}$. We conclude that $\tau_{\mathrm{el}} = \tau_{\mathrm{mag}}$ and obtain

$$\tau = -\frac{2\hbar cn}{\pi}\int_0^\infty \kappa^2 \mathrm{G}_s(\boldsymbol{r}, \boldsymbol{r}_0; i\kappa)\mathrm{d}\kappa, \qquad (10)$$

which sets the scene for calculating the Casimir stress σ from the electromagnetic Green function G in impedance-matched media.

The Green function for Maxwell's fish eye [Eq. (1)] has been obtained already [17,18]. We state the results we need here. For keeping our expressions uncluttered, we set

$$a = 1, \qquad n_1 = 1 \qquad (11)$$

in our calculation and then obtain the general result [Eq. (2)] by scaling arguments. Suppose the material [Eq. (1)] extends to infinity (without the mirror). In this case, the solution of Eq. (5) is [18]:

$$\mathrm{G}_0 = -\frac{\nabla \times n(r')\nabla \otimes \nabla_0 D(r') \times \overset{\leftarrow}{\nabla}_0}{n(\boldsymbol{r})n(\boldsymbol{r}_0)\kappa^2}, \qquad (12)$$

$$r' = \frac{|\boldsymbol{r} - \boldsymbol{r}_0|}{\sqrt{1 + 2\boldsymbol{r} \cdot \boldsymbol{r}_0 + r^2 r_0^2}}, \qquad (13)$$

$$D = \left(r' + \frac{1}{r'}\right)\frac{\sinh(2\kappa\,\mathrm{arccot}\,r')}{8\pi\sinh(\pi\kappa)}. \qquad (14)$$

In the geometrical picture behind Maxwell's fish eye [19] (Fig. 3), the electromagnetic wave propagates on the surface of the hypersphere from source $\boldsymbol{r}_0$ to spectator $\boldsymbol{r}$ in stereographic coordinates with distance $\arctan r'$, and D denotes the Green function of a conformally coupled scalar field [18]. The effect of the mirror is described by an adaptation of the method of images [9] on the hypersphere (Fig. 3). There, the mirror lies on the equator (a two-dimensional surface for the four-dimensional hypersphere). We subtract from G_0 the electromagnetic wave generated by the image source on the hypersphere (Fig. 3). This field is the mirror image of the original field. In stereographic projection [19], the reflection at the equator corresponds to the transformation $r \rightarrow r^{-1}$. To obtain the reflected wave G_0', we thus perform the coordinate transformation $\boldsymbol{r} = \boldsymbol{r}(\boldsymbol{r}')$ with $r' = r^{-1}$ and then replace $\boldsymbol{r}'$ by $\boldsymbol{r}$. Note that we also need to transform the field components of G_0, which is done by the Jacobian [24]

$$P = \left(\frac{\partial \boldsymbol{r}'}{\partial \boldsymbol{r}}\right) = \frac{\mathbb{1}}{r^2} - \frac{2\boldsymbol{r} \otimes \boldsymbol{r}}{r^4} \qquad (15)$$

such that

$$\mathrm{G}_0' = P\mathrm{G}_0(r^{-1}). \qquad (16)$$

In this way, we obtain for the Green function

$$\mathrm{G} = \mathrm{G}_0(r) - P\mathrm{G}_0(r^{-1}). \qquad (17)$$

One verifies that the transversal components of G vanish at $r = 1$, as they should at a perfectly reflecting electric mirror. Note that also for the reflected wave the magnetic Green function equals the electric one (up to an unimportant delta-function term) even if the mirror is not made by an impedance-matched material. To prove this, consider G_{mag} defined by Eq. (9) for the transformed Green function G_0' that describes the reflected wave and rename $\boldsymbol{r}$ as $\boldsymbol{r}'$ (we recall that G_0' is the result of a coordinate transformation). Then, we make use of two geometrical facts. First, $n(r)^{-1}\nabla \times \mathrm{G}_0$ defines a one-form with respect to the effective geometry with line element $n\mathrm{d}l$ [19]. Second, this one-form is invariant under the transformation $r' = r^{-1}$ because the line element $n\mathrm{d}l$ is invariant. Therefore, we can read $n(r')^{-1}\nabla' \times \mathrm{G}_0'$ as the coordinate-transformed $n(r)^{-1}\nabla \times \mathrm{G}_0$. Hence, we can also read the magnetic Green function of G_0' (with $\boldsymbol{r}$ renamed as $\boldsymbol{r}'$) as the coordinate-transformed G_{mag} of G_0. For G_0, the entire medium is impedanced-matched, and in such a case we have already established that the magnetic Green function agrees with the electric one up to a delta-function term. Consequently, the same must be true for the transformed Green function G_0' that describes the reflection at the mirror.

Now, we are ready to calculate the Casimir stress from the electric Green function according to Eqs. (3) and (10). The Green function G_0 of the infinitely extended fish-eye medium corresponds to the Green function on the entire surface of the hypersphere, which is a uniform space. It can only produce a uniform vacuum stress σ_0 that does not contribute to the Casimir-force density $\nabla \cdot \sigma$. As we are interested in contributions to σ that do generate a force, we take the uniform σ_0 as our regularizer. In this way, we are independent of actual regularization procedures that attempt to explain why the physical vacuum stress is finite. As an additional bonus, we only need to focus on the reflected part of the radiation field, similar to the Lifshitz theory [8] for the vacuum stress between conducting plates [2] or in other piecewise uniform planar materials [3]. We thus consider only $-PG_0(r^{-1})$ in the total Green function [Eq. (17)]. As the Casimir stress in a spherically symmetric medium must be spherically symmetric, we calculate $-PG_0(r^{-1})$ only in the x direction, i.e. we put $y = z = 0$ and $y_0 = z_0 = 0$ in the definitions [Eq. (12)–(15)] and evaluate $-PG_0(r^{-1})$ at $x_0 = x^{-1}$. We obtain, after some straightforward algebra,

$$-PG_0(r^{-1}) = \frac{(1+r^2)^2}{16\kappa^2 r^4 r'} \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_2 \end{pmatrix} \qquad (18)$$

with the matrix elements

$$d_1 = 2\frac{\mathrm{d}D}{\mathrm{d}r'}, \qquad d_2 = -\frac{\mathrm{d}D}{\mathrm{d}r'} - r'\frac{\mathrm{d}^2 D}{\mathrm{d}r'^2} \qquad (19)$$

where, in the x direction, $r = x$ and

$$r' = \tfrac{1}{2}\left(\frac{1}{r} - r\right). \qquad (20)$$

Equations (18)–(20) enter the formula for the correlation function [Eq. (10)] in place of G_s where G_s is integrated over all positive-imaginary wave numbers $i\kappa$. It is wise to perform the κ integration before the r' differentiations [Eq. (19)]. We obtain for the scalar Green function [Eq. (14)], using integral 2.4.4.1 of Ref. [25],

$$\int_0^\infty D\mathrm{d}\kappa = \frac{1 + r'^2}{16\pi r'^2}, \qquad (21)$$

which gives

$$\tau - \tau_0 = \hbar c \frac{1 + r^2}{16\pi^2 (r'r)^4} \mathbb{1}. \qquad (22)$$

All three eigenvalues of $\tau - \tau_0$ are identical in the x direction and, by virtue of spherical symmetry, they must be identical in all directions. Equation (22) is thus valid everywhere in the medium, and from Eqs. (3) and (20) we get our result [Eq. (2)] for $a = 1$ and $n_1 = 1$. Reinstating units for σ and r produces Eq. (2) for general a. For obtaining our result for general n_1, we notice that n_1 of Eq. (1) appears in the wave equation [Eq. (5)] as $n_1\kappa$ and a prefactor of n_1 of the source term. Consequently, we only need to replace $G(i\kappa)$ by $n_1 G(in_1\kappa)$ in Eq. (10) where n also carries the prefactor n_1, take $n_1\kappa$ as a new integration variable, and obtain Eq. (2) in full generality.

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