

CP^n supersymmetric mechanics in $U(n)$ background gauge fieldsStefano Bellucci,¹ Sergey Krivonos,² and Anton Sutulin²¹*INFN-Laboratori Nazionali di Frascati,**Via E. Fermi 40, 00044 Frascati, Italy*²*Bogoliubov Laboratory of Theoretical Physics,**JINR, 141980 Dubna, Russia*

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We construct a new $\mathcal{N} = 4$ supersymmetric mechanics describing the motion of a particle over a CP^n manifold in $U(n)$ background gauge fields.

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I. INTRODUCTION

The study of quantum Hall effect in higher (larger than two) dimensions has been of some interest in the last several years, following the analysis by Zhang and Hu [1]. They considered the Landau problem for charged fermions on S^4 with a background magnetic field which is $SU(2)$ instanton. A number of papers have extended the original idea in many aspects (see e.g. [2] and references therein). One of such extensions concerns the analysis of quantum Hall effect on complex projective spaces CP^n [3,4]. The corresponding bulk and edge actions were derived [3]. In addition, in [4] it has been shown that the bulk contribution coincides with the Chern-Simons action.

The geometry of the CP^n space is quite simple—this is just the coset $SU(n+1)/U(n)$. For any G/H coset one has at hands the analogue of a constant background field—the H -valued connection on G/H . Thus, the case of CP^n allows for both Abelian and non-Abelian background fields [5]. Moreover, the system describing the motion of particles over the CP^n manifold could be easily extended to possess $\mathcal{N} = 4$ supersymmetry [6–8] in the absence of background fields. Thus, it seems to be a proper task to include the coupling with the background gauge fields in the $\mathcal{N} = 4$ supersymmetric system on CP^n . That is just what we are doing in the present paper. We explicitly construct the $\mathcal{N} = 4$ supercharges and Hamiltonian which describe the motion of a particle over the CP^n manifold in the presence of background $U(n)$ fields. The corresponding gauge potential is proportional to the $U(n)$ -connection on $SU(n+1)/U(n)$. Surprisingly, this form of the gauge potential is dictated by $\mathcal{N} = 4$ supersymmetry. It turns out that $\mathcal{N} = 4$ supersymmetry demands the presence of additional pure potential terms in the Hamiltonian. In the simplest case of the CP^1 system this potential is just a harmonic oscillator one.

The paper is organized as follows. In Sec. II we review the $\mathcal{N} = 4$ supersymmetric mechanics on CP^n . The supercharges and the Hamiltonian are derived in Sec. III. We conclude our work with a short discussion.

II. CP^n MECHANICS WITH $\mathcal{N} = 4$ SUPERSYMMETRY

The construction of $\mathcal{N} = 4$ supersymmetric mechanics on CP^n manifold¹ is almost trivial. Indeed, if we take n complex $\mathcal{N} = 4$ chiral superfields $Z^\alpha(\theta_i, \bar{\theta}^i)$, $\bar{Z}_\alpha(\theta_i, \bar{\theta}^i)$ defined in $d = 1$, $N = 4$ superspace $\mathbb{R}^{(1,4)} = \{t, \theta_i, \bar{\theta}^i\}$, $i = 1, 2$ and obeying conditions

$$D^i Z^\alpha = 0, \quad \bar{D}_i \bar{Z}_\alpha = 0, \quad \alpha = 1 \dots n, \quad (2.1)$$

where $\mathcal{N} = 4$ covariant derivatives are defined as

$$\{D^i, \bar{D}_j\} = 2i\delta^i_j \partial_t, \quad (2.2)$$

then the superfields action S

$$S = \int dt d^4\theta \log[1 + Z^\alpha \bar{Z}_\alpha] \quad (2.3)$$

does all the job, completely defining the model. The explicit form of Lagrangian density in (2.3) immediately follows from invariance of the action with respect to the $SU(n+1)$ group, which is realized on the superfields Z, \bar{Z} as

$$\delta Z^\alpha = a^\alpha + Z^\alpha (Z^\beta \bar{a}_\beta), \quad \delta \bar{Z}_\alpha = \bar{a}_\alpha + \bar{Z}_\alpha (a^\beta \bar{Z}_\beta), \quad (2.4)$$

where a^α, \bar{a}_α are the parameters of the coset $SU(n+1)/U(n)$ transformations.

If we instead will not fix the integrand in the action S (2.3) leaving it to be an arbitrary function $\mathcal{L}(Z, \bar{Z})$, then the resulting system will describe a supersymmetric mechanics on an arbitrary n -dimensional Kähler manifold (see e.g. [6–8]). In the case of one superfield Z, \bar{Z} such a system has been firstly constructed in [9]. Recently, supersymmetric mechanics on complex manifolds has been considered in [10].

To fix our notations and for completeness, let us shortly discuss the Hamiltonian description of the $\mathcal{N} = 4$ supersymmetric CP^n mechanics which directly follows from

¹Let us stress that CP^n is a geometry of target space, while all fields depend on time t only.

(2.3) after passing to the components and removing the auxiliary fields.

So, our basic ingredients are bosonic variables $\{z^\alpha, \bar{z}_\alpha\}$ which parameterize the coset $SU(n+1)/U(n)$ and fermionic variables $\{\psi_i^\alpha, \bar{\psi}_i^\alpha\}$:

$$\begin{aligned} z^\alpha &= Z^\alpha|, & \bar{z}_\alpha &= \bar{Z}_\alpha|, \\ \psi_i^\alpha &= \bar{D}_i Z^\alpha|, & \bar{\psi}_i^\alpha &= D^i \bar{Z}_\alpha|, \end{aligned} \quad (2.5)$$

where $(\dots)|$ denotes $\theta_i = \bar{\theta}^i = 0$ limit. In what follows we will pay a great attention to $U(n)$ properties of our model. That is why we decided to keep the corresponding indices α, β of our fields (2.5) in a proper position. For the $SU(n+1)$ group we will fix the commutation relations to be

$$\begin{aligned} i[R_\alpha, \bar{R}^\beta] &= J_\alpha^\beta, \\ i[J_\alpha^\beta, J_\gamma^\sigma] &= \delta_\gamma^\beta J_\alpha^\sigma - \delta_\alpha^\sigma J_\gamma^\beta, \\ i[J_\alpha^\beta, R_\gamma] &= \delta_\gamma^\beta R_\alpha + \delta_\alpha^\beta R_\gamma, \\ i[J_\alpha^\beta, \bar{R}^\gamma] &= -\delta_\alpha^\gamma \bar{R}^\beta - \delta_\alpha^\beta \bar{R}^\gamma. \end{aligned} \quad (2.6)$$

Thus, the generators R_α, \bar{R}^α belong to the coset $SU(n+1)/U(n)$, while the J_α^β form $U(n)$. In addition we choose these generators to be antihermitian ones

$$(R_\alpha)^\dagger = -\bar{R}^\alpha, \quad (J_\alpha^\beta)^\dagger = -J_\beta^\alpha. \quad (2.7)$$

After introducing the momenta for all our variables and passing to Dirac brackets we will obtain the following set of relations²

$$\begin{aligned} \{\psi_i^\alpha, \bar{\psi}_j^\beta\} &= i\delta_j^i (g^{-1})_\beta^\alpha, \\ \{p_\alpha, \bar{p}^\beta\} &= -i(g_\alpha^\beta g_\mu^\nu + g_\alpha^\nu g_\mu^\beta) \bar{\psi}_\nu^i \psi_i^\mu, \\ \{p_\alpha, \psi_i^\beta\} &= -\frac{1}{(1+z \cdot \bar{z})} [\bar{z}_\alpha \psi_i^\beta + \delta_\alpha^\beta \psi_i^\gamma \bar{z}_\gamma], \\ \{\bar{p}^\alpha, \bar{\psi}_i^\beta\} &= -\frac{1}{(1+z \cdot \bar{z})} [z^\alpha \bar{\psi}_i^\beta + \delta_\beta^\alpha z^\gamma \bar{\psi}_i^\gamma]. \end{aligned} \quad (2.8)$$

Here, the CP^n metric g_α^β has the standard Fubini-Study form

$$g_\alpha^\beta = \frac{1}{(1+z \cdot \bar{z})} \left[\delta_\alpha^\beta - \frac{\bar{z}_\alpha z^\beta}{(1+z \cdot \bar{z})} \right], \quad z \cdot \bar{z} \equiv z^\alpha \bar{z}_\alpha. \quad (2.9)$$

Now, it is not too hard to check that the supercharges Q^i, \bar{Q}_i have the extremely simple form [6–8]

$$Q^i = \bar{p}^\alpha \bar{\psi}_i^\alpha, \quad \bar{Q}_i = \psi_i^\alpha p_\alpha. \quad (2.10)$$

They are perfectly anticommuting (in virtue of (2.8)) as

$$\{Q^i, \bar{Q}_j\} = i\delta_j^i H, \quad \{Q^i, Q^j\} = \{\bar{Q}_i, \bar{Q}_j\} = 0, \quad (2.11)$$

²As usual, the bosonic momenta are shifted by $\psi \cdot \bar{\psi}$ terms in this basis.

where the Hamiltonian H reads³

$$\begin{aligned} H &= \bar{p}^\alpha (g^{-1})_\alpha^\beta p_\beta + \frac{1}{4} (g_\mu^\alpha g_\rho^\sigma \\ &+ g_\mu^\sigma g_\rho^\alpha) \bar{\psi}_{\alpha i} \bar{\psi}_\sigma^i \psi^{\rho j} \psi_j^\mu. \end{aligned} \quad (2.12)$$

In principle, one may modify the supercharges and Hamiltonian by including potential terms [7,8], but here we will be interested in including the interaction with non-Abelian background fields which looks in itself rather complicated. Therefore we will ignore such possible modifications in what follows.

Finally, we will need the explicit expressions for the vielbeins e_α^β and $U(n)$ -connections ω_α^β on the CP^n manifold, which we choose as [11]

$$e_\alpha^\beta = \frac{1}{\sqrt{1+z \cdot \bar{z}}} \left[\delta_\alpha^\beta - \frac{\bar{z}_\alpha z^\beta}{\sqrt{1+z \cdot \bar{z}}(1+\sqrt{1+z \cdot \bar{z}})} \right], \quad (2.13)$$

$$\begin{aligned} \omega_\alpha^\beta &= \frac{1}{\sqrt{1+z \cdot \bar{z}}(1+\sqrt{1+z \cdot \bar{z}})} \\ &\times \left[\delta_\alpha^\beta - \frac{\bar{z}_\alpha z^\beta}{2\sqrt{1+z \cdot \bar{z}}(1+\sqrt{1+z \cdot \bar{z}})} \right]. \end{aligned} \quad (2.14)$$

With our definition of the $SU(n+1)$ algebra (2.6), these quantities enter the standard Cartan forms as

$$\begin{aligned} g^{-1} dg &= dz^\alpha e_\alpha^\beta R_\beta + \bar{R}^\alpha e_\alpha^\beta d\bar{z}_\beta \\ &+ iJ_\alpha^\beta (z^\alpha \omega_\beta^\gamma d\bar{z}_\gamma - dz^\gamma \omega_\gamma^\alpha \bar{z}_\beta), \end{aligned} \quad (2.15)$$

where

$$g = e^{x^\alpha R_\alpha + \bar{x}_\alpha \bar{R}^\alpha}, \quad \text{and} \quad z^\alpha \equiv \frac{\tan \sqrt{x \cdot \bar{x}}}{\sqrt{x \cdot \bar{x}}} x^\alpha. \quad (2.16)$$

III. GAUGE FIELDS: CONSTRUCTION

It is curious, but the simplest form of the supercharges (2.10) does not help in the coupling with background gauge fields. One may easily check that the standard coupling by shifting bosonic momenta in supercharges does not properly work. Our idea is to introduce the coupling simultaneously with all currents of the $SU(n+1)$ and/or $SU(1, n)$ groups. Thus, let us introduce the isospin currents spanning $SU(n+1)$ and/or $SU(1, n)$, respectively

$$\begin{aligned} \{R_\alpha, \bar{R}^\beta\} &= -AJ_\alpha^\beta, \\ \{J_\alpha^\beta, J_\gamma^\sigma\} &= \delta_\gamma^\beta J_\alpha^\sigma - \delta_\alpha^\sigma J_\gamma^\beta, \\ \{J_\alpha^\beta, R_\gamma\} &= \delta_\gamma^\beta R_\alpha + \delta_\alpha^\beta R_\gamma, \\ \{J_\alpha^\beta, \bar{R}^\gamma\} &= -\delta_\alpha^\gamma \bar{R}^\beta - \delta_\alpha^\beta \bar{R}^\gamma. \end{aligned} \quad (3.1)$$

³The $su(2)$ indices are raised and lowered as $A_i = \varepsilon_{ij} A^j$, $A^i = \varepsilon^{ij} A_j$ with $\varepsilon_{12} = \varepsilon^{21} = 1$.

The coefficient $A = \pm 1$ in the first line corresponds to the choice of $SU(1, n)$ or $SU(n + 1)$, respectively. It will be clear below, why we are going to consider both these cases.

Now, we are ready to write the Ansatz for the supercharges⁴

$$\begin{aligned} Q &= \bar{p}^\alpha \bar{\psi}_\alpha^i - z^\gamma J_\gamma^\beta h_\beta^\alpha \bar{\psi}_\alpha^i + \psi^{i\alpha} f_\alpha^\beta R_\beta, \\ \bar{Q}_i &= \psi_i^\alpha p_\alpha + \psi_i^\alpha h_\alpha^\beta J_\beta^\gamma \bar{z}_\gamma + \bar{R}^\beta f_\beta^\alpha \bar{\psi}_{i\alpha}. \end{aligned} \quad (3.2)$$

Here, h_α^β and f_α^β are arbitrary, for the time being, functions depending on the bosonic fields z^α, \bar{z}_α only. Moreover, due to the explicit $U(n)$ symmetry of our

$$\begin{aligned} \{Q, Q\} = 0 &\Rightarrow \begin{cases} f_1' = -(f_1 h_1 + x f_1 h_2), \\ h_1' = -(h_1^2 - h_2 + x h_1 h_2), \end{cases} & f_2' = -(2f_2 h_1 + f_1 h_2 + 2x f_2 h_2), \\ & & f_2 = -f_1 h_1 \\ \{Q^i, \bar{Q}_j\} = i\delta_j^i H &\Rightarrow h_2' = \frac{1}{2}(A f_1^2 h_1^2 + h_1^3), & h_2 = -\frac{1}{2} h_1^2, \quad A f_1^2 = (2h_1 - x h_1^2), \end{aligned} \quad (3.4)$$

where the derivatives are taken with respect to x .

The simplest, almost trivial solution of the Eqs. (3.4) reads

$$f_1 = f_2 = 0, \quad h_1 = \frac{1}{z \cdot \bar{z}}, \quad h_2 = -\frac{2}{(z \cdot \bar{z})^2}. \quad (3.5)$$

The functions h_1, h_2 in (3.5) have a singularity at $(z, \bar{z}) \rightarrow 0$. Moreover, they have no any geometric meaning within CP^n geometry. Thus, without R, \bar{R} terms in the Ansatz (3.2) the reasonable interaction can not be constructed.

In contrast, with nonzero f_1, f_2 functions the solution of (3.4) is fixed to be

$$\begin{aligned} f_1 &= \frac{1}{\sqrt{1 + Az \cdot \bar{z}}}, \\ f_2 &= -\frac{A}{(1 + Az \cdot \bar{z})(1 + \sqrt{1 + Az \cdot \bar{z}})}, \\ h_1 &= \frac{A}{\sqrt{1 + Az \cdot \bar{z}}(1 + \sqrt{1 + Az \cdot \bar{z}})}, \\ h_2 &= -\frac{1}{2(1 + Az \cdot \bar{z})(1 + \sqrt{1 + Az \cdot \bar{z}})^2}. \end{aligned} \quad (3.6)$$

Thus, we see that the matrix-valued function f_α^β perfectly coincides with the vielbeins for the CP^n manifold (2.13) if we choose $A = 1$. The background gauge field h_α^β is the part of the $U(n)$ -connection (2.14) for CP^n . It is worth to note that this field is identical to the one constructed in [13] as the solution of the Bogomol'nyi

construction, which we are going to keep unbroken, one may further restrict these functions as

$$h_\alpha^\beta = h_1 \delta_\alpha^\beta + h_2 \bar{z}_\alpha z^\beta, \quad f_\alpha^\beta = f_1 \delta_\alpha^\beta + f_2 \bar{z}_\alpha z^\beta, \quad (3.3)$$

where the scalar functions h_1, h_2, f_1, f_2 depend now on $x = z \cdot \bar{z}$ only.

The supercharges (3.2) have to obey the standard $\mathcal{N} = 4$ Poincaré superalgebra relations (2.11). Therefore, the closure of superalgebra is achieved if the following equations on functions in (3.3) are satisfied

equation for the Tchrakian's type of self-duality relations in $U(n)$ gauge theory [14,15].

The last step is to write the Hamiltonian

$$\begin{aligned} H &= (\bar{p}g^{-1}p) + (\bar{p}g^{-1}hJ\bar{z}) - (zJhg^{-1}p) \\ &\quad - (\bar{R}fg^{-1}fR) - (zJhg^{-1}hJ\bar{z}) \\ &\quad + i\left(\frac{1-A}{(1+z\cdot\bar{z})(1+Az\cdot\bar{z})}\right)((z\bar{\psi})^i(\bar{R}f\bar{\psi})_i \\ &\quad - (\psi fR)^i(\psi\bar{z})_i) - iA(\psi_i f J f \bar{\psi}^i) \\ &\quad + \frac{1}{4}(g_\mu^\alpha g_\rho^\sigma + g_\mu^\sigma g_\rho^\alpha)\bar{\psi}_{\alpha i}\bar{\psi}_\sigma^i\psi^{\rho j}\psi_j^\mu. \end{aligned} \quad (3.7)$$

Here, we used concise notations—all indices in parenthesis are in the proper positions and they are converted from top-left to down-right, e.g. $(\psi\bar{z})_i = \psi_i^\alpha \bar{z}_\alpha$, etc.

This Hamiltonian commutes with all our supercharges, as it should be. Its bosonic part (the first line in (3.7)) contains the terms describing the interaction with $U(n)$ background fields and a specific potential term. The parameter A takes two values $A = \pm 1$, according with the algebra (3.1). If we take $A = 1$, so the algebra of currents of the internal group is $SU(1, n)$, then the Hamiltonian drastically simplified to be

$$\begin{aligned} H_{A=1} &= (\bar{p}g^{-1}p) + (\bar{p}g^{-1}hJ\bar{z}) - (zJhg^{-1}p) \\ &\quad - (\bar{R}R) - (zJhg^{-1}hJ\bar{z}) - i(\psi_i f J f \bar{\psi}^i) \\ &\quad + \frac{1}{4}(g_\mu^\alpha g_\rho^\sigma + g_\mu^\sigma g_\rho^\alpha)\bar{\psi}_{\alpha i}\bar{\psi}_\sigma^i\psi^{\rho j}\psi_j^\mu. \end{aligned} \quad (3.8)$$

Clearly, the R, \bar{R} dependent term in the Hamiltonian (3.8) can be rewritten through the Casimir operator \mathcal{K} of $SU(1, n)$ algebra

⁴This Ansatz is a direct generalization of those supercharges for the $SU(2)$ case, which were explicitly constructed within the superspace approach in [12].

$$\mathcal{K} = \bar{R}^\alpha R_\alpha - \frac{1}{2} J_\alpha^\beta J_\beta^\alpha + \frac{1}{2(n+1)} J_\alpha^\alpha J_\beta^\beta. \quad (3.9)$$

Thus, the Hamiltonian depends only on $U(n)$ currents J_α^β and $SU(1, n)$ Casimir operator (3.9).

The $U(1)$ gauge potential presented in (3.2) and (3.8), has the standard form

$$\mathcal{A}_{U(1)} = i \frac{\dot{z} \bar{z} - z \dot{\bar{z}}}{2(1 + z \cdot \bar{z})}. \quad (3.10)$$

In the simplest case of CP^1 we have only this gauge potential in the theory, while the scalar potential term acquires the form⁵

$$\mathcal{V}_{CP^1} = -\bar{R}^\alpha R_\alpha - \frac{z \cdot \bar{z}}{4} J^2. \quad (3.11)$$

Let us remind that we choose the matrix-valued operators \bar{R} , R , J to be antihermitian (2.7). Thus, the potential (3.11) is positively defined.

Finally, we would like to say a few words about the explicit realization of the isospin groups $SU(n+1)$ and/or $SU(1, n)$ (3.1). The common way to involve the isospin variables in the supersymmetric theories is to introduce the set of semidynamical bosonic variables—harmonics and construct the currents from them (see e.g. [19] and references therein). The same strategy could be applied in the present model too.

IV. CONCLUSION AND DISCUSSION

In the present paper we have constructed a $\mathcal{N} = 4$ supersymmetric extension of mechanics describing the motion of a particle over CP^n manifold in the presence of background $U(n)$ fields. The gauge potential is proportional to the $U(n)$ -connection on $SU(n+1)/U(n)$. Such a type of background gauge field has been known for quite a long time in a purely bosonic case [20]. What is really nice is that this field appears in our system automatically, as a result of imposing $\mathcal{N} = 4$ supersymmetry. Moreover, in addition to gauge fields $\mathcal{N} = 4$ supersymmetry demands additional potential terms to be present in the Hamiltonian.

⁵This is just the example of superoscillator potential on CP^n manifolds constructed in [16,17]. See also [18].

In the simplest case of the CP^1 system, this potential is just a harmonic oscillator one.

One of the most unexpected features of the present model is a strange interplay between the isospin group which our background gauge fields are coupled to and the form of these fields. It turns out that the standard $SU(n+1)/U(n)$ $U(n)$ -connection appears as a gauge fields potential only in the case when isospin group is chosen to be $SU(1, n)$. Alternatively, the choice of the $SU(n+1)$ group for the isospin variables gives rise to a $U(n)$ -connection on the $SU(1, n)/U(n)$ group. At any rate, both cases are compatible with $\mathcal{N} = 4$ supersymmetry.

Another interesting peculiarity of our model is the presence of the isospin variables on the whole $SU(n+1)$ (or $SU(1, n)$) group, despite the fact that only $U(n)$ background fields appear in the Hamiltonian. Again, this situation is not new. The same effect has been noted in the recently constructed $\mathcal{N} = 4$ supersymmetric mechanics coupled to non-Abelian gauge fields [21–26].

One of the possible immediate applications of the constructed model is the analysis of the role the additional fermionic variables play in the quantum Hall effect on CP^n [3,4,18]. In this respect it could be important that $\mathcal{N} = 4$ supersymmetry insists on the simultaneous appearance of the gauge fields on $U(1)$ and $SU(n)$ with a proper fixed relative coefficient. The role of the special type of scalar potential which appears due to $\mathcal{N} = 4$ supersymmetry also has to be clarified.

Another interesting possibility to describe $\mathcal{N} = 4$ supersymmetric CP^n mechanics is to replace from the beginning the linear chiral supermultiplets by the nonlinear ones [27]. This case is under investigation at present.

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