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πN transition distribution amplitudes: Their symmetries and constraints from chiral dynamics

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Baryon to meson transition distribution amplitudes (TDAs) extend the concept of generalized parton distributions. Baryon to meson TDAs appear as building blocks in the collinear factorized description of amplitudes for a class of hard exclusive reactions, prominent examples of which being hard exclusive meson electroproduction off a nucleon in the backward region and baryon-antibaryon annihilation into a meson and a lepton pair. We study the general properties of these objects following the underlying symmetries of QCD. In particular, the Lorentz symmetry results in the polynomiality property of the Mellin moments in longitudinal momentum fractions. We present a detailed account of the isotopic and permutation symmetry properties of nucleon to pion (πN) TDAs. This restricts the number of independent leading twist πN TDAs to eight functions, providing description of all isotopic channels. Using chiral symmetry and the crossing relation between πN TDAs and πN generalized distribution amplitudes, we establish soft pion theorems for πN TDAs, which determine the magnitude of πN TDAs. Finally, we build a simple resonance exchange model for πN TDAs considering N and $\Delta(1232)$ exchange contributions into the isospin-$\frac{1}{2}$ and isospin-$\frac{3}{2}$ πN TDAs.

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I. INTRODUCING πN TDAS

Hadronic matrix elements of nonlocal light-cone operators are the conventional nonperturbative objects which arise in the description of hard exclusive electroproduction reactions within the collinear factorization approach. Factorization theorems for hard exclusive backward meson electroproduction argued in [1,2] and baryon-antibaryon annihilation into a pion and a high energy dilepton pair [3] lead to the introduction of baryon to meson transition distribution amplitudes (TDAs), non diagonal matrix elements of light-cone three-quark operators

$$\hat{O}^{\alpha\beta\gamma}_{\rho\tau\chi}(z_1, z_2, z_3) = \varepsilon_{c_1c_2c_3}\Psi^{c_1\alpha}_{\rho}(z_1)\Psi^{c_2\beta}_{\tau}(z_2)\Psi^{c_3\gamma}_{\chi}(z_3)|_{z_i^2=0} \tag{1}$$

between a baryon and a meson states. In (1) α, β, γ stand for quark flavor indices; ρ, τ and χ denote the Dirac indices and $c_{1,2,3}$ are indices of the color group. Throughout this paper we adopt the light-cone gauge $A^+ = 0$, so that the gauge link is equal to unity and we do not show it explicitly in the definition of the operator (1).

In accordance with the usual logic of the collinear factorization approach, baryon to meson TDAs have well established renormalization group behavior. The evolution properties of the three-quark nonlocal operator (1) on the light-cone [4–7] were extensively studied in the literature (see e.g. [8,9]) for the case of matrix elements between a baryon and the vacuum known as baryon distribution amplitudes (DAs). The definition of baryon to meson TDAs involves the same light-cone operator. Consequently, its evolution also determines the factorization scale dependence of TDAs [10].

Because of the nonperturbative nature of TDAs the initial conditions for evolution require modeling at low factorization scale. In particular, nucleon to pion (πN) TDAs at low scale were recently studied within a light-front quark model [11]. From the physics point of view πN TDAs may be seen as an essential object to probe the pion cloud content of the nucleon [12].

In [10,13,14] a factorized framework was introduced to describe the electroproduction process

$$\gamma^*(q) + N(p_1) \to \pi(p_\pi) + N(p_2) \tag{2}$$

in the generalized Bjorken limit ($Q^2 = -q^2$—large; $Q^2/(2p_1 \cdot q)$—fixed) in the so-called backward region $|u| \equiv |(p_\pi - p_1)^2| \ll Q^2$ in terms of πN TDAs. It is worth emphasizing that such kinematical regime essentially differs from the more conventional limit $-t \equiv -(p_2 - p_1)^2 \ll Q^2$ in which the factorization theorem for hard exclusive electroproduction of pion off a nucleon [15] applies for (2). In this later case the description of (2) involves standard nucleon generalized parton distributions (GPDs).

We introduce the standard Mandelstam variables for the reaction (2):

$$s = (p_1 + q)^2; \quad u = (p_\pi - p_1)^2; \quad t = (p_2 - p_1)^2. \tag{3}$$

Therefore, the t-channel of (2) corresponds to an exchange with quantum numbers of a meson while in the u-channel an intermediate state with baryon quantum numbers is involved.

Throughout this paper we adopt a reference frame in which the three-momenta $\vec{q}$ and $\vec{p}_1$ have only a third component. We define the light-cone vectors p and n such that $2p \cdot n = 1$ and introduce standard kinematical quantities: average momentum $P = \frac{1}{2}(p_1 + p_\pi)$, momentum transfer $\Delta = p_\pi - p_1$ and its transverse component Δ_T. The skewness parameter ξ is defined with respect to the u-channel momentum transfer in the usual way $\xi = -\frac{\Delta \cdot n}{2P \cdot n}$. The detailed description of kinematics of (2) in the backward regime is presented in [14].

The definition of the leading twist-3 πN TDA involved in the description of the reaction (2) in the backward regime can be symbolically written as

$$4(P \cdot n)^3 \int \left[\prod_{j=1}^{3} \frac{d\lambda_j}{2\pi} \right] e^{i \sum_{k=1}^{3} x_k \lambda_k (P \cdot n)} \langle \pi_a(p_\pi) | \hat{O}^{\alpha\beta\gamma}_{\rho\tau\chi}(\lambda_1 n, \lambda_2 n, \lambda_3 n) | N_\iota(p_1) \rangle$$
$$= \delta(x_1 + x_2 + x_3 - 2\xi) \sum_{s.f.} (f_a)^{\alpha\beta\gamma}_\iota s_{\rho\tau,\chi} H^{(\pi N)}_{s.f.}(x_1, x_2, x_3, \xi, \Delta^2). \tag{4}$$

The spin-flavor ($s.f.$) sum in (4) stands over all relevant independent flavor structures $(f_a)^{\alpha\beta\gamma}_\iota$ and the Dirac structures $s_{\rho\tau,\chi}$. The detailed account of the Dirac and flavor structure occurring in (4) is given in Secs. II and IV.

Nucleon to pion TDAs are conceptually much related to pion-nucleon generalized distribution amplitudes (GDAs) [16,17] which are defined through the cross-conjugated matrix element of the same three-quark operator (1). Indeed, a similar correspondance was established between pion GPD and 2π GDA [18,19].

Therefore, it is natural to simultaneously consider the cross-conjugated ($p'_\pi \leftrightarrow -p_\pi$, $q' \leftrightarrow -q$) reaction:

$$\pi(p'_\pi) + N(p_1) \to \gamma^*(q') + N(p_2). \tag{5}$$

The formal definition of πN GDA that arises in the description of (5) reads

$$4(P' \cdot n)^3 \int \left[\prod_{j=1}^{3} \frac{d\lambda_j}{2\pi} \right] e^{i \sum_{k=1}^{3} y_k \lambda_k (P' \cdot n)} \langle 0 | \hat{O}^{\alpha\beta\gamma}_{\rho\tau\chi}(\lambda_1 n, \lambda_2 n, \lambda_3 n) | N_\iota(p_1) \pi_a(p'_\pi) \rangle$$
$$= \delta(y_1 + y_2 + y_3 - 1) \sum_{s.f.} (f_a)^{\alpha\beta\gamma}_\iota s'_{\rho\tau,\chi} \Phi^{(\pi N)}_{s.f.}(y_1, y_2, y_3, \zeta, P'^2), \tag{6}$$

where we introduce $P' = p_1 + p'_\pi$ for the total momentum of πN state and $\Delta' = p_1 - p'_\pi$. The variable $\zeta = \frac{p_1 \cdot n}{P' \cdot n}$ characterizes the distribution of the plus momenta of the πN system. We choose the Dirac structures in (6) $s'_{\rho\tau,\chi}$ as being given by crossing of $s_{\rho\tau,\chi}$ in (4). πN TDA and GDA are interrelated by a crossing transformation

$$P' \leftrightarrow -\Delta; \qquad \Delta' \leftrightarrow 2P \tag{7}$$

and analytic continuation in the appropriate kinematical variables:

$$P'^2 \leftrightarrow \Delta^2; \quad 2\zeta + 1 \leftrightarrow \frac{1}{\xi}; \quad y_i \leftrightarrow \frac{x_i}{2\xi}, \quad i = \{1, 2, 3\}. \tag{8}$$

The physical domain in (Δ^2, ξ)-plane for both the direct channel (2) and cross-conjugated (5) reactions is determined by the requirement that the transverse momentum transfer $\Delta_T = -P'_T$ should be spacelike:

$$\Delta_T^2 = \frac{1-\xi}{1+\xi}\left(\Delta^2 - 2\xi\left[\frac{M^2}{1+\xi} - \frac{m^2}{1-\xi}\right]\right) \le 0. \tag{9}$$

Here M and m stand for nucleon and pion masses, respectively.

On the left panel of Fig. 1 we show physical domains for the reactions (2) and (5) for physical pion mass. One may distinguish two regimes: the direct channel regime with its threshold at $\Delta^2 = (M - m)^2$ and the cross-channel one with its threshold at $\Delta^2 = (M + m)^2$. The upper (lower) branch of the curve bordering the physical domain in the direct channel regime tends to $\xi = 1$ ($\xi = -1$) when $\Delta^2 \to -\infty$. Note that the physical domain of the direct channel of (2) includes both negative and positive values of Δ^2. Moreover, in the chiral limit ($m = 0$) the two thresholds stick together (see the right panel of Fig. 1). We exploit this fact later in Sec. V in order to work out the normalization for πN TDAs.

It is interesting to compare Fig. 1 to that in the case of equal masses of particles in $|\text{in}\rangle$ and $\langle\text{out}|$ states. On Fig. 2 we show the physical domains in (Δ^2, ξ) plane for pion GPD and 2π GDA occurring in the description of $\gamma^*\pi \to \gamma\pi$ and $\gamma^*\gamma \to \pi\pi$ (here Δ refers to the momentum transfer between initial and final pions in $\gamma^*\pi \to \gamma\pi$ and the usual definition of ξ is assumed). In particular, the physical domains for pion GPD and 2π GDA are symmetric under the reflection of ξ. The difference between Fig. 1 and 2 has purely kinematical origin and is not related to the nature of QCD operator in the matrix element in question.

The basic issue of the approach based on πN TDAs is that, contrary to the GPD case, πN TDAs lack an

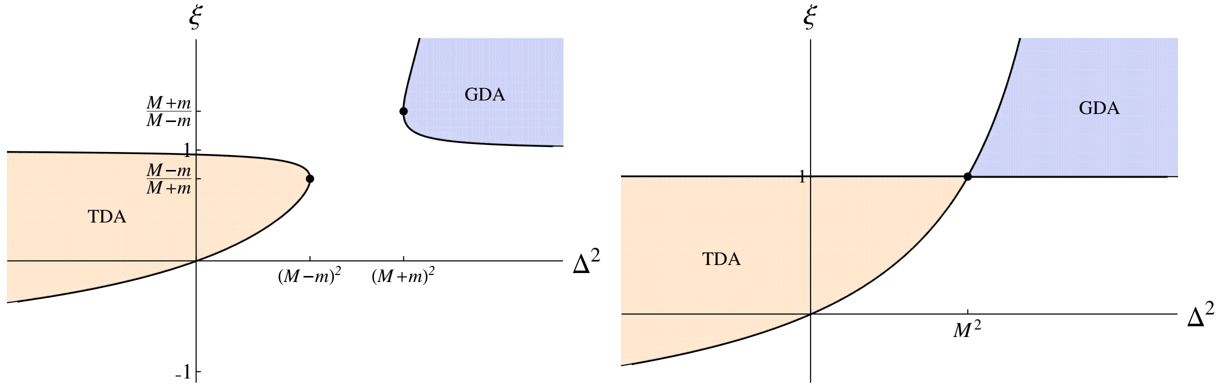

FIG. 1 (color online). Physical domains (bounded by the condition $\Delta_T^2 \leq 0$) in (Δ^2, ξ) plane for πN TDAs for the case $m \neq 0$ (left panel) and in the chiral limit $m = 0$ (right panel).

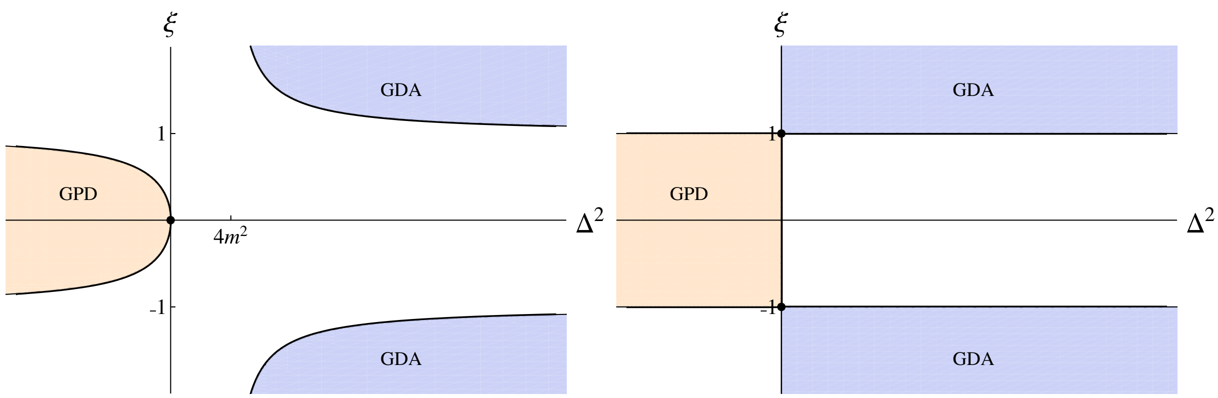

FIG. 2 (color online). Physical domains in (Δ^2, ξ) plane for pion GPD and 2π GDA for the case $m \neq 0$ (left panel) and in the chiral limit $m = 0$ (right panel).

intelligible forward limit $\xi \to 0$. However, the opposite limit $\xi \to 1$ turns out to be very illuminating. For simplicity, let us consider the pion to be massless. The point $\xi = 1$, $\Delta^2 = M^2$ (corresponding to both the direct and cross-channel threshold) belongs both to the physical regions for πN GDAs and TDAs. Moreover, it is for this very point that the soft pion theorem [20] applies for πN GDAs. As argued in [16,17], this allows to constrain πN GDAs at the threshold in terms of the nucleon DA. In the chiral limit the soft pion theorem for GDAs also constrains πN TDAs exactly as the soft pion theorem [18] for 2π GDA in the chiral limit links the isovector pion GPD at $\xi = 1$, $\Delta^2 = 0$ to the pion DA. Thus, in the chiral limit the soft pion theorem provides us with the desired reference point for πN TDAs. This valuable information may be used as input for realistic modeling of πN TDAs based on the spectral representation in terms of quadruple distributions [21]. In [22] we will argue that a possible approach consists in evolving from the $\xi = 1$ limit for πN TDAs through a procedure analogous to the one used for GPDs; in this latter case one employs the forward limit $\xi = 0$ to constrain GPDs through the successful Radyushkin's factorized Ansatz [23].

It is worth to mention that the simpler case of $\pi\gamma$ TDAs has already been discussed in details [24–28]. These TDAs share many features with πN TDAs, and are also subject to chiral symmetry constraints. Since the operator in this case is the same as in usual GPDs, polynomiality properties and isospin relations are straightforwardly extended from one case to the other.

In this paper, we analyze the constraints on πN TDAs imposed by the symmetries of QCD. First, we argue that the underlying Lorentz symmetry results in the polynomiality conditions which restrict the skewness parameter dependence of TDAs in a way similar to the well known GPD case. Second, we analyze in details the isospin decomposition of πN TDAs and establish the consequences of the isotopic and permutation symmetries. Third, we exploit the chiral symmetry of QCD to calculate πN GDAs and TDAs in the soft pion limit. Finally, we show how a model based on nucleon and $\Delta(1232)$ exchanges satisfies the revealed polynomiality and isospin constrains. The paper is organized as follows:

(i) In Sec. II we introduce the new parametrization for πN TDAs and show that, within this parametrization, πN TDAs satisfy the polynomiality conditions.

(ii) In Sec. III we consider the isospin structure of the three-quark operator and describe the general isospin parametrization of the leading twist baryon DAs. Next, we consider the consequences of isotopic and permutation symmetries of baryon DAs. We rederive the familiar isospin identities for the leading twist baryon DAs.

(iii) In Sec. IV we apply the isospin formalism to the case of πN TDAs and GDAs and derive the set of symmetry relations for πN TDAs and GDAs.

(iv) In Sec. V we derive the soft pion theorem for πN GDAs and discuss its consequences for πN TDAs.

(v) Section VI contains the calculation of u-channel nucleon and $\Delta(1232)$ exchange contributions into πN TDAs.

(vi) Our conclusions are presented in Sec. VII.

Let us stress that the main goal of the present paper is to provide the basic formalism for a consistent modelling of TDAs. The phenomenological applications of this formalism will be addressed in forthcoming publications. Measuring pion electroproduction at large angle is a challenging experimental problem. Some preliminary data are already available from J-Lab [29] and more data are expected from J-Lab at 12 GeV. A detailed proposal for measuring the reaction $\bar{p}p \to \gamma^* \pi N$ exists in the PANDA Physics program [30].

II. POLYNOMIALITY PROPERTY OF πN TDAS

In this section our goal is to show that, analogously to GPDs, nucleon to meson TDAs satisfy the polynomiality property, i.e. their Mellin moments in longitudinal momentum fractions x_i are polynomials of variable ξ of definite power. For definiteness we are going to consider the case of nucleon to pion TDAs. In this section we omit flavor indices in the operator $\hat{O}_{\rho\tau\chi}$ (1) since flavor symmetry is irrelevant for the present problem.

It turns out necessary to change the parametrization of πN TDA earlier proposed in Refs. [14,21]. The important drawback of our initial parametrization is that it involves the set of the Dirac structures which leads to spoiling of the polynomiality property of TDAs by the kinematical factors $\frac{1}{1+\xi}$.

In order to get rid of these kinematical singularities we suggest the following parametrization of the leading twist-3 πN TDAs[1]:

$$
\begin{aligned}
&4\mathcal{F}\langle \pi(p_\pi)|\hat{O}_{\rho\tau\chi}(\lambda_1 n, \lambda_2 n, \lambda_3 n)|N(p_1)\rangle \\
&\quad = \delta(x_1+x_2+x_3-2\xi) i \frac{f_N}{f_\pi M}[V_1^{\pi N}(x_1,x_2,x_3,\xi,\Delta^2)(\hat{P}C)_{\rho\tau}(\hat{P}U)_\chi + A_1^{\pi N}(x_1,x_2,x_3,\xi,\Delta^2)(\hat{P}\gamma^5 C)_{\rho\tau}(\gamma^5\hat{P}U)_\chi \\
&\qquad + T_1^{\pi N}(x_1,x_2,x_3,\xi,\Delta^2)(\sigma_{P\mu}C)_{\rho\tau}(\gamma^\mu\hat{P}U)_\chi + V_2^{\pi N}(x_1,x_2,x_3,\xi,\Delta^2)(\hat{P}C)_{\rho\tau}(\hat{\Delta}U)_\chi \\
&\qquad + A_2^{\pi N}(x_1,x_2,x_3,\xi,\Delta^2)(\hat{P}\gamma^5 C)_{\rho\tau}(\gamma^5\hat{\Delta}U)_\chi + T_2^{\pi N}(x_1,x_2,x_3,\xi,\Delta^2)(\sigma_{P\mu}C)_{\rho\tau}(\gamma^\mu\hat{\Delta}U)_\chi \\
&\qquad + \frac{1}{M}T_3^{\pi N}(x_1,x_2,x_3,\xi,\Delta^2)(\sigma_{P\Delta}C)_{\rho\tau}(\hat{P}U)_\chi + \frac{1}{M}T_4^{\pi N}(x_1,x_2,x_3,\xi,\Delta^2)(\sigma_{P\Delta}C)_{\rho\tau}(\hat{\Delta}U)_\chi],
\end{aligned}
\tag{10}
$$

where $\mathcal{F}$ stands for the Fourier transform

$$
\mathcal{F} \equiv \mathcal{F}(x_1,x_2,x_3)(\ldots) = (P\cdot n)^3 \int \left[\prod_{j=1}^{3}\frac{d\lambda_j}{2\pi}\right] e^{i\sum_{k=1}^{3} x_k\lambda_k(P\cdot n)}(\ldots);
\tag{11}
$$

f_π is the pion weak decay constant and f_N is a constant with the dimension of energy squared; U is the usual Dirac spinor and C is the charge conjugation matrix.

The price for avoiding the kinematical singularities in the invariant amplitudes in (10) is that apart from the leading twist contribution we have to keep some admixture of subleading twist (see Appendix C). The relationship between the parametrization involving pure twist-3 invariant amplitudes employed in Refs. [14,21] and that of Eq. (10) is given by Eq. (C11).

We introduce the following notations for the leading twist Dirac structures occurring in (10):

$$
\begin{aligned}
&(v_1^{\pi N})_{\rho\tau,\chi} = (\hat{P}C)_{\rho\tau}(\hat{P}U)_\chi; \quad (a_1^{\pi N})_{\rho\tau,\chi} = (\hat{P}\gamma^5 C)_{\rho\tau}(\gamma^5\hat{P}U)_\chi; \quad (t_1^{\pi N})_{\rho\tau,\chi} = (\sigma_{P\mu}C)_{\rho\tau}(\gamma^\mu\hat{P}U)_\chi; \\
&(v_2^{\pi N})_{\rho\tau,\chi} = (\hat{P}C)_{\rho\tau}(\hat{\Delta}U)_\chi; \quad (a_2^{\pi N})_{\rho\tau,\chi} = (\hat{P}\gamma^5 C)_{\rho\tau}(\gamma^5\hat{\Delta}U)_\chi; \quad (t_2^{\pi N})_{\rho\tau,\chi} = (\sigma_{P\mu}C)_{\rho\tau}(\gamma^\mu\hat{\Delta}U)_\chi; \\
&(t_3^{\pi N})_{\rho\tau,\chi} = \frac{1}{M}(\sigma_{P\Delta}C)_{\rho\tau}(\hat{P}U)_\chi; \quad (t_4^{\pi N})_{\rho\tau,\chi} = \frac{1}{M}(\sigma_{P\Delta}C)_{\rho\tau}(\hat{\Delta}U)_\chi.
\end{aligned}
\tag{12}
$$

[1]Throughout the text we employ Dirac's "hat" notation for the convolution of a 4-vector with the Dirac matrices: $\hat{a} \equiv \gamma_\mu a^\mu$. The following conventions are adopted: $\sigma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu]$; $\sigma^{v\nu} \equiv v_\mu \sigma^{\mu\nu}$, where v_μ is an arbitrary 4-vector.

We also employ the shortened notation for the whole set of twist-3 Dirac structures:

$$(s^{\pi N})_{\rho\tau,\chi} = \{(v^{\pi N}_{1,2})_{\rho\tau,\chi}, (a^{\pi N}_{1,2})_{\rho\tau,\chi}, (t^{\pi N}_{1,2,3,4})_{\rho\tau,\chi}\} \quad (13)$$

and for the corresponding invariant amplitudes

$$H^{\pi N}_s = \{V^{\pi N}_{1,2}, A^{\pi N}_{1,2}, T^{\pi N}_{1,2,3,4}\}. \quad (14)$$

Each invariant amplitude $V^{\pi N}_{1,2}$, $A^{\pi N}_{1,2}$, $T^{\pi N}_{1,2,3,4}$ of (10) is a function of the longitudinal momentum fractions x_i ($i = \{1, 2, 3\}$), skewness parameter $\xi = -\frac{(\Delta\cdot n)}{2(P\cdot n)}$ and the momentum transfer squared Δ^2. The support properties of πN TDAs in the longitudinal momentum fractions x_i were established in [21].

Now we are going to demonstrate that the πN TDAs defined in (10) satisfy the polynomiality property. Our demonstration generally repeats the usual way of arguing for the case of GPDs (see e.g. [31]).

The (n_1, n_2, n_3)-th ($n_1 + n_2 + n_3 = N$) Mellin moments of TDAs in x_1, x_2, x_3 lead to derivative operations acting on three-quark fields:

$$\begin{aligned} &4(P\cdot n)^{n_1+n_2+n_3+3}\int d^3x x_1^{n_1}x_2^{n_2}x_3^{n_3}\int\Big[\prod_{k=1}^{3}\frac{d\lambda_k}{2\pi}\Big]e^{i\sum_{k=1}^{3}x_k\lambda_k(P\cdot n)}\langle\pi(P+\Delta/2)|\hat{O}_{\rho\tau\chi}(\lambda_1 n, \lambda_2 n, \lambda_3 n)|N(P-\Delta/2)\rangle \\ &= (P\cdot n)^{n_1+n_2+n_3}\frac{if_N}{f_\pi M}\sum_s (s^{\pi N})_{\rho\tau,\chi}\int_{-1+\xi}^{1+\xi}dx_1\int_{-1+\xi}^{1+\xi}dx_2\int_{-1+\xi}^{1+\xi}dx_3 x_1^{n_1}x_2^{n_2}x_3^{n_3}\delta(x_1+x_2+x_3-2\xi)H_s^{\pi N}(x_1,x_2,x_3,\xi,\Delta^2) \\ &= 4(-1)^{n_1+n_2+n_3}\langle\pi(P+\Delta/2)|[(i\vec{\partial}^+)^{n_1}\Psi_\rho(0)][(i\vec{\partial}^+)^{n_2}\Psi_\tau(0)][(i\vec{\partial}^+)^{n_3}\Psi_\chi(0)]|N(P-\Delta/2)\rangle. \end{aligned} \quad (15)$$

Hence, the Mellin moments of nucleon to meson TDAs are expressed through the form factors of the local twist-3 operators:

$$\hat{O}^{\mu_1\ldots\mu_{n_1},\nu_1\ldots\nu_{n_2},\lambda_1\ldots\lambda_{n_3}}_{\rho\tau\chi}(0) = [i\vec{D}^{\mu_1}\ldots i\vec{D}^{\mu_{n_1}}\Psi_\rho][i\vec{D}^{\nu_1}\ldots i\vec{D}^{\nu_{n_2}}\Psi_\tau][i\vec{D}^{\lambda_1}\ldots i\vec{D}^{\lambda_{n_3}}\Psi_\chi], \quad (16)$$

where $\vec{D}^\mu = \vec{\partial}^\mu - \frac{ig}{2}A^{l\mu}\lambda^l$ is the covariant derivative (λ^l stand here for the Gell-Mann matrices). Note that in (15) and (16) we omit color indices.

Introducing the shortened notation

$$(\Delta^\mu)^i(P^\mu)^{n_1-i} \equiv \Delta^{\mu_1}\ldots\Delta^{\mu_i}P^{\mu_{i+1}}\ldots P^{\mu_{n_1}} \quad (17)$$

we write down the following parametrization for the πN matrix element of the local operator (16):

$$\begin{aligned} &4\langle\pi|\hat{O}^{\mu_1\ldots\mu_{n_1},\nu_1\ldots\nu_{n_2},\lambda_1\ldots\lambda_{n_3}}_{\rho\tau\chi}(0)|N\rangle \\ &= i\frac{f_N}{f_\pi M}\Big[\sum_s (s^{\pi N})_{\rho\tau,\chi}\sum_{i=0}^{n_1}\sum_{j=0}^{n_2}\sum_{k=0}^{n_3}A^{s(n_1,n_2,n_3)}_{ijk}(\Delta^2)(\Delta^\mu)^i(P^\mu)^{n_1-i}(\Delta^\nu)^j(P^\nu)^{n_2-j}(\Delta^\lambda)^i(P^\lambda)^{n_3-i} \\ &+ \Big\{(\hat{\Delta}C)_{\rho\tau}(\hat{P}U)_\chi C^{V_1(n_1,n_2,n_3)}_{N+1}(\Delta^2) + (\hat{\Delta}C)_{\rho\tau}(\hat{\Delta}U)_\chi C^{V_2(n_1,n_2,n_3)}_{N+1}(\Delta^2) + (\hat{\Delta}\gamma^5C)_{\rho\tau}(\gamma^5\hat{P}U)_\chi C^{A_1(n_1,n_2,n_3)}_{N+1}(\Delta^2) \\ &+ (\hat{\Delta}\gamma^5C)_{\rho\tau}(\gamma^5\hat{\Delta}U)_\chi C^{A_2(n_1,n_2,n_3)}_{N+1}(\Delta^2) + (\sigma_{\Delta\mu}C)_{\rho\tau}(\gamma^\mu\hat{P}U)_\chi C^{T_1(n_1,n_2,n_3)}_{N+1}(\Delta^2) \\ &+ (\sigma_{\Delta\mu}C)_{\rho\tau}(\gamma^\mu\hat{\Delta}U)_\chi C^{T_2(n_1,n_2,n_3)}_{N+1}(\Delta^2)\Big\}(\Delta^\mu)^{n_1}(\Delta^\nu)^{n_2}(\Delta^\lambda)^{n_3}\Big], \end{aligned} \quad (18)$$

where the sum in the first term is over all independent Dirac structures (12); $A^{s(n_1,n_2,n_3)}_{ijk}(\Delta^2)$ and $C^{V_{1,2},A_{1,2},T_{1,2}(n_1,n_2,n_3)}_{N+1}(\Delta^2)$ denote the appropriate invariant form factors.

We introduce the compact notation for the Mellin moments of TDAs:

$$\langle x_1^{n_1}x_2^{n_2}x_3^{n_3}H_s^{\pi N}\rangle = \int_{-1+\xi}^{1+\xi}dx_1\int_{-1+\xi}^{1+\xi}dx_2\int_{-1+\xi}^{1+\xi}dx_3\delta(x_1+x_2+x_3-2\xi)x_1^{n_1}x_2^{n_2}x_3^{n_3}H_s^{\pi N}(x_1,x_2,x_3,\xi,\Delta). \quad (19)$$

Now from (18) we establish the following relations for (n_1, n_2, n_3)-th ($n_1 + n_2 + n_3 = N$) Mellin moments of TDAs:

$$\langle x_1^{n_1} x_2^{n_2} x_3^{n_3} \{V_{1,2}, A_{1,2}, T_{1,2}\}\rangle = \sum_{n=1}^{N} (-1)^{N-n} (2\xi)^n \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} \sum_{k=0}^{n_3} \delta_{i+j+k,n} A_{ijk}^{\{V_{1,2},A_{1,2},T_{1,2}\}(n_1,n_2,n_3)}(\Delta^2) - (2\xi)^{N+1} C_{N+1}^{\{V_{1,2},A_{1,2},T_{1,2}\}(n_1,n_2,n_3)}(\Delta^2);$$
$$\langle x_1^{n_1} x_2^{n_2} x_3^{n_3} \{T_{3,4}\}\rangle = \sum_{n=1}^{N} (-1)^{N-n} (2\xi)^n \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} \sum_{k=0}^{n_3} \delta_{i+j+k,n} A_{ijk}^{\{T_{3,4}\}(n_1,n_2,n_3)}(\Delta^2). \tag{20}$$

Thus, we conclude that the πN TDAs defined in (10) indeed satisfy the polynomiality property. For $n_1 + n_2 + n_3 = N$ the highest power of ξ occurring in (n_1, n_2, n_3)-th Mellin moment of $\{V_{1,2}^{\pi N}, A_{1,2}^{\pi N}, T_{1,2}^{\pi N}\}$ is $N + 1$ while for $T_{3,4}^{\pi N}$ it is N. Consequently, the TDAs $\{V_{1,2}^{\pi N}, A_{1,2}^{\pi N}, T_{1,2}^{\pi N}\}$ include an analogue of the D-term contribution [19] which generates the highest possible power of ξ. Note that, exactly as in the case of GPDs, the spectral representation [21] cannot produces the highest possible power of ξ in the Mellin moments. Therefore, the complete parametrization of πN TDAs requires adding a separate D-term contribution to the spectral representation or a singular modification of corresponding spectral densities in the spirit of Ref. [32].

III. ISOSPIN PARAMETRIZATION FOR LEADING TWIST BARYON DISTRIBUTION AMPLITUDES

A. Notes on the operator in question

Below we review the group-theoretical properties of the three-quark operator (1) under the $SU(2)$ isospin symmetry group. Throughout the rest of the paper we adopt the following conventions:

(i) Letters from the beginning of the Greek alphabet are reserved for the $SU(2)$ isospin indices α, β, γ, ι, $\kappa = 1, 2$.

(ii) We have to distinguish between upper (contravariant) and lower (covariant) $SU(2)$ isospin indices. We introduce the totally antisymmetric tensor $\varepsilon_{\alpha\beta}$ for lowering indices and $\varepsilon^{\alpha\beta}$ for rising indices ($\varepsilon_{12} = \varepsilon^{12} = 1$): $\Psi^\alpha \varepsilon_{\alpha\beta} = \Psi_\beta$, $\Psi_\alpha \varepsilon^{\alpha\beta} = \Psi^\beta$ and $\delta^\alpha_\beta = -\varepsilon^\alpha_\beta = \varepsilon_\beta^\alpha$.

(iii) Letters from the middle of the Greek alphabet λ, μ, ν denote the Lorentz indices.

(iv) Letters from the second half of the Greek alphabet ρ, τ, χ are reserved for the Dirac indices.

(v) Letters from the beginning of the Latin alphabet $a, b, c \ldots$ are reserved for indices of the adjoint representation of the $SU(2)$ isospin group.

(vi) Letters c_1, c_2, c_3 stand for $SU(3)$ color indices.

To simplify our formulas we will often skip the color and the Dirac indices when they are irrelevant for the discussion. We will also often employ the shortened notation for the arguments of the operator (1): $\hat{O}^{\alpha\beta\gamma}_{\rho\tau\chi}(z_1, z_2, z_3) \equiv \hat{O}^{\alpha\beta\gamma}_{\rho\tau\chi}(1, 2, 3)$.

The operator (1) transforms according to the

$$2 \otimes 2 \otimes 2 = 4 \oplus 2 \oplus 2 \tag{21}$$

representation of the isospin $SU(2)$. To find out the operators transforming according to the isospin-$\frac{3}{2}$ and isospin-$\frac{1}{2}$ representations we single out the totally symmetric and totally antisymmetric parts of (1):

$$\hat{O}^{\alpha\beta\gamma} = \hat{O}^{[\alpha\beta\gamma]} + \hat{O}^{\{\alpha\beta\gamma\}} + \hat{\tilde{O}}^{\alpha\beta\gamma}. \tag{22}$$

The totally symmetric part

$$\hat{O}^{\{\alpha\beta\gamma\}} = \tfrac{1}{6}(\Psi^\alpha\Psi^\beta\Psi^\gamma + \Psi^\beta\Psi^\alpha\Psi^\gamma + \Psi^\alpha\Psi^\gamma\Psi^\beta + \Psi^\beta\Psi^\gamma\Psi^\alpha + \Psi^\gamma\Psi^\beta\Psi^\alpha + \Psi^\gamma\Psi^\alpha\Psi^\beta) \tag{23}$$

obviously transforms according to isospin-$\frac{3}{2}$ representation. The totally antisymmetric part

$$\hat{O}^{[\alpha\beta\gamma]} = \tfrac{1}{6}(\Psi^\alpha\Psi^\beta\Psi^\gamma - \Psi^\beta\Psi^\alpha\Psi^\gamma - \Psi^\alpha\Psi^\gamma\Psi^\beta + \Psi^\beta\Psi^\gamma\Psi^\alpha - \Psi^\gamma\Psi^\beta\Psi^\alpha + \Psi^\gamma\Psi^\alpha\Psi^\beta) \tag{24}$$

is zero in $SU(2)$. The explicit expression for the remaining part $\hat{\tilde{O}}^{\alpha\beta\gamma}$ reads:

$$\hat{\tilde{O}}^{\alpha\beta\gamma} = \tfrac{1}{3}(2\Psi^\alpha\Psi^\beta\Psi^\gamma - \Psi^\beta\Psi^\gamma\Psi^\alpha - \Psi^\gamma\Psi^\alpha\Psi^\beta). \tag{25}$$

One can represent $\hat{\tilde{O}}^{\alpha\beta\gamma}$ as a sum of three operators which are antisymmetric in pairs of indices $[\alpha, \beta]$, $[\alpha, \gamma]$ and $[\beta, \gamma]$:

$$\hat{\tilde{O}}^{\alpha\beta\gamma} = \hat{O}_1^{[\alpha\beta]\gamma} + \hat{O}_2^{[\alpha\check{\beta}\gamma]} + \hat{O}_3^{\alpha[\beta\gamma]}. \tag{26}$$

The explicit expressions for the operators $\hat{O}_{1,2,3}^{[..]}$ read

$$\hat{O}_1^{[\alpha\beta]\gamma} = \tfrac{1}{9}(2\Psi^\alpha\Psi^\beta\Psi^\gamma + \Psi^\alpha\Psi^\gamma\Psi^\beta - 2\Psi^\beta\Psi^\alpha\Psi^\gamma - \Psi^\beta\Psi^\gamma\Psi^\alpha - \Psi^\gamma\Psi^\alpha\Psi^\beta + \Psi^\gamma\Psi^\beta\Psi^\alpha);$$
$$\hat{O}_2^{[\alpha\check{\beta}\gamma]} = \tfrac{1}{9}(2\Psi^\alpha\Psi^\beta\Psi^\gamma + \Psi^\alpha\Psi^\gamma\Psi^\beta + \Psi^\beta\Psi^\alpha\Psi^\gamma - \Psi^\beta\Psi^\gamma\Psi^\alpha - \Psi^\gamma\Psi^\alpha\Psi^\beta - 2\Psi^\gamma\Psi^\beta\Psi^\alpha);$$
$$\hat{O}_3^{\alpha[\beta\gamma]} = \tfrac{1}{9}(2\Psi^\alpha\Psi^\beta\Psi^\gamma - 2\Psi^\alpha\Psi^\gamma\Psi^\beta + \Psi^\beta\Psi^\alpha\Psi^\gamma - \Psi^\beta\Psi^\gamma\Psi^\alpha - \Psi^\gamma\Psi^\alpha\Psi^\beta + \Psi^\gamma\Psi^\beta\Psi^\alpha). \tag{27}$$

Contracting operators (27) with the appropriate ε tensor we get a spinor transforming according to the fundamental representation of $SU(2)$. Note that only two operators in (27) are independent due to the relation

$$\varepsilon_{\alpha\beta}\hat{O}_1^{[\alpha\beta]\delta} - \varepsilon_{\alpha\gamma}\hat{O}_2^{[\alpha\check{\delta}\gamma]} + \varepsilon_{\beta\gamma}\hat{O}_3^{\delta[\beta\gamma]} = 0. \qquad (28)$$

Thus, in complete accordance with (21), the tensor decomposition of the three-quark operator (1) involves two copies of operators transforming according to the isospin-$\frac{1}{2}$ representations of the isospin group.

B. Case of nucleon DA

In this subsection we suggest convenient notations for the leading twist nucleon distribution amplitude. We introduce the isospin parametrization for the leading twist nucleon DA and rederive the familiar results [33,34] for the symmetry properties of the nucleon DA. This technique is applied in the next subsection to the analysis of the more involved cases of isospin structure and symmetry properties of $\Delta(1232)$ DA and πN TDAs and GDAs.

The leading twist nucleon DA [34] is defined through the matrix element of the three-quark operator $\hat{O}^{\alpha\beta\gamma}_{\rho\tau\chi}(1,2,3)$ between a nucleon state and the vacuum. Being guided by the principle of invariance under $SU(2)$ isospin we can put down the following isospin decomposition for the matrix element in question:

$$\begin{aligned}4\langle 0|\hat{O}^{\alpha\beta\gamma}_{\rho\tau\chi}(1,2,3)|N_\iota(p_N)\rangle\\ = \varepsilon^{\alpha\beta}\delta^\gamma_\iota M^N_{1\rho\tau\chi}(1,2,3) + \varepsilon^{\alpha\gamma}\delta^\beta_\iota M^N_{2\rho\tau\chi}(1,2,3)\\ + \varepsilon^{\beta\gamma}\delta^\alpha_\iota M^N_{3\rho\tau\chi}(1,2,3). \qquad (29)\end{aligned}$$

The three isospin invariant amplitudes are not independent due to the identity

$$\varepsilon^{\alpha\beta}\delta^\gamma_\iota + \varepsilon^{\beta\gamma}\delta^\alpha_\iota - \varepsilon^{\alpha\gamma}\delta^\beta_\iota = 0. \qquad (30)$$

It is worth emphasizing that the fact we have only two independent invariant isospin amplitudes meets with the consequences of the Wigner-Eckart theorem [35] for the matrix element of the three-quark operator $\hat{O}$. Indeed, as we checked in Sec. III A, the tensor decomposition of the operator $\hat{O}$ involves two independent copies of operators transforming according to the isospin-$\frac{1}{2}$ representations of the isospin group. However, it should be properly taken into account that the nucleon DA also possesses specific properties under group of permutations of three-quark fields occurring in the operator $\hat{O}$.

To address this issue we introduce the following notations for the combinations of the isotopic amplitudes defined in (29) which, as it is demonstrated below, are symmetric under permutation of the appropriate quark fields in the operator $\hat{O}^{\alpha\beta\gamma}_{\rho\tau\chi}(z_1,z_2,z_3)$:

$$\begin{aligned}M^N_{2\rho\tau\chi}(z_1,z_2,z_3) + M^N_{3\rho\tau\chi}(z_1,z_2,z_3) \equiv M^{N\{12\}}_{\rho\tau\chi}(z_1,z_2,z_3);\\ M^N_{1\rho\tau\chi}(z_1,z_2,z_3) - M^N_{3\rho\tau\chi}(z_1,z_2,z_3) \equiv M^{N\{13\}}_{\rho\tau\chi}(z_1,z_2,z_3);\\ -M^N_{1\rho\tau\chi}(z_1,z_2,z_3) - M^N_{2\rho\tau\chi}(z_1,z_2,z_3) \equiv M^{N\{23\}}_{\rho\tau\chi}(z_1,z_2,z_3).\end{aligned} \qquad (31)$$

Note that these combinations satisfy the identity

$$\begin{aligned}M^{N\{12\}}_{\rho\tau\chi}(z_1,z_2,z_3) + M^{N\{13\}}_{\rho\tau\chi}(z_1,z_2,z_3)\\ + M^{N\{23\}}_{\rho\tau\chi}(z_1,z_2,z_3) = 0. \qquad (32)\end{aligned}$$

In fact this is nothing but the familiar isospin identity for nucleon DA. Indeed, one may check that

$$\begin{aligned}&4\langle 0|\hat{O}^{uud}_{\rho\tau\chi}(z_1,z_2,z_3)|N_p(p_N)\rangle\\ &\quad = -4\langle 0|\hat{O}^{ddu}_{\rho\tau\chi}(z_1,z_2,z_3)|N_n(p_N)\rangle = M^{N\{12\}}_{\rho\tau\chi}(z_1,z_2,z_3);\\ &4\langle 0|\hat{O}^{udu}_{\rho\tau\chi}(z_1,z_2,z_3)|N_p(p_N)\rangle\\ &\quad = -4\langle 0|\hat{O}^{dud}_{\rho\tau\chi}(z_1,z_2,z_3)|N_n(p_N)\rangle = M^{N\{13\}}_{\rho\tau\chi}(z_1,z_2,z_3);\\ &4\langle 0|\hat{O}^{duu}_{\rho\tau\chi}(z_1,z_2,z_3)|N_p(p_N)\rangle\\ &\quad = -4\langle 0|\hat{O}^{udd}_{\rho\tau\chi}(z_1,z_2,z_3)|N_n(p_N)\rangle = M^{N\{23\}}_{\rho\tau\chi}(z_1,z_2,z_3)\end{aligned} \qquad (33)$$

and recover the usual form of the isospin identity from (32). Note that the neutron DA differs from that of the proton only by the overall sign, as it is well known.

To derive further symmetry properties of the nucleon DA under permutation of their arguments we employ the fact that quarks field operators in (1) anticommute. This allows in addition to (32) to establish the following relations for the isospin amplitudes:

$$\begin{aligned}M^{N\{12\}}_{\rho\tau\chi}(1,2,3) = M^{N\{12\}}_{\tau\rho\chi}(2,1,3);\\ M^{N\{13\}}_{\rho\tau\chi}(1,2,3) = M^{N\{13\}}_{\chi\tau\rho}(3,2,1);\\ M^{N\{23\}}_{\rho\tau\chi}(1,2,3) = M^{N\{23\}}_{\rho\chi\tau}(1,3,2);\\ M^{N\{23\}}_{\rho\tau\chi}(1,2,3) = M^{N\{12\}}_{\tau\chi\rho}(2,3,1);\\ M^{N\{13\}}_{\rho\tau\chi}(1,2,3) = M^{N\{12\}}_{\rho\chi\tau}(1,3,2);\end{aligned} \qquad (34)$$

For example, the last identity in (34) is the consequence of the relations

$$\begin{aligned}&\langle 0|\varepsilon_{c_1c_2c_3}\Psi^{c_1\alpha}_\rho(1)\Psi^{c_2\beta}_\tau(2)\Psi^{c_3\gamma}_\chi(3)|N_\iota(p_N)\rangle\\ &\quad = -\langle 0|\varepsilon_{c_1c_2c_3}\Psi^{c_1\alpha}_\rho(1)\Psi^{c_3\gamma}_\chi(3)\Psi^{c_2\beta}_\tau(2)|N_\iota(p_N)\rangle\\ &\quad = \langle 0|\varepsilon_{c_1c_2c_3}\Psi^{c_1\alpha}_\rho(1)\Psi^{c_2\gamma}_\chi(3)\Psi^{c_3\beta}_\tau(2)|N_\iota(p_N)\rangle. \qquad (35)\end{aligned}$$

The first three identities in (34) justify our definitions (31) while the two last ones further constrain isospin invariant amplitudes.

We choose to express all invariant isospin amplitudes through $M^{N\{12\}}$. This allows to write down the following invariant isospin parametrization for the nucleon DA:

$$\begin{aligned}4\langle 0|\hat{O}^{\alpha\beta\gamma}_{\rho\tau\chi}(1,2,3)|N_\iota(p_N)\rangle = \varepsilon^{\alpha\beta}\delta^\gamma_\iota M^{N\{12\}}_{\rho\chi\tau}(1,3,2)\\ + \varepsilon^{\alpha\gamma}\delta^\beta_\iota M^{N\{12\}}_{\rho\tau\chi}(1,2,3). \qquad (36)\end{aligned}$$

The next step is to consider the effect of the relations (32) and (34) for the nucleon DA. To the leading twist accuracy we neglect mass effects ($p_N \to p$, where p is lightlike) and employ the standard parametrization for the invariant amplitude symmetric under the exchange of the two first quark field operators:

$$M^{N\{12\}}_{\rho\tau\chi}(z_1,z_2,z_3)=f_N[V^p(z_1,z_2,z_3)v^N_{\rho\tau,\chi}+A^p(z_1,z_2,z_3)a^N_{\rho\tau,\chi}+T^p(z_1,z_2,z_3)t^N_{\rho\tau,\chi}], \tag{37}$$

where $\{v^N, a^N, t^N\}_{\rho\tau,\chi}$ are the conventional Dirac structures:

$$v^N_{\rho\tau,\chi}=(\hat{p}C)_{\rho\tau}(\gamma^5 U(p))_\chi; \quad a^N_{\rho\tau,\chi}=(\hat{p}\gamma^5 C)_{\rho\tau}(U(p))_\chi; \quad t^N_{\rho\tau,\chi}=(\sigma_{p\mu}C)_{\rho\tau}(\gamma^\mu\gamma^5 U(p))_\chi. \tag{38}$$

The symmetry relations (B3) for the Dirac structures (38) under the interchange of the two first Dirac indices together with (34) lead to the familiar symmetry properties:

$$V^p(1,2,3)=V^p(2,1,3); \quad T^p(1,2,3)=T^p(2,1,3); \quad A^p(1,2,3)=-A^p(2,1,3). \tag{39}$$

Next, using symmetry relations (34) and isospin identity (32) together with the Fierz transformation (B4) for the Dirac structures (38), one may establish the well known relation for twist-3 nucleon DAs [7,33,34]:

$$2T^p(1,2,3)=(V^p-A^p)(1,3,2)+(V^p-A^p)(2,3,1). \tag{40}$$

This reflects the fact that at leading twist there is only one independent nucleon DA, usually denoted as ϕ^N:

$$\phi^N\equiv V^p-A^p. \tag{41}$$

The DAs V^p, A^p and T^p are expressed through this latter function according to

$$\begin{aligned}2V^p(1,2,3)&=\phi^N(1,2,3)+\phi^N(2,1,3);\\ 2A^p(1,2,3)&=-\phi^N(1,2,3)+\phi^N(2,1,3);\\ 2T^p(1,2,3)&=\phi^N(1,3,2)+\phi^N(2,3,1).\end{aligned} \tag{42}$$

C. Case of $\Delta(1232)$ DA

In this subsection we introduce the invariant isospin notations for the leading twist DA of $\Delta(1232)$ resonance [36]. With respect to $SU(2)$ isospin group Δ resonance state represents a spin tensor with one covariant spinor index and one vector index:

$$I_a|\Delta_{b\iota}\rangle=\left\{i\varepsilon_{abc}\delta^\kappa{}_\iota+\frac{1}{2}(\sigma_a)^\kappa{}_\iota\delta_{bc}\right\}|\Delta_{c\kappa}\rangle. \tag{43}$$

It is natural to choose the isospin conventions for Δ resonance so that the isospin classification of Δ states coincide with that for isospin-$\frac{3}{2}$ πN states (A22).

With respect to the Lorentz group Δ resonance field is described with the help of the Rarita-Schwinger spin-tensor $\mathcal{U}^\mu_\iota(p_\Delta,s_\Delta)$. As usual, $\bar{\mathcal{U}}^\mu(p_\Delta,s_\Delta)=(\mathcal{U}^\mu(p_\Delta,s_\Delta))^\dagger\gamma_0$. For $p_\Delta^2=M_\Delta^2$ spin-tensor $\mathcal{U}^\mu_\iota$ satisfies the following auxiliary conditions:

$$\begin{aligned}(\hat{p}_\Delta-M_\Delta)\mathcal{U}^\mu(p_\Delta,s_\Delta)|_{p_\Delta^2=M_\Delta^2}&=0;\\ \bar{\mathcal{U}}^\mu(p_\Delta,s_\Delta)\mathcal{U}_\mu(p_\Delta,s_\Delta)|_{p_\Delta^2=M_\Delta^2}&=-2M_\Delta;\\ \gamma^\mu\mathcal{U}_\mu(p_\Delta,s_\Delta)|_{p_\Delta^2=M_\Delta^2}&=p^\mu_\Delta\mathcal{U}_\mu(p_\Delta,s_\Delta)|_{p_\Delta^2=M_\Delta^2}\\ &=0.\end{aligned} \tag{44}$$

Being guided by the invariance under the isospin group we may write the following tensor decomposition for the matrix element of the three-quark operator between Δ resonance state and vacuum:

$$4\langle 0|\hat{O}^{\{\alpha\beta\gamma\}}_{\rho\tau\chi}(z_1,z_2,z_3)|\Delta_{a\iota}(p_\Delta)\rangle=(f_a)^{\{\alpha\beta\gamma\}}{}_\iota M^\Delta_{\rho\tau\chi}(1,2,3). \tag{45}$$

Here $(f_a)^{\{\alpha\beta\gamma\}}{}_\iota$ stands for the only tensor totally symmetric in α, β, γ one can construct out of the existing structures:

$$\begin{aligned}(f_a)^{\{\alpha\beta\gamma\}}{}_\iota&=\frac{1}{6}((\sigma_a)^\alpha_\delta\varepsilon^{\delta\beta}\delta^\gamma_\iota+(\sigma_a)^\alpha_\delta\varepsilon^{\delta\gamma}\delta^\beta_\iota+(\sigma_a)^\beta_\delta\varepsilon^{\delta\alpha}\delta^\gamma_\iota\\ &\quad+(\sigma_a)^\beta_\delta\varepsilon^{\delta\gamma}\delta^\alpha_\iota+(\sigma_a)^\gamma_\delta\varepsilon^{\delta\alpha}\delta^\beta_\iota+(\sigma_a)^\gamma_\delta\varepsilon^{\delta\beta}\delta^\alpha_\iota)\\ &\equiv\frac{1}{3}((\sigma_a)^\alpha_\delta\varepsilon^{\delta\beta}\delta^\gamma_\iota+(\sigma_a)^\alpha_\delta\varepsilon^{\delta\gamma}\delta^\beta_\iota+(\sigma_a)^\beta_\delta\varepsilon^{\delta\gamma}\delta^\alpha_\iota);\\ &\text{since } (\sigma_a)^\alpha_\delta\varepsilon^{\delta\beta}=(\sigma_a)^\beta_\delta\varepsilon^{\delta\alpha}.\end{aligned} \tag{46}$$

One may check that the convolutions of the invariant tensor $(f_a)^{\{\alpha\beta\gamma\}}{}_\iota$ with the isospin projecting operators (A26) respect the following properties:

$$\begin{aligned}P^{3/2}{}_b{}^\kappa{}_{a\iota}(f_b)^{\{\alpha\beta\gamma\}}{}_\kappa&=(f_a)^{\{\alpha\beta\gamma\}}{}_\iota;\\ P^{1/2}{}_b{}^\kappa{}_{a\iota}(f_b)^{\{\alpha\beta\gamma\}}{}_\kappa&=0.\end{aligned} \tag{47}$$

We employ the following parametrization for the leading twist invariant amplitude $M^\Delta_{\rho\tau\chi}(1,2,3)$:

$$\begin{aligned}M^\Delta_{\rho\tau\chi}(1,2,3)=&-\frac{\lambda_\Delta^{1/2}}{\sqrt{2}}\{v^\Delta_{\rho\tau,\chi}V^\Delta(1,2,3)\\ &+a^\Delta_{\rho\tau,\chi}A^\Delta(1,2,3)+t^\Delta_{\rho\tau,\chi}T^\Delta(1,2,3)\}\\ &-\frac{f_\Delta^{3/2}}{\sqrt{2}}\varphi^\Delta_{\rho\tau,\chi}\phi^{\Delta_{3/2}}(1,2,3),\end{aligned} \tag{48}$$

where $\{v^\Delta, a^\Delta, t^\Delta, \varphi^\Delta\}_{\rho\tau,\chi}$ are the usual Dirac structures

$$\begin{aligned}v^\Delta_{\rho\tau,\chi}&=(\gamma_\mu C)_{\rho\tau}\mathcal{U}^\mu_\chi;\\ a^\Delta_{\rho\tau,\chi}&=(\gamma_\mu\gamma_5 C)_{\rho\tau}(\gamma_5\mathcal{U}^\mu)_\chi;\\ t^\Delta_{\rho\tau,\chi}&=\frac{1}{2}(\sigma_{\mu\nu}C)_{\rho\tau}(\gamma^\mu\mathcal{U}^\nu)_\chi;\\ \varphi^\Delta_{\rho\tau,\chi}&=(\sigma_{\mu\nu}C)_{\rho\tau}\left(p^\mu\mathcal{U}^\nu-\frac{1}{2}M_\Delta\gamma^\mu\mathcal{U}^\nu\right)_\chi\end{aligned} \tag{49}$$

and the constants $\lambda_\Delta^{1/2}$, $f_\Delta^{3/2}$ are defined in Ref. [36]. The factor $-\frac{1}{\sqrt{2}}$ in (48) ensures matching with the parametrization of [36] for the uuu DA of $|\Delta^{++}\rangle$ [c.f. Eq. (50)]. The

DAs V^Δ, A^Δ, T^Δ and $\phi^{\Delta_{3/2}}$ in (48) thus coincide with those of Refs. [9,36].

The isospin identities for the invariant amplitude $M^\Delta_{\rho\tau\chi}(z_1, z_2, z_3)$ are derived analogously to how this was done for the nucleon case in the previous subsection. Consider

$$4\langle 0|\varepsilon_{c_1c_2c_3} u^{c_1}_\rho(z_1) u^{c_2}_\tau(z_2) u^{c_3}_\chi(z_3)|\Delta^{++}\rangle = -\sqrt{2} M^\Delta_{\rho\tau\chi}(1, 2, 3). \quad (50)$$

The invariance under permutations of three u-quark fields in (50) leads to the complete symmetry of the invariant matrix element under simultaneous permutations of the arguments and of the Dirac indices:

$$M^\Delta_{\rho\tau\chi}(1,2,3) = M^\Delta_{\rho\chi\tau}(1,3,2) = M^\Delta_{\tau\rho\chi}(2,1,3) = M^\Delta_{\tau\chi\rho}(2,3,1) = M^\Delta_{\chi\tau\rho}(3,2,1) = M^\Delta_{\chi\rho\tau}(3,1,2). \quad (51)$$

Employing (51) together with the well-known symmetry relations (B6) for the Dirac structures (49) and the twist-3 Fierz transformations (B7) one establishes the familiar relations [36] for the invariant functions V^Δ, A^Δ, T^Δ and $\phi^{\Delta_{3/2}}$ defined in (48). Introducing the notation $\phi^{\Delta_{1/2}}(1, 2, 3) = V^\Delta(1, 2, 3) - A^\Delta(1, 2, 3)$ these relations can be written as

$$\begin{aligned} 2V^\Delta(1, 2, 3) &= \phi^{\Delta_{1/2}}(1, 2, 3) + \phi^{\Delta_{1/2}}(2, 1, 3); \\ 2A^\Delta(1, 2, 3) &= -\phi^{\Delta_{1/2}}(1, 2, 3) + \phi^{\Delta_{1/2}}(2, 1, 3); \\ T^\Delta(1, 2, 3) &= \phi^{\Delta_{1/2}}(2, 3, 1); \end{aligned} \quad (52)$$

together with the consistency condition

$$\phi^{\Delta_{1/2}}(1, 2, 3) = \phi^{\Delta_{1/2}}(3, 2, 1). \quad (53)$$

Meanwhile, $\phi^{\Delta_{3/2}}(1, 2, 3)$ turns out to be totally symmetric.

IV. ISOSPIN PARAMETRIZATION FOR πN TDA AND GDA

Let us consider now the matrix element of three-quark operator $\hat{O}^{\alpha\beta\gamma}_{\rho\tau\chi}(z_1, z_2, z_3)$ between πN states both in TDA and GDA regimes. From the point of view of the isospin symmetry the two regimes can be analyzed on the same footing since the pion field π_a transforms according to the adjoint representation of the isospin group. So below we present the isospin decomposition of πN TDA. The expression for πN GDA is exactly the same.

Isospin decomposition for πN TDA should involve both the isospin-$\frac{3}{2}$ and isospin-$\frac{1}{2}$ parts. Thus, analogously to the cases of Δ and nucleon DAs, we can write the following isospin decomposition:

$$\begin{aligned} 4\langle \pi_a|\hat{O}^{\alpha\beta\gamma}_{\rho\tau\chi}(z_1, z_2, z_3)|N_\iota\rangle &= (f_a)^{\{\alpha\beta\gamma\}}{}_\iota M^{(\pi N)_{3/2}}_{\rho\tau\chi}(1, 2, 3) + \varepsilon^{\alpha\beta}(\sigma_a)^\gamma{}_\iota M_1^{(\pi N)_{1/2}}{}_{\rho\tau\chi}(1, 2, 3) \\ &\quad + \varepsilon^{\alpha\gamma}(\sigma_a)^\beta{}_\iota M_2^{(\pi N)_{1/2}}{}_{\rho\tau\chi}(1, 2, 3) + \varepsilon^{\beta\gamma}(\sigma_a)^\alpha{}_\iota M_3^{(\pi N)_{1/2}}{}_{\rho\tau\chi}(1, 2, 3) \\ &= (f_a)^{\{\alpha\beta\gamma\}}{}_\iota M^{(\pi N)_{3/2}}_{\rho\tau\chi}(1, 2, 3) + \varepsilon^{\alpha\beta}(\sigma_a)^\gamma{}_\iota M^{(\pi N)_{1/2}\{13\}}_{\rho\tau\chi}(1, 2, 3) \\ &\quad + \varepsilon^{\alpha\gamma}(\sigma_a)^\beta{}_\iota M^{(\pi N)_{1/2}\{12\}}_{\rho\tau\chi}(1, 2, 3), \end{aligned} \quad (54)$$

where $(f_a)^{\{\alpha\beta\gamma\}}{}_\iota$ is the symmetric tensor defined in (46). To put down the last equality in (54) we employed the identity

$$\varepsilon^{\beta\gamma}(\sigma_a)^\alpha{}_\iota = -\varepsilon^{\alpha\beta}(\sigma_a)^\gamma{}_\iota + \varepsilon^{\alpha\gamma}(\sigma_a)^\beta{}_\iota \quad (55)$$

to eliminate the third structure corresponding to the isospin-$\frac{1}{2}$ representation. Analogously to (31) we introduce the notations

$$\begin{aligned} M_2^{(\pi N)_{1/2}}{}_{\rho\tau\chi}(1, 2, 3) + M_3^{(\pi N)_{1/2}}{}_{\rho\tau\chi}(1, 2, 3) &\equiv M^{(\pi N)_{1/2}\{12\}}_{\rho\tau\chi}(1, 2, 3); \\ M_1^{(\pi N)_{1/2}}{}_{\rho\tau\chi}(1, 2, 3) - M_3^{(\pi N)_{1/2}}{}_{\rho\tau\chi}(1, 2, 3) &\equiv M^{(\pi N)_{1/2}\{13\}}_{\rho\tau\chi}(1, 2, 3); \\ -M_1^{(\pi N)_{1/2}}{}_{\rho\tau\chi}(1, 2, 3) - M_2^{(\pi N)_{1/2}}{}_{\rho\tau\chi}(1, 2, 3) &\equiv M^{(\pi N)_{1/2}\{23\}}_{\rho\tau\chi}(1, 2, 3). \end{aligned} \quad (56)$$

Our present goal is to establish the isospin and permutation symmetry identities for πN TDA invariant isotopic amplitudes. We will show that isotopic and permutation symmetry reduces the number of independent πN TDAs from 16 functions [8 both for the isospin-$\frac{1}{2}$ and isospin-$\frac{3}{2}$ parts as in Eq. (10)] to just 8 independent functions.

A. Isospin-$\frac{1}{2}$ case

One may check that the isospin-$\frac{1}{2}$ invariant amplitudes satisfy the set of identities analogous to the isospin invariant amplitudes for nucleon DA. The isospin identity reads

$$M^{(\pi N)_{1/2}\{12\}}_{\rho\tau\chi}(1,2,3) + M^{(\pi N)_{1/2}\{13\}}_{\rho\tau\chi}(1,2,3) + M^{(\pi N)_{1/2}\{23\}}_{\rho\tau\chi}(1,2,3) = 0. \quad (57)$$

The permutation symmetry results in the set of identities analogous to (34):

$$M^{(\pi N)_{1/2}\{12\}}_{\rho\tau\chi}(1,2,3)=M^{(\pi N)_{1/2}\{12\}}_{\tau\rho\chi}(2,1,3);\qquad M^{(\pi N)_{1/2}\{13\}}_{\rho\tau\chi}(1,2,3)=M^{(\pi N)_{1/2}\{13\}}_{\chi\tau\rho}(3,2,1);$$
$$M^{(\pi N)_{1/2}\{23\}}_{\rho\tau\chi}(1,2,3)=M^{(\pi N)_{1/2}\{23\}}_{\rho\chi\tau}(1,3,2);\qquad M^{(\pi N)_{1/2}\{13\}}_{\rho\tau\chi}(1,2,3)=M^{(\pi N)_{1/2}\{12\}}_{\rho\chi\tau}(1,3,2);$$
$$M^{(\pi N)_{1/2}\{23\}}_{\rho\tau\chi}(1,2,3)=M^{(\pi N)_{1/2}\{12\}}_{\tau\chi\rho}(2,3,1). \tag{58}$$

For $M^{(\pi N)_{1/2}\{12\}}_{\rho\tau\chi}(1,2,3)$ we introduce the parametrization (10) and define 8 leading twist isospin-$\frac{1}{2}$ πN TDAs:

$$M^{(\pi N)_{1/2}\{12\}}_{\rho\tau\chi}(1,2,3)=i\frac{f_N}{f_\pi M}\Big[V_1^{(\pi N)_{1/2}}(1,2,3)(v_1^{\pi N})_{\rho\tau,\chi}+A_1^{(\pi N)_{1/2}}(1,2,3)(a_1^{\pi N})_{\rho\tau,\chi}+T_1^{(\pi N)_{1/2}}(1,2,3)(t_1^{\pi N})_{\rho\tau,\chi}$$
$$+V_2^{(\pi N)_{1/2}}(1,2,3)(v_2^{\pi N})_{\rho\tau,\chi}+A_2^{(\pi N)_{1/2}}(1,2,3)(a_2^{\pi N})_{\rho\tau,\chi}+T_2^{(\pi N)_{1/2}}(1,2,3)(t_2^{\pi N})_{\rho\tau,\chi}$$
$$+\frac{1}{M}T_3^{(\pi N)_{1/2}}(1,2,3)(t_3^{\pi N})_{\rho\tau,\chi}+\frac{1}{M}T_4^{(\pi N)_{1/2}}(1,2,3)(t_4^{\pi N})_{\rho\tau,\chi}\Big]. \tag{59}$$

One may check that the permutation symmetry relations (58) result in the familiar symmetry properties of the isospin-$\frac{1}{2}$ πN TDAs:

$$V_{1,2}^{(\pi N)_{1/2}}(1,2,3)=V_{1,2}^{(\pi N)_{1/2}}(2,1,3);\qquad T_{1,2,3,4}^{(\pi N)_{1/2}}(1,2,3)=T_{1,2,3,4}^{(\pi N)_{1/2}}(2,1,3);\qquad A_{1,2}^{(\pi N)_{1/2}}(1,2,3)=-A_{1,2}^{(\pi N)_{1/2}}(2,1,3). \tag{60}$$

We introduce two independent isospin-$\frac{1}{2}$ πN TDAs:

$$\phi_{1,2}^{(\pi N)_{1/2}}(1,2,3)\equiv V_{1,2}^{(\pi N)_{1/2}}(1,2,3)-A_{1,2}^{(\pi N)_{1/2}}(1,2,3). \tag{61}$$

Employing the Fierz transformations (B9) and (B10), one establishes the consequences of the isospin symmetry relation (57):

$$T_{3,4}^{(\pi N)_{1/2}}(1,2,3)+T_{3,4}^{(\pi N)_{1/2}}(1,3,2)+T_{3,4}^{(\pi N)_{1/2}}(2,3,1)=0 \tag{62}$$

and

$$2T_{1,2}^{(\pi N)_{1/2}}(1,2,3)=\phi_{1,2}^{(\pi N)_{1/2}}(1,3,2)+\phi_{1,2}^{(\pi N)_{1/2}}(2,3,1)+2g_{1,2}(\xi,\Delta^2)T_3^{(\pi N)_{1/2}}(1,2,3)+2h_{1,2}(\xi,\Delta^2)T_4^{(\pi N)_{1/2}}(1,2,3);$$
$$2V_{1,2}^{(\pi N)_{1/2}}(1,2,3)=\phi_{1,2}^{(\pi N)_{1/2}}(1,2,3)+\phi_{1,2}^{(\pi N)_{1/2}}(2,1,3);\qquad 2A_{1,2}^{(\pi N)_{1/2}}(1,2,3)=-\phi_{1,2}^{(\pi N)_{1/2}}(1,2,3)+\phi_{1,2}^{(\pi N)_{1/2}}(2,1,3), \tag{63}$$

where $g_{1,2}(\xi,\Delta^2)$, $h_{1,2}(\xi,\Delta^2)$ are defined in (B10). We conclude that the parametrization of the isospin-$\frac{1}{2}$ πN TDAs/GDAs require 4 independent functions: $\phi_{1,2}^{(\pi N)_{1/2}}$ and $T_{3,4}^{(\pi N)_{1/2}}$. The latter should satisfy the symmetry relations (62).

B. Isospin-$\frac{3}{2}$ case

The consequences of the isotopic and permutation symmetries for the isospin-$\frac{3}{2}$ invariant amplitude $M^{(\pi N)_{3/2}}$ are analogous to that for $\Delta(1232)$ DA (51). It turns out to be completely symmetric under simultaneous permutations of the arguments and the Dirac indices:

$$M^{(\pi N)_{3/2}}_{\rho\tau\chi}(1,2,3)=M^{(\pi N)_{3/2}}_{\rho\chi\tau}(1,3,2)=M^{(\pi N)_{3/2}}_{\tau\rho\chi}(2,1,3)=M^{(\pi N)_{3/2}}_{\tau\chi\rho}(2,3,1)=M^{(\pi N)_{3/2}}_{\chi\tau\rho}(3,2,1)=M^{(\pi N)_{3/2}}_{\chi\rho\tau}(3,1,2). \tag{64}$$

Again, in accordance with (10), we introduce the following parametrization for the leading twist isospin-$\frac{3}{2}$ πN TDAs:

$$M^{(\pi N)_{3/2}}_{\rho\tau\chi}(1,2,3)=i\frac{f_N}{f_\pi M}\Big[V_1^{(\pi N)_{3/2}}(1,2,3)(v_1^{\pi N})_{\rho\tau,\chi}+A_1^{(\pi N)_{3/2}}(1,2,3)(a_1^{\pi N})_{\rho\tau,\chi}+T_1^{(\pi N)_{3/2}}(1,2,3)(t_1^{\pi N})_{\rho\tau,\chi}$$
$$+V_2^{(\pi N)_{3/2}}(1,2,3)(v_2^{\pi N})_{\rho\tau,\chi}+A_2^{(\pi N)_{3/2}}(1,2,3)(a_2^{\pi N})_{\rho\tau,\chi}+T_2^{(\pi N)_{3/2}}(1,2,3)(t_2^{\pi N})_{\rho\tau,\chi}$$
$$+\frac{1}{M}T_3^{(\pi N)_{3/2}}(1,2,3)(t_3^{\pi N})_{\rho\tau,\chi}+\frac{1}{M}T_4^{(\pi N)_{3/2}}(1,2,3)(t_4^{\pi N})_{\rho\tau,\chi}\Big]. \tag{65}$$

Analogously to the isospin-$\frac{1}{2}$ case, employing the Fierz identities of Appendix B, one may check that the permutation symmetry relations (64) result in the following symmetry properties of the isospin-$\frac{3}{2}$ πN TDAs:

$$V_{1,2}^{(\pi N)_{3/2}}(1,2,3)=V_{1,2}^{(\pi N)_{3/2}}(2,1,3);\quad T_{1,2}^{(\pi N)_{3/2}}(1,2,3)=T_{1,2}^{(\pi N)_{3/2}}(2,1,3);\quad A_{1,2}^{(\pi N)_{3/2}}(1,2,3)=-A_{1,2}^{(\pi N)_{3/2}}(2,1,3), \tag{66}$$

while $T_{3,4}^{(\pi N)_{3/2}}$ is totally symmetric.

Introducing two independent isospin-$\frac{3}{2}$ πN TDAs:

$$\phi_{1,2}^{(\pi N)_{3/2}}(1,2,3)\equiv -V_{1,2}^{(\pi N)_{3/2}}(1,2,3)+A_{1,2}^{(\pi N)_{3/2}}(1,2,3), \tag{67}$$

and employing further consequences of permutation and isotopic symmetry relations (64) one may express isospin-$\frac{3}{2}$ TDAs as

$$\begin{aligned}&2T_{1,2}^{(\pi N)_{3/2}}(1,2,3)=\phi_{1,2}^{(\pi N)_{3/2}}(1,3,2)+2g_{1,2}(\xi,\Delta^2)T_3^{(\pi N)_{3/2}}(1,2,3)+2h_{1,2}(\xi,\Delta^2)T_4^{(\pi N)_{3/2}}(1,2,3);\\&2V_{1,2}^{(\pi N)_{3/2}}(1,2,3)=-\phi_{1,2}^{(\pi N)_{3/2}}(1,2,3)-\phi_{1,2}^{(\pi N)_{3/2}}(2,1,3);\quad 2A_{1,2}^{(\pi N)_{3/2}}(1,2,3)=\phi_{1,2}^{(\pi N)_{3/2}}(1,2,3)-\phi_{1,2}^{(\pi N)_{3/2}}(2,1,3),\end{aligned} \tag{68}$$

where $g_{1,2}(\xi,\Delta^2)$, $h_{1,2}(\xi,\Delta^2)$ are defined in (B11). The consistency condition for (68) may be established from (64):

$$\phi_{1,2}^{(\pi N)_{3/2}}(1,2,3)=\phi_{1,2}^{(\pi N)_{3/2}}(3,2,1). \tag{69}$$

Thus, we conclude that the parametrization of the isospin-$\frac{3}{2}$ πN TDAs/GDAs involves 4 independent functions: $T_{3,4}^{(\pi N)_{3/2}}$ (completely symmetric under permutation of their variables) and $\phi_{1,2}^{(\pi N)_{3/2}}$ [symmetric under permutation $1\leftrightarrow 3$ cf. Eq. (69)].

V. CHIRAL CONSTRAINTS FOR πN TDAS

In this section we rederive for πN GDAs the soft pion theorem [20] proposed in [16] to be valid at a scale $Q^2\gg\Lambda_{\rm QCD}^3/m$. Our technique of handling isospin developed in Sec. III and IV permits to distinguish between the isospin-$\frac{1}{2}$ and isospin-$\frac{3}{2}$ πN GDAs. This allows to fully take into account the consequences of the isotopic and permutation symmetries for πN GDAs. Using crossing between πN GDAs and πN TDAs discussed in Sec. I we simultaneously argue that the soft pion theorem for πN GDAs constrains πN TDAs in the chiral limit ($m\to 0$). The problem of validity of analytic continuation in Δ^2 existing for $m\neq 0$ has the same status as that for the case of pion GPDs vs 2π GDAs [18] (see also discussion in [37]). Assuming smallness of nonanalytic corrections to the relevant matrix element in the narrow domain in (Δ^2,ξ) plane defined by the inequalities

$$(M-m)^2<\Delta^2<(M+m)^2;\quad \frac{M-m}{M+m}<\xi<\frac{M+m}{M-m} \tag{70}$$

(see left panel of Fig. 1) one may argue that the soft pion limit provides us with the reference point for realistic modeling of πN TDAs.

Let us consider the matrix element of the three-quark operator $\hat{O}_{\rho\tau\chi}^{\alpha\beta\gamma}(z_1,z_2,z_3)$ in the regime of πN GDA:

$$\langle 0|\hat{O}_{\rho\tau\chi}^{\alpha\beta\gamma}(z_1,z_2,z_3)|\pi_a(-p_\pi)N_\iota(p_1)\rangle. \tag{71}$$

According to the partial conservation of axial current hypothesis (see e.g. [38]), a soft pion theorem [20] is valid for the matrix element (71):

$$\langle 0|\hat{O}_{\rho\tau\chi}^{\alpha\beta\gamma}(z_1,z_2,z_3)|\pi_aN_\iota\rangle = -\frac{i}{f_\pi}\langle 0|[\hat{Q}_5^a,\hat{O}_{\rho\tau\chi}^{\alpha\beta\gamma}(z_1,z_2,z_3)]|N_\iota\rangle. \tag{72}$$

The commutator of the chiral charge operator $\hat{Q}_5^a$ with the quark field operators is given by

$$[\hat{Q}_5^a,\Psi_\eta^\alpha]=-\tfrac{1}{2}(\sigma_a)_\delta^\alpha\gamma_{\eta\tau}^5\Psi_\tau^\delta, \tag{73}$$

where σ_a are the Pauli matrices.

Computing the commutator of the chiral charge with the operator $\hat{O}$ in (72) with the help of the chain rule $[A,BCD]=[A,B]CD+B[A,C]D+BC[A,D]$ we get:

$$\begin{aligned}4\langle 0|\hat{O}_{\rho\tau\chi}^{\alpha\beta\gamma}(z_1,z_2,z_3)|\pi_aN_\iota\rangle &= 4\frac{i}{2f_\pi}\Big\{(\sigma_a)^\alpha{}_\delta\gamma_{\rho\eta}^5\langle 0|\hat{O}_{\eta\tau\chi}^{\delta\beta\gamma}(z_1,z_2,z_3)|N_\iota\rangle+(\sigma_a)^\beta{}_\delta\gamma_{\tau\eta}^5\langle 0|\hat{O}_{\rho\eta\chi}^{\alpha\delta\gamma}(z_1,z_2,z_3)|N_\iota\rangle\\&\quad+(\sigma_a)^\gamma{}_\delta\gamma_{\chi\eta}^5\langle 0|\hat{O}_{\rho\tau\eta}^{\alpha\beta\delta}(z_1,z_2,z_3)|N_\iota\rangle\Big\}\\&=\frac{i}{2f_\pi}\Big\{(\sigma_a)^\alpha{}_\delta\varepsilon^{\delta\beta}\delta_\iota^\gamma\gamma_{\rho\eta}^5M_{\eta\tau\chi}^{N\{13\}}(1,2,3)+(\sigma_a)^\alpha{}_\delta\varepsilon^{\delta\gamma}\delta_\iota^\beta\gamma_{\rho\eta}^5M_{\eta\tau\chi}^{N\{12\}}(1,2,3)\\&\quad+(\sigma_a)^\beta{}_\delta\varepsilon^{\alpha\delta}\delta_\iota^\gamma\gamma_{\tau\eta}^5M_{\rho\eta\chi}^{N\{13\}}(1,2,3)+(\sigma_a)^\beta{}_\delta\varepsilon^{\alpha\gamma}\delta_\iota^\delta\gamma_{\tau\eta}^5M_{\rho\eta\chi}^{N\{12\}}(1,2,3)\\&\quad+(\sigma_a)^\gamma{}_\delta\varepsilon^{\alpha\beta}\delta_\iota^\delta\gamma_{\chi\eta}^5M_{\rho\tau\eta}^{N\{13\}}(1,2,3)+(\sigma_a)^\gamma{}_\delta\varepsilon^{\alpha\delta}\delta_\iota^\beta\gamma_{\chi\eta}^5M_{\rho\tau\eta}^{N\{12\}}(1,2,3)\Big\}.\end{aligned} \tag{74}$$

In the last equality we used the general isospin parametrization (36) for the nucleon DA.

Our present goal is to single out the contributions coming from (74) into the invariant isospin amplitudes $M_{\rho\tau\chi}^{(\pi N)_{3/2}}(1,2,3)$, $M_{\rho\tau\chi}^{(\pi N)_{1/2}\{12\}}(1,2,3)$ and $M_{\rho\tau\chi}^{(\pi N)_{1/2}\{13\}}(1,2,3)$ in the isospin decomposition for πN TDA/GDA (54).

Using the parametrization (37) for $M_{\rho\tau\chi}^{N\{12\}}(1,2,3)$, the isospin decomposition (29) and symmetry relations (34) for the nucleon DA together with the Fierz identities from Appendix B one may check that

$$\begin{aligned} M_{\rho\tau\chi}^{(\pi N)_{3/2}}(z_1,z_2,z_3) &= \frac{i}{2f_\pi}\{\gamma^5_{\rho\eta}M^{N\{12\}}_{\eta\tau\chi}(z_1,z_2,z_3) - \gamma^5_{\chi\eta}M^{N\{12\}}_{\rho\tau\eta}(z_1,z_2,z_3) + \gamma^5_{\rho\eta}M^{N\{13\}}_{\eta\tau\chi}(z_1,z_2,z_3) - \gamma^5_{\tau\eta}M^{N\{13\}}_{\rho\eta\chi}(z_1,z_2,z_3)\} \\ &= \frac{if_N}{f_\pi}\Big\{-(\gamma^5_{\chi\eta}v^N_{\rho\tau,\eta})\frac{1}{2}[\phi_N(1,2,3)+\phi_N(2,1,3)+\phi_N(3,2,1)+\phi_N(3,1,2)] \\ &\quad -(\gamma^5_{\chi\eta}a^N_{\rho\tau,\eta})\frac{1}{2}[-\phi_N(1,2,3)+\phi_N(2,1,3)-\phi_N(3,2,1)+\phi_N(3,1,2)] \\ &\quad -(\gamma^5_{\chi\eta}t^N_{\rho\tau,\eta})\frac{1}{2}[\phi_N(1,3,2)+\phi_N(2,3,1)]\Big\}, \end{aligned} \tag{75}$$

where ϕ_N is the leading twist nucleon DA (41). The invariant amplitude (75) satisfies the isospin-$\frac{3}{2}$ symmetry relations (64). This provides an additional cross-check.

$$\begin{aligned} M_{\rho\tau\chi}^{(\pi N)_{1/2}\{12\}}(z_1,z_2,z_3) &= \frac{i}{2f_\pi}\frac{1}{3}\{\gamma^5_{\rho\eta}M^{N\{12\}}_{\eta\tau\chi}(z_1,z_2,z_3) + 3\gamma^5_{\tau\eta}M^{N\{12\}}_{\rho\eta\chi}(z_1,z_2,z_3) - \gamma^5_{\chi\eta}M^{N\{12\}}_{\rho\tau\eta}(z_1,z_2,z_3) \\ &\quad - 2\gamma^5_{\rho\eta}M^{N\{13\}}_{\eta\tau\chi}(z_1,z_2,z_3) + 2\gamma^5_{\tau\eta}M^{N\{13\}}_{\rho\eta\chi}(z_1,z_2,z_3)\} \\ &= \frac{i}{f_\pi}\Big\{-(\gamma^5_{\chi\eta}v^N_{\rho\tau,\eta})\frac{1}{12}[-\phi_N(1,2,3)-\phi_N(2,1,3)-4(\phi_N(3,1,2)+\phi_N(3,2,1))] \\ &\quad -(\gamma^5_{\chi\eta}a^N_{\rho\tau,\eta})\frac{1}{12}[\phi_N(1,2,3)-\phi_N(2,1,3)-4(\phi_N(3,1,2)-\phi_N(3,2,1))] \\ &\quad -(\gamma^5_{\chi\eta}t^N_{\rho\tau,\eta})\frac{5}{12}[\phi_N(1,3,2)+\phi_N(2,3,1)]\Big\}. \end{aligned} \tag{76}$$

Analogously,

$$\begin{aligned} M_{\rho\tau\chi}^{(\pi N)_{1/2}\{13\}}(z_1,z_2,z_3) &= \frac{i}{2f_\pi}\frac{1}{3}\{\gamma^5_{\rho\eta}M^{N\{13\}}_{\eta\tau\chi}(z_1,z_2,z_3) + 3\gamma^5_{\chi\eta}M^{N\{13\}}_{\rho\tau\eta}(z_1,z_2,z_3) - \gamma^5_{\tau\eta}M^{N\{13\}}_{\rho\eta\chi}(z_1,z_2,z_3) \\ &\quad - 2\gamma^5_{\rho\eta}M^{N\{12\}}_{\eta\tau\chi}(z_1,z_2,z_3) + 2\gamma^5_{\chi\eta}M^{N\{12\}}_{\rho\tau\eta}(z_1,z_2,z_3)\}. \end{aligned} \tag{77}$$

Again one may check that $M^{(\pi N)_{1/2}\{12\}}$ $M^{(\pi N)_{1/2}\{13\}}$ (76) and (77) computed from the soft pion theorem satisfy the isospin-$\frac{1}{2}$ and permutation symmetry relations (57) and (58).

In particular for $p\pi^0$ GDA we get

$$\begin{aligned} 4\langle 0|u_\rho(1)u_\tau(2)d_\chi(3)|p\pi^0\rangle &= M^{(\pi N)_{1/2}\{12\}}_{\rho\tau\chi}(1,2,3) + \frac{2}{3}M^{(\pi N)_{3/2}}_{\rho\tau\chi}(1,2,3) = \frac{if_N}{f_\pi}\Big\{-(\gamma^5_{\chi\eta}v^N_{\rho\tau,\eta})\frac{1}{2}V^p(1,2,3) \\ &\quad -(\gamma^5_{\chi\eta}a^N_{\rho\tau,\eta})\frac{1}{2}A^p(1,2,3) - (\gamma^5_{\chi\eta}t^N_{\rho\tau,\eta})\frac{3}{2}T^p(1,2,3)\Big\}; \\ 4\langle 0|u_\rho(1)u_\tau(2)d_\chi(3)|n\pi^+\rangle &= -4\langle 0|d_\rho(1)d_\tau(2)u_\chi(3)|p\pi^-\rangle = \sqrt{2}M^{(\pi N)_{1/2}\{12\}}_{\rho\tau\chi}(1,2,3) - \frac{\sqrt{2}}{3}M^{(\pi N)_{3/2}}_{\rho\tau\chi}(1,2,3) \\ &= \frac{if_N}{f_\pi}\Big\{-(\gamma^5_{\chi\eta}v^N_{\rho\tau,\eta})\frac{1}{2\sqrt{2}}[-\phi_N(1,2,3)-\phi_N(2,1,3)-2(\phi_N(3,1,2)+\phi_N(3,2,1))] \\ &\quad -(\gamma^5_{\chi\eta}a^N_{\rho\tau,\eta})\frac{1}{2\sqrt{2}}[\phi_N(1,2,3)-\phi_N(2,1,3)-2(\phi_N(3,1,2)-\phi_N(3,2,1))] \\ &\quad -(\gamma^5_{\chi\eta}t^N_{\rho\tau,\eta})\frac{1}{2\sqrt{2}}[\phi_N(1,3,2)+\phi_N(2,3,1)]\Big\}. \end{aligned} \tag{78}$$

So we recover the result of [16].

Now we can establish the consequences of the soft pion theorem (72) for πN TDAs. Applying crossing to the matrix element (71) is trivial up to the problem of appropriate analytic continuation in Δ^2. The contributions to πN TDAs occurring in the parametrization (10) can be established with the help of the relations between the Dirac structures (12) and those of (75) and (76):

$$\gamma^5_{\chi\eta} v^N_{\rho\tau,\eta} = \frac{1}{M}\left((v_1{}^{\pi N})_{\rho\tau,\chi} - \frac{1}{2}(v_2{}^{\pi N})_{\rho\tau,\chi} \right); \qquad \gamma^5_{\chi\eta} a^N_{\rho\tau,\eta} = \frac{1}{M}\left((a_1{}^{\pi N})_{\rho\tau,\chi} - \frac{1}{2}(a_2{}^{\pi N})_{\rho\tau,\chi} \right);$$
$$\gamma^5_{\chi\eta} t^N_{\rho\tau,\eta} = -\frac{1}{M}\left((t_1{}^{\pi N})_{\rho\tau,\chi} - \frac{1}{2}(t_2{}^{\pi N})_{\rho\tau,\chi} \right). \tag{79}$$

One may check that in the chiral limit this results in the following contributions to the independent isospin-$\frac{1}{2}$ and isospin-$\frac{3}{2}$ πN TDAs (61) and (67) regular at $\Delta^2 = M^2$:

$$\begin{aligned}
\phi_1^{(\pi N)_{1/2}}(x_1, x_2, x_3, \xi = 1, \Delta^2 = M^2)|_{\substack{\text{soft}\\ \text{pion}}} &= \frac{1}{24}\phi^N\left(\frac{x_1}{2}, \frac{x_2}{2}, \frac{x_3}{2}\right) + \frac{1}{6}\phi^N\left(\frac{x_3}{2}, \frac{x_2}{2}, \frac{x_1}{2}\right);\\
\phi_2^{(\pi N)_{1/2}}(x_1, x_2, x_3, \xi = 1, \Delta^2 = M^2)|_{\substack{\text{soft}\\ \text{pion}}} &= -\frac{1}{2}\phi_1^{(\pi N)_{1/2}}(x_1, x_2, x_3, \xi = 1, \Delta^2 = M^2)|_{\substack{\text{soft}\\ \text{pion}}};\\
\phi_1^{(\pi N)_{3/2}}(x_1, x_2, x_3, \xi = 1, \Delta^2 = M^2)|_{\substack{\text{soft}\\ \text{pion}}} &= \frac{1}{4}\left(\phi^N\left(\frac{x_1}{2}, \frac{x_2}{2}, \frac{x_3}{2}\right) + \phi^N\left(\frac{x_3}{2}, \frac{x_2}{2}, \frac{x_1}{2}\right)\right);\\
\phi_2^{(\pi N)_{3/2}}(x_1, x_2, x_3, \xi = 1, \Delta^2 = M^2)|_{\substack{\text{soft}\\ \text{pion}}} &= -\frac{1}{2}\phi_1^{(\pi N)_{3/2}}(x_1, x_2, x_3, \xi = 1, \Delta^2 = M^2)|_{\substack{\text{soft}\\ \text{pion}}}.
\end{aligned} \tag{80}$$

The singular at $\Delta^2 = M^2$ contribution from the u-channel nucleon exchange pole is considered in the next Section.

VI. *u*-CHANNEL N AND Δ EXCHANGE CONTRIBUTION INTO πN TDAS

In this section, by employing the results of Secs. III and IV, we construct a simple resonance exchange model for the isospin-$\frac{1}{2}$ and isospin-$\frac{3}{2}$ πN TDAs. It represents a consistent model for πN TDAs in the Efremov-Radyushkin-Brodsky-Lepage (ERBL)-like region and satisfies the appropriate symmetry relations established in Sec. IV as well as the polynomiality conditions of Sec. II. It turns out, in particular, that the nucleon exchange results in a pure D-term contribution supplementary to the spectral representation of [21]. Let us also mention that the nucleon pole contribution may become dominant in the near to threshold kinematics of the reaction (2), where $\Delta^2 - M^2$ is small enough.

A. Nucleon exchange contribution

The effective Hamiltonian for $\pi \bar{N} N$ interaction can be written as (see e.g. [39]):

$$\mathcal{H}_{\text{eff}}(\pi NN) = i g_{\pi NN} \bar{N}_\alpha (\sigma_a)^\alpha{}_\beta \gamma_5 N^\beta \pi_a. \tag{81}$$

After the reduction the matrix element in question reads:

$$\begin{aligned}
&\langle \pi_a(p_\pi)|\hat{O}^{\alpha\beta\gamma}_{\rho\tau\chi}(\lambda_1 n, \lambda_2 n, \lambda_3 n)|N_\iota(p_1, s_1)\rangle\\
&\quad = \sum_{s_p} \langle 0|\hat{O}^{\alpha\beta\gamma}_{\rho\tau\chi}(\lambda_1 n, \lambda_2 n, \lambda_3 n)|N_\kappa(-\Delta, s_p)\rangle\\
&\qquad \times (\sigma_a)^\kappa{}_\iota \frac{i g_{\pi NN} \bar{U}_\varrho(-\Delta, s_p)}{\Delta^2 - M^2}(\gamma^5 U(p_1, s_1))_\varrho.
\end{aligned} \tag{82}$$

πN TDAs are computed form the matrix element (82) with the help of the Fourier transform (11).

Let us first consider isospin structure of (82). Employing the isospin decomposition of the nucleon DA (36) one may check that (82) contributes only into the invariant amplitudes $M^{(\pi N)_{1/2}\{12\}}_{\rho\tau\chi}$ and $M^{(\pi N)_{1/2}\{13\}}_{\rho\tau\chi}$.

The inverse Fourier transform allowing to express the matrix element in the second line of (82) through the nucleon DA $M^{N\{12\}}_{\rho\tau\chi}(y_1, y_2, y_3)$ reads:

$$\mathcal{F}^{-1}(\lambda_k(-\Delta\cdot n))(\ldots) = \int d^3y\, \delta(1 - y_1 - y_2 - y_3) e^{i(p\cdot n)2\xi \sum_{k=1}^3 y_k\lambda_k}(\ldots). \tag{83}$$

Calculation of the Fourier transform (11) of (82) gives:

$$\begin{aligned}
&4\mathcal{F}(x_1, x_2, x_3)\frac{1}{4}[\mathcal{F}^{-1}(\lambda_k(-\Delta\cdot n))[M^{N\{12\}}_{\rho\tau\chi}(y_1, y_2, y_3)]]\\
&\quad = (p\cdot n)^3 \int_0^1 dy_1 dy_2 dy_3 \delta(1 - y_1 - y_2 - y_3)\left[\prod_{k=1}^3 \frac{1}{2\pi}\int d\lambda_k e^{i\lambda_k(x_k - 2\xi y_k)(p\cdot n)}\right] M^{N\{12\}}_{\rho\tau\chi}(y_1, y_2, y_3)\\
&\quad = \frac{1}{(2\xi)^2}\delta(x_1 + x_2 + x_3 - 2\xi)\left[\prod_{k=1}^3 \theta(0 \le x_k \le 2\xi)\right] M^{N\{12\}}_{\rho\tau\chi}\left(\frac{x_1}{2\xi}, \frac{x_2}{2\xi}, \frac{x_3}{2\xi}\right).
\end{aligned} \tag{84}$$

Thus, we obtain the following result for the contribution of the matrix element (82) into πN TDA:

$$M_{\rho\tau\chi}^{(\pi N)_{1/2}\{12\}}(x_1,x_2,x_3)=\frac{1}{(2\xi)^2}\delta(x_1+x_2+x_3-2\xi)\Big[\prod_{k=1}^{3}\theta(0\le x_k\le 2\xi)\Big]f_N\sum_{s_p}\Big\{V^p\Big(\frac{x_1}{2\xi},\frac{x_2}{2\xi},\frac{x_3}{2\xi}\Big)(-\hat{\Delta}C)_{\rho\tau}(\gamma^5U(-\Delta,s_p))_\chi$$
$$+A^p\Big(\frac{x_1}{2\xi},\frac{x_2}{2\xi},\frac{x_3}{2\xi}\Big)(-\hat{\Delta}\gamma^5C)_{\rho\tau}U(-\Delta,s_p)_\chi+T^p\Big(\frac{x_1}{2\xi},\frac{x_2}{2\xi},\frac{x_3}{2\xi}\Big)(\sigma_{-\Delta\mu}C)_{\rho\tau}(\gamma^\mu\gamma^5U(-\Delta,s_p))_\chi\Big\}$$
$$\times\frac{ig_{\pi NN}\bar{U}_\varrho(-\Delta,s_p)}{\Delta^2-M^2}(\gamma^5U(p_1,s_1))_\varrho. \quad (85)$$

Now it is straightforward to trace the contribution of the nucleon exchange matrix element into the particular invariant functions occurring in the parametrization of πN TDA. For this issue, employing formulas given in the Appendix C, one has to express the Dirac structures in (85) in terms of standard ones. For example, let us consider the first term in (85). To the leading twist accuracy

$$\sum_{s_p}(-\hat{\Delta}C)_{\rho\tau}(\gamma^5U(-\Delta,s_p))_\chi(\bar{U}(-\Delta,s_p))_\varrho(\gamma^5U(p_1,s_1))_\varrho=2\xi(\hat{P}C)_{\rho\tau}((\hat{\Delta}U(p_1,s_1))_\chi+M(U(p_1,s_1))_\chi$$
$$=2\xi(\hat{P}C)_{\rho\tau}((\hat{P}U(p_1,s_1))_\chi+\xi(\hat{P}C)_{\rho\tau}((\hat{\Delta}U(p_1,s_1))_\chi. \quad (86)$$

Finally, one establishes the expressions for the contribution of the nucleon exchange into the isospin-$\frac{1}{2}$ πN TDAs

$$\{V_1,A_1,T_1\}^{(\pi N)_{1/2}}(x_1,x_2,x_3)=\Theta_{\rm ERBL}(x_1,x_2,x_3)(g_{\pi NN})\frac{Mf_\pi}{\Delta^2-M^2}2\xi\frac{1}{(2\xi)^2}\{V^p,A^p,T^p\}\Big(\frac{x_1}{2\xi},\frac{x_2}{2\xi},\frac{x_3}{2\xi}\Big);$$
$$\{V_2,A_2,T_2\}^{(\pi N)_{1/2}}(x_1,x_2,x_3)=\Theta_{\rm ERBL}(x_1,x_2,x_3)(g_{\pi NN})\frac{Mf_\pi}{\Delta^2-M^2}\xi\frac{1}{(2\xi)^2}\{V^p,A^p,T^p\}\Big(\frac{x_1}{2\xi},\frac{x_2}{2\xi},\frac{x_3}{2\xi}\Big), \quad (87)$$

where we introduced the notation

$$\Theta_{\rm ERBL}(x_1,x_2,x_3)\equiv\prod_{k=1}^{3}\theta(0\le x_k\le 2\xi). \quad (88)$$

Notice that (87) is a pure D- term contribution. It is nonzero only in the ERBL-like region and its (n_1,n_2,n_3)-th $(n_1+n_2+n_3=N)$ Mellin moments give rise to monomials of ξ of the maximal allowed power $N+1$.

B. Δ(1232) exchange contribution

The effective Hamiltonian for $\Delta N\pi$ interaction reads (see e.g. [40]):

$$\mathcal{H}_{\rm eff}(\pi N\Delta)=g_{\pi N\Delta}\bar{N}_\kappa P^{3/2}{}_b{}^\kappa{}_{a\iota}R^\iota_{\mu a}\partial^\mu\pi_b+{\rm h.c.}, \quad (89)$$

where $P^{(3/2)}$ denotes the isospin-$\frac{3}{2}$ projecting operator (A26). $g_{\pi N\Delta}$ is a dimensional coupling constant. As usual, the Δ resonance is described with the help of the Rarita-Schwinger spin-tensor $\mathcal{U}^\mu_\rho$ which satisfies the auxiliary conditions (44).

After the reduction the matrix element in question reads:

$$\langle\pi_a(p_\pi)|\hat{O}^{\alpha\beta\gamma}_{\rho\tau\chi}(\lambda_1n,\lambda_2n,\lambda_3n)|N_\iota(p_1,s_1)\rangle$$
$$=\sum_{s_\Delta}\langle 0|\hat{O}^{\alpha\beta\gamma}_{\rho\tau\chi}(\lambda_1n,\lambda_2n,\lambda_3n)|\Delta_{b\kappa}(-\Delta,s_\Delta)\rangle P^{3/2}{}_b{}^\kappa{}_{a\iota}$$
$$\times\frac{g_{\pi N\Delta}\bar{\mathcal{U}}^\nu_\varrho(-\Delta,s_\Delta)}{\Delta^2-M^2}(iP_\nu)(U(p_1,s_1))_\varrho. \quad (90)$$

For the matrix element involving Δ we employ the parametrization (45) with $M^\Delta_{\rho\tau\chi}$ given by (48). As a consequence of the identity (47) Δ exchange populates only the isospin-$\frac{3}{2}$ πN TDAs.

To compute the on-shell numerator of graph (90) we employ the method of contracted projectors [41] (see also Appendix I of Chapter I of [38]). We introduce the corresponding on-shell spin sum:

$$\Pi^\mu_{\nu\rho\tau}(-\Delta)|_{(-\Delta)^2=M^2_\Delta}\equiv\sum_{s_\Delta=-3/2}^{3/2}\mathcal{U}_{\rho\nu}(-\Delta,s_\Delta)\bar{\mathcal{U}}^\mu_\tau(-\Delta,s_\Delta), \quad (91)$$

which carries two Dirac indices as well as two Lorentz indices. The contracted projector is defined as

$$\mathcal{P}^{(3/2)}_{\rho\tau}(\Sigma,P,-\Delta)\equiv\Sigma^\nu\Pi^\mu_{\nu\rho\tau}(-\Delta)P_\mu, \quad (92)$$

where $P_\mu=\frac{1}{2}(p_1+p_\pi)_\mu$ and Σ in principle may be an arbitrary vector. In order to keep with the u-channel baryon resonance exchange picture of the $\gamma^*N\to N\pi$ reaction (2), Σ should be chosen as:

$$\Sigma_\mu=\frac{1}{2}(q+p_2)_\mu. \quad (93)$$

We also introduce the components of P_μ and Σ_μ transverse with respect to Δ_μ denoted as $\not{P}_\mu$ and $\not{\Sigma}_\mu$[2]:

$$\not{P}_\mu\equiv P_\mu-\frac{(P\cdot\Delta)}{\Delta^2}\Delta_\mu;\quad \not{\Sigma}_\mu\equiv\Sigma_\mu-\frac{(\Sigma\cdot\Delta)}{\Delta^2}\Delta_\mu; \quad (94)$$

[2]Not to be confused with the contraction with γ matrices. We rather adopt Dirac's "hat" notation for this issue.

Then the explicit expression for the on-shell contracted projector reads [38]:

$$\mathcal{P}^{(3/2)}_{\rho\tau}(\Sigma, P, -\Delta)|_{(-\Delta)^2=M_\Delta^2} = -\frac{1}{3}|\not{P}||\Sigma|\left\{P_2'\left(\frac{(\Sigma\cdot\not{P})}{|\Sigma||\not{P}|}\right) - \frac{\hat{\Sigma}\,\hat{\not{P}}}{|\Sigma||\not{P}|}P_1'\left(\frac{(\Sigma\cdot\not{P})}{|\Sigma||\not{P}|}\right)\right\}(-\hat{\Delta}+M_\Delta)$$
$$= -\frac{1}{3}|\Sigma||\not{P}|\left\{3\frac{(\Sigma\cdot\not{P})}{|\Sigma||\not{P}|} - \frac{1}{|\Sigma||\not{P}|}\hat{\Sigma}\,\hat{\not{P}}\right\}(-\hat{\Delta}+M_\Delta), \tag{95}$$

where $P_k'(\ldots)$ stands for the derivative of the k-th Legendre polynomial. Note that the argument of the polynomials is the cosine of the u-channel center-of-mass frame scattering angle at $(-\Delta)^2 = M_\Delta^2$:

$$\cos\theta_u = \frac{(\Sigma\cdot\not{P})}{|\Sigma||\not{P}|} = \frac{1-\xi\frac{M^2-m^2}{\Delta^2}}{\xi\sqrt{1+\frac{(M^2-m^2)^2}{(\Delta^2)^2}-\frac{2(M^2+m^2)}{\Delta^2}}} + O\left(\frac{1}{Q^2}\right), \tag{96}$$

For our purpose we also need the derivative of the contracted projector:

$$\frac{\partial}{\partial\Sigma^\mu}\mathcal{P}^{(3/2)}(\Sigma, P, -\Delta) = \left\{-\not{P}^\mu + \frac{1}{3}\left(\gamma^\mu - \frac{\Delta^\mu}{\Delta^2}\hat{\Delta}\right)\hat{\not{P}}\right\}(-\hat{\Delta}+M_\Delta). \tag{97}$$

The calculation of contributions of graph (90) into the appropriate invariant form factors is then straightforward and analogous to that for the case of nucleon exchange. For example to trace the contributions into $V_{1,2}^{(\pi N)_{3/2}}$ one has to decompose

$$(\gamma_\mu C)_{\rho\tau}\left(\sum_{s_\Delta}\mathcal{U}^\mu(-\Delta, s_\Delta)\bar{\mathcal{U}}_\nu(-\Delta, s_\Delta)P^\nu U(p_1, s_1)\right)_\chi = (\gamma_\mu C)_{\rho\tau}\left(\frac{\partial}{\partial\Sigma^\mu}\mathcal{P}^{(3/2)}(\Sigma, P, -\Delta)U(p_1, s_1)\right)_\chi \tag{98}$$

over the basis of the Dirac structures of (65).

After some algebra one may work out the following contributions of (90) into the invariant form factors (65) to the leading twist-3:

$$\{V_{1,2}^{(\pi N)_{3/2}}, A_{1,2}^{(\pi N)_{3/2}}\}(x_1,x_2,x_3,\xi,\Delta^2)|_{\Delta(1232)} = -\Theta_{\rm ERBL}(x_1,x_2,x_3)\frac{1}{(2\xi)^2}\{V^\Delta, A^\Delta\}\left(\frac{x_1}{2\xi},\frac{x_2}{2\xi},\frac{x_3}{2\xi}\right)$$
$$\times\frac{g_{\pi N\Delta}\lambda_\Delta^{1/2}Mf_\pi}{\sqrt{2}(\Delta^2-M_\Delta^2)f_N}R_{1,2}(\xi, M_\Delta);$$
$$T_{1,2}^{(\pi N)_{3/2}}(x_1,x_2,x_3,\xi,\Delta^2)|_{\Delta(1232)} = -\Theta_{\rm ERBL}(x_1,x_2,x_3)\left\{\frac{1}{(2\xi)^2}T^\Delta\left(\frac{x_1}{2\xi},\frac{x_2}{2\xi},\frac{x_3}{2\xi}\right)\frac{g_{\pi N\Delta}\lambda_\Delta^{1/2}Mf_\pi}{\sqrt{2}(\Delta^2-M_\Delta^2)f_N}R_{1,2}(\xi,M_\Delta)\right.$$
$$\left.+\frac{1}{(2\xi)^2}\phi^\Delta\left(\frac{x_1}{2\xi},\frac{x_2}{2\xi},\frac{x_3}{2\xi}\right)\frac{g_{\pi N\Delta}f_\Delta^{3/2}M^2f_\pi}{\sqrt{2}(\Delta^2-M_\Delta^2)f_N}\tilde{R}_{1,2}(\xi,M_\Delta)\right\};$$
$$T_{3,4}^{(\pi N)_{3/2}}(x_1,x_2,x_3,\xi,\Delta^2)|_{\Delta(1232)} = -\Theta_{\rm ERBL}(x_1,x_2,x_3)\frac{1}{(2\xi)^2}\phi^\Delta\left(\frac{x_1}{2\xi},\frac{x_2}{2\xi},\frac{x_3}{2\xi}\right)\frac{g_{\pi N\Delta}f_\Delta^{3/2}M^2f_\pi}{\sqrt{2}(\Delta^2-M_\Delta^2)f_N}R_{3,4}(M_\Delta). \tag{99}$$

Here $\lambda_\Delta^{1/2}$ is a dimensional constant with the dimension $[\mathrm{GeV}]^3$ and $f_\Delta^{3/2}$ is a dimensional constant with the dimension $[\mathrm{GeV}]^2$. In Ref. [36] the following numerical values are quoted:

$$|\lambda_\Delta^{1/2}| \equiv \sqrt{\frac{3}{2}}M_\Delta|f_\Delta^{1/2}| = (1.8\pm0.3)\times10^{-2}\ \mathrm{GeV}^3; \qquad |f_\Delta^{3/2}| = 1.4\times10^{-2}\ \mathrm{GeV}^2. \tag{100}$$

The functions $R_{1,2}$, $\tilde{R}_{1,2}$ are determined by residue at the pole $\Delta^2 = M_\Delta^2$. They read as

$$R_1(\xi, M_\Delta) = \frac{(\xi-3)M_\Delta^2 + 2MM_\Delta\xi + 4(M^2-m^2)\xi}{3MM_\Delta};$$
$$R_2(\xi, M_\Delta) = \frac{-4\xi M^3 + (4\xi m^2 + 6M_\Delta^2)M - M_\Delta^3(\xi-3) + 4m^2M_\Delta\xi}{6MM_\Delta^2}; \qquad R_3(M_\Delta) = -\frac{M_\Delta}{M}; \qquad R_4(M_\Delta) = 1+\frac{M_\Delta}{2M};$$
$$\tilde{R}_1(\xi, M_\Delta) = \frac{(M(M+M_\Delta)-m^2)\xi}{M^2} - M_\Delta^2\frac{(1-\xi)}{2M^2}; \qquad \tilde{R}_2(\xi, M_\Delta) = \frac{MM_\Delta + (m^2+M^2)\xi}{2M^2} + M_\Delta^2\frac{(1-\xi)}{4M^2}. \tag{101}$$

The crucial point is that the Δ exchange contribution into πN TDAs should satisfy symmetry relations for the isospin-$\frac{3}{2}$ TDAs established in Sec. IV. Employing the set of the Fierz identities, one may check that the part of (99) involving contributions of V^Δ, A^Δ, T^Δ decouples and satisfies the symmetry relations (64) as a consequence of symmetry relations (52) for Δ DAs.

The situation with the contribution involving ϕ^Δ is more complicated. This turns out to be due to the fact that the Fierz identities (B10) for the tensor structures $t_3^{\pi N}$, $t_4^{\pi N}$ involve coefficient functions with explicit dependence on ξ and Δ^2. In order to satisfy symmetry relation (64) we have to add a polynomial background to πN TDAs $T_{1,2}$. Symmetry relation (64) fixes the background uniquely. This provides the following final result for $\Delta(1232)$ exchange contributions into $T_{1,2}^{(\pi N)_{3/2}}$:

$$T_1^{(\pi N)_{3/2}}(x_1, x_2, x_3, \xi, \Delta^2)|_{\Delta(1232)} = -\Theta_{\rm ERBL}(x_1, x_2, x_3)\bigg\{\frac{1}{(2\xi)^2}T^\Delta\left(\frac{x_1}{2\xi}, \frac{x_2}{2\xi}, \frac{x_3}{2\xi}\right)\frac{g_{\pi N\Delta}\lambda_\Delta^{1/2}Mf_\pi}{\sqrt{2}f_N(\Delta^2 - M_\Delta^2)}R_1(\xi, M_\Delta)$$
$$+\frac{1}{(2\xi)^2}\phi^\Delta\left(\frac{x_1}{2\xi}, \frac{x_2}{2\xi}, \frac{x_3}{2\xi}\right)\frac{g_{\pi N\Delta}f_\Delta^{3/2}M^2f_\pi}{\sqrt{2}f_N}\left(\frac{\tilde{R}_1(\xi, M_\Delta)}{\Delta^2 - M_\Delta^2} - \frac{1-\xi}{2M^2}\right)\bigg\}; \quad (102)$$

$$T_2^{(\pi N)_{3/2}}(x_1, x_2, x_3, \xi, \Delta^2)|_{\Delta(1232)} = -\Theta_{\rm ERBL}(x_1, x_2, x_3)\bigg\{\frac{1}{(2\xi)^2}T^\Delta\left(\frac{x_1}{2\xi}, \frac{x_2}{2\xi}, \frac{x_3}{2\xi}\right)\frac{g_{\pi N\Delta}\lambda_\Delta^{1/2}Mf_\pi}{\sqrt{2}f_N(\Delta^2 - M_\Delta^2)}R_2(\xi, M_\Delta)$$
$$+\frac{1}{(2\xi)^2}\phi^\Delta\left(\frac{x_1}{2\xi}, \frac{x_2}{2\xi}, \frac{x_3}{2\xi}\right)\frac{g_{\pi N\Delta}f_\Delta^{3/2}M^2f_\pi}{\sqrt{2}f_N}\left(\frac{\tilde{R}_2(\xi, M_\Delta)}{\Delta^2 - M_\Delta^2} + \frac{1-\xi}{4M^2}\right)\bigg\}. \quad (103)$$

VII. CONCLUSIONS

We considered general symmetry properties of πN transition distribution amplitudes. We showed that the Lorentz invariance results in the polynomiality property of the Mellin moments of TDAs in the longitudinal momentum fractions. Analogously to the GPD case, we revealed the presence of a D-term contribution for the πN TDAs $V_{1,2}$, $A_{1,2}$ and $T_{1,2}$ generating the highest power monomials of the Mellin moments.

The detailed account of the isospin and permutation symmetries allowed us to provide a unified description of all isotopic channels in terms of eight independent πN TDAs. The general constraints derived here should be satisfied by any realistic model of TDAs.

The crossing relation between πN TDAs and GDAs lead us to establish a soft pion theorem for the isospin-$\frac{1}{2}$ and isospin-$\frac{3}{2}$ πN TDAs. This yields normalization conditions for πN TDAs.

We also presented a simple resonance exchange model for πN TDAs considering nucleon and $\Delta(1232)$ exchanges in the isospin-$\frac{1}{2}$ and isospin-$\frac{3}{2}$ channels, respectively. Nucleon exchange may be considered as a pure D-term contribution complementary to the spectral representation for TDAs in terms of quadruple distributions.

This work opens the way to various consistent models of baryon to meson TDAs to be confronted with experimental data.

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APPENDIX A: ISOTOPIC INVARIANCE AND ISOSPIN CLASSIFICATION OF πN STATES

Generators of the $SU(2)$ isospin group satisfy the familiar commutation relation:

$$[I_a, I_b] = i\varepsilon_{abc}I_c. \quad (A1)$$

When constructing spinor representations of $SU(2)$ one has to distinguish between the covariant and thev contravariant representations. We choose to transform $\bar{N}_\alpha$ field according to the covariant representation and to transform N^α field according to the contravariant representation

$$[I_a, \bar{N}_\alpha] = \frac{1}{2}(\sigma_a)^\beta{}_\alpha\bar{N}_\beta; \quad [I_a, N^\alpha] = -\frac{1}{2}(\sigma_a)^\beta{}_\alpha N^\alpha, \quad (A2)$$

where σ_a are the Pauli matrices.

We adopt the following standard convention upon the nucleon field [42]:

$$N^\alpha(x) = \int\frac{d^3k}{(2\pi)^3}\frac{M}{k_0}\sum_{s=1,2}\{e^{ikx}d^{\dagger\alpha}(k, s)V(k, s) + e^{-ikx}b^\alpha(k, s)U(k, s)\};$$
$$\bar{N}_\alpha(x) = \int\frac{d^3k}{(2\pi)^3}\frac{M}{k_0}\sum_{s=1,2}\{e^{ikx}b^\dagger{}_\alpha(k, s)\bar{U}(k, s) + e^{-ikx}d_\alpha(k, s)\bar{V}(k, s)\}. \quad (A3)$$

Here spinors $U(k,s)$ and $\bar{U}(k,s) \equiv U^\dagger(k,s)\gamma_0$ describe a nucleon, respectively, in the initial and final states, while spinors $\bar{V}(k,s) \equiv V^\dagger(k,s)\gamma_0$ and $V(k,s)$ describe an antinucleon in the initial and final states.

The creation and annihilation operators in (A3) satisfy the usual anticommutation relations for fermions [42]:

$$\{b^\alpha(p,s), b^\dagger{}_\beta(p',s')\} = (2\pi)^3 \frac{p_0}{M}\delta^3(p-p')\delta_{ss'}\delta^\alpha_\beta;$$
$$\{d_\alpha(p,s), d^{\dagger\beta}(p',s')\} = (2\pi)^3 \frac{p_0}{M}\delta^3(p-p')\delta_{ss'}\delta^\beta_\alpha. \quad (A4)$$

The "in" nucleon state $|N_\alpha\rangle$ is defined according to:

$$|N_1\rangle \equiv |N_p(p,s)\rangle = b_1^\dagger(p,s)|0\rangle;$$
$$|N_2\rangle \equiv |N_n(p,s)\rangle = b_2^\dagger(p,s)|0\rangle. \quad (A5)$$

Analogously, the "in" antiparticle state $|\bar{N}^\alpha\rangle$ is defined as:

$$|\bar{N}^1\rangle \equiv |N^{\bar{p}}(p,s)\rangle = d^{\dagger 1}(p,s)|0\rangle;$$
$$|\bar{N}^2\rangle \equiv |N^{\bar{n}}(p,s)\rangle = d^{\dagger 2}(p,s)|0\rangle. \quad (A6)$$

In order to check the consistency of our conventions (A2) and (A3) we should explicitly construct the isospin and hypercharge operators and make sure that the nucleon and antinucleon states (A5) and (A6) have the proper quantum numbers.

With the help of Noether's theorem from the free nucleon Lagrangian

$$\mathcal{L} = \frac{i}{2}[\bar{N}_\alpha\gamma^\mu(\partial_\mu N^\alpha) - (\partial_\mu \bar{N}_\alpha)\gamma^\mu N^\alpha] - m\bar{N}_\alpha N^\alpha \quad (A7)$$

employing (A2) we construct the explicit expression for the nucleon isospin operator:

$$I_a^{(N)} = \int d^3x {:} N_\alpha^\dagger(x)\frac{(\sigma_a)^\alpha_\beta}{2}N^\beta(x){:}$$
$$= \int \frac{d^3k}{(2\pi)^3}\frac{M}{k_0}\sum_s\Big[b_\alpha^\dagger(k,s)\frac{(\sigma_a)^\alpha_\beta}{2}b^\beta(k,s) - d^{\dagger\beta}(k,s) \times \frac{(\sigma_a)^\alpha_\beta}{2}d_\alpha(k,s)\Big]. \quad (A8)$$

Thus, the isospin operator acts on the incoming nucleon state according to

$$I_a^{(N)}|N_\alpha\rangle = \frac{1}{2}(\sigma_a)^\beta{}_\alpha|N_\beta\rangle; \quad I_a^{(N)}|\bar{N}^\alpha\rangle = -\frac{1}{2}(\sigma_a)^\alpha{}_\beta|\bar{N}^\beta\rangle. \quad (A9)$$

We also introduce the hypercharge operator Y according to

$$[Y, \bar{N}_\alpha] = \bar{N}_\alpha; \qquad [Y, N^\alpha] = -N^\alpha. \quad (A10)$$

The explicit expression for the nucleon hypercharge operator reads

$$Y^{(N)} = \int \frac{d^3k}{2\pi^3}\frac{M}{k_0}\sum_s[b_\alpha^\dagger(k,s)b^\alpha(k,s) - d^{\dagger\alpha}(k,s)d_\alpha(k,s)]. \quad (A11)$$

It acts on the nucleon states according to

$$Y_a^{(N)}|N_\alpha\rangle = |N_\alpha\rangle; \qquad Y_a^{(N)}|\bar{N}^\alpha\rangle = -|\bar{N}^\alpha\rangle. \quad (A12)$$

Now we construct the nucleon charge operator employing the Gell-Mann-Nishijima formula

$$Q^{(N)} = I_3^{(N)} + \frac{Y^{(N)}}{2} \quad (A13)$$

and perform the classification of states (A5) and (A6) to check the consistency of our conventions.

The case of pion field is simpler since for the adjoint representation of $SU(2)$ there is no difference between covariant and contravariant representation. Indeed, using

$$[I_a, \pi_b] = i\varepsilon_{abc}\pi_c \equiv (t_a)_{cb}\pi_c \quad (A14)$$

one may check that $(t_a)_{bc} = -(t_a)_{cb}$.

We describe pions with the help of real pseudoscalar field π_a:

$$\pi_a(x) = \int \frac{d^3k}{(2\pi)^3 2k_0}(e^{ikx}a_a^+(k) + e^{-ikx}a_a^-(k)) \quad (A15)$$

and adopt the usual conventions of [42] for the commutation relations of the corresponding creation/annihilation operators $a_a^\pm$. Pion states are defined as $|\pi_a\rangle = a_a^+|0\rangle$.

The expression for the pion isospin operator reads:

$$I_a^{(\pi)} = i\int d^3x {:}\pi_b(x)(t_a)_{bc}\partial_0\pi_c(x){:}$$
$$= \int \frac{d^3k}{(2\pi)^3 2k_0}(t_a)_{bc}a_b^+(k)a_c^-(k). \quad (A16)$$

Pion isospin operator acts on the pion state according to

$$I_a^{(\pi)}|\pi_b\rangle = (t_a)_{cb}|\pi_c\rangle. \quad (A17)$$

We may construct the usual charged combinations

$$|\pi^\pm\rangle = \left|\frac{\pi_1 \pm i\pi_2}{\sqrt{2}}\right\rangle; \qquad |\pi^0\rangle = |\pi_3\rangle. \quad (A18)$$

Now we perform the isospin classification of pion-nucleon states. Let us consider the action of the isospin operator I_a on the pion-nucleon state

$$I_a|\pi_b N_\iota\rangle = \{I_a^{(N)}\cdot \mathbf{1}^{(\pi)} + I_a^{(\pi)}\cdot\mathbf{1}^{(N)}\}|\pi_b N_\iota\rangle$$
$$= \Big\{i\varepsilon_{abc}\delta^\kappa{}_\iota + \frac{1}{2}(\sigma_a)^\kappa{}_\iota\delta_{bc}\Big\}|\pi_c N_\kappa\rangle. \quad (A19)$$

The action of the operator of the total isospin I^2 on the pion-nucleon state then reads:

$$I^2|\pi_a N_\iota\rangle = \Big\{\frac{11}{4}\delta_{ab}\delta^\kappa{}_\iota - i\varepsilon_{bac}(\sigma_c)^\kappa{}_\iota\Big\}|\pi_b N_\kappa\rangle. \quad (A20)$$

This allows to classify the pion-nucleon states with respects to total isospin I^2 and its third projection I_3 and compute the Clebsch-Gordan coefficients:

$$|I, I_3\rangle = \sum_{a,\iota} \mathcal{C}_a{}^{\iota}(I, I_3)|\pi_a N_\iota\rangle. \tag{A21}$$

Let us emphasize that the calculation of the Clebsch-Gordan coefficients is subject of adopting a particular phase convention. There is much controversy on this point in the literature (see discussion in [35]). To be consistent we prefer to fix our own phase convention which turns out to be different from the so-called Condon-Shortley and Wigner phase convention adopted, e.g. in the tables in [43].

The calculation within our phase convention gives the following result for the isospin-$\frac{3}{2}$ πN states:

$$\begin{aligned} \left|\frac{3}{2};\frac{3}{2}\right\rangle &= \left|\frac{\pi_1 N_1 + i\pi_2 N_1}{\sqrt{2}}\right\rangle; \\ \left|\frac{3}{2};\frac{1}{2}\right\rangle &= -\sqrt{\frac{2}{3}}|\pi_3 N_1\rangle + \sqrt{\frac{1}{3}}\left|\frac{\pi_1 N_2 + i\pi_2 N_2}{\sqrt{2}}\right\rangle; \\ \left|\frac{3}{2};-\frac{1}{2}\right\rangle &= \sqrt{\frac{2}{3}}|\pi_3 N_2\rangle + \sqrt{\frac{1}{3}}\left|\frac{\pi_1 N_1 - i\pi_2 N_1}{\sqrt{2}}\right\rangle; \\ \left|\frac{3}{2};-\frac{3}{2}\right\rangle &= \left|\frac{\pi_1 N_2 - i\pi_2 N_2}{\sqrt{2}}\right\rangle. \end{aligned} \tag{A22}$$

The expansion of the isospin-$\frac{1}{2}$ πN states reads:

$$\begin{aligned} \left|\frac{1}{2};\frac{1}{2}\right\rangle &= \sqrt{\frac{1}{3}}|\pi_3 N_1\rangle + \sqrt{\frac{2}{3}}\left|\frac{\pi_1 N_2 + i\pi_2 N_2}{\sqrt{2}}\right\rangle; \\ \left|\frac{1}{2};-\frac{1}{2}\right\rangle &= -\sqrt{\frac{1}{3}}|\pi_3 N_2\rangle + \sqrt{\frac{2}{3}}\left|\frac{\pi_1 N_1 - i\pi_2 N_1}{\sqrt{2}}\right\rangle. \end{aligned} \tag{A23}$$

The inverse expansion reads

$$|\pi_a N_\iota\rangle = \sum_{I,I_3} \mathcal{D}_{a\iota}(I, I_3)|I, I_3\rangle, \quad \text{where } \mathcal{D}_{a\iota}(I, I_3) = (\mathcal{C}_a{}^{\iota}(I, I_3))^\dagger. \tag{A24}$$

Note that the last equality in (A24) should be understood as the equality of the corresponding numerical values (and not as that of $SU(2)$ spin-tensors).

We also compute the isospin projecting operators:

$$P^I{}_b{}^\kappa{}_{a\iota} = \sum_{I_3,I_3'} \mathcal{C}_b{}^\kappa(I, I_3)\mathcal{D}_{a\iota}(I, I_3')|I, I_3'\rangle\langle I, I_3|. \tag{A25}$$

The explicit expressions for the isospin projecting operators read [40]:

$$\begin{aligned} P^{3/2}{}_b{}^\kappa{}_{a\iota} &= \frac{2}{3}\left(\delta_{ba}\delta^\kappa{}_\iota - \frac{i}{2}\varepsilon_{bac}(\sigma_c)^\kappa{}_\iota\right); \\ P^{1/2}{}_b{}^\kappa{}_{a\iota} &= \frac{1}{3}\left(\delta_{ba}\delta^\kappa{}_\iota + i\varepsilon_{bac}(\sigma_c)^\kappa{}_\iota\right). \end{aligned} \tag{A26}$$

APPENDIX B: FIERZ IDENTITIES

Employing the Fierz identity for γ matrices (see e.g. [44]) one may establish the following useful identity for arbitrary Dirac structures Γ, Γ':

$$\begin{aligned} (\Gamma C)_{\rho\tau}(\Gamma' U)_\chi &= \frac{1}{4}\Big\{C_{\chi\tau}(\Gamma\Gamma' U)_\rho + (\gamma^\mu C)_{\chi\tau}(\Gamma\gamma^\mu\Gamma' U)_\rho \\ &\quad + (\gamma^5 C)_{\chi\tau}(\Gamma\gamma^5\Gamma' U)_\rho - (\gamma^5\gamma^\mu C)_{\chi\tau}(\Gamma\gamma^5\gamma^\mu\Gamma' U)_\rho \\ &\quad - \frac{1}{2}(\sigma^{\mu\nu} C)_{\chi\tau}(\Gamma\sigma_{\mu\nu}\Gamma' U)_\rho\Big\}. \end{aligned} \tag{B1}$$

Here U stands for an arbitrary spin-tensor with one Dirac index and C is the charge conjugation matrix.

1. Nucleon DA

To the leasing twist-3 the parametrization of the nucleon DA involves the following Dirac structures

$$v^N_{\rho\tau,\chi} = (\hat{p}C)_{\rho\tau}(\gamma^5 U)_\chi; \qquad a^N_{\rho\tau,\chi} = (\hat{p}\gamma^5 C)_{\rho\tau}(U)_\chi; \qquad t^N_{\rho\tau,\chi} = (\sigma_{p\mu} C)_{\rho\tau}(\gamma^\mu\gamma^5 U)_\chi. \tag{B2}$$

The Dirac structures (B2) satisfy symmetry relations:

$$v^N_{\rho\tau,\chi} = v^N_{\tau\rho,\chi}; \quad a^N_{\rho\tau,\chi} = -a^N_{\tau\rho,\chi}; \quad t^N_{\rho\tau,\chi} = t^N_{\tau\rho,\chi}. \tag{B3}$$

With the help of (B1) one may establish the following set of the Fierz identities valid to the leading twist-3 accuracy:

$$\begin{aligned} v^N_{\rho\tau,\chi} &= \frac{1}{2}(v^N - a^N - t^N)_{\chi\tau,\rho}; \\ a^N_{\rho\tau,\chi} &= \frac{1}{2}(-v^N + a^N - t^N)_{\chi\tau,\rho}; \\ t^N_{\rho\tau,\chi} &= (-v^N - a^N)_{\chi\tau,\rho}. \end{aligned} \tag{B4}$$

2. Δ(1232) DA

Leading twist Dirac structures employed in the parametrization (48) of Δ(1232) resonance DA read:

$$\begin{aligned} v^\Delta_{\rho\tau,\chi} &= (\gamma_\mu C)_{\rho\tau}\mathcal{U}^\mu_\chi; \\ a^\Delta_{\rho\tau,\chi} &= (\gamma_\mu\gamma_5 C)_{\rho\tau}(\gamma_5\mathcal{U}^\mu)_\chi; \\ t^\Delta_{\rho\tau,\chi} &= \frac{1}{2}(\sigma_{\mu\nu} C)_{\rho\tau}(\gamma^\mu\mathcal{U}^\nu)_\chi; \\ \varphi^\Delta_{\rho\tau,\chi} &= (\sigma_{\mu\nu} C)_{\rho\tau}\Big(p^\mu\mathcal{U}^\nu - \frac{1}{2}M_\Delta\gamma^\mu\mathcal{U}^\nu\Big)_\chi. \end{aligned} \tag{B5}$$

The Dirac structures (B5) satisfy symmetry relations:

$$\begin{aligned} v^\Delta_{\rho\tau,\chi} &= v^\Delta_{\tau\rho,\chi}; \qquad a^\Delta_{\rho\tau,\chi} = -a^\Delta_{\tau\rho,\chi}; \\ t^\Delta_{\rho\tau,\chi} &= t^\Delta_{\tau\rho,\chi}; \qquad \varphi^\Delta_{\rho\tau,\chi} = \varphi^\Delta_{\tau\rho,\chi}; \end{aligned} \tag{B6}$$

The set of the corresponding Fierz identities valid to the leading twist-3 accuracy reads:

$$\begin{aligned} v^{\Delta}_{\rho\tau,\chi} &= \Big(\frac{1}{2}v^{\Delta} - \frac{1}{2}a^{\Delta} + t^{\Delta}\Big)_{\chi\tau,\rho}; \\ a^{\Delta}_{\rho\tau,\chi} &= \Big(-\frac{1}{2}v^{\Delta} + \frac{1}{2}a^{\Delta} + t^{\Delta}\Big)_{\chi\tau,\rho}; \\ t^{\Delta}_{\rho\tau,\chi} &= \Big(\frac{1}{2}v^{\Delta} + \frac{1}{2}a^{\Delta}\Big)_{\chi\tau,\rho}; \qquad \varphi^{\Delta}_{\rho\tau,\chi} = \varphi^{\Delta}_{\chi\tau,\rho}. \end{aligned} \tag{B7}$$

3. πN TDA

Below we consider the properties of the Dirac structures (12) occurring in the parametrization of the πN TDA. First, one may check that the Dirac structures (12) satisfy symmetry relations:

$$(v^{\pi N}_{1,2})_{\rho\tau,\chi} = (v^{\pi N}_{1,2})_{\tau\rho,\chi}; \qquad (a^{\pi N}_{1,2})_{\rho\tau,\chi} = -(a^{\pi N}_{1,2})_{\tau\rho,\chi}; \quad (t^{\pi N}_{1,2,3,4})_{\rho\tau,\chi} = (t^{\pi N}_{1,2,3,4})_{\tau\rho,\chi}. \tag{B8}$$

The set of the corresponding Fierz identities for the structures $s^{\pi N}_{1,2}$ is similar to that for the case of the nucleon DA (B4):

$$\begin{aligned} (v^{\pi N}_{1,2})_{\rho\tau,\chi} &= \frac{1}{2}(v^{\pi N}_{1,2})_{\chi\tau,\rho} - \frac{1}{2}(a^{\pi N}_{1,2})_{\chi\tau,\rho} - \frac{1}{2}(t^{\pi N}_{1,2})_{\chi\tau,\rho}; \\ (a^{\pi N}_{1,2})_{\rho\tau,\chi} &= -\frac{1}{2}(v^{\pi N}_{1,2})_{\chi\tau,\rho} + \frac{1}{2}(a^{\pi N}_{1,2})_{\chi\tau,\rho} - \frac{1}{2}(t^{\pi N}_{1,2})_{\chi\tau,\rho}; \\ (t^{\pi N}_{1,2})_{\rho\tau,\chi} &= -(v^{\pi N}_{1,2})_{\chi\tau,\rho} - (a^{\pi N}_{1,2})_{\chi\tau,\rho}. \end{aligned} \tag{B9}$$

The result for $(t^{\pi N}_{3,4})$ is a bit more involved:

$$\begin{aligned} (t^{\pi N}_{3})_{\rho\tau,\chi} &= (t^{\pi N}_{3})_{\chi\tau,\rho} + g_1(\xi,\Delta^2)(v^{\pi N}_1 + a^{\pi N}_1 + t^{\pi N}_1)_{\chi\tau,\rho} \\ &\quad + g_2(\xi,\Delta^2)(v^{\pi N}_2 + a^{\pi N}_2 + t^{\pi N}_2)_{\chi\tau,\rho}; \\ (t^{\pi N}_{4})_{\rho\tau,\chi} &= (t^{\pi N}_{4})_{\chi\tau,\rho} + h_1(\xi,\Delta^2)(v^{\pi N}_1 + a^{\pi N}_1 + t^{\pi N}_1)_{\chi\tau,\rho} \\ &\quad + h_2(\xi,\Delta^2)(v^{\pi N}_2 + a^{\pi N}_2 + t^{\pi N}_2)_{\chi\tau,\rho}, \end{aligned} \tag{B10}$$

where

$$\begin{aligned} g_1(\xi,\Delta^2) &= \frac{-\Delta^2(1-\xi) - 2(m^2+M^2)\xi}{4M^2}; \\ g_2(\xi,\Delta^2) &= \frac{2\xi m^2 + 2M^2(\xi-2) + \Delta^2(1-\xi)}{8M^2}; \\ h_1(\xi,\Delta^2) &= \frac{-\Delta^2(1-\xi) - 2(m^2-M^2)\xi}{2M^2}; \\ h_2(\xi,\Delta^2) &= \frac{2(m^2+M^2)\xi + \Delta^2(1-\xi)}{4M^2}. \end{aligned} \tag{B11}$$

APPENDIX C: CHOICE OF INDEPENDENT DIRAC STRUCTURES

Keeping the first-order corrections in the masses and Δ_T^2 one can establish the following Sudakov decomposition for the momenta of reaction (2) [14]:

$$\begin{aligned} p_1 &= (1+\xi)p + \frac{M^2}{1+\xi}n; \\ q &\simeq -2\xi\Big(1 + \frac{(\Delta_T^2 - M^2)}{Q^2}\Big)p + \frac{Q^2}{2\xi(1+\frac{(\Delta_T^2-M^2)}{Q^2})}n; \\ p_\pi &= (1-\xi)p + \frac{m^2-\Delta_T^2}{1-\xi}n + \Delta_T; \\ \Delta &= -2\xi p + \Big[\frac{m-\Delta_T^2}{1-\xi} - \frac{M^2}{1+\xi}\Big]n + \Delta_T. \end{aligned} \tag{C1}$$

Because of the Dirac equation

$$\hat{p}_1 U(p_1,s_1) = MU(p_1,s_1) \tag{C2}$$

one has two following relations for the large ($U^+ = \hat{p}\,\hat{n}\,U$) and small ($U^- = \hat{n}\,\hat{p}\,U$) components of the nucleon Dirac spinor:

$$\begin{aligned} &\hat{p}U(p_1,s_1) = \frac{M}{1+\xi}U^+(p_1,s_1) \\ &\Big(p = \frac{1}{1+\xi}p_1 - \frac{M^2}{(1+\xi)^2}n\Big); \\ &\hat{n}U(p_1,s_1) = \frac{1+\xi}{M}U^-(p_1,s_1) \\ &\Big(n = \frac{1+\xi}{M^2}p_1 - \frac{(1+\xi)^2}{M^2}p\Big). \end{aligned} \tag{C3}$$

As the consequence of the Dirac equation we establish the following identities:

$$\begin{aligned} (\hat{P}U(p_1,s_1))_\gamma &= \frac{M}{1+\xi}U^+(p_1,s_1)_\gamma + \frac{1}{2}(\hat{\Delta}_T U(p_1,s_1))_\gamma \\ &\quad + \frac{(1+\xi)}{2M}\Big[\frac{M^2}{1+\xi} + \frac{m^2-\Delta_T^2}{1-\xi}\Big]U^-(p_1,s_1)_\gamma; \end{aligned} \tag{C4}$$

$$\begin{aligned} (\hat{\Delta}U(p_1,s_1))_\gamma &= -2\xi\frac{M}{1+\xi}U^+(p_1,s_1) + (\hat{\Delta}_T U(p_1,s_1))_\gamma \\ &\quad - \frac{(1+\xi)}{M}\Big[\frac{M^2}{1+\xi} - \frac{m^2-\Delta_T^2}{1-\xi}\Big]U^-(p_1,s_1)_\gamma. \end{aligned} \tag{C5}$$

The last term in (C4) and (C5) is of subleading twist while the two first terms are of the leading twist. Thus, among the four structures containing the leading twist contribution which can be written as

$$\begin{aligned} &(\hat{P}U(p_1,s_1))_\gamma; \qquad (\hat{\Delta}U(p_1,s_1))_\gamma; \\ &U(p_1,s_1)_\gamma; \quad \text{and} \quad (\hat{\Delta}_T U(p_1,s_1))_\gamma \end{aligned} \tag{C6}$$

only two are independent. In order to keep the traditional formulation of the polynomiality condition for πN TDAs and avoid the appearing of singular $\frac{1}{1+\xi}$ we choose $(\hat{P}U(p_1,s_1))_\gamma$ and $(\hat{\Delta}U(p_1,s_1))_\gamma$ to be the independent Dirac structures.

We establish the following useful relations:

$$2\xi(\hat{P}U)_\gamma + (\hat{\Delta}U)_\gamma = (1+\xi)(\hat{\Delta}_T U)_\gamma + \left(\frac{4\xi M^2 + (m^2 - M^2 - \Delta^2)(1+\xi)}{M}\right)U^-_\gamma. \tag{C7}$$

From (C4) we also establish the relation:

$$MU^+(p_1,s_1)_\gamma = (1+\xi)(\hat{P}U(p_1,s_1))_\gamma - \frac{1+\xi}{2}(\hat{\Delta}_T U(p_1,s_1))_\gamma - \frac{(1+\xi)^2}{2M}\left[\frac{M^2}{1+\xi} + \frac{m^2-\Delta_T^2}{1-\xi}\right]U^-(p_1,s_1)_\gamma. \tag{C8}$$

This results in

$$MU(p_1,s_1)_\gamma = (\hat{P}U(p_1,s_1))_\gamma - \frac{1}{2}(\hat{\Delta}U(p_1,s_1))_\gamma + \left\{\text{Twist-4 terms}\right\}. \tag{C9}$$

One may also check that

$$\begin{aligned}(\hat{P}\,\hat{\Delta}\,U(p_1,s_1))_\gamma &= \frac{2(P^2-M^2)}{M}(\hat{P}U(p_1,s_1))_\gamma - \frac{P^2}{M}(\hat{\Delta}U(p_1,s_1))_\gamma + \left\{\text{Twist-4 terms}\right\};\\ (\hat{\Delta}\,\hat{P}\,U(p_1,s_1))_\gamma &= 2(P\cdot\Delta)(U(p_1,s_1))_\gamma - (\hat{P}\,\hat{\Delta}\,U(p_1,s_1))_\gamma\\ &= \frac{\Delta^2}{2M}(\hat{P}U(p_1,s_1))_\gamma + \left(M - \frac{\Delta^2}{4M}\right)(\hat{\Delta}U(p_1,s_1))_\gamma + \left\{\text{Twist-4 terms}\right\}.\end{aligned} \tag{C10}$$

The relation of new definition (10) of πN TDAs to that of [14,21] is given by

$$\begin{aligned}\{V_1,A_1,T_1\}^{\pi N}|_{[14,12]} &= \left(\frac{1}{1+\xi}\{V_1,A_1,T_1\}^{\pi N} - \frac{2\xi}{1+\xi}\{V_2,A_2,T_2\}^{\pi N}\right)\Big|_{\text{This work}};\\ \{V_2,A_2\}^{\pi N}|_{[14,21]} &= \left(\{V_2,A_2\}^{\pi N} + \frac{1}{2}\{V_1,A_1\}^{\pi N}\right)\Big|_{\text{This work}};\\ T_3^{\pi N}|_{[14,21]} &= T_2^{\pi N}|_{\text{This work}} + \frac{1}{2}T_1^{\pi N}|_{\text{This work}}\\ T_2^{\pi N}|_{[14,21]} &= \left(\frac{1}{2}T_1^{\pi N} + T_2^{\pi N} + T_3^{\pi N} - 2\xi T_1^{\pi N}\right)\Big|_{\text{This work}};\\ T_4^{\pi N}|_{[14,21]} &= \left(\frac{1+\xi}{2}T_1^{\pi N} + (1+\xi)T_4^{\pi N}\right)\Big|_{\text{This work}}.\end{aligned} \tag{C11}$$

These relations can be easily established employing Eqs. (C4) and (C5) and the identity $\gamma^\mu\hat{\Delta}_T = \Delta_T^\mu + \sigma^{\mu\Delta_T}$. Note the appearance of $\frac{1}{1+\xi}$ factors that are of pure kinematical origin and, in particular, lead to violation of polynomiality property of TDAs.

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