# Meson transition form factors in light-front holographic QCD

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We study the photon-to-meson transition form factors (TFFs) $F_{M\gamma}(Q^2)$ for $\gamma\gamma^* \to M$ using light-front holographic methods. The Chern-Simons action, which is a natural form in five-dimensional anti–de Sitter (AdS) space, is required to describe the anomalous coupling of mesons to photons using holographic methods and leads directly to an expression for the photon-to-pion TFF for a class of confining models. Remarkably, the predicted pion TFF is identical to the leading order QCD result where the distribution amplitude has asymptotic form. The Chern-Simons form is local in AdS space and is thus somewhat limited in its predictability. It only retains the $q\bar{q}$ component of the pion wave function, and further, it projects out only the asymptotic form of the meson distribution amplitude. It is found that in order to describe simultaneously the decay process $\pi^0 \to \gamma\gamma$ and the pion TFF at the asymptotic limit, a probability for the $q\bar{q}$ component of the pion wave function $P_{q\bar{q}} = 0.5$ is required, thus giving indication that the contributions from higher Fock states in the pion light-front wave function need to be included in the analysis. The probability for the Fock state containing four quarks $P_{q\bar{q}q\bar{q}} \sim 10\%$, which follows from analyzing the hadron matrix elements for a dressed current model, agrees with the analysis of the pion elastic form factor using light-front holography including higher Fock components in the pion wave function. The results for the TFFs for the $\eta$ and $\eta'$ mesons are also presented. The rapid growth of the pion TFF exhibited by the *BABAR* data at high $Q^2$ is not compatible with the models discussed in this article, whereas the theoretical calculations are in agreement with the experimental data for the $\eta$ and $\eta'$ TFFs.

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## I. INTRODUCTION

The anti–de Sitter/conformal field theory (AdS/CFT) correspondence between an effective gravity theory on a higher dimensional AdS space and conformal field theories in physical space-time [1–3] has led to a remarkably accurate semiclassical approximation for strongly coupled QCD, and it also provides physical insights into its nonperturbative dynamics. Incorporating the AdS/CFT correspondence as a useful guide, light-front holographic methods were originally introduced [4,5] by matching the electromagnetic (EM) current matrix elements in AdS space [6] to the corresponding Drell-Yan-West (DYW) expression [7–9], using light-front (LF) theory in physical space-time. One obtains the identical holographic mapping using the matrix elements of the energy-momentum tensor [10] by perturbing the AdS metric,

$$ds^2 = \frac{R^2}{z^2}(\eta_{\mu\nu}dx^\mu dx^\nu - dz^2), \tag{1}$$

around its static solution [11].

A precise gravity dual to QCD is not known, but color confinement can be incorporated in the gauge/gravity correspondence by modifying the AdS geometry in the large infrared (IR) domain $z \sim 1/\Lambda_{\rm QCD}$, which also sets the mass scale of the strong interactions in a class of confining models. The modified theory generates the pointlike hard behavior expected from QCD, such as constituent counting rules [12–14] from the ultraviolet (UV) conformal limit at the AdS boundary at $z \to 0$, instead of the soft behavior characteristic of extended objects [15].

One can also study the gauge/gravity duality starting from the light-front Lorentz-invariant Hamiltonian equation for the relativistic bound-state system $P_\mu P^\mu|\psi(P)\rangle = (P^+P^- - \mathbf{P}_\perp^2)|\psi(P)\rangle = \mathcal{M}^2|\psi(P)\rangle$, $P^\pm = P^0 \pm P^3$, where the light-front time evolution operator $P^-$ is determined canonically from the QCD Lagrangian [16]. To a first semiclassical approximation, where quantum loops and quark masses are not included, this leads to a LF Hamiltonian equation which describes the bound-state dynamics of light hadrons in terms of an invariant impact variable $\zeta$ [17] which measures the separation of the partons within the hadron at equal light-front time $\tau = x^0 + x^3$ [18]. This allows us to identify the holographic variable $z$ in AdS space with the impact variable $\zeta$ [4,10,17].

The pion transition form factor (TFF) between a photon and pion measured in the $e^+e^- \to e^+e^-\pi^0$ process, with one tagged electron, is the simplest bound-state process in QCD. It can be predicted from first principles in the asymptotic $Q^2 \to \infty$ limit [13]. More generally, the pion TFF at large $Q^2$ can be calculated at leading twist as a convolution of a perturbative hard scattering amplitude $T_H(\gamma\gamma^* \to q\bar{q})$ and a gauge-invariant meson distribution amplitude (DA) which incorporates the nonperturbative dynamics of the QCD bound-state [13].

 

The *BABAR* Collaboration has reported measurements of the transition form factors from the $\gamma^*\gamma \to M$ process for the $\pi^0$ [19], $\eta$, and $\eta'$ [20,21] pseudoscalar mesons for a momentum transfer range much larger than previous measurements [22,23]. Surprisingly, the *BABAR* data for the $\pi^0 - \gamma$ TFF exhibit a rapid growth for $Q^2 > 15\ \mathrm{GeV}^2$, which is unexpected from QCD predictions. In contrast, the data for the $\eta - \gamma$ and $\eta' - \gamma$ TFFs are in agreement with previous experiments and theoretical predictions. Many theoretical studies have been devoted to explaining *BABAR*'s experimental results [24–35].

Motivated by the conflict of theory with experimental results we have examined in a recent paper [34] existing models and approximations used in the computation of pseudoscalar meson TFFs in QCD, incorporating the evolution of the pion distribution amplitude [13,36] which controls the meson TFFs at large $Q^2$. In this article we will study the anomalous coupling of mesons to photons which follows from the Chern-Simons (CS) action present in the dual higher dimensional gravity theory [3,37], which is required to describe the meson transition form factor using holographic principles. A simple analytical form is found which satisfies both the low-energy theorem for the decay $\pi^0 \to \gamma\gamma$ and the QCD predictions at large $Q^2$, thus allowing us to encompass the perturbative and nonperturbative spacelike regimes in a simple model. We choose the soft-wall approach to modify the infrared AdS geometry to include confinement, but the general results are not expected to be sensitive to the specific model chosen to deform AdS space in the IR since the Chern-Simons action is a topological invariant.

After a brief review of EM meson form factors in the framework of light-front holographic QCD in Sec. II, we discuss the Chern-Simons structure of the meson transition form factor in AdS space in Sec. III. The pion transition form factors calculated with the free and dressed currents are presented in Sec IV. The higher Fock state contributions to the pion transition form factor are studied in Sec. V for a dressed EM current model. The results for the $\eta$ and $\eta'$ transition form factors are given in Sec. VI. Some conclusions are presented in Sec. VII. Different forms of the pion light-front wave functions (LFWFs) from holographic mappings are discussed in the Appendix.

## II. MESON ELECTROMAGNETIC FORM FACTOR

In the higher dimensional gravity theory, the hadronic transition matrix element corresponds to the coupling of an external electromagnetic field $A^M(x, z)$ for a photon propagating in AdS space with the extended field $\Phi_P(x, z)$ describing a meson in AdS [6] and it is given by

$$\int d^4x \int dz \sqrt{g} A^M(x, z) \Phi^*_{P'}(x, z) \overleftrightarrow{\partial}_M \Phi_P(x, z) \sim (2\pi)^4 \delta^{(4)}(P' - P - q) \epsilon_\mu (P + P')^\mu F_M(q^2), \quad (2)$$

where the coordinates of $\mathrm{AdS}_5$ are the Minkowski coordinates $x^\mu$ and $z$ labeled $x^M = (x^\mu, z)$, with $M = 1, \ldots, 5$, and $g$ is the determinant of the metric tensor. The pion has initial and final four-momenta $P$ and $P'$, respectively, and $q$ is the four-momentum transferred to the pion by the photon with polarization $\epsilon_\mu$. The expression on the right-hand side of (2) represents the spacelike QCD electromagnetic transition amplitude in physical space-time $\langle P' | J^\mu(0) | P \rangle = (P + P')^\mu F_M(q^2)$. It is the EM matrix element of the quark current $J^\mu = e_q \bar{q} \gamma^\mu q$, and represents a local coupling to pointlike constituents. Although the expressions for the transition amplitudes look very different, one can show that a precise mapping of the matrix elements can be carried out at fixed light-front time [4,5].

The form factor is computed in the light front from the matrix elements of the plus-component of the current $J^+$, in order to avoid coupling to Fock states with different numbers of constituents. Expanding the initial and final mesons states $|\psi_M(P^+, \mathbf{P}_\perp)\rangle$ in terms of Fock components, $|\psi_M\rangle = \sum_n \psi_{n/M} |n\rangle$, we obtain DYW expression [7,8] upon the phase space integration over the intermediate variables in the $q^+ = 0$ frame,

$$F_M(q^2) = \sum_n \int [dx_i][d^2\mathbf{k}_{\perp i}] \sum_j e_j \psi^*_{n/M}(x_i, \mathbf{k}'_{\perp i}, \lambda_i) \times \psi_{n/M}(x_i, \mathbf{k}_{\perp i}, \lambda_i), \quad (3)$$

where the variables of the light-cone Fock components in the final state are given by $\mathbf{k}'_{\perp i} = \mathbf{k}_{\perp i} + (1 - x_i)\mathbf{q}_\perp$ for a struck constituent quark and $\mathbf{k}'_{\perp i} = \mathbf{k}_{\perp i} - x_i \mathbf{q}_\perp$ for each spectator. The formula is exact if the sum is over all Fock states $n$. The $n$-parton Fock components $\psi_{n/M}(x_i, \mathbf{k}_{\perp i}, \lambda_i)$ are independent of $P^+$ and $\mathbf{P}_\perp$ and depend only on the relative partonic coordinates: the momentum fraction $x_i = k_i^+/P^+$, the transverse momentum $\mathbf{k}_{\perp i}$, and spin component $\lambda_i^z$. Momentum conservation requires $\sum_{i=1}^n x_i = 1$ and $\sum_{i=1}^n \mathbf{k}_{\perp i} = 0$. The light-front wave functions $\psi_n$ provide a frame-independent representation of a hadron which relates its quark and gluon degrees of freedom to their asymptotic hadronic state. The form factor can also be conveniently written in impact space as a sum of overlap of LFWFs of the $j = 1, 2, \cdots, n - 1$ spectator constituents [9]

$$F_M(q^2) = \sum_n \prod_{j=1}^{n-1} \int dx_j d^2\mathbf{b}_{\perp j} \times \exp\left(i\mathbf{q}_\perp \cdot \sum_{j=1}^{n-1} x_j \mathbf{b}_{\perp j}\right) |\psi_{n/M}(x_j, \mathbf{b}_{\perp j})|^2, \quad (4)$$

corresponding to a change of transverse momentum $x_j \mathbf{q}_\perp$ for each of the $n - 1$ spectators with $\sum_{i=1}^n \mathbf{b}_{\perp i} = 0$.

For definiteness we shall consider the $\pi^+$ valence Fock state $|u\bar{d}\rangle$ with charges $e_u = \frac{2}{3}$ and $e_{\bar{d}} = \frac{1}{3}$. For $n = 2$, there are two terms which contribute to Eq. (4). Exchanging $x \leftrightarrow 1 - x$ in the second integral we find

$$F_{\pi^+}(q^2) = 2\pi \int_0^1 \frac{dx}{x(1-x)} \times \int \zeta d\zeta J_0\left(\zeta q \sqrt{\frac{1-x}{x}}\right) |\psi_{u\bar{d}/\pi}(x, \zeta)|^2, \quad (5)$$

where $\zeta^2 = x(1-x)\mathbf{b}_\perp^2$ and $F_{\pi^+}(q = 0) = 1$.

We now compare this result with the electromagnetic form factor in AdS space-time. The incoming electromagnetic field propagates in AdS according to $A_\mu(x^\mu, z) = \epsilon_\mu(q)e^{-iq\cdot x}V(q^2, z)$, where $V(q^2, z)$, the bulk-to-boundary propagator, is the solution of the AdS wave equation given by

$$V(Q^2, z) = zQK_1(zQ), \quad (6)$$

with $Q^2 = -q^2 > 0$ and boundary conditions $V(q^2 = 0, z) = V(q^2, z = 0) = 1$ [6]. The propagation of the pion in AdS space is described by a normalizable mode $\Phi_P(x^\mu, z) = e^{-iP\cdot x}\Phi(z)$ with invariant mass $P_\mu P^\mu = \mathcal{M}_\pi^2$ and plane waves along Minkowski coordinates $x^\mu$. In the chiral limit for massless quarks $\mathcal{M}_\pi = 0$. Extracting the overall factor $(2\pi)^4\delta^4(P' - P - q)$ from momentum conservation at the vertex from integration over Minkowski variables in (2) we find [6]

$$F(Q^2) = R^3 \int \frac{dz}{z^3} V(Q^2, z)\Phi^2(z), \quad (7)$$

where $F(Q^2 = 0) = 1$. Using the integral representation of $V(Q^2, z)$

$$V(Q^2, z) = \int_0^1 dx J_0\left(zQ\sqrt{\frac{1-x}{x}}\right), \quad (8)$$

we write the AdS electromagnetic form factor as

$$F(Q^2) = R^3 \int_0^1 dx \int \frac{dz}{z^3} J_0\left(zQ\sqrt{\frac{1-x}{x}}\right)|\Phi(z)|^2. \quad (9)$$

To compare with the light-front QCD form factor expression (5) we write the LFWF as

$$\psi(x, \zeta, \varphi) = e^{iM\varphi}X(x)\frac{\phi(\zeta)}{\sqrt{2\pi\zeta}}, \quad (10)$$

thus factoring out the angular dependence $\varphi$ in the transverse LF plane, the longitudinal $X(x)$, and transverse mode $\phi(\zeta)$.[1] If both expressions for the form factor are identical for arbitrary values of $Q$, we obtain $\phi(\zeta) = (\zeta/R)^{3/2}\Phi(\zeta)$ and $X(x) = \sqrt{x(1-x)}$ [4], where we identify the transverse impact LF variable $\zeta$ with the holographic variable $z$, $z \to \zeta = \sqrt{x(1-x)}|\mathbf{b}_\perp|$. We choose the normalization $\langle\phi|\phi\rangle = \int d\zeta|\langle\zeta|\phi\rangle|^2 = P_{q\bar{q}}$, where $P_{q\bar{q}}$ is the probability of finding the $q\bar{q}$ component in the pion light-front wave function. The longitudinal mode is thus normalized as $\int_0^1 \frac{X^2(x)}{x(1-x)} = 1$.[2] Identical results follow from mapping the matrix elements of the energy-momentum tensor [10].

### A. Elastic form factor with a dressed current

The results for the elastic form factor described above correspond to a free current propagating on AdS space. It is dual to the electromagnetic pointlike current in the Drell-Yan-West light-front formula [7,8] for the pion form factor. The DYW formula is an exact expression for the form factor. It is written as an infinite sum of an overlap of LF Fock components with an arbitrary number of constituents. This allows one to map state-by-state to the effective gravity theory in AdS space. However, this mapping has the shortcoming that the multiple pole structure of the timelike form factor cannot be obtained in the timelike region unless an infinite number of Fock states is included. Furthermore, the moments of the form factor at $Q^2 = 0$ diverge term-by-term; for example, one obtains an infinite charge radius [38].

Alternatively, one can use a truncated basis of states in the LF Fock expansion with a limited number of constituents, and the nonperturbative pole structure can be generated with a dressed EM current as in the Heisenberg picture, i.e., the EM current becomes modified as it propagates in an IR deformed AdS space to simulate confinement. The dressed current is dual to a hadronic EM current which includes any number of virtual $q\bar{q}$ components.

Conformal invariance can be broken analytically by the introduction of a confining dilaton profile $\varphi(z)$ in the action, $S = \int d^4x \int dz\sqrt{g}e^{\varphi(z)}\mathcal{L}$, thus retaining conformal AdS metrics as well as introducing a smooth IR cutoff. It is convenient to scale away the dilaton factor in the action by a field redefinition [39,40]. For example, for a scalar field we shift $\Phi \to e^{-\varphi/2}\Phi$, and the bilinear component in the action is transformed into the equivalent problem of a free kinetic part plus an effective potential $V(\Phi, \varphi)$.

A particularly interesting case is a dilaton profile $\exp(\pm\kappa^2z^2)$ of either sign, since it leads to linear Regge trajectories consistent with the light-quark hadron spectroscopy [41]. It avoids the ambiguities in the choice of boundary conditions at the infrared wall. In this case the effective potential takes the form of a harmonic oscillator confining potential $\kappa^4z^2$, and the normalizable solution for

[1]The factorization of the LFWF given by (10) is a natural factorization in the light-front formalism since the corresponding canonical generators, the longitudinal and transverse generators $P^+$ and $\mathbf{P}_\perp$, and the $z$-component of the orbital angular momentum $J^z$, are kinematical generators which commute with the LF Hamiltonian generator $P^-$ [18].

[2]Extension of the results to arbitrary $n$ follows from the $x$-weighted definition of the transverse impact variable of the $n-1$ spectator system [4]: $\zeta = \sqrt{\frac{x}{1-x}}|\sum_{j=1}^{n-1} x_j\mathbf{b}_{\perp j}|$, where $x = x_n$ is the longitudinal momentum fraction of the active quark. In general the mapping relates the AdS density $\Phi^2(z)$ to an effective LF single-particle transverse density [4].

a meson of a given twist $\tau$, corresponding to the lowest radial $n = 0$ node, is given by

$$\Phi^\tau(z) = \sqrt{\frac{2P_\tau}{\Gamma(\tau - 1)}}\kappa^{\tau-1} z^\tau e^{-\kappa^2 z^2/2}, \quad (11)$$

with normalization

$$\langle \Phi^\tau | \Phi^\tau \rangle = \int \frac{dz}{z^3} e^{-\kappa^2 z^2} \Phi^\tau(z)^2 = P_\tau, \quad (12)$$

where $P_\tau$ is the probability for the twist $\tau$ mode (11). This agrees with the fact that the field $\Phi^\tau$ couples to a local hadronic interpolating operator of twist $\tau$ defined at the asymptotic boundary of AdS space, and thus the scaling dimension of $\Phi^\tau$ is $\tau$.

In the case of soft-wall potential [41], the EM bulk-to-boundary propagator is [5,42]

$$V(Q^2, z) = \Gamma\left(1 + \frac{Q^2}{4\kappa^2}\right) U\left(\frac{Q^2}{4\kappa^2}, 0, \kappa^2 z^2\right), \quad (13)$$

where $U(a, b, c)$ is the Tricomi confluent hypergeometric function. The modified current $V(Q^2, z)$, Eq. (13), has the same boundary conditions as the free current (6), and reduces to (6) in the limit $Q^2 \to \infty$. Equation (13) can be conveniently written in terms of the integral representation [42]

$$V(Q^2, z) = \kappa^2 z^2 \int_0^1 \frac{dx}{(1 - x)^2} x^{(Q^2/4\kappa^2)} e^{-\kappa^2 z^2 x/(1-x)}. \quad (14)$$

Hadronic form factors for the harmonic potential $\kappa^2 z^2$ have a simple analytical form [5]. Substituting in (7) the expression for a hadronic state (11) with twist $\tau = N + L$ ($N$ is the number of components) and the bulk-to-boundary propagator (14) we find that the corresponding elastic form factor for a twist $\tau$ Fock component $F_\tau(Q^2)$ ($Q^2 = -q^2 > 0$)

$$F_\tau(Q^2) = \frac{P_\tau}{\left(1 + \frac{Q^2}{M_\rho^2}\right)\left(1 + \frac{Q^2}{M_{\rho'}^2}\right) \cdots \left(1 + \frac{Q^2}{M_{\rho^{\tau-2}}^2}\right)}, \quad (15)$$

which is expressed as a $\tau - 1$ product of poles along the vector meson Regge radial trajectory. For a pion, for example, the lowest Fock state—the valence state—is a twist-2 state, and thus the form factor is the well-known monopole form [5]. The remarkable analytical form of (15), expressed in terms of the $\rho$ vector meson mass and its radial excitations, incorporates the correct scaling behavior from the constituent's hard scattering with the photon and the mass gap from confinement. It is also apparent from (15) that the higher-twist components in the Fock expansion are relevant for the computation of hadronic form factors, particularly for the timelike region which is particularly sensitive to the detailed structure of the amplitudes [43]. For a confined EM current in AdS a precise mapping can also be carried out to the DYW expression for the form factor. In this case we find an effective LFWF, which corresponds to a superposition of an infinite number of Fock states. This is discussed in the Appendix for the soft-wall model.

## III. THE CHERN-SIMONS STRUCTURE OF THE MESON TRANSITION FORM FACTOR IN ADS SPACE

To describe the pion transition form factor within the framework of holographic QCD we need to explore the mathematical structure of higher-dimensional forms in the five-dimensional action, since the amplitude (2) can only account for the elastic form factor $F_M(Q^2)$. For example, in the five-dimensional compactification of type IIB supergravity [44,45], there is a Chern-Simons term in the action in addition to the usual Yang-Mills term $F^2$ [3]. In the case of the $U(1)$ gauge theory the CS action is of the form $\epsilon^{LMNPQ} A_L \partial_M A_N \partial_P A_Q$ in the five-dimensional Lagrangian [37]. The CS action is not gauge-invariant: under a gauge transformation it changes by a total derivative which gives a surface term.

The Chern-Simons form is the product of three fields at the same point in five-dimensional space corresponding to a local interaction. Indeed the five-dimensional CS action is responsible for the anomalous coupling of mesons to photons and has been used to describe, for example, the $\omega \to \pi\gamma$ [46] decay as well as the $\gamma\gamma^* \to \pi^0$ [47] and $\gamma^*\rho^0 \to \pi^0$ [48] processes.[3]

The hadronic matrix element for the anomalous electromagnetic coupling to mesons in the higher gravity theory is given by the five-dimensional CS amplitude

$$\int d^4x \int dz \epsilon^{LMNPQ} A_L \partial_M A_N \partial_P A_Q \sim (2\pi)^4 \delta^{(4)}(P - q - k) F_{\pi\gamma}(q^2) \epsilon^{\mu\nu\rho\sigma} \epsilon_\mu(q) P_\nu \epsilon_\rho(k) q_\sigma, \quad (16)$$

which includes the pion field as well as the external photon fields by identifying the fifth component of $A$ with the meson mode in AdS space [50]. In the right-hand side of (16) $q$ and $k$ are the momenta of the virtual and on-shell incoming photons, respectively, with corresponding polarization vectors $\epsilon_\mu(q)$ and $\epsilon_\mu(k)$ for the amplitude $\gamma\gamma^* \to \pi^0$. The momentum of the outgoing pion is $P$.

The pion transition form factor $F_{\pi\gamma}(Q^2)$ can be computed from first principles in QCD. To leading order in $\alpha_s(Q^2)$ and leading twist, the result is [13] ($Q^2 = -q^2 > 0$)

$$Q^2 F_{\pi\gamma}(Q^2) = \frac{4}{\sqrt{3}} \int_0^1 dx \frac{\phi(x, \bar{x}Q)}{\bar{x}} \left[1 + O\left(\alpha_s, \frac{m^2}{Q^2}\right)\right], \quad (17)$$

[3]The anomalous EM couplings to mesons in the Sakai and Sugimoto model is described in Ref. [49].

where $x$ is the longitudinal momentum fraction of the quark struck by the virtual photon in the hard scattering process and $\bar{x} = 1 - x$ is the longitudinal momentum fraction of the spectator quark. The pion distribution amplitude $\phi(x, Q)$ in the light-front formalism [13] is the integral of the valence $q\bar{q}$ LFWF in light-cone gauge $A^+ = 0$

$$\phi(x, Q) = \int_0^{Q^2} \frac{d^2\mathbf{k}_\perp}{16\pi^3} \psi_{q\bar{q}/\pi}(x, \mathbf{k}_\perp), \tag{18}$$

and has the asymptotic form [13] $\phi(x, Q \to \infty) = \sqrt{3} f_\pi x(1 - x)$; thus the leading order QCD result for the TFF at the asymptotic limit is obtained [13],

$$Q^2 F_{\pi\gamma}(Q^2 \to \infty) = 2f_\pi. \tag{19}$$

We now compare the QCD expression on the right-hand side of (16) with the AdS transition amplitude on the left-hand side. As for the elastic form factor discussed in Sec. II, the incoming off-shell photon is represented by the propagation of the non-normalizable electromagnetic solution in AdS space, $A_\mu(x^\mu, z) = \epsilon_\mu(q)e^{-iq\cdot x}V(q^2, z)$, where $V(q^2, z)$ is the bulk-to-boundary propagator with boundary conditions $V(q^2 = 0, z) = V(q^2, z = 0) = 1$ [6]. Since the incoming photon with momentum $k$ is on its mass shell, $k^2 = 0$, its wave function is $A_\mu(x^\mu, z) = \epsilon_\mu(k)e^{-ik\cdot x}$. Likewise, the propagation of the pion in AdS space is described by a normalizable mode $\Phi_P(x^\mu, z) = e^{-iP\cdot x}\Phi_\pi(z)$ with invariant mass $P_\mu P^\mu = \mathcal{M}_\pi^2 = 0$ in the chiral limit for massless quarks. The normalizable mode $\Phi(z)$ scales as $\Phi(z) \to z^{\tau=2}$ in the limit $z \to 0$, since the leading interpolating operator for the pion has twist-2. A simple dimensional analysis implies that $A_z \sim \Phi_\pi(z)/z$, matching the twist scaling dimensions: two for the pion and one for the EM field. Substituting in (16) the expression given above for the pion and the EM fields propagating in AdS, and extracting the overall factor $(2\pi)^4\delta^{(4)}(P - q - k)$ upon integration over Minkowski variables in (16) we find ($Q^2 = -q^2 > 0$)

$$F_{\pi\gamma}(Q^2) = \frac{1}{2\pi}\int_0^\infty \frac{dz}{z}\Phi_\pi(z)V(Q^2, z), \tag{20}$$

where the normalization is fixed by the asymptotic QCD prediction (19). We have defined our units such that the AdS radius $R = 1$.

Since the LF mapping of (20) to the asymptotic QCD prediction (19) only depends on the asymptotic behavior near the boundary of AdS space, the result is independent of the particular model used to modify the large $z$ IR region of AdS space. At large enough $Q$, the important contribution to (19) only comes from the region near $z \sim 1/Q$ where $\Phi(z) = 2\pi f_\pi z^2 + \mathcal{O}(z^4)$. Using the integral

$$\int_0^\infty dx x^\alpha K_1(x) = 2^{\alpha-2}\alpha\left[\Gamma\left(\frac{\alpha}{2}\right)\right]^2, \quad \mathrm{Re}(\alpha) > 1, \tag{21}$$

we recover the asymptotic result (19)

$$Q^2 F_{\pi\gamma}(Q^2 \to \infty) = 2f_\pi + \mathcal{O}\left(\frac{1}{Q^2}\right), \tag{22}$$

with the pion decay constant $f_\pi$ (see the Appendix)

$$f_\pi = \frac{1}{4\pi}\frac{\partial_z\Phi_\pi(z)}{z}\bigg|_{z=0}. \tag{23}$$

Since the pion field is identified as the fifth component of $A_M$, the CS form $\epsilon^{LMNPQ}A_L\partial_M A_N\partial_P A_Q$ is similar in form to an axial current; this correspondence can explain why the resulting pion distribution amplitude has the asymptotic form.

In Ref. [47] the pion TFF was studied in the framework of a CS extended hard-wall AdS/QCD model with $A_z \sim \partial_z\Phi(z)$. The expression for the TFF which follows from (16) then vanishes at $Q^2 = 0$, and has to be corrected by the introduction of a surface term at the IR wall [47]. However, this procedure is only possible for a model with a sharp cutoff. The pion TFF has also been studied using the holographic approach to QCD in Refs. [51–53].

## IV. A SIMPLE HOLOGRAPHIC CONFINING MODEL

QCD predictions of the TFF correspond to the local coupling of the free electromagnetic current to the elementary constituents in the interaction representation [13]. To compare with QCD results, we first consider a simplified model where the non-normalizable mode $V(Q^2, z)$ for the EM current satisfies the "free" AdS equation subject to the boundary conditions $V(Q^2 = 0, z) = V(Q^2, z = 0) = 1$; thus the solution $V(Q^2, z) = zQK_1(zQ)$, dual to the free electromagnetic current [4]. To describe the normalizable mode representing the pion we take the soft-wall exponential form (11). Its LF mapping has also a convenient exponential form and has been studied considerably in the literature [34]. The exponential form of the LFWF in momentum space has important support only when the virtual states are near the energy shell, and thus it implements in a natural way the requirements of the bound-state dynamics. From (11) we have for twist $\tau = 2$

$$\Phi_{q\bar{q}/\pi}(z) = \sqrt{2P_{q\bar{q}}}\kappa z^2 e^{-\kappa^2 z^2/2}, \tag{24}$$

with normalization

$$\langle\Phi_{q\bar{q}/\pi}|\Phi_{q\bar{q}/\pi}\rangle = \int\frac{dz}{z^3}\Phi^2_{q\bar{q}/\pi}(z) = P_{q\bar{q}}, \tag{25}$$

where $P_{q\bar{q}}$ is the probability for the valence state. From (23) the pion decay constant is

$$f_\pi = \sqrt{P_{q\bar{q}}}\frac{\kappa}{\sqrt{2}\pi}. \tag{26}$$

It is not possible in this model to introduce a surface term as in Ref. [47] to match the value of the TFF at $Q^2 = 0$ derived from the decay $\pi^0 \to \gamma\gamma$. Instead, higher Fock components which modify the pion wave function at large distances are required to satisfy this low-energy constraint naturally. Since the higher-twist components have a faster falloff at small distances, the asymptotic results are not modified.

Substituting the pion wave function (24) and using the integral representation for $V(Q^2, z)$

$$zQK_1(zQ) = 2Q^2 \int_0^\infty \frac{tJ_0(zt)}{(t^2 + Q^2)^2} dt, \quad (27)$$

we find upon integration

$$F_{\pi\gamma}(Q^2) = \frac{\sqrt{2P_{q\bar{q}}}Q^2}{\pi\kappa} \int_0^\infty \frac{tdt}{(t^2 + Q^2)^2} e^{-t^2/2\kappa^2}. \quad (28)$$

Changing variables as $x = \frac{Q^2}{t^2+Q^2}$ one obtains

$$F_{\pi\gamma}(Q^2) = \frac{P_{q\bar{q}}}{2\pi^2 f_\pi} \int_0^1 dx \exp\left(-\frac{(1-x)P_{q\bar{q}}Q^2}{4\pi^2 f_\pi^2 x}\right). \quad (29)$$

Upon integration by parts, Eq. (29) can also be written as

$$Q^2 F_{\pi\gamma}(Q^2) = \frac{4}{\sqrt{3}} \int_0^1 dx \frac{\phi(x)}{1-x}\left[1 - \exp\left(-\frac{(1-x)P_{q\bar{q}}Q^2}{4\pi^2 f_\pi^2 x}\right)\right], \quad (30)$$

where $\phi(x) = \sqrt{3} f_\pi x(1-x)$ is the asymptotic QCD distribution amplitude with $f_\pi$ given by (26).

Remarkably, the pion transition form factor given by (30) for $P_{q\bar{q}} = 1$ is identical to the results for the pion TFF obtained with the exponential light-front wave function model of Musatov and Radyushkin [54], consistent with the leading order QCD result [13] for the TFF at the asymptotic limit, $Q^2 F_{\pi\gamma}(Q^2 \to \infty) = 2f_\pi$.[4,5] The leading-twist result (30) does not include nonleading order $\alpha_s$ corrections in the hard scattering amplitude nor gluon exchange in the evolution of the distribution amplitude, since the semiclassical correspondence implied in the gauge/gravity duality does not contain quantum effects such as particle emission and absorption.

The transition form factor at $Q^2 = 0$ can be obtained from Eq. (30),

$$F_{\pi\gamma}(0) = \frac{1}{2\pi^2 f_\pi} P_{q\bar{q}}. \quad (31)$$

The form factor $F_{\pi\gamma}(0)$ is related to the decay width for the $\pi^0 \to \gamma\gamma$ decay,

$$\Gamma_{\pi^0\to\gamma\gamma} = \frac{\alpha^2 \pi m_\pi^3}{4} F_{\pi\gamma}^2(0), \quad (32)$$

where $\alpha = 1/137$. The form factor $F_{\pi\gamma}(0)$ is also well described by the Schwinger, Adler, Bell, and Jackiw anomaly [55] which gives

$$F_{\pi\gamma}^{\mathrm{SABJ}}(0) = \frac{1}{4\pi^2 f_\pi}, \quad (33)$$

in agreement within a few percent of the observed value obtained from the decay $\pi^0 \to \gamma\gamma$.

Taking $P_{q\bar{q}} = 0.5$ in (31) one obtains a result in agreement with (33). This suggests that the contribution from higher Fock states vanishes at $Q = 0$ in this simple holographic confining model (see Sec. V for further discussion). Thus (30) represents a description on the pion TFF which encompasses the low-energy nonperturbative and the high-energy hard domains, but includes only the asymptotic DA of the $q\bar{q}$ component of the pion wave function at all scales. The results from (30) are shown as dotted curves in Figs. 1 and 2 for $Q^2 F_{\pi\gamma}(Q^2)$ and $F_{\pi\gamma}(Q^2)$, respectively. The calculations agree reasonably well with the experimental data at low- and medium-$Q^2$ regions ($Q^2 < 10$ GeV$^2$), but disagree with *BABAR*'s large $Q^2$ data.

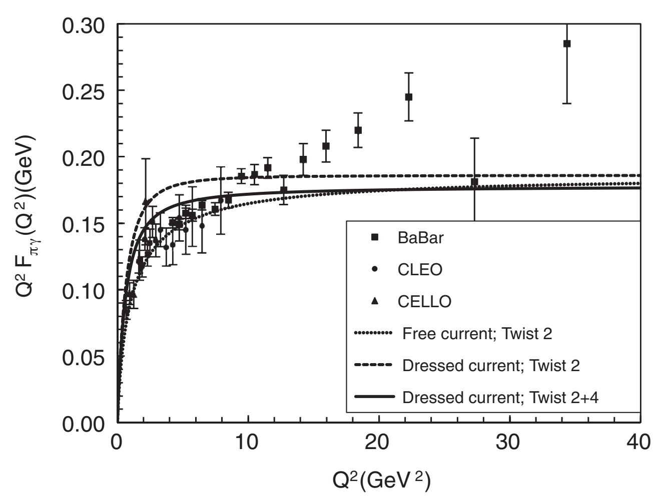


FIG. 1. The $\gamma\gamma^* \to \pi^0$ transition form factor shown as $Q^2 F_{\pi\gamma}(Q^2)$ as a function of $Q^2 = -q^2$. The dotted curve is the asymptotic result predicted by the Chern-Simons form. The dashed and solid curves include the effects of using a confined EM current for twist-2 and twist-2 plus twist-4, respectively. The data are from [19,22,23].

[4]The expression (30) is not appropriate to describe the timelike region where the exponential factor in (30) grows exponentially. It is important to study the behavior of the pion TFF in other kinematical regions to describe, for example, the process $e^+ + e^- \to \gamma^* \to \pi^0 + \gamma$. This also would test the *BABAR* anomaly.

[5]A similar mapping can be done for the case when the two photons are virtual $\gamma^*\gamma^* \to \pi^0$. In the case where at least one of the incoming photons has large virtuality the transition form factor can be expressed analytically in a simple form. The result is $F_{\pi\gamma^*}(q^2, k^2) = -\frac{4}{\sqrt{3}} \int_0^1 dx \frac{\phi(x)}{xq^2 + (1-x)k^2}$, with $\phi(x)$ the asymptotic DA. See Ref. [47].

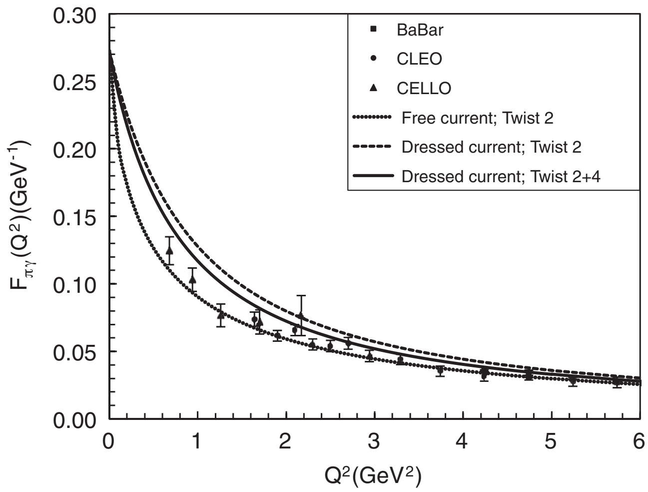


FIG. 2. Same as Fig. 1 for $F_{\pi\gamma}(Q^2)$.

### A. Transition form factor with the dressed current

The simple valence $q\bar{q}$ model discussed above should be modified at small $Q^2$ by introducing the dressed current which corresponds effectively to a superposition of Fock states (see the Appendix). Inserting the valence pion wave function (24) and the confined EM current (14) in the amplitude (20) one finds

$$F_{\pi\gamma}(Q^2) = \frac{P_{q\bar{q}}}{\pi^2 f_\pi} \int_0^1 \frac{dx}{(1+x)^2} x^{Q^2 P_{q\bar{q}}/(8\pi^2 f_\pi^2)}. \tag{34}$$

Equation (34) gives the same value for $F_{\pi\gamma}(0)$ as (31), which was obtained with the free current. Thus the anomaly result $F_{\pi\gamma}(0) = 1/(4\pi^2 f_\pi)$ is reproduced if $P_{q\bar{q}} = 0.5$ is also taken in (34). Upon integration by parts, Eq. (34) can also be written as

$$Q^2 F_{\pi\gamma}(Q^2) = 8 f_\pi \int_0^1 dx \frac{1-x}{(1+x)^3} (1 - x^{Q^2 P_{q\bar{q}}/(8\pi^2 f_\pi^2)}). \tag{35}$$

Noticing that the second term in Eq. (35) vanishes at the limit $Q^2 \to \infty$, one recovers Brodsky-Lepage's asymptotic prediction for the pion TFF: $Q^2 F_{\pi\gamma}(Q^2 \to \infty) = 2 f_\pi$ [13].

The results calculated with (34) for $P_{q\bar{q}} = 0.5$ are shown as dashed curves in Figs. 1 and 2. One can see that the calculations with the dressed current are larger as compared with the results computed with the free current and the experimental data at low- and medium-$Q^2$ regions ($Q^2 < 10$ GeV$^2$). The new results again disagree with *BABAR*'s data at large $Q^2$.

## V. HIGHER-TWIST COMPONENTS TO THE TRANSITION FORM FACTOR

In a previous light-front QCD analysis of the pion TFF [56] it was argued that the valence Fock state $|q\bar{q}\rangle$ provides only half of the contribution to the pion TFF at $Q^2 = 0$, while the other half comes from diagrams where the virtual photon couples inside the pion (strong interactions occur between the two photon interactions). This leads to a surprisingly small value for the valence Fock state probability $P_{q\bar{q}} = 0.25$. More importantly, this raises the question on the role played by the higher Fock components of the pion LFWF,

$$|\pi\rangle = \psi_2|q\bar{q}\rangle + \psi_3|q\bar{q}g\rangle + \psi_4|q\bar{q}q\bar{q}\rangle + \cdots, \tag{36}$$

in the calculations for the pion TFF.

The contributions to the transition form factor from these higher Fock states are suppressed, compared with the valence Fock state, by the factor $1/(Q^2)^n$ for $n$ extra $q\bar{q}$ pairs in the higher Fock state, since one needs to evaluate an off-diagonal matrix element between the real photon and the multiquark Fock state [13]. We note that in the case of the elastic form factor, the power suppression is $1/(Q^2)^{2n}$ for $n$ extra $q\bar{q}$ pairs in the higher Fock state. These higher Fock state contributions are negligible at high $Q^2$. On the other hand, it has long been argued that the higher Fock state contributions are necessary to explain the experimental data at the medium $Q^2$ region for exclusive processes [57,58]. The contributions from the twist-3 components of the two-parton pion distribution amplitude to the pion elastic form factors were evaluated in Ref. [59]. The three-parton contributions to the pion elastic form factor were studied in Ref. [60]. The contributions from diagrams where the virtual photon couples inside the pion to the pion transition form factor were estimated using light-front wave functions in Refs. [26,61]. The higher twist (twist-4 and twist-6) contributions to the pion transition form factor [62] were evaluated using the method of light-cone sum rules in Refs. [33,35], but opposite claims were made on whether the *BABAR* data could be accommodated by including these higher twist contributions.

It is also not very clear how the higher Fock states contribute to decay processes, such as $\pi^0 \to \gamma\gamma$ [63], due to the long-distance nonperturbative nature of decay processes. Second-order radiative corrections to the triangle anomaly do not change the anomaly results as they contain one internal photon line and two vertices on the triangle loop. Upon regulation no new anomaly contribution occurs. In fact, the result is expected to be valid at all orders in perturbation theory [64,65]. It is thus generally argued that in the chiral limit of QCD (i.e., $m_q \to 0$), one needs only the $q\bar{q}$ component to explain the anomaly, but as shown below, the higher Fock state components can also contribute to the decay process $\pi^0 \to \gamma\gamma$ in the chiral limit.

As discussed in the last two sections, matching the AdS/QCD results computed with the free and dressed currents for the TFF at $Q^2 = 0$ with the anomaly result requires a probability $P_{q\bar{q}} = 0.5$. Thus it is important to investigate the contributions from the higher Fock states. In AdS/QCD there are no dynamic gluons and confinement is realized via an effective instantaneous interaction in light-front time, analogous to the instantaneous gluon exchange [16]. The effective confining potential also creates quark-antiquark pairs from the amplitude $q \to q\bar{q}q$. Thus in AdS/QCD higher Fock states can have any number of extra $q\bar{q}$ pairs. These higher Fock states lead to higher-twist contributions to the pion transition form factor.

To illustrate this observation consider the two diagrams in Fig. 3. In the leading process, Fig. 3(a), where both photons couple to the same quark, the valence $|q\bar{q}\rangle$ state has $J^z = S^z = L^z = 0$,

$$|q\bar{q}\rangle = \frac{1}{\sqrt{2}}\left(\left|+\frac{1}{2}, -\frac{1}{2}\right\rangle - \left|-\frac{1}{2}, +\frac{1}{2}\right\rangle\right). \tag{37}$$

Equation (37) represents a $J^{PC} = 0^{-+}$ state with the quantum numbers of the conventional $\pi$ meson axial vector interpolating operator $\mathcal{O} = \bar{\psi}\gamma^+\gamma^5\psi$.

In the process involving the four-quark state $|q\bar{q}q\bar{q}\rangle$ of the pion, Fig. 3(b), where each photon couples directly to a $q\bar{q}$ pair, the four-quark state also satisfies $J^z = S^z = L^z = 0$ and is represented by

$$|q\bar{q}q\bar{q}\rangle = \frac{1}{2}\left(\left|+\frac{1}{2}, -\frac{1}{2}, +\frac{1}{2}, -\frac{1}{2}\right\rangle + \left|+\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, +\frac{1}{2}\right\rangle - \left|-\frac{1}{2}, +\frac{1}{2}, +\frac{1}{2}, -\frac{1}{2}\right\rangle - \left|-\frac{1}{2}, +\frac{1}{2}, -\frac{1}{2}, +\frac{1}{2}\right\rangle\right). \tag{38}$$

The four-quark state in Eq. (38) has also quantum numbers $J^{PC} = 0^{-+}$ corresponding to the quantum numbers of the local interpolating operators $\mathcal{O} = \bar{\psi}\gamma^+\gamma^5\psi\bar{\psi}\psi$, where the scalar interpolating operator $\bar{\psi}\psi$ has quantum numbers $J^{PC} = 0^{++}$.

We note that for the Compton scattering $\gamma H \to \gamma H$ process, similar higher-twist contributions, as illustrated in Fig. 3(b), are proportional to $\sum_{e_i \neq e_j} e_i e_j$ and are necessary to derive the low-energy amplitude for Compton scattering which is proportional to the total charge squared $e_H^2 = (e_i + e_j)^2$ of the target [66].

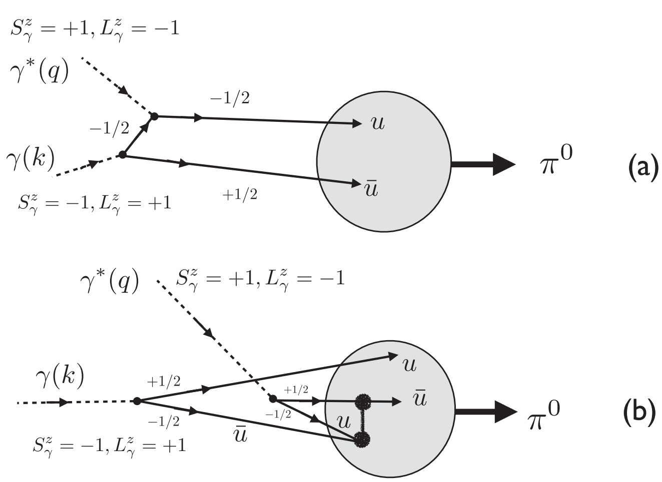


FIG. 3. Leading-twist contribution (a) and twist-4 contribution (b) to the process $\gamma\gamma^* \to \pi^0$.

Both processes illustrated in Fig. 3 make contributions to the two photon process $\gamma^*\gamma \to \pi^0$. Time reversal invariance means that the four-quark state $|q\bar{q}q\bar{q}\rangle$ should also contribute to the decay process $\pi^0 \to \gamma\gamma$. In a semiclassical model without dynamic gluons, Fig. 3(b) represents the only higher twist term which contribute to the $\gamma^*\gamma \to \pi^0$ process. The twist-4 contribution vanishes at large $Q^2$ compared to the leading-twist contribution, thus maintaining the asymptotic predictions while only modifying the large distance behavior of the wave function.

To investigate the contributions from the higher Fock states in the pion LFWF, we write the twist-2 and twist-4 hadronic AdS components from (11),

$$\Phi_\pi^{\tau=2}(z) = \frac{\sqrt{2}\kappa z^2}{\sqrt{1+\alpha^2}} e^{-\kappa^2 z^2/2}, \tag{39}$$

$$\Phi_\pi^{\tau=4}(z) = \frac{\alpha\kappa^3 z^4}{\sqrt{1+\alpha^2}} e^{-\kappa^2 z^2/2}, \tag{40}$$

with normalization

$$\int_0^\infty \frac{dz}{z^3}[|\Phi_\pi^{\tau=2}(z)|^2 + |\Phi_\pi^{\tau=4}(z)|^2] = 1, \tag{41}$$

and probabilities $P_{q\bar{q}} = 1/(1+|\alpha|^2)$ and $P_{q\bar{q}q\bar{q}} = \alpha^2/(1+|\alpha|^2)$. The pion decay constant follows from the short-distance asymptotic behavior of the leading contribution and is given by

$$f_\pi = \frac{1}{\sqrt{1+\alpha^2}}\frac{\kappa}{\sqrt{2}\pi}. \tag{42}$$

Using (39) and (40) together with (14) in Eq. (20) we find the total contribution from twist-2 and twist-4 components for the dressed current,

$$F_{\pi\gamma}(Q^2) = \frac{1}{\pi^2 f_\pi}\frac{1}{(1+\alpha^2)^{3/2}}\int_0^1 \frac{dx}{(1+x)^2} x^{Q^2/[8\pi^2 f_\pi^2(1+\alpha^2)]} \times\left[1 + \frac{4\alpha}{\sqrt{2}}\frac{1-x}{1+x}\right]. \tag{43}$$

The transition form factor at $Q^2 = 0$ is given by

$$F_{\pi\gamma}(0) = \frac{1}{2\pi^2 f_\pi}\frac{1+\sqrt{2}\alpha}{(1+\alpha^2)^{3/2}}. \tag{44}$$

The Brodsky-Lepage asymptotic prediction for the pion TFF can be recovered from Eq. (43) by noticing that the second term vanishes at $Q^2 \to \infty$ and the similarity between Eq. (35) and the first term in Eq. (43).

Imposing the anomaly result (33) on (44) we find two possible real solutions for $\alpha$: $\alpha_1 = -0.304$ and $\alpha_2 = 1.568$.[6] The larger value $\alpha_2 = 1.568$ yields $P_{q\bar{q}} = 0.29$, $P_{q\bar{q}q\bar{q}} = 0.71$, and $\kappa = 1.43$ GeV. The resulting value of $\kappa$ is about 4 times larger than the value obtained from the AdS/QCD analysis of the hadron spectrum and the pion elastic form factor [43], and thereby should be discarded. The other solution $\alpha_1 = -0.304$ gives $P_{q\bar{q}} = 0.915$, $P_{q\bar{q}q\bar{q}} = 0.085$, and $\kappa = 0.432$ GeV—results that are similar to that found from an analysis of the space and timelike behavior of the pion form factor using LF holographic methods, including higher Fock components in the pion wave function [43]. Semiclassical holographic methods, where dynamical gluons are not presented, are thus compatible with a large probability for the valence state of the order of 90%. On the other hand, QCD analyses including multiple gluons on the pion wave function favor a small probability (25%) for the valence state [56]. Both cases (and examples in between) are examined in Ref. [34].

The results for the transition form factor are shown as solid curves in Figs. 1 and 2. The agreements with the experimental data at low- and medium-$Q^2$ regions ($Q^2 < 10$ GeV$^2$) are greatly improved compared with the results obtained with only twist-2 component computed with the dressed current. However, the rapid growth of the pion-photon transition form factor exhibited by the *BABAR* data at high $Q^2$ still cannot be reproduced. So we arrive at a similar conclusion as we did in a QCD analysis of the pion TFF in Ref. [34]: it is difficult to explain the rapid growth of the form factor exhibited by the *BABAR* data at high $Q^2$ within the current framework of QCD.

## VI. TRANSITION FORM FACTORS FOR THE $\eta$ AND $\eta'$ MESONS

The $\eta$ and $\eta'$ mesons result from the mixing of the neutral states $\eta_8$ and $\eta_1$ of the $SU(3)_F$ quark model. The transition form factors for the latter have the same expression as the pion transition form factor, except an overall multiplying factor $c_P = 1, \frac{1}{\sqrt{3}}$, and $\frac{2\sqrt{2}}{\sqrt{3}}$ for the $\pi^0$, $\eta_8$, and $\eta_1$, respectively. By multiplying Eqs. (30), (34), and (43) by the appropriate factor $c_P$, one obtains the corresponding expressions for the transition form factors for the $\eta_8$ and $\eta_1$.

The transition form factors for the physical states $\eta$ and $\eta'$ are a superposition of the transition form factors for the $\eta_8$ and $\eta_1$

$$\begin{pmatrix} F_{\eta\gamma} \\ F_{\eta'\gamma} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} F_{\eta_8\gamma} \\ F_{\eta_1\gamma} \end{pmatrix}, \quad (45)$$

where $\theta$ is the mixing angle for which we adopt $\theta = -14.5^o \pm 2^o$ [67]. The results for the $\eta$ and $\eta'$ transitions form factors are shown in Figs. 4 and 5 for $Q^2F_{M\gamma}(Q^2)$, and Figs. 6 and 7 for $F_{M\gamma}(Q^2)$. The calculations agree very well with available experimental data over a large range of $Q^2$. We note that other mixing schemes were proposed in

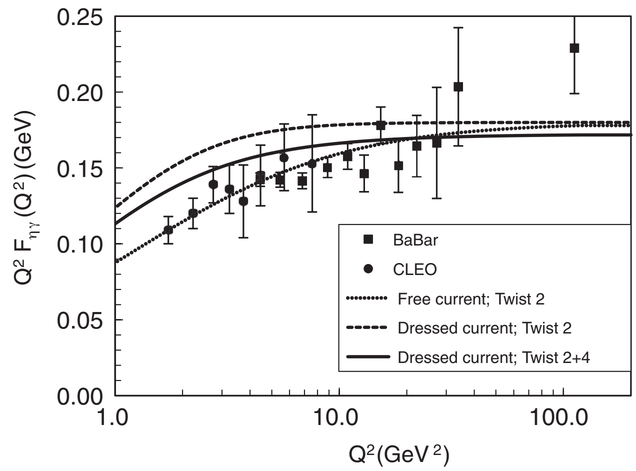


FIG. 4. The $\gamma\gamma^* \to \eta$ transition form factor shown as $Q^2F_{\eta\gamma}(Q^2)$ as a function of $Q^2 = -q^2$. The dotted curve is the asymptotic result. The dashed and solid curves include the effects of using a confined EM current for twist-2 and twist-2 plus twist-4, respectively. The data are from [19,22,23].

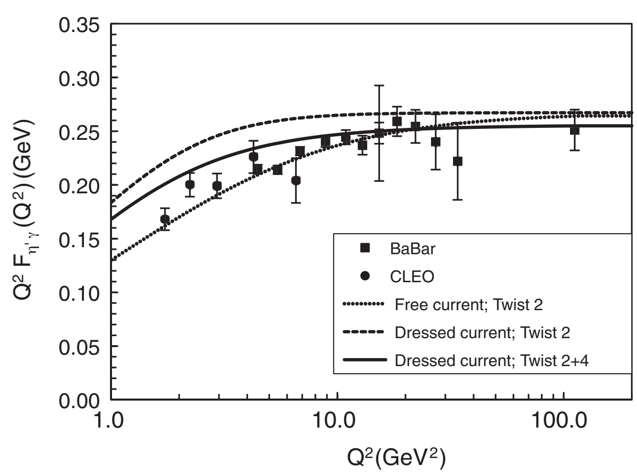


FIG. 5. Same as Fig. 4 for the $\gamma\gamma^* \to \eta'$ transition form factor shown as $Q^2F_{\eta'\gamma}(Q^2)$.

[6]If we impose the condition that the twist 4 contribution at $Q^2 = 0$ is exactly half the value of the twist 2 contribution one obtains $\alpha = -\frac{1}{2\sqrt{2}}$, which is very close to the value of $\alpha$ which follows by imposing the triangle anomaly constraint. In this case the pion TFF has a very simple form $F_{\pi\gamma}(Q^2) = \frac{8}{3\pi\kappa}\int_0^1 \frac{dx}{(1+x)^3} x^{Q^2/4\kappa^2+1}$.

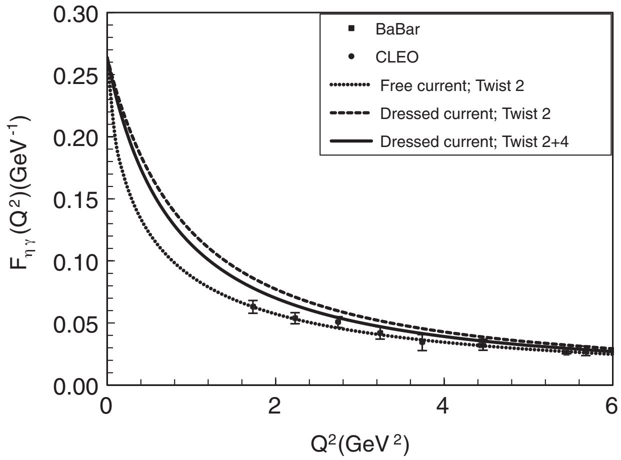


FIG. 6. Same as Fig. 4 for the $\gamma\gamma^* \to \eta$ transition form factor shown as $F_{\eta\gamma}(Q^2)$.

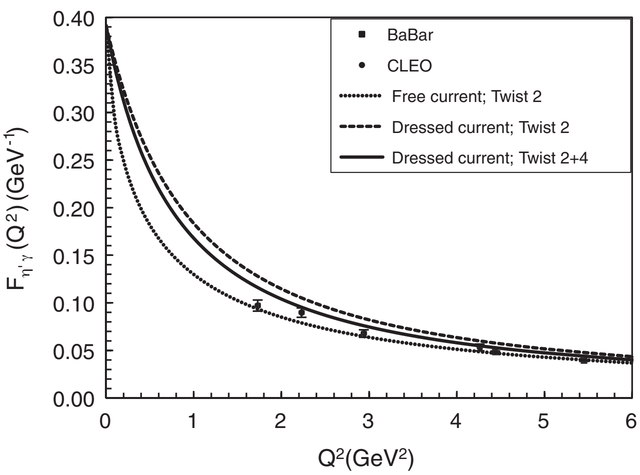


FIG. 7. Same as Fig. 4 for the $\gamma\gamma^* \to \eta'$ transition form factor shown as $F_{\eta'\gamma}(Q^2)$.

studying the mixing behavior of the decay constants and states of the $\eta$ and $\eta'$ mesons [68–70]. Since the transition form factors are the primary interest in this study, it is appropriate to use the conventional single-angle mixing scheme for the states. Furthermore, the predictions for the $\eta$ and $\eta'$ transition form factors remain largely unchanged if other mixing schemes are used in the calculation.

## VII. CONCLUSIONS

The light-front holographic approach provides a direct mapping between an effective gravity theory defined in a fifth-dimensional warped space-time and a corresponding semiclassical approximation to strongly coupled QCD quantized on the light-front. In addition to predictions for hadron spectroscopy, important outputs are the elastic form factors of hadrons and constraints on their light-front bound-state wave functions. The soft-wall holographic model is particularly successful.

We have studied the photon-to-meson transition form factors $F_{M\gamma}(Q^2)$ for $\gamma^*\gamma \to M$ using light-front holographic methods. The Chern-Simons action, which is a natural form in five-dimensional AdS space, is required to describe the anomalous coupling of mesons to photons using holographic methods and leads directly to an expression for the photon-to-pion transition form factor for a class of confining models. Remarkably, the pion transition form factor given by Eq. (30) derived from the CS action is identical to the leading order QCD result, where the distribution amplitude has the asymptotic form $\phi(x) \propto x(1-x)$.

The Chern-Simons form is local in AdS space and is thus somewhat limited in its predictability. It only retains the $q\bar{q}$ component of the pion wave function, and further, it projects out only the asymptotic form of the meson distribution amplitude $\phi(x) \propto x(1-x)$. In contrast, the holographic light-front mapping of electromagnetic and gravitational form factors gives the full form of the distribution amplitude $\phi(x) \propto \sqrt{x(1-x)}$ for arbitrary values of $Q^2$. This apparently contradictory result was first found in Ref. [47] in a hard-wall AdS extended model. This contradiction indicates that the local interaction from the CS action can only represent the pointlike asymptotic form. The asymptotic result coincides with the CS amplitude which is only sensitive to short-distance physics. If the QCD evolution for the distribution amplitude $\phi \propto \sqrt{x(1-x)}$ is included, the asymptotic DA is recovered at very large $Q$ [34].

It is found that in order to describe simultaneously the decay process $\pi^0 \to \gamma\gamma$ and the pion TFF at the asymptotic limit a probability for the $q\bar{q}$ component of the pion wave function $P_{q\bar{q}} = 0.5$ is required for the calculations with the free and dressed AdS currents.

We have argued that the contributions from the higher Fock components in the pion light-front wave function also need to be included in the analysis of exclusive processes. In fact, just as in $1+1$ QCD, the confining interaction of the LF Hamiltonian in light-front holography leads to Fock states with any number of extra $q\bar{q}$ pairs. These contributions lead to higher-twist contributions to the hadron form factor. We have shown how the effect of the higher Fock states in form factors can be obtained by analyzing the hadron matrix elements of the confined dressed electromagnetic Heisenberg current from the gauge/gravity duality. The probability for the four-quark states obtained in this work $P_{q\bar{q}q\bar{q}} = 0.085$ is similar to that found from an analysis of the space- and timelike behavior of the pion form factor using LF holographic methods, including higher Fock components in the pion wave function [43].

The results obtained for the $\eta$- and $\eta'$-photon transition form factors are consistent with all currently available experimental data. However, the rapid growth of the pion-photon transition form factor exhibited by the

*BABAR* data at high $Q^2$ is not compatible with the models discussed in this paper, and in fact is very difficult to explain within the current framework of QCD.

## ACKNOWLEDGMENTS

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## APPENDIX: LIGHT-FRONT WAVE FUNCTIONS FROM HOLOGRAPHIC MAPPING

For a two-parton bound state, light-front holographic mapping relates the light-front wave function $\psi(x, \zeta, \varphi)$ in physical space-time with its dual field $\Phi(z)$ in AdS space. The precise relation is given by (10)

$$\psi(x, \zeta, \varphi) = e^{iM\varphi} X(x) \frac{\phi(\zeta)}{\sqrt{2\pi\zeta}}, \tag{A1}$$

where we have factored out the angular dependence $\varphi$ and the longitudinal, $X(x)$, and transverse mode [4,5,10,17]

$$\phi(\zeta) = \zeta^{-3/2}\Phi(\zeta). \tag{A2}$$

The holographic variable $z$ is related to the light-front invariant variable $\zeta$ which represents the transverse separation of the quarks within the pion

$$z \to \zeta = \sqrt{x(1-x)}|\mathbf{b}_\perp|. \tag{A3}$$

The LF variable $x$ is the longitudinal light-cone momentum fraction $x = k^+/P^+$ and $\mathbf{b}_\perp$ is the impact separation and Fourier conjugate to $\mathbf{k}_\perp$, the relative transverse momentum coordinate.

The LFWF is normalized according to

$$\langle \psi_{q\bar{q}/\pi} | \psi_{q\bar{q}/\pi} \rangle = P_{q\bar{q}}, \tag{A4}$$

where $P_{q\bar{q}}$ is the probability of finding the $q\bar{q}$ component in the pion light-front wave function. We choose the normalization of the LF mode $\phi(\zeta) = \langle \zeta | \psi \rangle$ as

$$\langle \phi | \phi \rangle = \int d\zeta |\langle \zeta | \phi \rangle|^2 = P_{q\bar{q}}, \tag{A5}$$

and thus the longitudinal mode is normalized as

$$\int_0^1 \frac{X^2(x)}{x(1-x)} = 1. \tag{A6}$$

As we have shown in Sec. II the factorization (A1) is required to map the elastic electromagnetic form factors for arbitrary values of the transverse momentum $Q$ with the result $X(x) = \sqrt{x(1-x)}$ [4,5] for the longitudinal mode. Identical results follow from the mapping to the gravitational form factor [10]. The longitudinal mode $X(x)$ cannot be determined from the mapping of the Hamiltonian equation for bound states as it decouples in the ultrarelativistic limit $m_q \to 0$ [17].

For a harmonic confining potential $U(z) \sim \kappa^4 z^2$ we have from (11)

$$\Phi_{q\bar{q}/\pi}(z) = \sqrt{2P_{q\bar{q}}}\kappa z^2 e^{-\kappa^2 z^2/2}, \tag{A7}$$

for a twist $\tau = 2$ mode propagating in AdS space. From Eqs. (A1)–(A3) we find the LFWF ($L^z = M = 0$)

$$\psi_{q\bar{q}/\pi}(x, \mathbf{b}_\perp) = \frac{\kappa}{\sqrt{\pi}} \sqrt{P_{q\bar{q}}} \sqrt{x(1-x)} e^{-(1/2)\kappa^2 x(1-x)\mathbf{b}_\perp^2} \tag{A8}$$

in physical space time.

The pion distribution amplitude in the light-front formalism [13] is the integral of the valence $q\bar{q}$ light-front wave function

$$\phi(x) = \int \frac{d^2\mathbf{k}_\perp}{16\pi^3} \psi_{q\bar{q}/\pi}(x, \mathbf{k}_\perp), \tag{A9}$$

and satisfies the normalization condition which follows from the decay process $\pi \to \mu\nu$ ($N_C = 3$)

$$\int_0^1 dx \phi(x) = \frac{f_\pi}{2\sqrt{3}}, \tag{A10}$$

where $f_\pi = 92.4$ MeV is the pion decay constant. From (A8) we find the distribution amplitude

$$\phi(x) = \frac{4}{\sqrt{3}\pi} \sqrt{x(1-x)}, \tag{A11}$$

and the pion decay constant

$$f_\pi = \sqrt{P_{q\bar{q}}} \frac{\sqrt{3}}{8} \kappa. \tag{A12}$$

As discussed in the paper, the CS mapping gives the asymptotic distribution amplitude since the CS maps a pointlike pion. The corresponding longitudinal mode in the LFWF is $X(x) = \sqrt{6}x(1-x)$ and thus the LFWF

$$\psi_{q\bar{q}/\pi}(x, \mathbf{b}_\perp) = \frac{\kappa}{\sqrt{\pi}} \sqrt{P_{q\bar{q}}} \sqrt{6} x(1-x) e^{-(1/2)\kappa^2 x(1-x)\mathbf{b}_\perp^2}. \tag{A13}$$

The pion decay constant in this case is

$$f_\pi = \sqrt{P_{q\bar{q}}} \frac{\kappa}{\sqrt{2}\pi}, \tag{A14}$$

consistent with (26).

The evolution of the pion distribution amplitude in $\log Q^2$ is governed by the Efremov-Radyushkin-Brodsky-Lepage (ERBL) evolution equation [13,36]. It can be expressed in terms of the anomalous dimensions of the Gegenbauer polynomial projection of the DA. If we normalize the full LFWF of the pion by $\langle \psi | \psi \rangle = 1$, we can compute the probability to find the pion in a given component of a Gegenbauer polynomial expansion $X(x) = x(1-x)\sum_n \alpha_n C_n^{(3/2)}(2x-1)$. We find

$$P_n = \frac{(n+2)(n+1)}{4(2n+3)} \alpha_n^2, \tag{A15}$$

where $\sum_n P_n = 1$. For the AdS solution $X(x) = \sqrt{x(1-x)}$ the asymptotic component $\alpha_0 = 3\pi/4$ and the probability to find the pion in its asymptotic state is $P_0 = 3\pi^2/32 \simeq 92.5\%$, not too far from the asymptotic result. Notice that $P_n$ in (A15) are the probabilities related to the Gegenbauer projection of the valence state of the pion. They are not related to the probabilities discussed in the section below, which are the probabilities of higher particle number Fock states in the pion.

The asymptotic form has zero anomalous dimension. The distribution amplitude $\phi(x) \propto \sqrt{x(1-x)}$ derived from LF holographic methods is sensitive to soft physics $1 - x \sim \kappa/Q^2$, and has Gegenbauer polynomial components with nonzero anomalous dimensions which are driven to zero for large values of $Q^2$. Expanding the distribution amplitude at any finite scale as $x(1-x)$ times Gegenbauer polynomials, only its projection on the lowest Gegenbauer polynomial with zero anomalous moment survives at large $Q^2$.

### 1. Effective light-front wave function from holographic mapping of a confined electromagnetic current

It is also possible to find a precise mapping of a confined EM current propagating in a warped AdS space to the light-front QCD Drell-Yan-West expression for the form factor. In this case the resulting LFWF incorporates nonvalence higher Fock states generated by the "dressed" confined current. For the soft-wall model this mapping can be done analytically.

The form factor in light-front QCD can be expressed in terms of an effective single-particle density [9]

$$F(Q^2) = \int_0^1 dx \rho(x, Q), \tag{A16}$$

where

$$\rho(x, Q) = 2\pi \int_0^\infty b db J_0(bQ(1-x)) |\psi(x, b)|^2, \tag{A17}$$

for a two-parton state ($b = |\mathbf{b}_\perp|$).

We can also compute an effective density on the gravity side corresponding to a twist $\tau$ hadronic mode $\Phi_\tau$ in a modified AdS space. For the soft-wall model the expression is [5]

$$\rho(x, Q) = (\tau - 1)(1-x)^{\tau-2} x^{(Q^2/4\kappa^2)}. \tag{A18}$$

To compare (A18) with the QCD expression (A17) for twist-2 we use the integral

$$\int_0^\infty u du J_0(\alpha u) e^{-\beta u^2} = \frac{1}{2\beta} e^{-\alpha^2/4\beta}, \tag{A19}$$

and the relation $x^\gamma = e^{\gamma \ln(x)}$. We find the effective two-parton LFWF

$$\psi(x, \mathbf{b}_\perp) = \kappa \frac{(1-x)}{\sqrt{\pi \ln(\frac{1}{x})}} e^{-(1/2)\kappa^2 \mathbf{b}_\perp^2 (1-x)^2/\ln(1/x)}, \tag{A20}$$

in impact space. The momentum space expression follows from the Fourier transform of (A20) and it is given by

$$\psi(x, \mathbf{k}_\perp) = 4\pi \frac{\sqrt{\ln(\frac{1}{x})}}{\kappa(1-x)} x^{\mathbf{k}_\perp^2/2\kappa^2(1-x)^2}. \tag{A21}$$

The effective LFWF encodes nonperturbative dynamical aspects that cannot be learned from a term-by-term holographic mapping, unless one adds an infinite number of terms. Furthermore, it has the right analytical properties to reproduce the bound-state vector meson pole in the timelike EM form factor. Unlike the "true" valence LFWF, the effective LFWF, which represents a sum of an infinite number of Fock components, is not symmetric in the longitudinal variables $x$ and $1-x$ for the active and spectator quarks, respectively.

As we have discussed in Secs. IV and IV A for the free and dressed currents, respectively, a simple model with only a twist-2 valence pion state requires a 50% probability, $P_{q\bar{q}} = \frac{1}{2}$, to reproduce the decay process $\pi^0 \to \gamma\gamma$. We recall that for the soft-wall model the EM form factor is given by (15), and thus for $\tau = 2$ its asymptotic normalization is given by

$$Q^2 F(Q^2 \to \infty) = P_{q\bar{q}} M_\rho^2. \tag{A22}$$

One of the unsolved difficulties of the holographic approach to QCD is that the vector mesons masses obtained from the spin-1 equation of motion does not match the poles of the dressed current when computing a form factor. The discrepancy is an overall factor of $\sqrt{2}$.[7] Light-front holography provides a precise relation of the fifth-dimensional mass $\mu$ with the total and orbital angular momentum of a hadron in the transverse LF plane $(\mu R)^2 = -(2-J)^2 + L^2$, $L = |L^z|$ [17]. Thus the $\rho$ meson mass computed from the AdS wave equations for a conserved current $\mu R = 0$, corresponds to a $J = L = 1$ twist-3 state. In fact, the twist-3 computation of the spacelike form factor, involves the current $J^+$, and the poles do not correspond to the physical poles of the twist-2 transverse

[7]This discrepancy is also present in the gap scale if one computes the spectrum and form factors without recourse to holographic methods, for example, using the semiclassical approximation of Ref. [17]. In this case a discrepancy of a factor $\sqrt{2}$ is also found between the spectrum and the computation of spacelike form factors.

current $\mathbf{J}_\perp$ present in the annihilation channel, namely, the $J = 1$, $L = 0$ state [71].

If we define the physical vector meson mass by the relation $\bar{M}_\rho = P_{q\bar{q}} M_\rho = M_\rho/\sqrt{2}$ the asymptotic result (A22) becomes

$$Q^2 F(Q^2 \to \infty) = \bar{M}_\rho^2. \tag{A23}$$

We can thus define a form factor $\bar{F}(Q^2)$ shifting the poles in (15) to their physical locations but keeping the same analytical structure. Thus for $\tau = 2$

$$\bar{F}(Q^2) = \frac{1}{1 + \frac{Q^2}{\bar{M}_\rho^2}}, \tag{A24}$$

which satisfies the asymptotic normalization (A23) and charge normalization at $Q = 0$, $F(0) = 1$. For arbitrary twist $\tau$ the expression is

$$\bar{F}_\tau(Q^2) = \frac{\bar{P}_\tau}{\left(1 + \frac{Q^2}{\bar{M}_\rho^2}\right)\left(1 + \frac{Q^2}{\bar{M}_{\rho'}^2}\right)\cdots\left(1 + \frac{Q^2}{\bar{M}_{\rho^{\tau-2}}^2}\right)}. \tag{A25}$$

It is important to notice that the values of the probabilities $\bar{P}$ corresponding to the physical vector masses $\bar{M}$ are markedly different from the probabilities $P$ obtained from the formulas with the unphysical masses $M$. For example for a pion $\bar{P}_{q\bar{q}} \simeq 90\%$ and $\bar{P}_{q\bar{q}q\bar{q}} \simeq 10\%$ [43]. When the vector meson masses are shifted to their physical values the agreement of the predictions with observed data is very good [43,71]. Although the arguments presented above are not rigorous, they can help explain why a systematic difference of a factor $\sqrt{2}$ in the gap scale is found when comparing predictions with the spectrum or form factor data.


[1] J. M. Maldacena, Int. J. Theor. Phys. **38**, 1113 (1999).
[2] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, Phys. Lett. B **428**, 105 (1998).
[3] E. Witten, Adv. Theor. Math. Phys. **2**, 253 (1998).
[4] S. J. Brodsky and G. F. de Teramond, Phys. Rev. Lett. **96**, 201601 (2006).
[5] S. J. Brodsky and G. F. de Teramond, Phys. Rev. D **77**, 056007 (2008).
[6] J. Polchinski and M. J. Strassler, J. High Energy Phys. 05 (2003) 012.
[7] S. D. Drell and T. M. Yan, Phys. Rev. Lett. **24**, 181 (1970).
[8] G. B. West, Phys. Rev. Lett. **24**, 1206 (1970).
[9] D. E. Soper, Phys. Rev. D **15**, 1141 (1977).
[10] S. J. Brodsky and G. F. de Teramond, Phys. Rev. D **78**, 025032 (2008).
[11] Z. Abidin and C. E. Carlson, Phys. Rev. D **77**, 095007 (2008).
[12] S. J. Brodsky and G. R. Farrar, Phys. Rev. Lett. **31**, 1153 (1973).
[13] G. P. Lepage and S. J. Brodsky, Phys. Rev. D **22**, 2157 (1980).
[14] V. A. Matveev, R. M. Muradian, and A. N. Tavkhelidze, Lett. Nuovo Cimento Soc. Ital. Fis. **7**, 719 (1973).
[15] J. Polchinski and M. J. Strassler, Phys. Rev. Lett. **88**, 031601 (2002).
[16] S. J. Brodsky, H. C. Pauli, and S. S. Pinsky, Phys. Rep. **301**, 299 (1998).
[17] G. F. de Teramond and S. J. Brodsky, Phys. Rev. Lett. **102**, 081601 (2009).
[18] P. A. M. Dirac, Rev. Mod. Phys. **21**, 392 (1949).
[19] B. Aubert *et al.* (*BABAR* Collaboration), Phys. Rev. D **80**, 052002 (2009).
[20] P. A. Sanchez *et al.* (*BABAR* Collaboration), Phys. Rev. D **84**, 052001 (2011).
[21] V. P. Druzhinin *et al.* (*BABAR* Collaboration), Proc. Sci., ICHEP (2010) 144 [arXiv:1011.6159].
[22] H.-J. Behrend *et al.* (CELLO Collaboration), Z. Phys. C **49**, 401 (1991).
[23] J. Gronberg *et al.* (CLEO Collaboration), Phys. Rev. D **57**, 33 (1998).
[24] H.-N. Li and S. Mishima, Phys. Rev. D **80**, 074024 (2009).
[25] S. V. Mikhailov and N. G. Stefanis, Mod. Phys. Lett. A **24**, 2858 (2009).
[26] X.-G. Wu and T. Huang, Phys. Rev. D **82**, 034024 (2010).
[27] H. L. L. Roberts, C. D. Roberts, A. Bashir, L. X. Gutierrez-Guerrero, and P. C. Tandy, Phys. Rev. C **82**, 065202 (2010).
[28] W. Broniowski and E. R. Arriola, Proc. Sci., LC2010 (2010) 062 [arXiv:1008.2317].
[29] T. N. Pham and X. Y. Pham, arXiv:1101.3177; arXiv:1103.0452 [Int. J. Mod. Phys. A (to be published)].
[30] P. Kroll, Eur. Phys. J. C **71**, 1623 (2011).
[31] M. Gorchtein, P. Guo, and A. P. Szczepaniak, arXiv:1102.5558.
[32] A. E. Dorokhov, arXiv:1003.4693; JETP Lett. **92**, 707 (2010).
[33] S. S. Agaev, V. M. Braun, N. Oen, and F. A. Porkert, Phys. Rev. D **83**, 054020 (2011).
[34] S. J. Brodsky, F.-G. Cao, and G. F. de Teramond, Phys. Rev. D **84**, 033001 (2011) and references therein.
[35] A. P. Bakulev, S. V. Mikhailov, A. V. Pimikov, and N. G. Stefanis, Phys. Rev. D **84**, 034014 (2011).
[36] A. V. Efremov and A. V. Radyushkin, Phys. Lett. B **94**, 245 (1980).
[37] C. T. Hill, Phys. Rev. D **73**, 085001 (2006); **73**, 126009 (2006).
[38] G. F. de Teramond and S. J. Brodsky, J. Phys. Conf. Ser. **287**, 012007 (2011).
[39] S. S. Afonin, Int. J. Mod. Phys. A **26**, 3615 (2011).
[40] V. E. Lyubovitskij, T. Gutsche, I. Schmidt, and A. Vega, arXiv:1108.0533.

[41] A. Karch, E. Katz, D. T. Son, and M. A. Stephanov, Phys. Rev. D **74**, 015005 (2006).
[42] H. R. Grigoryan and A. V. Radyushkin, Phys. Rev. D **76**, 095007 (2007).
[43] G. F. de Teramond and S. J. Brodsky, Proc. Sci., LC2010 (2010) 029 [arXiv:1010.1204].
[44] M. Pernici, K. Pilch, and P. van Nieuwenhuizen, Nucl. Phys. **B259**, 460 (1985).
[45] M. Gunaydin, L. J. Romans, and N. P. Warner, Nucl. Phys. **B272**, 598 (1986).
[46] A. Pomarol and A. Wulzer, Nucl. Phys. **B809**, 347 (2009).
[47] H. R. Grigoryan and A. V. Radyushkin, Phys. Rev. D **77**, 115024 (2008).
[48] F. Zuo, Y. Jia, and T. Huang, Eur. Phys. J. C **67**, 253 (2010).
[49] T. Sakai and S. Sugimoto, Prog. Theor. Phys. **114**, 1083 (2005).
[50] C. T. Hill and C. K. Zachos, Phys. Rev. D **71**, 046002 (2005).
[51] L. Cappiello, O. Cata, and G. D'Ambrosio, Phys. Rev. D **83**, 093006 (2011).
[52] A. Stoers and I. Zahed, Phys. Rev. C **84**, 025202 (2011).
[53] D. Rodriguez-Gomez and J. Ward, J. High Energy Phys. 09 (2008) 103.
[54] I. V. Musatov and A. V. Radyushkin, Phys. Rev. D **56**, 2713 (1997).
[55] J. S. Schwinger, Phys. Rev. **82**, 664 (1951); S. L. Adler, Phys. Rev. **177**, 2426 (1969); J. S. Bell and R. Jackiw, Nuovo Cimento A **60**, 47 (1969).
[56] S. J. Brodsky, T. Huang, and G. P. Lepage, in *Particles and Fields-2, Proceedings of the Ban Summer Institute, Ban, Alberta, 1981*, edited by A. Z. Capri and A. N. Kamal (Plenum, New York, 1983), p. 143; Report No. SLAC-PUB-2540, 1980 (unpublished).
[57] F.-G. Cao, T. Huang, and B.-Q. Ma, Phys. Rev. D **53**, 6582 (1996).
[58] F.-G. Cao, J. Cao, T. Huang, and B.-Q. Ma, Phys. Rev. D **55**, 7107 (1997).
[59] F.-G. Cao, Y.-B. Dai, and C.-S. Huang, Eur. Phys. J. C **11**, 501 (1999).
[60] Y.-C. Chen and H.-N. Li, Phys. Rev. D **84**, 034018 (2011).
[61] T. Huang and X.-G. Wu, Int. J. Mod. Phys. A **22**, 3065 (2007).
[62] A. S. Gorsky, Report No. ITEP-87-85 (unpublished); Report No. ITEP-87-86 (unpublished).
[63] R. Alkofer and H. Reinhardt, arXiv:hep-ph/9212238.
[64] S. L. Adler and W. A. Bardeen, Phys. Rev. **182**, 1517 (1969).
[65] A. Zee, Phys. Rev. Lett. **29**, 1198 (1972).
[66] S. J. Brodsky and J. R. Primack, Ann. Phys. (N.Y.) **52**, 315 (1969).
[67] F.-G. Cao and A. I. Signal, Phys. Rev. D **60**, 114012 (1999).
[68] H. Leutwyler, Nucl. Phys. B, Proc. Suppl. **64**, 223 (1998).
[69] T. Feldmann and P. Kroll, Eur. Phys. J. C **5**, 327 (1998).
[70] T. Feldmann, P. Kroll, and B. Stech, Phys. Rev. D **58**, 114006 (1998).
[71] G. F. de Teramond and S. J. Brodsky, arXiv:1108.0965.