Swaying oscillons in the signum-Gordon model

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We present a new class of oscillons in the (1 + 1)-dimensional signum-Gordon model. The oscillons periodically move to and fro in space. They have finite total energy, finite size, and are strictly periodic in time. The corresponding solutions of the scalar field equation are explicitly constructed from the second order polynomials in the time and position coordinates.

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I. INTRODUCTION AND PRELIMINARIES

Scalar fields play an essential role in many branches of physics, from cosmology to condensed matter physics to particle physics—there is an unremitting interest in models of self-interacting scalar fields. The rich variety of such models includes some that have been studied only recently, e.g., the so-called K fields with a nonstandard kinetic part [1], or models with a nonsmooth V-shaped self-interaction [2]. The signum-Gordon model considered in the present paper is probably the simplest example from the latter class. The pertinent field potential has the form $U(\varphi) = g|\varphi|$, where $g > 0$ is a coupling constant and $|\varphi|$ is the modulus of the real scalar field φ. It is V-shaped with the sharp minimum at the vacuum field $\varphi = 0$. Subsequent investigations [3] have revealed that the V-shaped form of the potential has very interesting consequences for the dynamics of the scalar field. One of them is the existence of strictly periodic oscillons [4].

The present brief report is a follow-up to the paper [4]. The oscillons described in that paper did not move in space (apart from the trivial uniform motion obtained by applying Lorentz boosts). Rather unexpectedly, we have found that there exist also oscillons that periodically move to and fro in space with arbitrary constant velocity $\pm v$, where $0 < |v| \leq 1$. For the oscillons presented in [4], $v = 0$.

In comparison with other oscillons discussed in literature [5], several differences should be pointed out. First, our oscillons are strictly periodic in time; in particular, they do not emit any radiation. Second, they have strictly finite size because the field assumes the vacuum value at a finite distance exactly. Third, they have a relatively simple exact analytic form composed of several linear and quadratic functions of the time t and the spatial coordinate x.

The swaying oscillon is reminiscent of the wobbling kink in the φ^4 model [6]. However, one should note that the wobbling kink is an excitation of the static kink, while all the swaying oscillons are degenerate in energy, and moreover there is no static oscillon—even for the nonswaying one presented in [4] the field oscillates in time.

The Lagrangian of the signum-Gordon model has the form

$$L = \tfrac{1}{2}(\partial_t\varphi\partial_t\varphi - \partial_x\varphi\partial_x\varphi) - g|\varphi|, \quad (1)$$

where φ is a real scalar field, and t, x are time and position coordinates in the two-dimensional Minkowski space-time M. For convenience, t, x, φ, g are dimensionless—this can always be achieved by suitable redefinitions. The signum-Gordon equation

$$\partial_t^2\varphi - \partial_x^2\varphi + \mathrm{sign}(\varphi(x,t)) = 0 \quad (2)$$

is the Euler-Lagrange equation corresponding to the Lagrangian (1) (from now on we put $g = 1$). The sign function has the values ± 1 for $\varphi \neq 0$ and sign(0) = 0. The simplest way to obtain Eq. (2) from the Lagrangian (1) is first to regularize the field potential $U(\varphi) = |\varphi|$, e.g., $U(\varphi) = \sqrt{\epsilon^2 + \varphi^2}$ or $U(\varphi) = \epsilon \ln(\cosh(\varphi/\epsilon))$, and to take the limit $\epsilon \to 0_+$ in the Euler-Lagrange equation obtained from the regularized Lagrangian. Direct computation of the variation of the action $S = \int dtdxL$ is more subtle because of the $|\varphi|$ term, but it too gives the signum-Gordon equation (2).

The left-hand side of Eq. (2) is not continuous with respect to φ. Because such equations are not very common in field theory, let us briefly comment on the related mathematical aspects. First, it is clear that in general one should expect nonsmooth solutions: the value of at least one of the second derivatives $\partial_t^2\varphi$, $\partial_x^2\varphi$ has to jump when the function sign(φ) changes its value. Second, the use of the stationary action principle implies that in general we consider the so-called weak solutions of the Euler-Lagrange equation, [7]. For the weak ones it is sufficient that

$$\delta S = \int_M dtdx\left(\frac{\partial L}{\partial \phi}\delta\phi(x,t) + \frac{\partial L}{\partial(\partial_\mu\phi)}\partial_\mu\delta\phi(x,t)\right) = 0$$

for all test functions $\delta\phi(x,t)$ from a certain class [typically one uses the $D(M)$ class of smooth functions on M with compact support]. This condition is equivalent to $\int_M dtdx\mathcal{E}\mathcal{L}\delta\varphi = 0$, where $\mathcal{E}\mathcal{L} = \partial L/\partial\phi - \partial_\mu(\partial L/\partial(\partial_\mu\varphi))$, only if the derivative $\partial_\mu(\partial L/\partial(\partial_\mu\varphi))$ exists for the given probed function $\varphi(x,t)$. Then the Euler-Lagrange equation $\mathcal{E}\mathcal{L} = 0$ has to be satisfied at almost all points

(x, t) in the two-dimensional space-time M, but not necessarily at all points as would be the case with strong solutions. Of course, the set of weak solutions contains the strong ones as a subset.

In the case of the signum-Gordon equation the weak solutions that are not strong ones are ubiquitous. For instance, $\varphi_0 = x^2/2$ is a smooth static solution of Eq. (2) in the weak sense, but not in the strong sense. The point is that $\partial_t^2\varphi_0 - \partial_x^2\varphi_0 + \text{sign}(\varphi_0) = 0$ everywhere in M except the line $x=0$. On this line $\partial_t^2\varphi_0 - \partial_x^2\varphi_0 + \text{sign}(\varphi_0) = -1$ because $\partial_t^2\varphi_0 = 0$, $\partial_x^2\varphi_0 = 1$, $\text{sign}(0) = 0$. Nevertheless,

$$\int_M dtdx[\partial_t^2\varphi_0 - \partial_x^2\varphi_0 + \text{sign}(\varphi_0)]\delta\phi(x, t) = 0$$

for the arbitrary test function $\delta\phi$.

In general, the weak solutions are physically relevant. To see this, consider the following simple example from classical mechanics of a point particle on a plane with Cartesian coordinates (x, y). The particle is free except when on the y axis, where it is subjected to a finite constant force $\vec{F}_0$ parallel to the y axis. Thus, the force $\vec{F} = 0$ at all points (x, y) with $x \neq 0$, and $\vec{F} = \vec{F}_0$ when $x = 0$. It is clear that integrating the Newton's equation $d\vec{p}/dt = \vec{F}$ we obtain $\vec{p} = \text{const}$ even if the trajectory crosses the y axis. The physical reason is that the finite force $\vec{F}_0$ acts on the particle only during the infinitesimally short time when the particle is exactly on the y axis; hence, it is not able to perturb the free motion. Such trajectories are the weak solutions of Newton's equation [now the test functions are denoted as $\delta\vec{r}(t)$ and we integrate over t]. On the other hand, the trajectories which do not intersect the y axis are solutions in the strong sense. Notice that our Newton's equation is not equivalent to the free equation, in which $\vec{F} = 0$ everywhere, because our particle is accelerated if it moves along the y axis.

Because the function $\text{sign}(\varphi)$ is piecewise constant, it is natural first to solve Eq. (2) in the regions in which $\text{sign}(\varphi)$ is constant. For instance, if $\varphi < 0$, Eq. (2) acquires the form

$$\partial_t^2\varphi - \partial_x^2\varphi - 1 = 0. \tag{3}$$

The oscillon solutions are constructed from second order polynomials in x, t. The most general second order polynomial that obeys Eq. (3) has the form

$$\varphi_2(x, t) = a_0x^2 + a_1tx + (a_0 + \tfrac{1}{2})t^2 + b_0x + b_1t + c_0, \tag{4}$$

where a_0, a_1, b_0, b_1, c_0 are constants (beware that they are not completely arbitrary because of the condition $\varphi_2 < 0$). The class of functions of the form (4) is invariant with respect to Lorentz boosts, space-time translations, and the reflections $x \rightarrow -x$, $t \rightarrow -t$. It contains the static solutions of the form

$$\varphi_s = -\tfrac{1}{2}(x - b_0)^2 + c_0 + \tfrac{1}{2}b_0^2, \tag{5}$$

where $c_0 + b_0^2/2 < 0$ in order to keep $\varphi_s < 0$.

The oscillons are constructed by patching together several such polynomial solutions. The patching conditions have the following standard form: the field φ is continuous all over M. Also the derivatives $\partial_t\varphi$, $\partial_x\varphi$ are continuous functions of x, t, with the possible exception when the border line between two patches is a (segment of) the characteristic line ($x = \pm t + \text{const}$) for the signum-Gordon equation; in this case the derivative in the direction perpendicular to that line does not have to be continuous—i.e., a finite jump is allowed.

II. THE SWAYING OSCILLONS

A hint that new oscillons may exist comes from the following procedure for constructing periodic solutions of the signum-Gordon equation. Let $\varphi_-(x, t)$ be a solution of Eq. (3) negative for all t from an open interval $(0, T)$, >0, and such that

$$\varphi_-(x, 0) = 0 = \varphi_-(x, T). \tag{6}$$

Then the function φ_+ defined by

$$\varphi_+(x, t) = -\varphi_-(x, -t) \tag{7}$$

is a positive solution of the equation $\partial_t^2\varphi - \partial_x^2\varphi + 1 = 0$ for all $t \in (-T, 0)$. The functions φ_-, φ_+ as well as their time derivatives match each other at the time $t = 0$:

$$\varphi_+(x, 0) = 0 = \varphi_-(x, 0),$$
$$\lim_{t\rightarrow 0_-}\partial_t\varphi_+(x, t) = \lim_{s\rightarrow 0_+}\partial_s\varphi_-(x, s),$$

where s stands for $-t$, and $t \in (-T, 0)$. The crucial observation is that also $\varphi_+(x, -T)$, $\varphi_-(x, T)$ match each other:

$$\varphi_+(x, -T) = 0 = \varphi_-(x, T),$$
$$\lim_{t\rightarrow -T_+}\partial_t\varphi_+(x, t) = \lim_{s\rightarrow T_-}\partial_s\varphi_-(x, s).$$

Therefore, we may extend our partial solutions $\varphi_\pm$ to all times $t \geq T$ and $t \leq -T$ just by applying time translations (by multiples of $\pm T$) to $\varphi_\pm$. In this way we obtain periodic solutions of the signum-Gordon equation (2) with the period equal to $2T$. The solution $\varphi_-(x, t)$ with the property (6) can be constructed by patching together several solutions of the form (4). Also the trivial solution $\varphi = 0$ is involved. The schematic picture of such a "patchwork" for the swaying oscillon is presented in Fig. 1.

In order to ensure finiteness of the total energy we assume that $\varphi_-(x, t) = 0$ outside a certain compact interval. Thus, our first task is to find the polynomials of the form (4) which match the trivial solution $\varphi = 0$. The matching conditions imposed on a line $x(t)$ in M can be written in the form

$$\varphi_2(x(t), t) = 0, \qquad \partial_x\varphi_2(x, t)|_{x=x(t)} = 0,$$

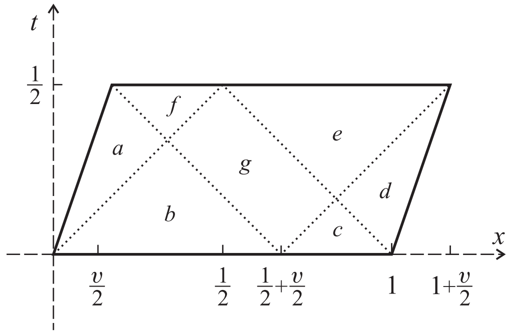

FIG. 1. The support of the solution $\varphi_-(x,t)$. The field $\varphi_-(x,t)$ vanishes on the continuous lines that form the boundary of the parallelogram. In each sector $a \div f$ the function φ_- is given by a different formula. The matching conditions that relate the functions in neighboring sectors are imposed along the dotted lines. These four lines with the slopes ± 1 are characteristic lines of the signum-Gordon equation.

provided that $x(t)$ does not coincide with a characteristic line.

They give the following two equations:

$$a_0x^2(t) + a_1tx(t) + (a_0 + \tfrac{1}{2})t^2 + b_0x(t) + b_1t + c_0 = 0,$$
$$2a_0x(t) + a_1t + b_0 = 0.$$

Their solution $x(t)$ exists only if $a_0 \neq 0$, and then

$$x(t) = vt + x_0, \qquad \varphi_2(x,t) = -\frac{(x - x(t))^2}{2(1 - v^2)}, \quad (8)$$

where $v = -a_1/(2a_0)$, $x_0 = -b_0/(2a_0)$, and $v^2 < 1$ in order to satisfy the condition $\varphi_2 < 0$.

Thus we have found that the boundary of our oscillon has to move with the constant velocity v, and close to the boundary the field has the parabolic shape (as expected, see [2]). Note that φ given by formula (8) coincides with the Lorentz boosted and translated in space static solution

$$\varphi_s = \begin{cases} 0 & x \leq 0, \\ -\frac{x^2}{2} & x > 0. \end{cases}$$

The structure of the solution φ_- is shown in Fig. 1, in which the support of φ_- for the swaying oscillon of unit length and vanishing total momentum is depicted. The period of this oscillon is equal to its spatial size, i.e., to 1. As discussed in [4], we may use the symmetries of the signum-Gordon equation, such as Poincaré or scaling transformations, in order to obtain more general oscillons. The interior of the parallelogram is divided into seven sectors $a \div f$ by the four characteristic lines drawn from its corners. Each sector has a different causal neighborhood. For instance, the field in the triangular sector c is determined by Cauchy data on the segment $[1/2 + v/2, 1]$ of the x axis; the sector e is controlled by Cauchy data in the future, i.e., on the segment $[1/2, 1 + v/2]$ of the $t = 1/2$ line which lies in the future of the sector e; the sectors a and d are controlled by the boundaries of the oscillon, etc. The parallelogram has height equal to one-half of its length. In this case the characteristic lines drawn from the lower (upper) corners meet at a point lying on the upper (lower) edge.

In the case of the nonswaying oscillon presented in [4] we have $v = 0$ and a rectangle in Fig. 1. The parallelogram is the simplest deformation of that rectangle consistent with the conditions $\varphi_-(x,0) = 0 = \varphi_-(x,\frac{1}{2})$, and with the fact that both the ends of the oscillon have to move with a constant velocity. Such a generalization—the parallelogram instead of the rectangle—is very suggestive in the "patchwork approach" adopted in the present paper, but it is not obvious at all in that based on the d'Alembert formula approach used in [4]. Let us also note that a Lorentz boost of the oscillon considered in [4] gives a uniform rectilinear motion, and not the swaying one.

The building blocks of φ_- are denoted as $\varphi_a, \ldots, \varphi_g$ after the sectors of the parallelogram. The fields φ_a, φ_d in the sectors a and d have the form (8) with $x(t) = vt$ and $x(t) = vt + 1$, respectively, i.e.,

$$\varphi_a = -\frac{(x - vt)^2}{2(1 - v^2)}, \qquad \varphi_d = -\frac{(x - vt - 1)^2}{2(1 - v^2)}. \quad (9)$$

In the regions $x \leq vt$ and $x \geq vt + 1$, i.e., on both sides of the parallelogram, the field has the vacuum value $\varphi = 0$.

The functions φ_b, φ_c, φ_e, φ_f are determined by imposing on the solution (4) the condition $\varphi_2 = 0$ on the lines $t = 0$ and $t = 1/2$, and the conditions of matching with φ_a or φ_d on the characteristic lines. As an example let us determine φ_b. The condition $\varphi_2(x,0) = 0$ gives $a_0 = b_0 = c_0 = 0$. Next, $\varphi_2 = t^2/2 + a_1tx + b_1t$ is compared to φ_a on the part of the characteristic line $x = t$ with $t \in (0, 1/4 + v/4)$:

$$\frac{t^2}{2} + a_1t^2 + b_1t = -\frac{1 - v}{2(1 + v)}t^2.$$

Therefore, $b_1 = 0$, $a_1 = -1/(1 + v)$, and $\varphi_b = t^2/2 - tx/(1 + v)$. Similar calculations give φ_c, φ_e, φ_f. Finally, we compute φ_g by comparing φ_2 to φ_b, φ_c, φ_e, φ_f along the four characteristic lines that form the boundary of the sector g. We obtain

$$\varphi_b = \frac{t^2}{2} - \frac{tx}{1 + v}, \qquad \varphi_c = \frac{t^2}{2} + \frac{t(x - 1)}{1 - v}, \quad (10)$$

$$\varphi_e = \frac{1}{2}\left(t - \frac{1}{2}\right)\left(\frac{1}{2} + t + \frac{1 - 2x}{1 + v}\right), \quad (11)$$

$$\varphi_f = \frac{1}{2}\left(t - \frac{1}{2}\right)\left(\frac{1}{2} + t + \frac{2x - 1}{1 - v}\right), \quad (12)$$

$$\varphi_g = \frac{(vx+t)^2}{2(1-v^2)} + \frac{x^2+t^2}{2} + \frac{1+v-4(x+t)}{8(1-v)}. \quad (13)$$

All these functions are negative inside their domains.

The evolution of our oscillon is described by the function $\varphi_-(x, t)$ in the time interval $[0, 1/2]$, and by $\varphi_+(x, t)$, formula (7), for $t \in [-1/2, 0]$. In particular, the field φ_+ at the boundaries of the oscillon has the form

$$\varphi_{+,a}(x, t) = \frac{(x+vt)^2}{2(1-v^2)}, \qquad \varphi_{+,d}(x, t) = \frac{(x+vt-1)^2}{2(1-v^2)}.$$

We see that now the boundaries of the oscillon move with the velocity $-v$. The world sheet of the swaying oscillon is depicted in Fig. 2. At the times $t = k/2$, k integer, when the sharp turns take place, the field φ vanishes everywhere.

In the case $x(t)$ is a characteristic line [see the derivation of formula (8)], we have $x(t) = vt + x_0$, where $|v| = 1$. There is just one matching condition $\varphi_2(x(t), t) = 0$. Solving it we obtain certain relations between the constant coefficients present in φ_2. The next steps are similar to those described above, but now the situation is much simpler. When $v = 1$, the left- and right-hand sides of the parallelogram in Fig. 1 coincide with characteristic lines. Therefore, the sectors a, f, c, d, g are absent. The remaining sectors b, e meet at the line $x = 1 - t$. The corresponding functions φ_b, φ_e are given by formulas (10) and (11) with $v = 1$, and they correctly match each other on that line.

The total energy E and momentum P of the oscillon are calculated from formulas

$$E = \frac{1}{2}\int_{-\infty}^{\infty} dx[(\partial_t\varphi)^2 + (\partial_x\varphi)^2] + \int_{-\infty}^{\infty} dx|\varphi|,$$

$$P = -\int_{-\infty}^{\infty} dx \partial_t\varphi\partial_x\varphi,$$

considered at the time $t = 0$ when $\varphi = 0 = \partial_x\varphi$. We see that $P = 0$, in spite of the swaying motion of the oscillon. This can be understood if we regard the swaying oscillon as a nonlinear bound state of the basic oscillon, that is, the one with $v = 0$, with a wave packet traveling inside the basic oscillon. Because the swaying oscillon has $P = 0$, the nonzero momentum of the wave packet is compensated by the momentum of the basic oscillon. Thus the basic oscillon has to move accordingly when the wave packet bounces in its interior.

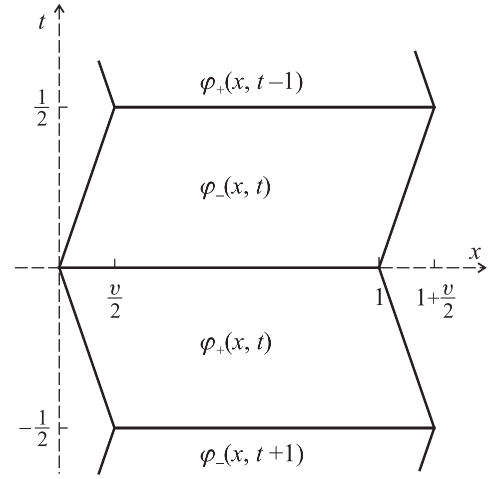

FIG. 2. The world sheet of the swaying oscillon. In the interior of the parallelograms $\varphi_+ > 0$ and $\varphi_- < 0$, whereas on their boundaries (the thick continuous lines) $\varphi_\pm = 0$.

In order to compute the total energy we need $\partial_t\varphi|_{t=0}$. Formulas (9) and (10) give $\partial_t\varphi_b|_{t=0} = -x/(1+v)$ for $x \in [0, (1+v)/2]$, and $\partial_t\varphi_c|_{t=0} = (x-1)/(1-v)$ for $x \in [(1+v)/2, 1]$. In the case $v = \pm 1$ the part with φ_c is absent. Simple integration gives $E = 1/24$. Thus all our swaying oscillons have the same energy. We have not found any explanation for such a degeneracy.

III. CONCLUSION

We have shown that oscillons in the $(1+1)$-dimensional signum-Gordon model can periodically move to and fro in space (the x line) with a constant speed v from the interval $[0, 1]$. The amplitude of such a swaying motion is equal to $vl/2$, where l is the length of the oscillon. The pertinent exact analytic solutions of the field equation have been constructed from the second order polynomials in t and x.

Our findings contribute to the already substantial evidence that the models of the signum-Gordon type have rather amazing properties. In particular, it is quite surprising that one can find simple, explicit solutions that describe very nontrivial objects like the oscillons, or Q balls [8], and this happens in spite of the unpleasant form of the nonlinear term in the field equation.

[1] C. Armendariz-Picon, T. Damour, and V. F. Mukhanov, Phys. Lett. B 458, 209 (1999); C. Adam, P. Klimas, J. Sanchez-Guillen, and A. Wereszczyński, J. Math. Phys. (N.Y.) 50, 102 303 (2009).
[2] H. Arodź, Acta Phys. Pol. B 33, 1241 (2002); 35, 625 (2004).
[3] H. Arodź, P. Klimas, and T. Tyranowski, Acta Phys. Pol. B 36, 3861 (2005); Phys. Rev. E 73, 046609 (2006).
[4] H. Arodź, P. Klimas, and T. Tyranowski, Phys. Rev. D 77, 047701 (2008).

[5] See, e.g., I. L. Bogolyubskii and V. G. Makhankov, Sov. Phys. JETP Lett. **24**, 12 (1976); M. Gleiser, Phys. Rev. D **49**, 2978 (1994); Phys. Lett. B **600**, 126 (2004); M. Hindmarsh and P. Salmi, Phys. Rev. D **74**, 105005 (2006); G. Fodor, P. Forgacs, P. Grandclément, and I. Rácz, Phys. Rev. D **74**, 124003 (2006); M. Gleiser and J. Thorarinson, Phys. Rev. D **76**, 041701(R) (2007); M. Gleiser and D. Sicilia, Phys. Rev. Lett. **101**, 011602 (2008).

[6] See, e.g., I. V. Barashenkov and O. F. Oxtoby, Phys. Rev. E **80**, 026608 (2009); O. F. Oxtoby and I. V. Barashenkov, Theor. Math. Phys. **159**, 863 (2009).

[7] See, e.g., R. D. Richtmyer, *Principles of Advanced Mathematical Physics* (Springer-Verlag, New York, 1978), Sec. 17.3; L. C. Evans, *Partial Differential Equations* (American Math. Society, Providence, RI, 1998).

[8] H. Arodź and J. Lis, Phys. Rev. D **77**, 107702 (2008).